On the Dual Darboux Rotation Axis of the Timelike Dual Space Curve

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Abstract

In this paper, the Dual Darboux rotation axis for timelike dual space curve in the semi-dual space D_1^3 is decomposed in two simultaneous rotation. The axis of these simultaneous rotations are joined by a simple mechanism. This study is original corresponding in the semi-dual space D_1^3 of the article entitled On the Dual Darboux Rotation Axis of the Dual Space Curve [6].

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Key words: dual Darboux vector, Darboux rotation axis, timelike dual curve, Semi-dual space.

1 Introduction

Theory of space curves of Riemannian manifold M is fully developed and its local and global geometry is well-known. In case M is proper semi-Riemannian, then, there are three categories of curves, namely, spacelike, timelike and null, depending on their causal character.

In the Euclidean 3−dimensional space R^3, lines combined with one of their two directions can be represented by unit dual vectors over the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines R^3 are in one to-one correspondence with the points of dual unit sphere D^3 [13]. Similarly, Oriented timelike and spacelike lines in R_1^3 may be represented by timelike and spacelike unit vectors with three components in the semi-dual space D_1^3, respectively.

If φ ve φ^* are real numbers and $\xi^2 = 0$ the combination

$$\hat{\varphi} = \varphi + \xi\varphi^* \tag{1}$$

is called a dual number. The symbol ξ designates the dual unit with the property $\xi^2 = 0$. In analogy with the complex numbers W. K. Clifford defined the dual numbers and showed that they form an algebra, not a field. The pure dual numbers are ξa^*.

According to the definition pure dual numbers ξa^* are zero divisors. No number ξa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers $(a +$

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$ib, i^2 = -1$). Later, E. Study introduced the dual angle subtended by two nonparallel lines in R^3 and defined it as $\hat{\varphi} = \varphi + \xi\varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines [8].

2 Preliminaries

Let L^2 be the vector space R^2 provided with Lorentzian inner product

$$< x, y >= x_1y_1 - x_2y_2 \quad , \quad x = (x_1, x_2),\ y = (y_1, y_2). \tag{2}$$

We denote by G the proper Lorentz group $SO^+(1,1)$ consisting of all matrices of the form

$$A(u) \ = \begin{bmatrix} chu & shu \\ shu & chu \end{bmatrix} \quad , u \in R \tag{3}$$

where we are writing ch and sh for hyperbolic functions $cosh$ and $sinh$ [3].

In L^2, the norm $\|x\|$ is defined to be $\sqrt{|< x, x >|}$. We also consider the time orientation as follows. Let $e = (0, 1)$. A timelike vector $x = (x_1, x_2)$ is future-pointing (resp., past-pointing) if $< x, e >< 0$ (resp., $< x, e >> 0$). So a vector $x = (x_1, x_2)$ is timelike and future-pointing if and only if $x_1^2 - x_2^2 < 0$ and $x_2 > 0$; in other words, if and only if $|x_1| < x_2$. We note that the group G is in fact the group of all linear transformations of L^2 which preserve inner product, orientation and time-orientation [3].

The set

$$D = \{\hat{x} = x + \xi x^* | x, x^* \in R\} \tag{4}$$

of dual numbers is a commutative ring with respect to the operations

$i)$ $(x + \xi x^*) + (y + \xi y^*) = (x + y) + \xi\,(x^* + y^*)$

$ii)$ $(x + \xi x^*) \cdot (y + \xi y^*) = xy + \xi\,(xy^* + yx^*)$.

The division $\frac{\hat{x}}{\hat{y}}$ is possible and unambiguous if $y \neq 0$ and it easily see that

$$\frac{\hat{x}}{\hat{y}} = \frac{x + \xi x^*}{y + \xi y^*} = \frac{x}{y} + \xi\,\frac{x^*y - xy^*}{y^2}. \tag{5}$$

The set

$$D^3 = D \times D \times D = \{\hat{x} | \hat{x} = (x_1 + \xi x_1^*,\, x_2 + \xi x_2^*,\, x_3 + \xi x_3^*) \tag{6}$$

$$= (x_1,\, x_2,\, x_3) + \xi\,(x_1^*,\, x_2^*,\, x_3^*) \tag{7}$$

$$= x + \xi x^*, \quad x \in R^3,\, x^* \in R^3\}. \tag{8}$$

is a module over the ring D.

Let $\hat{x} = x + \xi x^*$, $\widehat{y} = y + \xi y^*$. The Lorentzian inner product of $\hat{x}$ and $\hat{y}$ is defined by

$$< \hat{x},\, \hat{y} >=< x,\ y > +\xi(< x,\ y^* > + < x^*,\ y >).$$

We call the dual space D^3 together with this Lorentzian inner product as dual Lorentzian space and show by D_1^3. It is clear that any dual vector $\hat{x}$ in D_1^3, consists of any two real vector x and x^* in R_1^3 which are expressed in the natural right handed orthonormal frame in the 3−dimensional semi-Euclidean space R_1^3. We call the elements of D_1^3 the dual vectors [12]. If $x \neq 0$ the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by $\|\hat{x}\| = (|< \hat{x}\,, \hat{x} >|)^{\frac{1}{2}}$.

Let $\hat{x}$ be dual vector. $\hat{x}$ is said to be spacelike, timelike and lightlike (null) if the vector x is spacelike, timelike and lightlike (null), respectively.

For any $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in R_1^3$, the pseudo vector product of x and y is defined by

$$x \times y = (x_2y_3 - x_3y_2,\, x_1y_3 - x_3y_1,\, x_2y_1 - x_1y_2), \tag{9}$$

where $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = -e_2$ [1].

Definition 1. Let $\hat{x}$, $\hat{y} \in D_1^3$. We define the Lorentzian vectoral product of $\hat{x}$ and $\hat{y}$ by

$$\hat{x} \times \hat{y} = (\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2,\, \hat{x}_1\hat{y}_3 - \hat{x}_3\hat{y}_1,\, \hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2),$$

where $\hat{x} = (x_1,\ x_2,\ x_3)$, $\hat{y} = (y_1,\ y_2,\ y_3)$ and $\widehat{e}_1 \times \widehat{e}_2 = -\widehat{e}_3$, $\widehat{e}_2 \times \widehat{e}_3 = \widehat{e}_1$ and $\widehat{e}_3 \times \widehat{e}_1 = -\widehat{e}_2$.

Definition 2. Let $n \geq 2$ and $0 \leq \nu \leq n$. Then

i) The pseudosphere of radius $r > 0$ in R_υ^{n+1} is the hyperquadric

$$S_\nu^n(r) = q^{-1}(r^2) = \{p \in R_\upsilon^{n+1} \big| < p,\ p >= r^2\},$$

with dimension n and indeks ν.

ii) The pseudohyperbolic space of radius $r > 0$ in $R_{\upsilon+1}^{n+1}$ is the hyperquadric

$$H_\nu^n(r) = q^{-1}(-r^2) = \{p \in R_{\upsilon+1}^{n+1} \big| < p,\ p >= -r^2\},$$

with dimension n and indeks ν [11].

Definition 3. Let $\hat{x} = x + \xi x^* \in D_1^3$.

i) The set

$$S_1^2 = \{\, \hat{x} = x + \xi x^* |\ \|\hat{x}\| = (1,\, 0)\,;\ x,\ x^* \in R_1^3 \text{ and the vector } x \text{ is spacelike } \}$$

is called the dual Lorentzian unit sphere in D_1^3.

ii) The set

$$H_0^2 = \{\, \hat{x} = x + \xi x^* |\ \|\hat{x}\| = (1,\, 0);\ x,\ x^* \in R_1^3 \text{ and the vector } x \text{ is timelike } \}$$

is called the dual hyperbolic unit sphere in D_1^3 [14].

3 On the dual Darboux rotation axis of the timelike dual space curve

Let $\{\hat{t}, \hat{n}, \hat{b}\}$ be the Dual Frenet trihedron of the differentiable timelike dual space curve in the semi-dual space D_1^3. Then the Frenet equations are

$$\begin{array}{rcl} \hat{t}' & = & \kappa n + \xi(\kappa^* n + \kappa n^*) \\ \hat{n}' & = & \kappa t + \tau b + \xi(\kappa^* t + \kappa t^* + \tau^* b + \tau b^*) \\ \hat{b}' & = & -\tau n - \xi(\tau^* n + \tau n^*), \end{array} \tag{10}$$

where $\hat{\kappa} = \kappa + \xi\kappa^*$ is nowhere pure dual curvature and $\hat{\tau} = \tau + \xi\tau^*$ is nowhere pure dual torsion.

These equations form a dual rotation motion with dual Darboux vector,

$$\hat{\partial} = \partial + \xi\partial^* = -\tau t - \kappa b - \xi(\tau^* t + \tau t^* + \kappa b^* + \kappa^* b). \tag{11}$$

Also momentum dual rotation vector is expressed as follows:

$$\begin{array}{rcl} \hat{t}' & = & \hat{\partial} \times \hat{t} \\ \hat{n}' & = & \hat{\partial} \times \hat{n} \\ \hat{b}' & = & \hat{\partial} \times \hat{b}. \end{array} \tag{12}$$

Dual Darboux rotation of dual Frenet frame can be seperated into two rotation motions:

$\hat{t}$ tangent vectors rotates with a $\hat{\kappa}$ angular speed round $\hat{b}$ binormal vector, that is

$$\hat{t}' = (-\hat{\kappa}\hat{b}) \times \hat{t} \tag{13}$$

and $\hat{b}'$ binormal vector rotates with a $\hat{\tau}$ angular speed round $\hat{t}$ tangent vector, that is

$$\hat{b}' = (-\hat{\tau}\hat{t}) \times \hat{b}. \tag{14}$$

The seperation of the rotation motion of the momentum dual Darboux axis into two rotation motions can be indicated as such;

$\frac{\partial}{\|\hat{\partial}\|}$ vector rotates with a $\hat{w} = \frac{\hat{\tau}'\hat{\kappa} - \hat{\tau}\hat{\kappa}'}{-\hat{\tau}^2 + \hat{\kappa}^2}$ angular speed round $\hat{n}$ principal normal, also

$$\left(\frac{\hat{\partial}}{\left\|\hat{\partial}\right\|}\right)' = (\hat{w}\hat{n}) \times \frac{\hat{\partial}}{\left\|\hat{\partial}\right\|}, \tag{15}$$

and $\hat{n}$ principal normal vector rotates with a $\left\|\hat{\partial}\right\|$ angular speed round $\frac{\hat{\partial}}{\|\hat{\partial}\|}$ dual Darboux axis, also

$$\hat{n}' = \hat{\partial} \times \hat{n}. \tag{16}$$

From now on we shall do a further study of momentum dual Darboux axis. We obtain the unit vector $\hat{e}$ from dual Darboux vector,

$$\hat{e} = \frac{\hat{\partial}}{\left\|\hat{\partial}\right\|} = \frac{-\tau t - \kappa b - \xi(\tau^* t + \tau t^* + \kappa b^* + \kappa^* b)}{\sqrt{|-\tau^2 + \kappa^2 + 2\,\xi(-\tau\tau^* + \kappa\kappa^*)|}}. \tag{17}$$

From $\hat{\partial}' = -\hat{\tau}'\hat{t} - \hat{\kappa}'\hat{b}$, differentation of $\hat{e}$

$$\hat{e}' = \left(\frac{\hat{\partial}}{\left\|\hat{\partial}\right\|}\right)' = \frac{\left\|\hat{\partial}\right\|\hat{\partial}' - \hat{\partial}\left\|\hat{\partial}\right\|'}{\left\|\hat{\partial}\right\|^2} = \frac{\hat{\tau}'\hat{\kappa} - \hat{\tau}\hat{\kappa}'}{-\hat{\tau}^2 + \hat{\kappa}^2} \cdot \frac{-\hat{\kappa}\hat{t} - \hat{\tau}\hat{b}}{\sqrt{|-\tau^2 + \kappa^2 + 2\,\xi(-\tau\tau^* + \kappa\kappa^*)|}} \tag{18}$$

is found. From this,

$$\hat{e}' = -\varepsilon(\hat{e})\,\hat{w}\,(\hat{e} \times \hat{n}) \tag{19}$$

is written. According to the 2.Frenet formula,

$$\hat{n}' = \varepsilon(\hat{e})\,\left\|\hat{\partial}\right\|\,(\hat{e} \times \hat{n}) \tag{20}$$

and

$$(\hat{e} \times \hat{n})' = -\varepsilon(\hat{e})\,\varepsilon(\hat{e} \times \hat{n})\,\left\|\hat{\partial}\right\|\,\hat{n} + \varepsilon(\hat{e} \times \hat{n})\,\hat{w}\,\hat{e} \tag{21}$$

are obtained. These three equations are in the form of the Frenet formulas that is

$$\begin{array}{rcl} \hat{n}' & = & \varepsilon(\hat{e})\,\left\|\hat{\partial}\right\|\,(\hat{e} \times \hat{n}) \\ (\hat{e} \times \hat{n})' & = & -\varepsilon(\hat{e})\,\varepsilon(\hat{e} \times \hat{n})\,\left\|\hat{\partial}\right\|\,\hat{n} + \varepsilon(\hat{e} \times \hat{n})\,\hat{w}\,\hat{e} \\ \hat{e}' & = & -\varepsilon(\hat{e})\,\hat{w}\,(\hat{e} \times \hat{n}), \end{array} \tag{22}$$

where the first coefficient $\left\|\hat{\partial}\right\|$ is nowhere pure dual and second coefficient

$$\hat{w} = \frac{\hat{\tau}'\hat{\kappa} - \hat{\tau}\hat{\kappa}'}{-\hat{\tau}^2 + \hat{\kappa}^2} = \frac{(\frac{\hat{\tau}}{\hat{\kappa}})'}{1 - (\frac{\hat{\tau}}{\hat{\kappa}})^2} \tag{23}$$

is related only to $\frac{\hat{\tau}}{\hat{\kappa}}$ harmonic curvature. Thus, the vectors $\hat{n}$, $(\hat{e} \times \hat{n})$, $\hat{e}$ define a dual rotation motion together the dual rotation vector,

$$\hat{\partial}_1 = \hat{w}\hat{n} + \left\|\hat{\partial}\right\|\hat{e} = \hat{w}\hat{n} + \hat{\partial}. \tag{24}$$

Also momentum dual rotation vector is expressed as follows:

$$\hat{n}' = \hat{\partial}_1 \times \hat{n} \tag{25}$$

$$(\hat{e} \times \hat{n})' = \hat{\partial}_1 \times (\hat{e} \times \hat{n}) \tag{26}$$

$$(\hat{e})' = \hat{\partial}_1 \times \hat{e}. \tag{27}$$

This dual rotation motion of dual Darboux axis can be seperated into two dual rotation motions again. Here $\hat{\partial}_1$ dual rotation vector is the addition of the dual rotation vectors of the dual rotation motions.

When continued in the similar way, the dual rotation motion of dual Darboux axis is done in a consecutive manner. In this way the series of dual Darboux vectors are obtained.

That is

$$\hat{\partial}_0 = \hat{\partial}, \hat{\partial}_1, \ldots \;. \tag{28}$$

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