Three-Dimensional Complex Homogeneous Complex Contact Manifolds

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Abstract

In this paper, we classify all three-dimensional complex contact manifolds which have global complex contact forms and which are complex homogeneous with transitive groups of holomorphic contactomorphisms. We compare these results to previous results by W. Boothby and J. Wolf.

1 Introduction

A *complex contact manifold* is a complex manifold M of complex dimension $2n + 1$ with an open atlas $\mathcal{U} = \{\mathcal{O}\}$ such that

1. On each $\mathcal{O} \in \mathcal{U}$, there is a holomorphic 1-form η with $\eta \wedge (d\eta)^n \neq 0$ everywhere on $\mathcal{O}$.

2. On $\mathcal{O} \cap \mathcal{O}'$, there exists a holomorphic function $f : \mathcal{O} \cap \mathcal{O}' \to \mathbf{C}^*$ such that $\eta = f\eta'$.

In particular, $\mathcal{H}^{\infty,'} = \cup_{\mathcal{O} \in \mathcal{U}}(\| \rceil \nabla \ \eta)$ is a well-defined holomorphic subbundle of $T^{1,0}M$ with maximal rank. We say that M is *complex homogeneous,* if there is a complex Lie group G acting transitively as a space of biholomorphic contactomorphisms (i.e. preserving $\mathcal{H}$) on M. For a general reference on these types of manifolds, see [3] and [13].

In [4] and [5], W. Boothby classified all compact, simply-connected complex homogeneous complex contact manifolds. By later work of J. Wolf [20], S. Salamon [17], and B. Bérard-Bergery [1], it was found that this list consisted of the twistor spaces of homogeneous quaternionic-Kähler manifolds (See [2]). However, this list excluded a sizable portion of complex contact manifolds. In particular, those with global complex contact forms are left out, since their universal covers are not compact.

In this paper we will be classifying these manifolds. We will make heavy use of the fact that all homogeneous complex 2-manifolds and 3-manifolds have been

classified in [16] and [19], respectively. In the first section, we discuss an important class of complex contact manifolds, the so-called complex Boothby-Wang fibrations. In the third section, we present the main results. Finally, we end the paper with some remarks.

2   Complex Boothby-Wang fibrations

In this section, we describe a newly discovered class of complex contact manifolds, called the *complex Boothby-Wang fibrations*. Recall that in [6], W. Boothby and H.C. Wang proved the following theorems.

Theorem 2.1 (Boothby-Wang Fibration, part i) *Let B be a symplectic manifold with fundamental 2-form Ω such that $[\Omega] \in H^2(M, \mathbf{Z})$. Then the principal S^1-bundle M corresponding to $[\Omega]$ has a connection form η such that $d\eta$ is the pull-back of Ω and η is a global real contact form on M.*

Theorem 2.2 (Boothby-Wang Fibration, part ii) *Let M be a compact, regular contact manifold with contact form η'. Then M is the total space of an principal S^1-bundle $\pi : M \to B$, where B is a symplectic manifold with symplectic form Ω. Also, there is a nowhere-zero function τ on M such that $\eta = \tau\eta'$ is a global contact form on M and a connection form for the fibration such that $d\eta = (\pi)^*(\Omega)$. Lastly, the free S^1-action on M is defined by the characteristic vector field ξ of η, given by the equations $\eta(\xi) = 1$ and $\iota(\xi)d\eta = 0$.*

The resulting real contact manifolds are called *real Boothby-Wang fibrations.*

In [9], the author proved complex analogues of these results.

Theorem 2.3 (Complex Boothby-Wang Fibration, part i) *Let B be a complex-symplectic manifold with a complex symplectic form $\Omega = \Omega_1 + i\Omega_2$ such that both Ω_1 and Ω_2 are integral classes. Then M the $S^1 \times S^1$ -bundle defined by $([\Omega_1], [\Omega_2]) \in H^2(B, \mathbf{Z}) \oplus H^2(B, \mathbf{Z})$ (or $[\Omega] \in H^2(B, \mathbf{Z} + i\mathbf{Z})$) has an integrable complex structure and also a complex contact structure given by a holomorphic connection form whose curvature form is given by ω.*

Theorem 2.4 (Complex Boothby-Wang Fibration, part ii) *Let M be a $(2n+1)$-dimensional compact complex contact manifold with a global holomorphic contact form π such that the resulting vertical vertical vector fields U and JU are regular in M. Then π generates a free $S^1 \times S^1$-action on M, and M is an principal $S^1 \times S^1$-bundle over a complex symplectic manifold B such that π is a connection form of this fibration and the symplectic form Ω on B is given by $\pi^*\Omega = d\pi$.*

In our context, we are curious as to which of the above manifolds (called *complex Boothby-Wang Fibrations*) are complex homogeneous. In particular, we need to know which complex symplectic surfaces are complex homogeneous.

In [10], H. Geiges classified all complex symplectic surfaces.

Theorem 2.5 (Geiges) *Let M be a closed 4-manifold. Then M admits a complex symplectic structure if and only if M is diffeomorphic to one of the following manifolds:*

1. A complex torus,
2. A primary Kodaira surface,
3. A K3 Surface.

Furthermore, in [16], K. Oeljeklaus and W. Richthofer give a list of all complex homogeneous surfaces:

Theorem 2.6 *Let M be a complex homogeneous surface.*

1. **(Tits [18])** *If M is compact, then it is diffeomorhphic to one of the following:*

   (a) $\mathbf{C}P^2$

   (b) $\mathbf{C}P^1 \times \mathbf{C}P^1$

   (c) A complex torus

   (d) A homogeneous Hopf surface

   (e) The product of an elliptic curve with $\mathbf{C}P^1$

2. **(Huckleberry, Livorni [11])** *If M is non-compact, then it is one of the following:*

   (a) A product of complex homogeneous Riemann surfaces

   (b) A topologically trivial $\mathbf{C}^*$-bundle over an elliptic curve

   (c) An elliptic curve bundle over $\mathbf{C}$

   (d) A certain nontrivial $\mathbf{C}^*$-bundle over $\mathbf{C}^*$

   (e) A $\mathbf{C}^*$-bundle over $\mathbf{C}P^1$

   (f) A positive line bundle over $\mathbf{C}P^1$

   (g) The affine quadric

   (h) The complement of the quadric curve in $\mathbf{C}P^2$ with the affine quadric as its universal cover

The intersection of these two lists is simply the set of all complex torii. In [9], the author shows that only a certain class of complex torii have complex Boothby-Wang fibrations and that the universal cover of all such fibrations is the complex Heisenberg group

$$H_{\mathbf{C}} = \left\{ \begin{pmatrix} 1 & z_2 & z_1 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbf{C} \right\}.$$

The complex contact structure of this manifold is given by the left-invariant form $\omega = dz_1 - z_2 dz_3$. For further details on this group, see [7].

3   Main theorem

Given a complex contact manifold M with a global complex contact form ω, we define $U \in T^{1,0}M$ to be the unique holomorphic vector field given by the equations:

$$\omega(U) = 1, \ \iota(U)d\omega = 0.$$

We then let $\mathcal{V} = \mathrm{span}_{\mathbf{C}}(\mathcal{U})$. Then $\mathcal{V}$ is a holomorphic subbundle of $T^{1,0}M$, which is transerve to $\mathcal{H}$, i.e. $T^{1,0}M = \mathcal{V} \oplus \mathcal{H}$. In general, the author shows in [8] that such transerve subbundles can be constructed on any complex contact manifold, although the resulting "vertical bundle" $\mathcal{V}$ in these cases will not necessarily split $T^{1,0} = \mathcal{V} \oplus \mathcal{H}$ holomorphically.

Theorem 3.1 *If M is a three-dimensional complex-homogeneous complex contact manifold with global complex contact form, then M is of the form $M = G/\Gamma$ where G is a simply-connected three-dimensional complex Lie group and $\Gamma \subset G$ is a discrete subgroup.*

1. *Suppose G is unimodular. Then*

    (a) *$G = Sl(2, \mathbf{C})$, if $rk(ad(\mathcal{V})) = \in$,*

    (b) *$G = \tilde{E}_{\mathbf{C}}(2)$, the universal cover of $E_{\mathbf{C}}(2)$ the rigid motions of the complex euclidean plane, if $rk(ad(\mathcal{V})) = \infty$,*

    (c) *$G = H_{\mathbf{C}}$, if $rk(ad(\mathcal{V})) = \iota$.*

2. *Suppose G is not unimodular. Then G is necessarily solvable; $rk(ad(\mathcal{V})) = \infty$; and G is one of the following complex Lie groups:*

    (a) *The semidirect product $G_\alpha = \mathbf{C} \times_{\tau_\alpha} \mathbf{C}^2$, for any $\alpha \in \mathbf{C}^* \setminus 1$, where τ_α is a certain representation of $\mathbf{C}$ in $Gl(2, \mathbf{C})$.*

    (b) *$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbf{C} \right\}$*

Proof. We prove this theorem in two lemmas. The first lemma will classify all left-invariant complex contact structures on complex Lie groups; the last will show that there are no other possible three-dimensional complex-homogeneous complex contact manifolds with a global complex contact form.

Lemma 3.2 *Suppose G is a three-dimensional, simply-connected complex Lie group with a left-invariant complex contact structure $\mathcal{H}$. Then G is one of the complex Lie groups as described above.*

Proof. Suppose G is a 3-dimensional complex Lie group with a left-invariant complex contact structure. Let $\mathbf{g}$ be the Lie algebra of G. Then there is a 2-dimensional complex subbundle $\mathcal{H} = \langle]_\in,]_\ni \rangle \subset \mathbf{g}$ such that $[e_2, e_3] = 2f_1 \notin \mathcal{H}$. Define the left-invariant 1-form $\eta \in \mathbf{g}^*$ by $\eta = \frac{1}{2}f_1^{\;*}$ with respect to the dual basis $\{f_1, e_2, e_3\}$ of $\mathbf{g}^*$. Then there is a unique left-invariant vector field $U \in \mathbf{g}$ such that $\eta(U) = \frac{1}{2}$ and $\iota(U)d\eta = 0$. And, thus, we have $\eta = \frac{1}{2}U^*$ with respect to the dual basis $\{U^*, e_2^*, e_3^*\}$.

Set $e_1 = U$. Then, for any $Z \in \mathbf{g}$,

$$0 = (\iota(e_1)d\eta)(Z) = -\frac{1}{2}\eta([e_1, Z]).$$

Hence,

$$\begin{array}{rcrcrcr} [e_2, e_3] & = & 2e_1 & - & \alpha_1 e_2 & - & \beta_1 e_3 \\ [e_3, e_1] & = & & & \alpha_2 e_2 & + & \beta_2 e_3 \\ [e_1, e_2] & = & & & \alpha_3 e_2 & + & \beta_3 e_3. \end{array}$$

An application of Jacobi's identity gives that

$$\alpha_3 = \beta_2.$$

and also that

$$\begin{cases} 0 & = & -\alpha_1\alpha_3 & + & \beta_1\alpha_2 \\ 0 & = & -\alpha_1\beta_3 & + & \beta_1\beta_2. \end{cases}$$

These last two equations can be put in the form

$$\begin{pmatrix} -\beta_2 & \alpha_2 \\ -\beta_3 & \beta_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have

$$ad(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha_2 & -\beta_2 \\ 0 & \beta_2 & \beta_3 \end{pmatrix}.$$

We now split into the various possible cases of $ad(e_1)$ according to its rank.
Case 1: $rank(ad(e_1)) = 2$.

In this case,

$$det \begin{pmatrix} -\alpha_2 & -\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} \neq 0,$$

so that the equation

$$\begin{pmatrix} -\alpha_2 & -\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ -\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a unique solution, namely $\alpha_1 = \beta_1 = 0$. So, we have

$$\begin{array}{rcll} [e_2, e_3] & = & 2e_1 \\ [e_3, e_1] & = & & \alpha_2 e_2 & + & \beta_2 e_3 \\ [e_1, e_2] & = & & \beta_2 e_2 & + & \beta_3 e_3. \end{array}$$

Since $det(ad(e_1)) \neq 0$, we know that $ad(e_1)$ has at least one eigenvector X with corresponding nonzero eigenvalue λ. Furthermore, $ad(e_1)\mathbf{g} \subset \langle e_2, e_3 \rangle$, so that $X \in \mathcal{H}$. Also,

$$\begin{array}{rcl} ad(e_1) & = & \dfrac{1}{2}ad([e_2, e_3]) \\[2mm] & = & \dfrac{1}{2}ad(e_2) \circ ad(e_3) - \dfrac{1}{2}ad(e_3) \circ ad(e_2). \end{array}$$

In particular, $tr(ad(e_1)) = \frac{1}{2}tr(ad(e_2) \circ ad(e_3) - ad(e_3) \circ ad(e_2)) = 0$. So, $ad(e_1)$ has another eigenvector Y in $\mathcal{H}$ with eigenvalue $-\lambda$. By rescaling and renaming if necessary, we then have:

$$\begin{array}{rcll} [e_2, e_3] & = & 2e_1 \\ [e_3, e_1] & = & & \lambda e_3 & . \\ [e_1, e_2] & = & -\lambda e_2 \end{array}$$

Thus, $\mathbf{g}$ is isomorphic to

$$\mathbf{sl}(2,\mathbf{C}) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\lambda}{2} \\ 0 & 0 \end{pmatrix} \right\rangle.$$

Thus, $G = Sl(2,\mathbf{C})$ and hence is unimodular.

Case 2: $rank(ad(e_2)) = 1$

In this case,

$$det \begin{pmatrix} -\alpha_2 & -\beta_2 \\ \beta_2 & \beta_3 \end{pmatrix} = 0,$$

yet one of the rows is non-zero. Let us assume that $\begin{pmatrix} \beta_2 \\ \beta_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then

$$\begin{pmatrix} -\alpha_2 \\ -\beta_2 \end{pmatrix} = k \begin{pmatrix} \beta_2 \\ \beta_3 \end{pmatrix},$$

so that $-\alpha_2 = k\beta_2 = k^2\beta_3$. Also,

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -k\beta_3 & -k^2\beta_3 \\ \beta_3 & k\beta_3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \\ &= \beta_3 \begin{pmatrix} -k(\alpha_1 + k\beta_1) \\ \alpha_1 + k\beta_1 \end{pmatrix}. \end{aligned}$$

Thus, $0 = \alpha_1 + k\beta_1$.

Hence, $\mathbf{g}$ has the following Lie algebra structure:

$$\begin{aligned} [e_2, e_3] &= 2e_1 & - & k\beta_1 e_2 & + & \beta_1 e_3 \\ [e_3, e_1] &= & & k^2\beta_3 e_2 & + & -k\beta_3 e_3 \\ [e_1, e_2] &= & & -k\beta_3 e_2 & + & \beta_3 e_3. \end{aligned}$$

with $\beta_3 \neq 0$. Thus, $[\mathbf{g}, \mathbf{g}] = \langle e_1, -ke_2 + e_3 \rangle$. Clearly, if we choose e_2 and e_3 well enough, we can assume $k = 0$. This will give us this Lie algebra structure:

$$\begin{aligned} [e_2, e_3] &= 2e_1 & - & \beta_1 e_3 \\ [e_3, e_1] &= 0 \\ [e_1, e_2] &= & & \beta_3 e_3. \end{aligned}$$

Set $f_1 = 2e_1 + \beta_1 e_3$, $f_3 = \beta_3 e_3$. So, $[\mathbf{g}, \mathbf{g}] = \langle f_1, f_3 \rangle$. It is then easily checked that

$$\begin{aligned} [e_2, f_1] &= \beta_1 f_1 & - & 2f_3 \\ [e_2, f_3] &= \beta_3 f_1 & & \\ [f_1, f_3] &= 0 \end{aligned},$$

so that, with respect to the basis $\{f_1, f_3\}$, $ad(e_2)$ on $[\mathbf{g}, \mathbf{g}]$ is given by the matrix

$$\tilde{T} = \begin{pmatrix} \beta_1 & \beta_3 \\ -2 & 0 \end{pmatrix}.$$

The characteristic polynomial $p(X)$ of $\tilde{T}$ is given by $p(X) = X^2 - \beta_1 X + 2\beta_3$. Let γ_1 and γ_2 be the roots of p. We then have two possibilities:

1. $\gamma_1 = \gamma_2$.

2. $\gamma_1 \neq \gamma_2$.

Subcase 1: Suppose $\gamma_1 = \gamma_2 = \gamma$.

Then $\beta_1 = -2\gamma$ and $\gamma^2 = 2\beta_3$. Thus,

$$\tilde{T} = \begin{pmatrix} -2\gamma & \frac{1}{2}\gamma^2 \\ -2 & 0 \end{pmatrix}.$$

Set $f_2 = \dfrac{1}{\gamma} e_2$. Then with respect to $\{f_1, f_3\}$, $ad(f_2)$ on $[\mathbf{g}, \mathbf{g}]$ is given by the matrix

$$T = \begin{pmatrix} -2 & \frac{1}{2}\gamma \\ -\frac{2}{\gamma} & 0 \end{pmatrix}.$$

The characteristic polynomial of *this* matrix is $q(X) = (X-1)^2$. Furthermore, the eigenspace of $ad(f_2)$ corresponding to 1 is given by:

$$E_1 = \left\langle -\left(\frac{\gamma}{2}\right) f_1 + f_3 \right\rangle.$$

Finally, it is easily checked that

$$ad(f_2)\left(-\left(\frac{\gamma}{2}\right) f_1\right) = -\left(\frac{\gamma}{2}\right) f_1 + \left(-\left(\frac{\gamma}{2}\right) f_1 + f_3\right).$$

In particular, if we set

$$\begin{aligned} v_1 &= -\left(\frac{\gamma}{2}\right) f_1 + f_3 \\ v_2 &= f_2 \\ v_3 &= -\left(\frac{\gamma}{2}\right) f_1, \end{aligned}$$

then

$$\begin{aligned} [v_1, v_2] &= v_1, \\ [v_2, v_3] &= v_1 + v_3, \\ [v_1, v_3] &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} v_1 &= -\gamma e_1 + \frac{3}{2}\gamma^2 e_3, \\ v_2 &= \frac{1}{\gamma} e_2, \\ v_3 &= \frac{1}{2}\gamma^2 e_3. \end{aligned}$$

Thus, $\mathcal{H} = \langle \sqsubseteq_\in, \sqsubseteq_\ni \rangle$.

Now, it easily checked that the Lie algebra given by

$$\mathbf{g} = \left\{ \begin{pmatrix} x & x & y+z \\ 0 & x & z \\ 0 & 0 & 0 \end{pmatrix} : x, y, z \in \mathbf{C} \right\}$$

satisfies the above criterion with

$$v_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, having no center, the Lie group

$$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbf{C} \right\}$$

is the unique *connected* complex Lie algebra having $\mathbf{g}$ as its Lie algebra. Note that, using $\{v_1, v_2, v_3\}$ as above, we have that $tr(ad(v_1)) = 1$, meaning that G is not unimodular (by theorem in [15])

Case 2 Suppose $\gamma_1 \neq \gamma_2$.

The characteristic polynomial $p(X)$ of $ad(e_2)$ is then given by $p(X) = (X-\gamma_1)(X-\gamma_2)$, i.e. $\beta_1 = \gamma_1 + \gamma_2$ and $2\beta = \gamma_1\gamma_2$.

Set $f_2 = \dfrac{1}{\gamma_1}e_2$. Then, with respect to the basis $\{f_1, f_3\}$, $ad(f_2)$ on $[\mathbf{g}, \mathbf{g}]$ is given by the matrix

$$T = \begin{pmatrix} \dfrac{\gamma_1 + \gamma_2}{\gamma_1} & \dfrac{\gamma_1}{2} \\ -\dfrac{2}{\gamma_1} & 0 \end{pmatrix}.$$

The characteristic polynomial of this matrix is given by $p(X) = (X - 1)(X - \alpha)$, where $\alpha = \dfrac{\gamma_1}{\gamma_2}$. Furthermore, the eigenspaces of $ad(f_2)$ are given by

$$E_1 = \left\langle -\left(\frac{\gamma_1}{2}\right)f_1 + f_3 \right\rangle, \quad E_\alpha = \left\langle -\left(\frac{\gamma_2}{2}\right)f_1 + f_3 \right\rangle.$$

Set

$$v_1 = f_1 - \left(\frac{2}{\gamma_1}\right)f_3 \in E_1$$

$$v_2 = -f_1 + \left(\frac{2}{\gamma_2}\right)f_3 \in E_\alpha$$

$$v_3 = f_2.$$

Then

$$
\begin{aligned}
[v_3, v_1] &= v_1 \\
[v_3, v_2] &= \alpha v_2 \\
[v_1, v_2] &= 0.
\end{aligned}
$$

Finally, $\mathcal{H} = \langle \sqsubseteq_\infty + \sqsubseteq_\in, \sqsubseteq_\ni \rangle$.

For any $\alpha \in \mathbf{C}$, not equal to 1, define the action $\tau_\alpha : \mathbf{C} \to Gl(2, \mathbf{C})$ by

$$
t \mapsto \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-\alpha t} \end{pmatrix}.
$$

Let $G_\alpha = \mathbf{C} \times_{\tau_\alpha} \mathbf{C}^2$ be the semi-direct product, whose multplication is given by

$$
(t_1, w_1) \cdot (t_2, w_2) = (t_1 + t_2, w_1 + \tau_\alpha(t_1) w_2) \text{ for } t_1, t_2 \in \mathbf{C}, \ w_1, w_2 \in \mathbf{C}^2.
$$

Set

$$
\begin{aligned}
v_1 &= \frac{d}{dt}\left(t, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)_{|t=0}, \\
v_2 &= \frac{d}{dt}\left(0, \begin{pmatrix} t \\ 0 \end{pmatrix}\right)_{|t=0}, \\
v_3 &= \frac{d}{dt}\left(0, \begin{pmatrix} 0 \\ t \end{pmatrix}\right)_{|t=0}.
\end{aligned}
$$

Then it is easily checked that the Lie algebra $\mathbf{g}_\alpha$ of G_α is given by $\mathbf{g}_\alpha = \langle v_1, v_2, v_3 \rangle$ that v_1, v_2, and v_3 satisfy the Lie brackets given in the previous paragraph. Lastly note that

$$
\begin{aligned}
tr(ad(v_1)) &= 0 \\
tr(ad(v_2)) &= 0 \\
tr(ad(v_3)) &= \alpha.
\end{aligned}
$$

So, $\mathbf{g}_\alpha$ is unimodular if and only if $\alpha = -1$, in which case G_α is the universal cover of $E(2, \mathbf{C})$, the space of rigid motions of the complex euclidean plane.

Case 3: $rank(ad(e_1)) = 0$

In this case, $\alpha_2 = \beta_2 = \beta_3 = 0$, so that $\mathbf{g}$ is the Lie algebra given by:

$$
\begin{aligned}
[e_2, e_3] &= 2e_1 \\
[e_3, e_1] &= 0 \\
[e_1, e_2] &= 0.
\end{aligned}
$$

That is, G is the complex Heisenberg group.

Lemma 3.3 *Suppose M is a three-dimensional complex-homogeneous complex contact manifold with global complex contact form η. Then $M = G/\Gamma$ for some complex Lie group G with discrete subgroup $\Gamma \subset G$ such that G preserves η.*

Proof. Let G be a complex Lie group of complex-contactomorphisms of M whose action on M is transitive. Let $G = S \times_f R$ be the Levi decomposition of G where S is a semi-simple Lie group acting on a solvable group R via a representation $f : S \to Aut(R)$. Then $M = G/L$ for some closed subgroup $L \subset G$.

Case 1. Suppose G is semi-simple, i.e. $R = (0)$.

By the proof of the theorem A on page 147 in [4], we know that being semi-simple, G preserves the global contact form η and hence also $d\eta$. Setting $p : G \to M = G/L$ to be the obvious submersion, we have the standard statements for this situation:

1. If ω is a G-invariant form on G/L, then $\omega^* = p^*(\omega)$ is an element of g^* such that:

   (a) $\omega^*(\mathbf{l}) = 0$,

   (b) $ad(x)\omega^* = \omega^*$ for all $x \in \mathbf{l}$.

2. Conversely from the above statements, any form in g^* satisfying conditions a) and b) is the image under p^* of a G-invariant form on G/L.

3. If ω is a G-invariant form on G/L with $\omega^* = p^*\omega$, then $(d\omega)^k \neq 0$ if and only if $(d\omega^*)^k \neq 0$.

Let $(,)$ denote the Killing-Cartan form of g. Then there exists a unique $Z \in \mathbf{g}$ such that $\omega^*(X) = (Z, X)$. Set $c(Z) = \{X \in \mathbf{g} : [X, Z] = 0\}$. For any $X, Y \in \mathbf{g}$,

$$
\begin{aligned}
d\omega^*(X, Y) &= -\frac{1}{2}\omega^*([X, Y]) \\
&= -\frac{1}{2}(Z, [X, Y]) \\
&= -\frac{1}{2}([Z, X], Y).
\end{aligned}
$$

So, $\iota(X)d\omega^* = 0$ if and only if $[Z, X] = 0$ (since $(,)$ is nondegenerate). Hence, $c(Z) = \{X \in \mathbf{g} : \iota(X)d\omega^* = 0\}$. Furthermore, $\mathbf{l} \subset c(Z)$.

Set $N = dim_{\mathbf{C}}G$ so that $dim_{\mathbf{C}}L = N - 3$. We know that $d\omega^* \neq 0$ and that $(d\omega^*)^2 = 0$ by statement b) above. So, with respect to some dual basis $\{e_1^*, \ldots, e_N^*\}$ of $\mathbf{g}^*$, $d\omega^* = e_1^* \wedge e_2^*$, which means that $c(Z) = \langle e_3, \ldots, e_N \rangle_{\mathbf{C}}$. i.e. $dim_{\mathbf{C}}c(Z) = dim_{\mathbf{C}}G - 2$. Therefore, $c(Z) = \mathbf{l} \oplus \langle Z \rangle_{\mathbf{C}}$.

With respect to the nilpotent subalgebra $\langle Z \rangle_{\mathbf{C}} \subset \mathbf{g}$, $\mathbf{g}$ has the root decomposition

$$
\mathbf{g} = \underbrace{\langle Z \rangle_{\mathbf{C}} \oplus \mathbf{l}}_{=\mathbf{g}_0} \oplus \mathbf{g}_\lambda \oplus \mathbf{g}_{-\lambda},
$$

for nonzero root $\lambda \in (\langle Z \rangle_{\mathbf{C}})^*$, where $\mathbf{g}_\lambda$ and $\mathbf{g}_{-\lambda}$ are the one-dimensional subspaces of $\mathbf{g}$ satisfying:

1. $[\mathbf{g}_0, \mathbf{g}_\lambda] \subset \mathbf{g}_\lambda$,

2. $[\mathbf{g}_0, \mathbf{g}_{-\lambda}] \subset \mathbf{g}_{-\lambda}$,

3. $[Z, \mathbf{l}] = 0$,

4. $[\mathbf{g}_\lambda, \mathbf{g}_{-\lambda}] \cap \langle Z \rangle \neq (0)$.

Since both of the nonzero root spaces are one-dimensional, we can set $\mathbf{g}_\lambda = \langle X_1 \rangle$ and $\mathbf{g}_{-\lambda} = \langle X_2 \rangle$. Then, for some $a \neq 0$ and $W \in l$,

$$[X_1, X_2] = aZ + W$$

$$[X_1, Z] = \lambda(Z)X_1$$

$$[X_2, Z] = -\lambda(Z)X_2.$$

Since $[\mathbf{g}_0, \mathbf{g}_{\pm\lambda}] \subset \mathbf{g}_{\pm\lambda}$, we also know that there are some numbers μ_1 and μ_2 such that

$$[X_1, W] = \mu_1 X_1,$$

$$[X_2, W] = \mu_2 X_2.$$

However, an easy application of the Jacobi identity tells us that $\mu_2 = -\mu_1$. Thus, by adjusting and renaming the coefficients properly, we have

$$[X_1, Z] = \lambda X_1, \ [X_2, Z] = -\lambda X_2,$$

$$[X_1, X_2] = Z + W, \ [Z, W] = 0,$$

$$[X_1, W] = \mu X_1, \ [X_2, W] = -\mu X_2.$$

In particular, $\langle X_1, X_2, Z, W \rangle$ is a Lie subalgebra of $\mathbf{g}$, and we can thus assume that G is 4-dimensional with $\mathbf{g}$ as described above.

We also note that $\mathbf{h} = \langle Z, W \rangle$ is a nilpotent subalgebra of $\mathbf{g}$, which induces a root space decomposition of $\mathbf{g}$ given by:

$$\mathbf{g} = \mathbf{g}_0 \oplus \mathbf{g}_\nu \oplus \mathbf{g}_{-\nu},$$

where $\nu \in \mathbf{h}^*$ is given by the equations:

$$\nu(Z) = \lambda, \ \nu(W) = \mu,$$

and $\mathbf{g}_0 = \mathbf{h}$, $\mathbf{g}_\nu = \mathbf{g}_\lambda$, $\mathbf{g}_{-\nu} = \mathbf{g}_{-\lambda}$. Thus, $\mathbf{h}$ is a Cartan subalgebra of $\mathbf{g}$. However, it is known that for a Cartan subalgebra $\mathbf{h}$, the set of nonzero roots Δ of $\mathbf{h}$ spans $\mathbf{h}^*$ (For reference, see [11]). In our case, the set of nonzero roots consists of $\{\pm\nu\}$, so that $\mathbf{h}^*$ is one-dimensional. Thus, $L \subset G$ is a discrete subgroup, and the universal cover G of M is a three-dimensional semi-simple Lie group.

Case 2. Suppose G is solvable, i.e. $S = (0)$.

To prove the lemma for this case, we need to make use of the following proposition due to Winkelmann [18].

Proposition 3.4. Let G be a solvable, connected Lie group acting transitively on a complex manifold X with $dim_{\mathbf{C}} X \leq 3$. Then there exists a solvable, connected complex Lie group K acting transitively on X with $dim_{\mathbf{C}} K = dim_{\mathbf{C}} X$.

Thus, in our situation, we know that $M = K/\Gamma$ for some solvable complex Lie group K and some discrete subgroup $\Gamma \subset K$. The problem is that it is not clear from the proof of this proposition that the original complex contact structure is preserved by this group K. In fact, in general, it won't be. In order to remedy this, we will construct a possibly new complex contact structure which is preserved by K.

We assume that $M = K$. At $e \in M$, we choose linearly independent vectors $X, Y \in \mathcal{H}_1^{\infty,\prime}$. We then extend these to form linearly independent holomorphic vector fields in $\mathcal{H}^{\infty,\prime}$ near e. Also, let X' and Y' be the K-left-invariant holomorphic vector fields on K given by $X'_e = X_e$, $Y'_e = Y_e$. So, we have a K-left-invariant subbundle $\mathcal{H}' = \langle \mathcal{X}', \mathcal{Y}' \rangle$ in $T^{1,0}M$.

Both $\mathcal{H}^{\infty,\prime}$ and $\mathcal{H}'$ are 2-dimensional subspaces of a 3-dimensional vector space, so that $\mathcal{H}^{\infty,\prime} \cap \mathcal{H}' \neq (\prime)$ at each point on M. Thus, we can assume that locally $\mathcal{H}^{\infty,\prime} \cap \mathcal{H}' = \langle \mathcal{X} \rangle$, i.e. $X = X'$.

Since $\mathcal{H}^{\infty,\prime}$ is a complex contact structure, we know that $[X, Y] = W \notin \mathcal{H}^{\infty,\prime}$. So, $\{X, Y, W\}$ is a local holomorphic basis of $T^{1,0}M$. Also, since we're assuming $X = X'$, we can also assume that $Y' = Y + \epsilon W$ for some local holomorphic function ϵ with $\epsilon(e) = 0$. Furthermore,

$$[X, Y'] - [X, Y] = [X, \epsilon W] = (X\epsilon)W + \epsilon[X, W].$$

In particular, $[X, Y']_e - [X, Y]_e = (X\epsilon)(e)W_e$. So, $[X, Y']_e \in \langle W \rangle$. Since $\mathcal{H}'_1 = \mathcal{H}^{\infty,\prime}$, we see that $[X, Y'] \notin \mathcal{H}'$. But $[X, Y']$ and $\mathcal{H}'$ are both K-left-invariant structures on M. So, $[X, Y'] \notin \mathcal{H}'$ at all points on M. Thus, $\mathcal{H}'$ is a K-left-invariant complex contact structure on $M = K$.

Case 3. Suppose G is mixed, i.e. $S \neq (0) \neq R$.

By Winkelmann's list in [19], we know that all three-dimensional complex homogeneous manifolds in this category are $\mathbf{C}, \mathbf{C}^*$, or $\mathbf{C}^2$ bundles over complex homogeneous manifolds. Let $\pi : M \to N$ be the corresponding bundle map in our case with structure group $K \subset G$. Since this map π preserves the actions of G on M and N, we know that $ker(\pi_*)$ is G-invariant. Set $\mathcal{H}' = \mathcal{H} \cap \|\| \nabla(\pi_*)$. So, $\mathcal{H}'$ is a K-invariant subbundle of M.

We have three possibilities:

1. $\mathcal{H} \pitchfork \|\| \nabla(\pi_*)$,

2. $\mathcal{H} \subset \|\| \nabla(\pi_*)$,

3. $\mathcal{H} \cap \|\| \nabla(\pi_*) \neq \emptyset$, but $\mathcal{H} \not\subset \|\| \nabla(\pi_*)$.

If (1) were true, then M would be a complex Boothby-Wang fibration over a complex-homogeneous complex symplectic manifold, which as we have seen must be a complex torus. In particular, the universal cover of M is the complex Heisenberg group. If (2) were true, then by dimension-counting, $\mathcal{H} = \|\| \nabla(\pi_*)$. This is clearly impossible, since the vertical subbundle of any submersion is integrable.

Finally, we come to case (3). There exists a K-connection $\mathcal{D}$ of the fibration $M \to N$ such that $\mathcal{H}'' = \mathcal{D} \cap \mathcal{H} \neq (\prime)$. Then $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$. Let $X \in \mathcal{H}', \mathcal{Y} \in \mathcal{H}''$ be local K-invariant vector fields. Then $\pi_*(X) = 0$, and $\pi_*(Y)$ is a well-defined vector field on N. Thus, $[\pi_*(X), \pi_*(Y)] = \pi_*[X, Y] = 0$. So, $[X, Y] \in ker(\pi_*)$. This means that $ker(\pi_*)$ is 2-dimensional. By Winkelmann's list, then, we know that M is the $\mathbf{C}^2$-bundle over $\mathbf{C}P^1$ given by the transition functions:

$$
\begin{aligned}
w_1 &= -w_0 \left(\frac{z_0}{z_1}\right)^n \\
v_1 &= v_0 \left(\frac{z_0}{z_1}\right)^{np+n-2} - w_0^{p+1} \left(\frac{z_0}{z_1}\right)^{np+n-1} .
\end{aligned}
$$

Furthermore, although the subbundle $\mathcal{V} = \langle \mathcal{U} \rangle$ may not live in $ker(\pi_*)$, we can set $pr : TM \to ker(\pi_*)$ to be the holomorphic projection with respect to the holomorphic splitting $TM = ker(\pi_*) \oplus \mathcal{D}$. Then we will have that $U' = pr(U)$ is a nowhere-zero holomorphic vector field in $ker(\pi_*)$, which can be taken as a nowhere-zero holomorphic section of M as a $\mathbf{C}^2$-bundle over $\mathbf{C}P^1$. This is a contradiction, since it is known that this bundle has no holomorphic sections. Thus, we have proven our lemma.

4 Final remarks

Given a 3-dimensional complex contact manifold M with local complex contact forms $\{\omega\}$, we can define a $\mathbf{C}^*$-bundle B over M locally by

$$B = \{f\omega : f \in \mathbf{C}^*\} \subset \Lambda^{1,0}M.$$

Let $\pi : B \to M$ be the projection. It is well-known that there exists a globally defined 1-form Ω on B such that

1. $(d\Omega)^2 \neq 0$.

2. $\Omega = 0$ on vectors tangent to the fibres created by the fibration $B \to M$.

3. $(R_z)^*\Omega = z\Omega$ for all $z \in \mathbf{C}^*$, where R_z is the right-handed $\mathbf{C}^*$-action on B.

 Finally, it also well-known that any contactomorphism of M determines a unique biholomorphism of B which (1) commutes with right translations, and (2) preserves Ω. Conversely, any biholomorphism of B satisfying these conditions determines a unique contactomorphism of M. For a given $x \in M$ and $b \in \pi^{-1}(x)$, set $L = \{g \in G : g(x) = x\}$ and $L_1 = \{g \in G : g(b) = b\}$. Then $L_1 \subset L$; and either $L = L_1$ or $L \setminus L_1 = \mathbf{C}^*$. All of these facts are documented in [4]. It is clear that M has a global complex contact form if and only if B is holomorphically trivial.

Using this terminology, our main theorem gives us the following corollary.

Corollary 4.1 *Let M be a three-dimensional complex contact manifold with transitive group of holomorphic contactomorphisms G. If B is trivial, then (1) G preserves each global complex contact form and hence (2) the subgroup of G preserving a fibre of B over M acts trivially on that fibre.*

This corollary allows us to extend one of Boothby's results, using a similar proof as he does in [4].

Theorem 4.2 *Let M be a three-dimensional complex homogeneous complex contact manifold with transitive group of holomorphic contactomorphisms G. Then there are two possible cases:*

1. *The following equivalent statments hold:*

   (a) *G is transitive on B;*

   (b) *B is non-trivial as a bundle over M;*

   (c) *$L \setminus L_1 = \mathbf{C}^*$;*

2. *The following equivalent statments hold:*

    (a) *The subset of G preserving a fibre of B acts as the identity on that fibre;*

    (b) *B is trivial as a bundle over M;*

    (c) *$L = L_1$.*

Boothby's original theorem assumed that M was compact and simply-connected and also that G was semi-simple. Furthermore, in [5], Boothby showed that under these same circumstances, the second category in his theorem is null. It is unknown to the author whether there are any known examples of manifolds in category (1) which are not homogeneous twistor spaces.

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