Differential Geometry - Dynamical Systems
Monographs # 1

Constantin Udrişte Vladimir Balan

ANALYTIC AND DIFFERENTIAL GEOMETRY

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Differential Geometry - Dynamical Systems
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Preface

This book is intended for an introductory course in analytic and differential geometry. It is aimed to acquaint the students with the various fundamental concepts of these fields, as well as to enable them to attain some degree of proficiency in the manipulation and application of the mathematical tools of geometry. It involves the didactic experience of the authors as members of Department of Mathematics at University Politehnica of Bucharest, enhanced further by the lecturing in English since 1990. The text is designed for a 1-semester geometry course, having as basis the lectures taught and seminarized by the authors at the Department of Engineering Sciences. The topics and the style of the book are similar to those in the books of the first author [46], [48], [49], [51], [52].

Throughout the volume, we placed a great emphasis on intuitive understanding via numerous geometric illustrations. We stress the development of basic techniques and computational skills and deliberately de-emphasize highly complex proofs. At the end of each chapter there are exercises of varying degree of difficulty to test students' comprehension on the subject matter presented and to make the students proficient in the basic computational techniques of geometry.

As a pre-requisite for a course based on this book, the students must be familiar with the topics covered in high-school algebra and geometry. Specifically, the students must know the basic facts and techniques of synthetic geometry, matrix algebra, plane analytic geometry and algebraic structures.

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June, 2000 The authors

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Part I

ANALYTIC GEOMETRY

Chapter 1

Free vectors

Lesson 1

1.1 Free vectors

Let E_3 be the 3-dimensional space of elementary geometry. Fixing two distinct points $A \neq B \in E_3$ and the oriented segment $\overrightarrow{AB}$, we shall call A *the origin* and B *the extremity* of the oriented segment. If $A = B$, then $\overrightarrow{AA}$ shall be *the null oriented segment.*

Fig. 1

If $A \neq B$, then the straight line AB will be called *the supporting line* of the oriented segment (see fig. 1).

1.1 Definition. Two oriented segments are called *equipolent* if they have the same direction, same sense, same length.

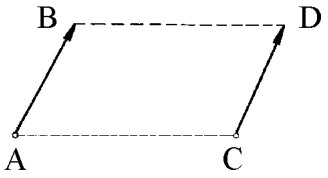


Fig. 2

If $\overrightarrow{AB}$ is equipolent to $\overrightarrow{CD}$ (see fig. 2), then we write

$$\overrightarrow{AB} \sim \overrightarrow{CD}.$$

1.2 Theorem. *The equipolence relation is an equivalence relation.*

Proof. Straightforward verification of the three properties of an equivalence relation: the equipolence is *reflexive, symmetric* and *transitive* (homework).

The equipolence relation splits the set of all oriented segments into equivalence classes.

1.3 Definition. The equivalence classes with respect to the equipolence relation $\sim$ are called *free vectors*. The direction, sense and length which are common for all oriented segments defining a free vector, are called *the direction, sense and length of the free vector*. An oriented segment defining a free vector is called a *representative* of the free vector.

Notations. We shall denote the free vectors as

$$\bar{a}, \bar{b}, \ldots; \overline{AB}, \overline{CD}, \ldots,$$

where, e.g., $\overrightarrow{AB}$ is a representative of $\overline{AB}$; their lengths (norms) will be denoted as a, AB or $||\bar{a}||, ||\overline{AB}||$. A free vector whose length is 1 will be called *unit vector* or *unit vector* and will be denoted by $\bar{e}$.

The free vector whose length is 0 is *the null vector*, denoted by $\bar{0}$; its direction and sense are undetermined. Two free vectors which have the same direction are called *collinear vectors*; two collinear vectors with the same length but with opposite sense are called *opposite vectors*, and are denoted, e.g., as $\bar{a}$ and $-\bar{a}$ (see fig. 3); $\bar{0}$ is collinear to any vector. Two vectors are considered to be *equal* if their representatives are equipolent (same direction, sense and length). Three vectors are called *coplanar* if their representatives with the common origin determine a plane.

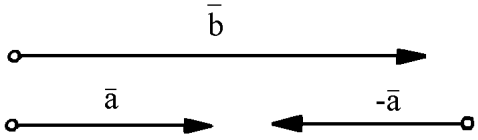


Fig. 3

Let V be the set of all free vectors in E_3. We fix the origin $O \in E_3$. To any point $M \in E_3$ corresponds a vector and only one $\bar{r} = \overline{OM} \in V$ whose representative is $\overrightarrow{OM}$; the converse is also true. Hence it follows the bijection

$$f : E_3 \to V, \quad f(M) = \bar{r}.$$

1.2 Addition of free vectors

Let V be the set of all free vectors; V can be organized as a commutative additive group if defining the addition by the *triangle law (parallelogram law).*

2.1 Definition (triangle law). Let $\bar{a}, \bar{b} \in V$; let $\overrightarrow{OA}$ be a representative of $\bar{a}$ and $\overrightarrow{AB}$ be a representative of $\bar{b}$. The free vector $\bar{c}$ whose representative is $\overrightarrow{OB}$, is called *the sum of the free vectors* $\bar{a}$ and $\bar{b}$. We write $\bar{c} = \bar{a} + \bar{b}$ (see fig. 4).

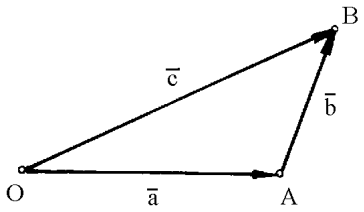


Fig. 4

Obviously $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar vectors, and $\bar{c} = \bar{a} + \bar{b}$ does not depend on the choice of the origin O.

2.2 Theorem. *The addition of free vectors has the following properties:*
1) associativity, $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$;
2) $\bar{0}$ *is the neutral element,* $\bar{a} + \bar{0} = \bar{0} + \bar{a} = \bar{a}$;
3) the opposite $(-\bar{a})$ *of* $\bar{a}$ *is the symmetric of* $\bar{a}$, $\bar{a} + (-\bar{a}) = (-\bar{a}) + \bar{a} = \bar{0}$;
4) commutativity, $\bar{a} + \bar{b} = \bar{b} + \bar{a}$.

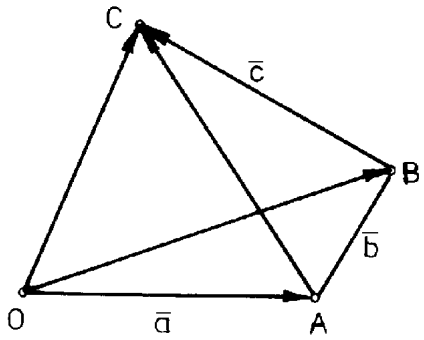


Fig. 5

Proof. 1) We consider the general case and we refer to the figure (see fig. 5): $\overrightarrow{OB}$ is the representative of the sum $\bar{a}+\bar{b}$, and $\overrightarrow{OC}$ is the representative of the sum $(\bar{a}+\bar{b})+\bar{c}$; $\overrightarrow{AC}$ is the representative of the sum $\bar{b}+\bar{c}$, and $\overrightarrow{OC}$ is the representative of the sum $\bar{a}+(\bar{b}+\bar{c})$. It follows

$$(\bar{a}+\bar{b})+\bar{c}=\bar{a}+(\bar{b}+\bar{c}).$$

Remarks. 1) The commutativity leads to the parallelogram law.

2) the associativity permits a generalization to the law of spacial polygonal contour (see fig. 6) for $n \geq 3$ vectors.

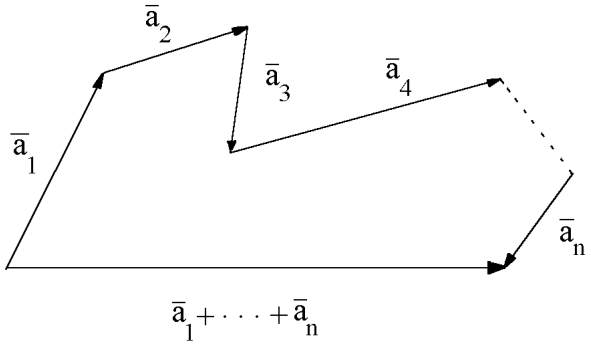


Fig. 6

3) $(V,+)$ is a commutative group. In this group we define *the subtraction*, as

$$\bar{a}-\bar{b}=\bar{a}+(-\bar{b});$$

also, the equation

$$\bar{b} + \bar{x} = \bar{a}$$

has a unique solution, $\bar{x} = \bar{a} - \bar{b}$ which is called *the difference* between $\bar{a}$ and $\bar{b}$. Geometrically we can represent this as in the following figure (see fig. 7).

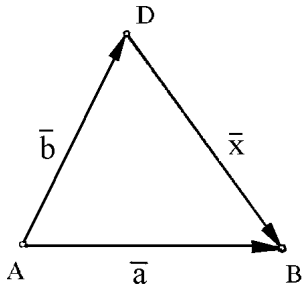


Fig. 7

1.3 Multiplication of a vector by a scalar

Let $\mathbb{R}$ be the set of real numbers (the scalar field) and V the additive commutative group of free vectors. We introduce a function defined on $\mathbb{R} \times V$ with values in V which is called *the multiplication of a vector by a scalar.*

3.1 Definition. Let $t \in \mathbb{R}$ and $\bar{a} \in V$. By $t\bar{a}$ we understand the free vector defined as follows:

1) if $\bar{a} \neq \bar{0}$ and $t \neq 0$, then $t\bar{a}$ is the vector which has the same direction with $\bar{a}$, the same sense with $\bar{a}$ if $t > 0$; opposite sense if $t < 0$; the length $|t| \cdot ||\bar{a}||$.

2) If $\bar{a} = 0$ or $t = 0$, then $t\bar{a} = \bar{0}$.

Obviously, $t\bar{a}$ is collinear to $\bar{a}$ (see fig. 8).

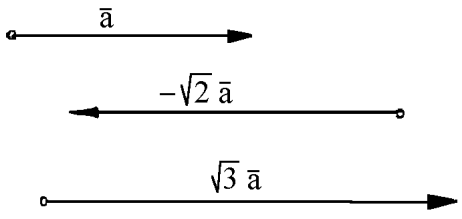


Fig. 8

3.2 Theorem. *The multiplication of a free vector with a scalar has the following properties*

1) $1 \cdot \bar{a} = \bar{a}$.
2) $s(t\bar{a}) = (st)\bar{a}$, *associativity property.*
3) $(s+t)\bar{a} = s\bar{a} + t\bar{a}$, *distributivity with respect to the addition of scalars.*
4) $t(\bar{a} + \bar{b}) = t\bar{a} + t\bar{b}$, *distributivity with respect to the addition of vectors.*

Proof. 3) The equality is obvious if at least one of the numbers s, t is zero, or the vector $\bar{a}$ is equal to the zero vector. Therefore, we may consider that s, t, $\bar{a}$ are non-zero. If s, t are of the same sign, then $s\bar{a}, t\bar{a}$ are of the same direction and sense, and we have

$$||s\bar{a} + t\bar{a}|| = |s|||\bar{a}|| + |t|||\bar{a}|| = (|s| + |t|)||\bar{a}|| = |s + t|\,||\bar{a}||.$$

The other cases can be proved in a similar way.

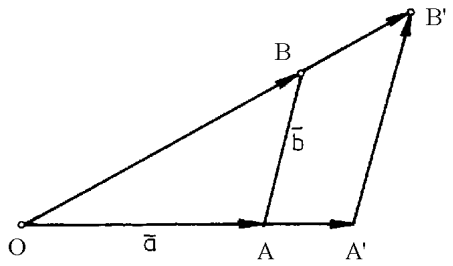


Fig. 9

4) Use similar triangles (see fig. 9) (homework).

Theorems 2.2, 3.2 show that the set V of all free vectors is a vector space over the field of real numbers.

1.4 Collinearity, coplanarity

Let V be the real vector space of free vectors; the notions of vector subspace, linear independence, dimension, basis, isomorphism are known from linear algebra. Now we want to show that V is 3-dimensional, and any basis of V consists of three non-coplanar free vectors.

To any vector $\bar{a}$ with length $||\bar{a}|| > 0$ we can associate a unit vector $\bar{e} = \frac{1}{||\bar{a}||} \cdot \bar{a}$.

4.1 Theorem. *Let* $\bar{a} \neq \bar{0}$. *If* $\bar{b}$ *is collinear to* $\bar{a}$, *then there exits a unique real number* t *such that* $\bar{b} = t\bar{a}$.

Proof. $\bar{a} = ||\bar{a}||\ \bar{a}_0, \quad \bar{b} = ||\bar{b}||\ \bar{b}_0; \quad \bar{b}_0 = \pm\bar{a}_0$. If $\bar{b}_0 = \bar{a}_0$, then $\bar{b} = ||\bar{b}||\bar{a}_0 = ||\bar{b}|| \cdot \frac{1}{||\bar{a}||} \cdot a_0$.

4.2 Corrolary. *Suppose $\bar{a} \neq \bar{0}$. Then the set*

$$V_1 = \{\bar{b} \in V |\ \exists t \in \mathbb{R}, \bar{b} = t\bar{a}\},$$

for $\bar{a} \neq \bar{0}$, is an 1-dimensional vector space.

The collinearity of two free vectors is equivalent to their linear dependence. Noncollinearity is equivalent to their linear independence.

We consider now three free vectors $\bar{a}, \bar{b}, \bar{c}$. We remind that these vectors are called *coplanar* if their representatives with common origin determine a plane.

4.3 Theorem. *Three vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if and only if they are linearly dependent.*

Proof. Suppose there exist $r, s, t \in \mathbb{R}$, such that $r\bar{a} + s\bar{b} + t\bar{c} = \bar{0}$, with $t \neq 0$. Then chosing for $\bar{a}, \bar{b}, \bar{c}$ the representatives $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ respectively, we have

$$\bar{c} = \alpha\bar{a} + \beta\bar{b} \Rightarrow \overline{OC} = \alpha\overline{OA} + \beta\overline{OB},$$

i.e., $\overrightarrow{OC}$ belongs to the plane generated by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ (see fig. 10). The converse is obvious.

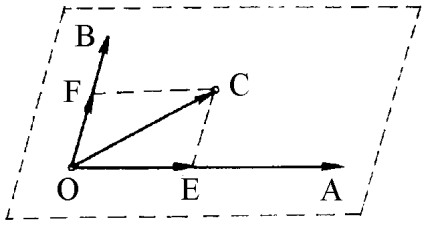


Fig. 10

4.4 Corrolary. *Suppose $\bar{a}$ and $\bar{b}$ are non-collinear vectors. Then the set*

$$V_2 = \{\bar{c} \in V |\ \exists r, s \in \mathbb{R}, \bar{c} = r\bar{a} + s\bar{b}\},$$

is a 2-dimensional vector space.

4.5 Theorem. *The real vector space V of all free vectors has dimension 3.* Therefore, we shall denote $V = V_3$.

Proof. Let $\bar{a}, \bar{b}, \bar{c}$ be three linearly independent (non-coplanar) vectors. Let $\bar{d} \in V$, and let $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}, \overrightarrow{OD}$ be the representatives of these vectors. We draw a parallelipiped with edges on OD_1, OD_2, OD_3 and with the diagonal OD (see fig. 11). Then we have

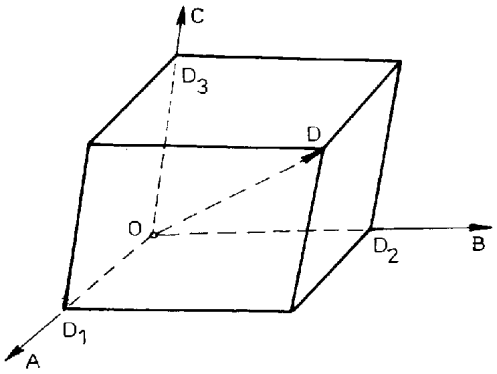


Fig. 11

$$\overrightarrow{OD} = \overrightarrow{OD}_1 + \overrightarrow{OD}_2 + \overrightarrow{OD}_3 = r\overrightarrow{OA} + s\overrightarrow{OB} + t\overrightarrow{OC},$$

whence $\bar{d} = r\bar{a} + s\bar{b} + t\bar{c}$. We conclude that $\{\bar{a}, \bar{b}, \bar{c}\}$ generates V. Therefore $\{\bar{a}, \bar{b}, \bar{c}\}$ is a basis for V, and $\dim V = 3$.

Let (r, s, t) be the coordinates (the components) of $\bar{d}$ with respect to a basis $\{\bar{a}, \bar{b}, \bar{c}\} \subset V_3$. We shall write briefly, $\bar{d} = (r, s, t)$. Let be $\bar{d}_i = (r_i, s_i, t_i), i \in \{1, 2, 3\}$ three vectors in V_3. Then we have

1) $\bar{d}_1 = \bar{d}_2 \Leftrightarrow r_1 = r_2, s_1 = s_2, t_1 = t_2$.
2) $\bar{d}_1 + \bar{d}_2 = (r_1 + r_2, s_1 + s_2, t_1 + t_2)$.
3) $\alpha\bar{d} = (\alpha r, \alpha s, \alpha t)$.
4) $\bar{d}_1$ collinear to $\bar{d}_2 \Leftrightarrow \frac{r_1}{r_2} = \frac{s_1}{s_2} = \frac{t_1}{t_2}$.
5) $\bar{d}_1, \bar{d}_2$ and $\bar{d}_3$ are *coplanar* iff

$$\begin{vmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{vmatrix} = 0.$$

Lesson 2

1.5 Orthogonal projections

5.1 Let D be a straight line and $\overline{AB}$ a free vector. We use the planes P and Q which contain A and B and are orthogonal to D. Denoting $\{A'\} = D \cap P$, $\{B'\} = D \cap Q$, we obtain the projection $\overrightarrow{A'B'}$. It can be shown (see fig. 12) that $\overline{A'B'}$ is well defined (does not depend on the representative $\overrightarrow{AB}$).

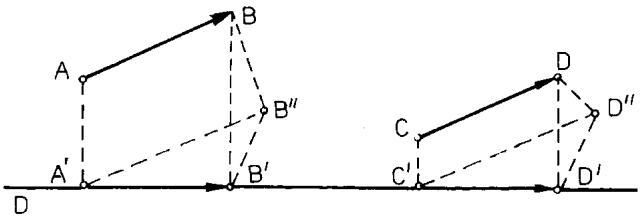


Fig. 12

Then the free vector $\overline{A'B'}$ is called *the projection of* $\overline{AB}$ *on the straight line* D and is denoted by $\pi_D\overline{AB}$. If the vector $\bar{u} \in V_3\backslash\{\bar{0}\}$ has the direction of D, we shall write $\pi_{\bar{u}}\overline{AB} = \pi_D\overline{AB}$. It can be easily seen that the definition does not depend on D but on its direction $\bar{u}$, and that the following properties hold true:

$$\pi_{\bar{u}}(\bar{a} + \bar{b}) = \pi_{\bar{u}}(\bar{a}) + \pi_{\bar{u}}(\bar{b}), \quad \pi_{\bar{u}}(t\bar{a}) = t\pi_{\bar{u}}(\bar{a}),$$

for any free vectors $\bar{a}, \bar{b} \in V_3$ and $t \in \mathbb{R}$.

Let be the free vectors $\bar{a}, \bar{b} \in V_3$, $\bar{u} \in V_3\backslash\{\bar{0}\}$, and let $\bar{u}_0$ be the associated unit vector of $\bar{u}$ defined by $\bar{u} = ||\bar{u}||\bar{u}_0$. The real number $pr_{\bar{u}}\bar{a}$ defined by the relation (see fig. 13)

$$\pi_{\bar{u}}(\bar{a}) = (pr_{\bar{u}}\bar{a})\bar{u}_0$$

is called *the algebraic measure of the orthogonal projection* $\pi_{\bar{u}}(\bar{a})$.

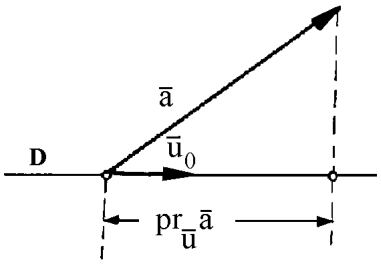


Fig. 13

We have the properties

$$pr_{\bar{u}}(\bar{a}+\bar{b}) = pr_{\bar{u}}(\bar{a}) + pr_{\bar{u}}(\bar{b}) \;\; pr_{\bar{u}}(t\bar{a}) = tpr_{\bar{u}}(\bar{a}),$$

for any free vectors $\bar{a}, \bar{b} \in V_3$ and $t \in \mathbb{R}$.

5.2 For $\bar{a} = \overline{OA}$, $\bar{b} = \overline{OB} \in V_3 \backslash \{\bar{0}\}$, the angle $\varphi \in [0, \pi]$ determined by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is called *the angle between* $\bar{a}$ *and* $\bar{b}$ (see fig. 14).

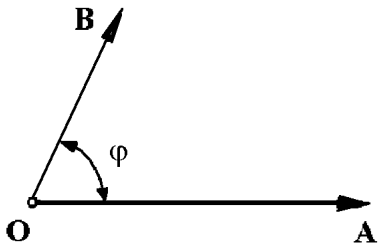


Fig. 14

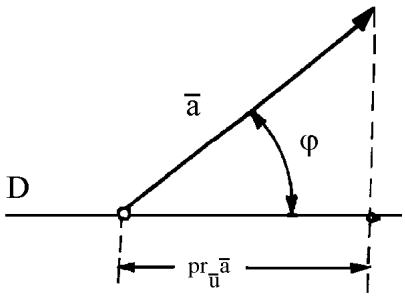


Fig. 15

We say that $\bar{a}$ and $\bar{b}$ are *orthogonal* iff $\varphi = \pi/2$, and we admit that $\bar{0}$ is orthogonal to any vector. We have the relation (see fig. 15):

$$pr_{\bar{u}}\bar{a} = ||\bar{a}|| \cos\varphi.$$

5.3 In a similar manner, for $\bar{a} = \overline{AB}$ and a given plane P, denoting by A' and B' the projections of the points A and B on P, we can show that the free vector $\overline{A'B'}$ does not depend on the representative of $\bar{a}$, so it depends on $\bar{a}$ and P alone. Hence we call $\overline{A'B'}$ *the projection of* $\bar{a}$ *on* P, and denote it by $\pi_P(\bar{a})$ (see fig. 16).

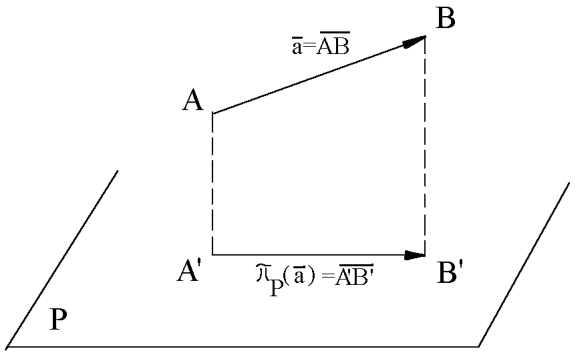


Fig. 16

1.6 Scalar (dot, inner) product

Let $\bar{a}, \bar{b} \in V_3$. If $\bar{a} \neq \bar{0}, \bar{b} \neq \bar{0}$, then *the angle between* $\bar{a}$ *and* $\bar{b}$ is defined as the angle between the representatives of these vectors reduced to a common origin.

6.1 Theorem. *The function* $(\,\cdot\,,\,\cdot\,) : V_3 \times V_3 \to \mathbb{R}$,

$$(\bar{a}, \bar{b}) = \begin{cases} ||\bar{a}||\,||\bar{b}|| \cos\varphi, & \text{for } \bar{a} \neq 0 \text{ and } \bar{b} \neq 0 \\ 0, & \text{for } \bar{a} = 0 \text{ or } \bar{b} = 0, \end{cases}$$

is a scalar product on V_3.

Proof. We check the properties of a scalar product, namely
1) $(\bar{a}, \bar{b}) = (\bar{b}, \bar{a})$, symmetry;
2) $t(\bar{a}, \bar{b}) = (t\bar{a}, \bar{b}) = (\bar{a}, t\bar{b})$, homogeneity;
3) $(\bar{a}, \bar{a}) = ||\bar{a}||^2 \geq 0$, for all $\bar{a} \in V_3$ and
$(\bar{a}, \bar{a}) = 0 \Leftrightarrow \bar{a} = \bar{0}$, positive definiteness;
4) $(\bar{a}, \bar{b} + \bar{c}) = (\bar{a}, \bar{b}) + (\bar{a}, \bar{c})$, distributivity.
For the proof of (e.g.) 4) we use the fact that the algebraic measure of the orthogonal projection of a vector $\bar{v}$ on the unit vector $\bar{e}$ can be written as $(\bar{e}, \bar{v})$.

If $\bar{a} = \bar{0}$, then the relation holds true, since

$$(\bar{a}, \bar{b} + \bar{c}) = 0 = (\bar{a}, \bar{b}) + (\bar{a}, \bar{c}).$$

For $\bar{a} \neq \bar{0}$, choosing a unit vector having the direction of $\bar{a}$, we obtain $\bar{a} = ||\bar{a}||\bar{e}$. Hence from

$$(\bar{e}, \bar{b} + \bar{c}) = (\bar{e}, \bar{b}) + (\bar{e}, \bar{c}),$$

we infer by multiplication to $||\bar{a}||$ the desired result.

Remarks. The following properties hold true
1) V_3 is a Euclidean space;
2) $||\bar{a}|| = \sqrt{(\bar{a}, \bar{a})}$;
3) $|\cos\varphi| \leq 1 \Longrightarrow |(\bar{a}, \bar{b})| \leq ||\bar{a}||||\bar{b}||$;
4) $\bar{a} \perp \bar{b} \Longleftrightarrow (\bar{a}, \bar{b}) = 0$.

6.2 Let $\{\bar{a}, \bar{b}, \bar{c}\}$ be a basis in V_3 and let be the vectors

$$\bar{u} = r_1\bar{a} + s_1\bar{b} + t_1\bar{c}, \;\; \bar{v} = r_2\bar{a} + s_2\bar{b} + t_2\bar{c} \in V_3.$$

Then we have

$$(\bar{u}, \bar{v}) = (r_1, s_1, t_1) \begin{pmatrix} (\bar{a}, \bar{a}) & (\bar{a}, \bar{b}) & (\bar{a}, \bar{c}) \\ (\bar{b}, \bar{a}) & (\bar{b}, \bar{b}) & (\bar{b}, \bar{c}) \\ (\bar{c}, \bar{a}) & (\bar{c}, \bar{b}) & (\bar{c}, \bar{c}) \end{pmatrix} \begin{pmatrix} r_2 \\ s_2 \\ t_2 \end{pmatrix}.$$

The matrix in the relation above is called *the table of the scalar product*. We say that $\{\bar{a}, \bar{b}, \bar{c}\}$ is an *orthogonal basis* iff the matrix above is a diagonal matrix; if the vectors of such a basis are unit vectors, then the basis is said to be *orthonormal*, and the matrix is the identity. We usually denote such a basis by $\{\bar{i}, \bar{j}, \bar{k}\}$, and the components of a vector relative to such a basis are called *Euclidean components* of the vector. Also, with respect to an orthonormal basis, the scalar product has a simple analytic expression,

$$(\bar{a}, \bar{b}) = r_1r_2 + s_1s_2 + t_1t_2,$$

for all $\bar{a} = r_1\bar{i} + s_1\bar{j} + t_1\bar{k}, \; \bar{b} = r_2\bar{i} + s_2\bar{j} + t_2\bar{k} \in V_3$.
In this case, we have the norm given by

$$||\bar{a}|| = \sqrt{r_1^2 + s_1^2 + t_1^2}$$

and the angle φ between the free vectors $\bar{a}$ and $\bar{b}$ is

$$\cos\varphi = \frac{(\bar{a}, \bar{b})}{||\bar{a}||||\bar{b}||}, \;\; \varphi \in [0, \pi].$$

Lesson 3

1.7 Vector (cross, exterior) product

Let V_3 be the vector space of free vectors. If the fingers of the right hand point in the direction of the shortest rotation of $\bar{a}$ to $\bar{b}$, then the thumb points in the direction of $\bar{a} \times \bar{b}$ (*the right hand rule*).

7.1 Definition. We call *the vector product* of the vectors $\bar{a}$ and $\bar{b}$, the vector

$$\bar{a} \times \bar{b} = \begin{cases} ||\bar{a}||\ ||\bar{b}|| \sin\varphi\ \bar{e}, & \text{for } \bar{a}, \bar{b} \text{ noncollinear} \\ \bar{0}, & \text{for } \bar{a}, \bar{b} \text{ collinear} \end{cases}$$

where $\bar{e}$ is a unit vector orthogonal on $\bar{a}$ and $\bar{b}$, with the sense given by the the right-hand law for the triple $(\bar{a}, \bar{b}, \bar{e})$ (see fig. 17).

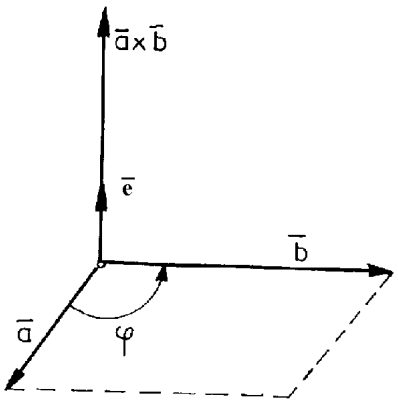


Fig. 17

From the definition, it directly follows that the vector product satisfies the following *properties*
1) $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$, *anticommutativity*;
2) $t(\bar{a} \times \bar{b}) = (t\bar{a}) \times \bar{b} = \bar{a} \times (t\bar{b})$, *homogeneity*;
3) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$, *distributivity*;
4) $\bar{a} \times \bar{a} = \bar{0}$, $\bar{a} \times \bar{0} = \bar{0}$;
5) $\bar{a} \times \bar{b} = \bar{0} \quad \Leftrightarrow \quad \bar{a}, \bar{b}$ are collinear;

6) $||\bar{a} \times \bar{b}||^2 = ||\bar{a}||^2||\bar{b}||^2 - (\bar{a}, \bar{b})^2$, *the Lagrange identity.*

Proof. 3) We use the projection of a vector on a plane; let $\bar{b}'$ be the projection of $\bar{b}$ on the plane perpendicular to $\bar{a}$. Then $\bar{a} \times \bar{b} = \bar{a} \times \bar{b}'$. Supposing that $||\bar{a}|| = 1$ and denoting by $\bar{b}'$, $\bar{c}'$ respectively the projections of $\bar{b}, \bar{c}$ on the plane perpendicular to $\bar{a}$, we get that $\bar{a} \times \bar{b}', \bar{a} \times \bar{c}'$ and $\bar{a} \times (\bar{b}' + \bar{c}')$ are obtained from the vectors $\bar{b}', \bar{c}', \bar{b}' + \bar{c}'$ respectively by a rotation of angle $\pi/2$. Consequently $\bar{a} \times (\bar{b}' + \bar{c}') = \bar{a} \times \bar{b}' + \bar{a} \times \bar{c}'$, whence $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$ (see fig. 18).

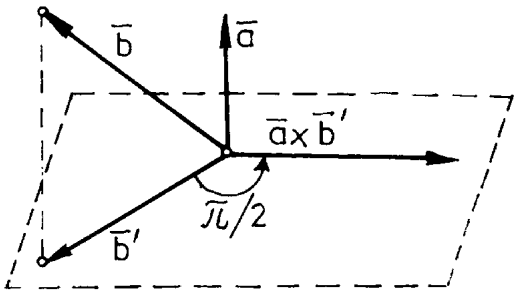


Fig. 18

6) Let φ be the angle between $\bar{a}$ and $\bar{b}$; then we have

$$||\bar{a} \times \bar{b}||^2 = ||\bar{a}||^2||\bar{b}||^2 \sin^2 \varphi = ||\bar{a}||^2||\bar{b}||^2(1 - \cos^2 \varphi) = ||\bar{a}||^2||\bar{b}||^2 - (\bar{a}, \bar{b})^2.$$

Remark. From the definition of the vector product, we have the relation

$$||\bar{a} \times \bar{b}|| = ||\bar{a}||||\bar{b}|| \sin \varphi,$$

where φ is the angle between $\bar{a}$ and $\bar{b}$. This is the area of the parallelogram built on representatives of $\bar{a}, \bar{b}$ with common origin.

Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be an orthonormal basis in V_3. Then we have the relations

$$\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = \bar{0}, \qquad \bar{i} \times \bar{j} = \bar{k},\ \bar{j} \times \bar{k} = \bar{i},\ \bar{k} \times \bar{i} = \bar{j},$$

and the vector product of two vectors

$$\bar{a} = r_1\bar{i} + s_1\bar{j} + t_1\bar{k} \text{ and } \bar{b} = r_2\bar{i} + s_2\bar{j} + t_2\bar{k} \in V_3$$

is given by the formal determinant

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \end{vmatrix}.$$

The double vector product $\bar{a} \times (\bar{b} \times \bar{c})$, satisfies the identity

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}, \bar{c})\bar{b} - (\bar{a}, \bar{b})\bar{c}.$$

1.8 Mixed (triple) product

8.1 Let $\bar{a}, \bar{b}, \bar{c} \in V_3$ be three free vectors. The number $(\bar{a}, \bar{b} \times \bar{c})$ is called *the mixed product* of the three vectors $\bar{a}, \bar{b}$ and $\bar{c}$. We remark that since $\bar{b} \times \bar{c} = ||\bar{b}||||\bar{c}|| \sin\varphi\, \bar{e}$, we can write

$$(\bar{a}, \bar{b} \times \bar{c}) = (||\bar{a}|| \cos\theta)(||\bar{b}||||\bar{c}|| \sin\varphi).$$

Hence the absolute value of $(\bar{a}, \bar{b} \times \bar{c})$ is the volume of the parallelepiped for which the representatives of $\bar{a}, \bar{b}$ and $\bar{c}$ are co-terminal sides (see fig. 19).

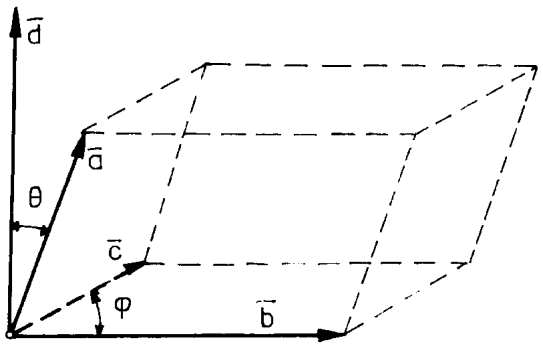


Fig. 19

8.2 The mixed product has the following properties:
1) $(\bar{a}, \bar{b} \times \bar{c}) = (\bar{b}, \bar{c} \times \bar{a}) = (\bar{c}, \bar{a} \times \bar{b})$;
2) an interchange of any two vectors reverses the sign of the triple product, i.e.,

$$\begin{aligned} (\bar{a}, \bar{b} \times \bar{c}) &= -(\bar{a}, \bar{c} \times \bar{b}) = (\bar{c}, \bar{a} \times \bar{b}) = \\ &= -(\bar{c}, \bar{b} \times \bar{a}) = (\bar{b}, \bar{c} \times \bar{a}) = -(\bar{b}, \bar{a} \times \bar{c}); \end{aligned}$$

3) $(t\bar{a}, \bar{b} \times \bar{c}) = (\bar{a}, t\bar{b} \times \bar{c}) = (\bar{a}, \bar{b} \times t\bar{c})$
4) $(\bar{a} + \bar{b}, \bar{c} \times \bar{d}) = (\bar{a}, \bar{c} \times \bar{d}) + (\bar{b}, \bar{c} \times \bar{d})$;
5) $(\bar{a} \times \bar{b}, \bar{c} \times \bar{d}) = \begin{vmatrix} (\bar{a}, \bar{c}) & (\bar{a}, \bar{d}) \\ (\bar{b}, \bar{c}) & (\bar{b}, \bar{d}) \end{vmatrix}$, *the Lagrange identity*;
6) $(\bar{a}, \bar{b} \times \bar{c}) = 0$ iff at least one vector is null, or two of them are collinear, or the three vectors are coplanar. Otherwise, the three vectors are non-coplanar, and hence they determine a basis in V_3.

8.3 Let $\{\bar{i}, \bar{j}, \bar{k}\}$ be an orthonormal basis. Then the mixed product of any three vectors

$$\bar{a} = r_1\bar{i} + s_1\bar{j} + t_1\bar{k}, \;\; \bar{b} = r_2\bar{i} + s_2\bar{j} + t_2\bar{k}, \;\; \bar{c} = r_3\bar{i} + s_3\bar{j} + t_3\bar{k} \in V_3$$

is given by the determinant

$$(\bar{a}, \bar{b} \times \bar{c}) = \begin{vmatrix} r_1 & s_1 & t_1 \\ r_2 & s_2 & t_2 \\ r_3 & s_3 & t_3 \end{vmatrix}.$$

Hence, we remark that the properties of the mixed product are inferred from those of the determinants of order three.

Any orthonormal basis is called positive oriented if it satisfies the right-hand rule. Generally, a basis $\{\bar{a}, \bar{b}, \bar{c}\}$ is *positive oriented* iff its vectors satisfy $(\bar{a}, \bar{b} \times \bar{c}) > 0$.

1.9 Problems

1. Let be the polygon $ABCD$, such that

$$AD||BC, \; \sphericalangle A = \sphericalangle C = \pi/2, \; \sphericalangle B = 5\pi/6, \; \overline{AD} = \bar{a}, \; \overline{AB} = \bar{b}.$$

Decompose $\overline{BC}$, $\overline{DC}$, $\overline{AC}$, $\overline{BD}$ with respect to the vectors $\bar{a}$ and $\bar{b}$.

2. Let be the vectors

$$\bar{d}_1 = \bar{a} - \alpha\bar{b} + 3\bar{c}, \; \bar{d}_2 = \alpha\bar{a} - \bar{b} + \bar{c}, \; \bar{d}_3 = 3\bar{a} + \bar{b} - \bar{c},$$

where $\{\bar{a}, \bar{b}, \bar{c}\}$ is a basis in V_3. Find $\alpha \in \mathbb{R}$ such that the vectors $\bar{d}_1$, $\bar{d}_2$, $\bar{d}_3$ are coplanar. For this value of α, decompose the vector $\bar{d}_2$ with respect to the vectors $\bar{d}_1$ and $\bar{d}_3$.

3. Let be the points A, B, C given by their position vectors

$$\overline{OA} = 14\bar{i} - 7\bar{j} + 2\bar{k},\ \ \overline{OB} = 2\bar{i} + 2\bar{j} - 7\bar{k},\ \ \overline{OC} = -2\bar{i} + 7\bar{j} + 2\bar{k}.$$

Show that the triangle AOB has a right angle and that the triangle BOC has two sides of equal lengths. Compute the perimeter of the triangle ABC and the angle $\sphericalangle BAC$. Write the analytic expression of the bisector of the angle $\sphericalangle BAC$.

4. Let be the vectors

$$\bar{a} = \bar{i} + 2\lambda\bar{j} - (\lambda - 1)\bar{k},\ \ \bar{b} = (3 - \lambda)\bar{i} + \bar{j} + 3\bar{k},\ \ \lambda \in \mathbb{R}.$$

Find λ such that the vectors $\bar{a}$ and $\bar{b}$ be orthogonal. For this value of λ find the projection of the vector $\bar{a}$ on the vector $\bar{a} + \bar{b}$.

5. Compute the area of the parallelogram built on the representatives with common origin of the free vectors

$$\bar{r}_1 = \frac{1 + \cos v}{\cos^2 u}\bar{j} + \frac{\sin u \sin v}{\cos^2 u}\bar{k},\quad \bar{r}_2 = -\sin v\bar{i} - \sin v\ \mathrm{tg}\ u\ \bar{j} + \frac{\cos v}{\cos u}\bar{k},$$

using the Lagrange identity.

6. Let be the vectors

$$\bar{a} = \bar{i} - 5\bar{j} - 7\bar{k},\ \ \bar{b} = 2\bar{i} - 3\bar{j} + 6\bar{k},\ \ \bar{c} = -\bar{i} + 2\bar{j} - 2\bar{k}.$$

Compute $\bar{w} = \bar{a} \times (\bar{b} \times \bar{c})$ and check the linear dependence of the vectors $\bar{w}, \bar{b}, \bar{c}$.

7. Let be the frame $\{O; \bar{a}, \bar{b}, \bar{c}\}$. The vectors defined by

$$\bar{a}' = \frac{\bar{b} \times \bar{c}}{(\bar{a}, \bar{b} \times \bar{c})},\ \ \bar{b}' = \frac{\bar{c} \times \bar{a}}{(\bar{a}, \bar{b} \times \bar{c})},\ \ \bar{c}' = \frac{\bar{a} \times \bar{b}}{(\bar{a}, \bar{b} \times \bar{c})}$$

are called *the reciprocal vectors* of the triple $\{\bar{a}, \bar{b}, \bar{c}\}$, and the frame $\{O; \bar{a}, \bar{b}, \bar{c}\}$ is called the *reciprocal frame*. What relations can be established between $\{\bar{a}, \bar{b}, \bar{c}\}$ and $\{\bar{a}', \bar{b}', \bar{c}'\}$?

8. Let be the vectors

$$\bar{a} = \bar{i} - \alpha\bar{j} + 3k,\ \bar{b} = \alpha\bar{i} - \bar{j} + \bar{k},\ \bar{c} = 3\bar{i} + \bar{j} - \bar{k}.$$

Find the value of α such that the vectors $\{\bar{a}, \bar{b}, \bar{c}\}$ are coplanar.

For $\alpha = 2$, find the height of the parallelepiped which is built on the representatives of the vectors $\bar{a}, \bar{b}, \bar{c}$, knowing that it corresponds to the basis formed by the representatives of the vectors $\bar{a}$ and $\bar{b}$.

9. Show that the points

$$A(1,1,1),\ B(3,-1,4),\ C(0,7,-3),\ D(5,7,2)$$

are coplanar.

Chapter 2

Equations of straight line and plane in space

Lesson 4

2.1 Cartesian frame

Let E_3 be the 3-dimensional geometric space, V_3 the 3-dimensional vector space of free vectors and $\mathbb{R}^3$ the vector space of triples of real numbers. Fixing an origin $O \in E_3$, we obtain a bijection between E_3 and V_3, which associates to any point $M \in E_3$ the free vector $\overline{OM} \in V_3$. Moreover, fixing a basis $\{\bar{i}, \bar{j}, \bar{k}\} \subset V_3$, we obtain a bijection between V_3 and $\mathbb{R}^3$, which associates to any vector $\bar{a} = r\bar{i} + s\bar{j} + t\bar{k} \in V_3$ the triple $(r, s, t) \in \mathbb{R}^3$; this bijection is an isomorphism of vector spaces.

The set $\{O; \bar{i}, \bar{j}, \bar{k}\}$ is called a *Cartesian frame* in E_3, O is called *the origin of the frame* and $\{\bar{i}, \bar{j}, \bar{k}\}$ is called *basis of the frame*.

Hence, any vector $\overline{OM}$ admits a unique representation of the form

$$\overline{OM} = x\bar{i} + y\bar{j} + z\bar{k}.$$

The numbers x, y, z are called *the Cartesian coordinates of the point* M, and we write $M(x, y, z)$. The bijection between E_3 and $\mathbb{R}^3$ which associates to $M \in E_3$ the triple $(x, y, z) \in \mathbb{R}^3$ is called *Cartesian system of coordinates*.

The origin O and $\bar{i}$ determine the straight line Ox; similarly, O and $\bar{j}, \bar{k}$ determine respectively the lines Oy, Oz. These lines are called *the coordinate axes*; Ox is also called *the* x *- axis*, similarly Oy and Oz. The planes xOy, yOz, zOx are called

coordinate planes; the plane xOy is also called *the xy - plane*, similarly yOz and zOx.

Geometrically the coordinates of the point M are obtained in the following way (see fig. 20): we draw through M a straight line parallel to Oz which intersects xOy at some point M'; then a parallel through M' to Oy intersects Ox at the point whose algebraic distance to O is x, a parallel through M' to Ox intersects Oy at the point whose algebraic distance to O is y; the parallel to OM' through M intersects Oz at the point whose algebraic distance to O is z.

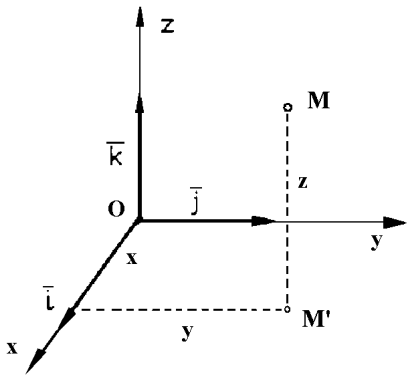


Fig. 20

Having in view the construction above, the Cartesian frame $\{O; \bar{i}, \bar{j}, \bar{k}\}$ is usually denoted by $Oxyz$.

2.2 Equations of straight line in space

Any straight line can be specified in the following way:
1) by a point and a director vector;
2) by two points;
3) by the intersection of two planes.
Now we shall transfer these geometrical conditions into equations.

2.1 Straight line determined by a point and a (non-zero) director vector. Let D be a straight line, containing the point $M_0(x_0, y_0, z_0)$ and having the director vector $\bar{a} \equiv (l, m, n)$; let us denote $\bar{r}_0 = \overline{OM_0}$ (see fig. 21).

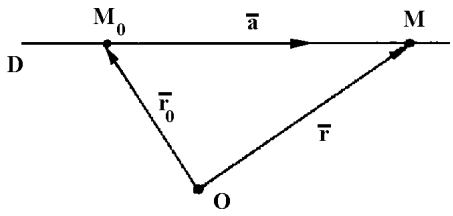


Fig. 21

Then, for a point $M \in E_3$, denoting $\bar{r} = \overline{OM}$, we have $M \in D$ iff $\bar{r} - \bar{r}_0$ is collinear to $\bar{a}$, i.e., iff we have

$$(\bar{r} - \bar{r}_0) \times \bar{a} = \bar{0}.$$

This equality is called the *vector equation* (in V_3) of D; the numbers l, m, n are called *director parameters of the straight line*. The previous equation can be rewritten

$$\bar{r} = \bar{r}_0 + t\bar{a}, \quad t \in \mathbb{R} \quad \Leftrightarrow \quad \begin{cases} x = x_0 + tl \\ y = y_0 + tm \\ z = z_0 + tn \end{cases}, \quad t \in \mathbb{R};$$

the last equations are called *the parametric equations of the straight line* D.

Eliminating t from the last three equations, we obtain the equivalent pair of equations

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

with the convention that if a denominator vanishes, then the corresponding numerator must also be zero. A straight line is a closed, unbounded subset of $\mathbb{R}^3$. It is a 1-dimensional vector subspace of $\mathbb{R}^3$ iff it contains the origin.

2.2 Straight line given by two distinct points. Let

$$M_i(x_i, y_i, z_i), \quad i = 1, 2$$

be two distinct points on D. We apply the procedure 2.1, choosing $M_0 = M_1$ (the point on D) and the director vector $\bar{a} = \overline{M_1 M_2}$ (see fig. 22). We obtain *the Cartesian equations* of D

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

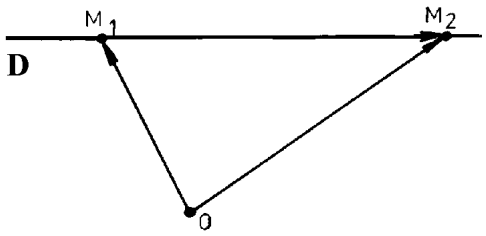


Fig. 22

2.3 Oriented straight line. Let D be a straight line. On D we have two senses of way. If $\bar{a}$ is the director vector of D, then the positive sense of D is the sense of $\bar{a}$ and is denoted by "+". A straight line with a sense of way, i.e., the pair $(D, \bar{a})$ is called *oriented line*. The unit vector $\bar{e} = \frac{1}{||\bar{a}||}\bar{a}$ is called *oriented direction*. The angles between $\bar{e}$ and $\bar{i}, \bar{j}, \bar{k}$ are called *director angles of* D; if we denote them respectively with α, β and γ, then we have (see fig. 23)

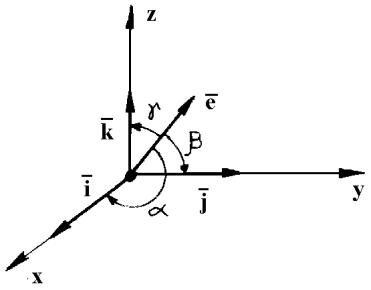


Fig. 23

$$\bar{e} = \cos\alpha\bar{i} + \cos\beta\bar{j} + \cos\gamma\bar{k},$$

and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The components $\cos\alpha,\ \cos\beta,\ \cos\gamma$ of the unit vector $\bar{e}$ are called *director cosines*.

2.4 Angle between two oriented straight lines. Let $\bar{a}, \bar{b}$ the director vectors of the two lines D_1 and D_2 (see fig. 24). Then *the angle* φ *between* $(D_1, \bar{a})$ *and* $(D_2, \bar{b})$ is the angle between their two director vectors, whence

$$cos\varphi = \frac{(\bar{a}, \bar{b})}{||\bar{a}||\ ||\bar{b}||},\quad \varphi \in [0, \pi].$$

We remark that

$$\begin{aligned} D_1 || D_2 &\Leftrightarrow \bar{a} \times \bar{b} = \bar{0} \\ D_1 \perp D_2 &\Leftrightarrow (\bar{a}, \bar{b}) = 0. \end{aligned}$$

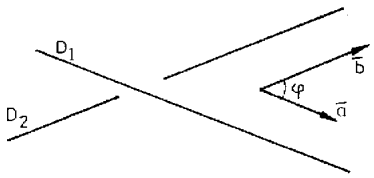


Fig. 24

2.3 Equation of plane in space

Among the geometrical conditions which fix a plane, we have the following
- three noncollinear points;
- two concurrent straight lines;
- two parallel lines;
- a straight line and an exterior point;
- a point and a normal vector.

3.1 Plane determined by a point and a normal vector.

a) A point $M(x, y, z) \in E_3$ belongs to the plane P determined by a point $M_0(x_0, y_0, z_0)$ and a normal (nonzero) vector $\bar{n} \equiv (a, b, c)$ iff it satisfies the condition (see fig. 25) $\overline{M_0M} \perp \bar{n}$, i.e.,

$$(\overline{M_0M}, \bar{n}) = 0.$$

This relation is called *the vector equation of the plane* P. On the other hand, since we have

$$\overline{M_0M} = (x - x_0)\bar{i} + (y - y_0)\bar{j} + (z - z_0)\bar{k},$$

the equation rewrites

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

and is called *the Cartesian equation of the plane.*

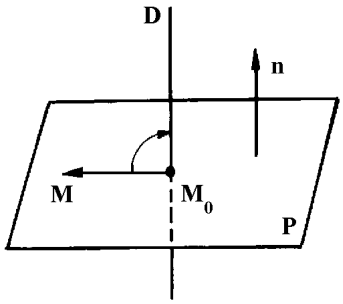


Fig. 25

b) We remark that a point $M(x, y, z) \in E_3$ belongs to *the straight line D which is normal to P and contains the point M_0* iff it satisfies

$$\overline{M_0M} = t\bar{n}, t \in \mathbb{R} \Leftrightarrow \overline{M_0M} \times \bar{n} = \bar{0},$$

i.e., it satisfies the Cartesian equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

c) We remark that the Cartesian equation of the plane can be rewritten in the form

$$ax + by + cz + d = 0,$$

which is called *the general Cartesian equation of a plane.* The numbers a, b, c do not vanish simultaneously. Also, only three of the four coefficients (parameters) are essential, since the multiplication of the equation with a non-zero number doesn't alter its solution, i.e., the described plane P remains the same.

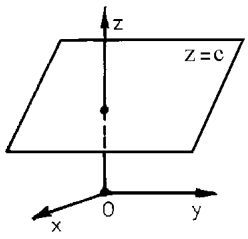


Fig. 26

3.2 Particular planes. 1) The plane $xOy : z = 0$ has the normal vector $\bar{n} \equiv (0, 0, 1)$; a plane perpendicular to Oz has the equation $z = c$ (see fig. 26)

2) a plane perpendicular to xOy has the equation of the form $ax + by + c = 0$ and is parallel to Oz;

3) a plane which contains an axis, e.g., the axis Oz, has the equation of the form $ax + by = 0$;

4) a plane through the origin $O(0, 0, 0)$ has the equation of the form $ax+by+cz = 0$.

Lesson 5

3.3 Plane determined by three non-collinear points. Let be three distinct non-collinear points $M_i(x_i, y_i, z_i), i = 1, 2, 3$ (see fig. 27).

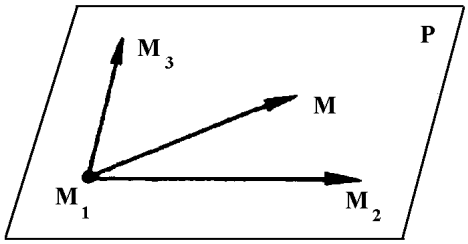


Fig. 27

Using the general equation of the plane

$$P: \quad ax + by + cz + d = 0,$$

and imposing the condition $M_i \in P, i = 1, 2, 3$, we obtain, together with the general equation, the homogeneous linear system

$$\begin{cases} ax + by + cz + d = 0 \\ ax_1 + by_1 + cz_1 + d = 0 \\ ax_2 + by_2 + cz_2 + d = 0 \\ ax_3 + by_3 + cz_3 + d = 0 \end{cases}$$

with unknowns a, b, c, d and the condition $a^2 + b^2 + c^2 > 0$. The compatibility condition of the system, namely

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0,$$

provides *the equation of the plane which contains the non-collinear points* M_1, M_2, M_3.

We note that this equation is of first degree in x, y, z, and is obviously verified by the triples $(x_i, y_i, z_i), i = 1, 2, 3$.

Remarks. 1) As a consequence, the coplanarity of four points $M_i(x_i, y_i, z_i), i = 0, 1, 2, 3$ can be expressed by the condition

$$\begin{vmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

2) The equation obtained above becomes equivalent to

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0,$$

which expresses in fact the condition that the point $M(x, y, z) \in E_3$ belongs to the plane determined by M_1, M_2, M_3, i.e., the vectors $\overline{M_1M}$, $\overline{M_1M_2}$ and $\overline{M_1M_3}$ are coplanar.

3) From the equation above, one can infer *the intercept form of the equation of a plane*. Namely, if the plane determines on the three axes Ox, Oy and Oz segments of algebraic length α, β and γ (see fig. 28),

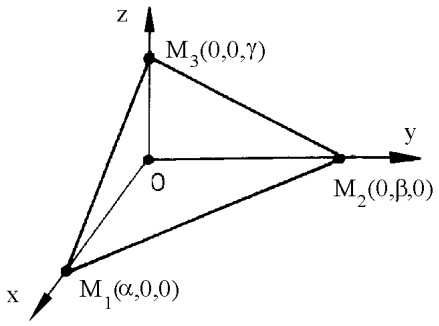


Fig. 28

i.e., it passes through the points $M_1(\alpha, 0, 0)$, $M_2(0, \beta, 0)$, $M_3(0, 0, \gamma)$ respectively, then it follows its equation

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1.$$

4) Topologically, the plane is a closed unbounded subset of $\mathbb{R}^3$. It is a 2-dimensional vector subspace of $\mathbb{R}^3$ iff it contains the origin.

3.4 Plane determined by a point and two non-collinear vectors. Let $\bar{u}(l_1, m_1, n_1)$, $\bar{v}(l_2, m_2, n_2)$ be two non-collinear vectors, and

$$M_1(x_1, y_1, z_1) \in E_3$$

a point in space. A point $M(x, y, z) \in E_3$ belongs to the plane which is parallel to $\bar{u}$ and $\bar{v}$ and which contains M_1, iff the vectors $\bar{u}, \bar{v}$ and $\overline{M_1M}$ are coplanar (see fig. 29), i.e., iff the components of these vectors satisfy the relation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

which is *the equation of the plane determined by a point and two non-collinear directions*. This rewrites, providing *the parametric equations of the plane*

$$\begin{cases} x = x_1 + rl_1 + sl_2 \\ y = y_1 + rm_1 + sm_2 \\ z = z_1 + rn_1 + sn_2 \end{cases}, \quad r, s \in \mathbb{R}.$$

The numbers $r, s \in \mathbb{R}$ are called *parameters*.

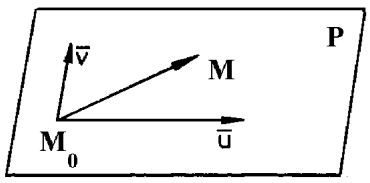


Fig. 29

3.5 Oriented plane. A plane $P : ax + by + cz + d = 0$ together with a sense on the normal straight line given by a normal vector $\bar{n}(a, b, c)$, is called *oriented plane* (see fig. 30).

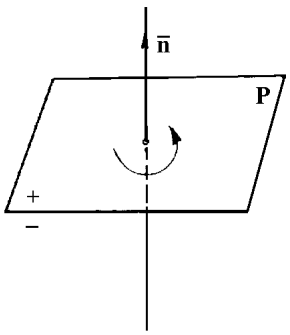


Fig. 30

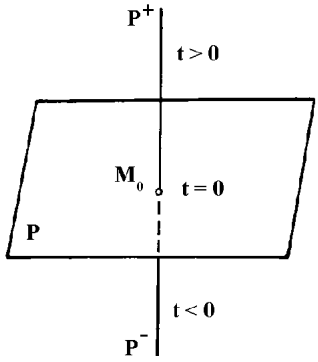


Fig. 31

3.6 Semispaces. Let be the plane

$$P: \quad f(x,y,z) \equiv ax + by + cz + d = 0.$$

Then we define *the positive semispace* (P^+) and *the negative semispace* (P^-) respectively, as

$$P^+: \quad f(x,y,z) > 0, \qquad P^-: \quad f(x,y,z) < 0.$$

These are, topologically, open and convex sets in $\mathbb{R}^3$. Also, we have the disjoint union (see fig. 31)

$$E_3 = P^+ \cup P \cup P^-.$$

Indeed, for some point $M_0(x_0, y_0, z_0) \in P$, the line

$$D: \quad \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}, \; t \in \mathbb{R}$$

is normal to P and passes through M_0. The points of D are divided into into three disjoint subsets according to the values of t (for $t < 0,\ t = 0$ and $t > 0$, respectively). Hence, if M_0 describes P, then

$$f(x,y,z) = t(a^2 + b^2 + c^2),$$

and we have that for $t > 0$, the points $M \in D$ describe P^+, and for $t < 0$, they describe P^-. So, we have

$$P^- = f^{-1}((-\infty, 0)), \quad P^+ = f^{-1}((0, \infty)),$$

and hence they are open, convex sets. Together with P, these sets cover the whole space E_3.

To decide the subspace where a given point $M(x, y, z) \in E_3$ lies, is therefore enough to find the sign of the value $f(x, y, z)$.

3.7 Union and intersection of planes. Let be the planes

$$P_i : f(x, y, z) \equiv a_i x + b_i y + c_i z + d_i = 0, \quad i = 1, 2.$$

Then *the union* $P_1 \cup P_2$ *of the planes* P_1 *and* P_2 (see fig. 32) is the set of points $M(x, y, z)$ (degenerate quadric) which satisfy the equation

$$(a_1 x + b_1 y + c_1 z + d_1)(a_2 x + b_2 y + c_2 z + d_2) = 0,$$

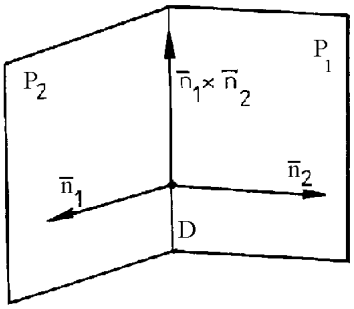


Fig. 32

and *the intersection* $P_1 \cap P_2$ *of the planes* P_1 *and* P_2 is the set of points which satisfy the system

$$\begin{cases} a_1 x + b_1 y + c_1 z + d_1 = 0 \\ a_2 x + b_2 y + c_2 z + d_2 = 0. \end{cases}$$

We remark that if

$$\operatorname{rank} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2,$$

then the intersection is a straight line. In this case, the set of all planes through $P_1 \cap P_2$ is called *pencil of planes*. The equation of an arbitrary plane in the pencil is of the form

$$r(a_1 x + b_1 y + c_1 z + d_1) + s(a_2 x + b_2 y + c_2 z + d_2) = 0, \qquad r, s \in \mathbb{R}, r^2 + s^2 > 0,$$

and is called *the equation of the pencil*.

Also, for a given normal direction $\bar{n}(a, b, c)$, the planes which are normal to $\bar{n}$ are parallel to each other and form a *pencil of parallel planes*; the equation of an arbitrary plane in the pencil of parallel planes is of the form

$$ax + by + cz + \lambda = 0, \quad \lambda \in \mathbb{R}.$$

3.8 Angle between two oriented planes. Let be the planes

$$P_i : f(x, y, z) \equiv a_i x + b_i y + c_i z + d_i = 0, \quad i = 1, 2.$$

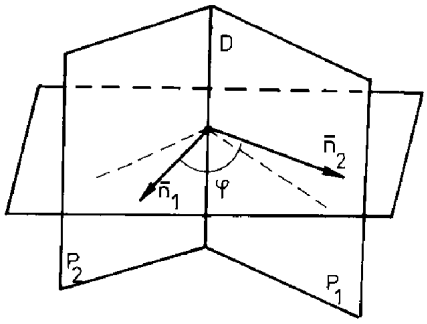


Fig. 33

Then *the diedral angle of* P_1 *and* P_2 is the plane angle determined by a plane orthogonal to P_1 and P_2 having as edges the lines of intersection of P with these planes. This angle is equal or supplementary to the angle of the normals $\bar{n}_1$ and $\bar{n}_2$ of the two planes (see fig. 33).

We define *the angle between the oriented planes* $(P_1, \bar{n}_1)$ *and* $(P_2, \bar{n}_2)$ as the angle φ between the vectors $\bar{n}_1$ and $\bar{n}_2$, given by

$$\cos\varphi = \frac{(\bar{n}_1, \bar{n}_2)}{||\bar{n}_1||\ ||\bar{n}_2||}, \quad \varphi \in [0, \pi].$$

Lesson 6

2.4 Distances and angles

4.1 Distance between a point and a straight line. Let be the straight line

$$D : \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}.$$

Then D contains the point $M_0(x_0, y_0, z_0)$ and has the director vector $\bar{a}(l, m, n)$. Let $A \in E_3$ and let A' be the orthogonal projection of the point A on D; then *the distance between A and D* is exactly the length of the segment AA' (see fig. 34),

$$d(A; D) = d(A, A') = \frac{||\bar{a} \times \overline{M_0 A}||}{||\bar{a}||}.$$

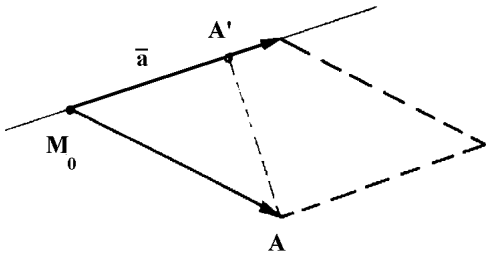


Fig. 34

We remark that $d(A; D) = \inf_{M \in D} d(A, M)$.

4.2 Distance between a point and a plane. Let be the plane

$$P : \quad ax + by + cz + d = 0;$$

its normal vector is $\bar{n}(a, b, c)$. Let $M_0(x_0, y_0, z_0)$ be an exterior point, and let $M_1(x_1, y_1, z_1)$ be its orthogonal projection on P. Then the distance from M_0 to P is exactly the distance between M_0 and M_1 (see fig. 35), and is given by the formula

$$d(M_0, P) = d(M_0, M_1) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

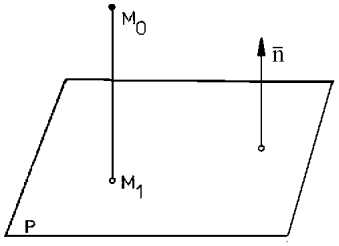


Fig. 35

Indeed, since $\overline{M_0M_1}, \bar{n}$ are collinear, we obtain the relation

$$(\bar{n}, \overline{M_0M_1}) = a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) = \pm\sqrt{a^2 + b^2 + c^2} \cdot d(M_0, P),$$

where, using that $M_1 \in P \Leftrightarrow ax_1 + by_1 + cz_1 = -d$, we infer the desired formula for $d(M_0, P)$.

4.3 Angle between an oriented straight line and an oriented plane. Let be the plane $P: \quad ax + by + cz + d = 0$; its normal vector is $\bar{n}(a, b, c)$. Let be the straight line $D: \frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$; its director vector is $\bar{a}(l, m, n)$. Then the angle θ between the oriented straight line $(D, \bar{a})$ and the oriented plane $(P, \bar{n})$ is the angle between the vectors $\bar{n}$ and $\bar{a}$ (see fig. 36), given by the relation

$$cos\theta = \frac{(\bar{a}, \bar{n})}{||\bar{a}||||\bar{n}||}, \quad \theta \in [0, \pi].$$

We remark that $D||P \Leftrightarrow (\bar{n}, \bar{a}) = 0$ and $D \perp P \Leftrightarrow \bar{n} \times \bar{a} = \bar{0}$.

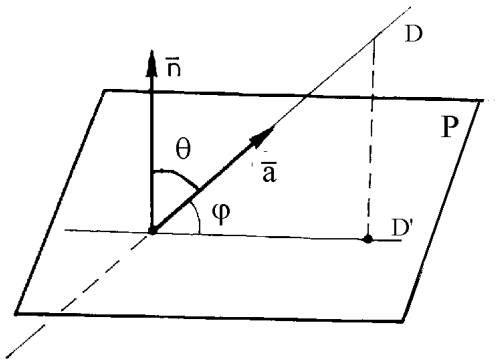


Fig. 36

4.4 Common perpendicular of two straight lines. Let be the two distinct non-parallel lines (see fig. 37)

$$D_1 : \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1};\ \ D_2 : \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}.$$

Their director vectors are $\bar{a}_i(l_i, m_i, n_i), i = 1, 2$. Let

$$\bar{n}(l_3, m_3, n_3) = \bar{a}_1 \times \bar{a}_2;$$

then $\bar{n}$ is orthogonal both to $\bar{a}_1$ and $\bar{a}_2$. Let P_1 be the plane determined by D_1 and $\bar{n}$, and let P_2 be the plane determined by D_2 and $\bar{n}$.

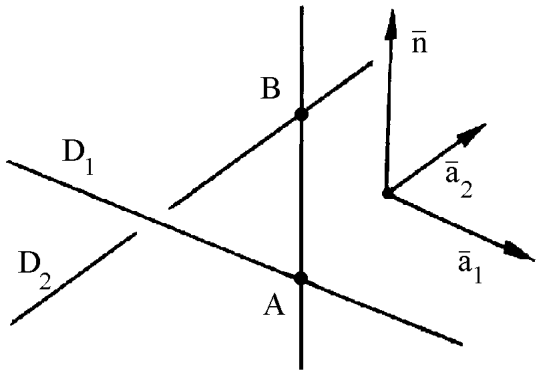


Fig. 37

Then the straight line $D = P_1 \cap P_2$ (see fig. 37) is called *the common perpendicular of the lines* D_1 *and* D_2. Its equations are

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

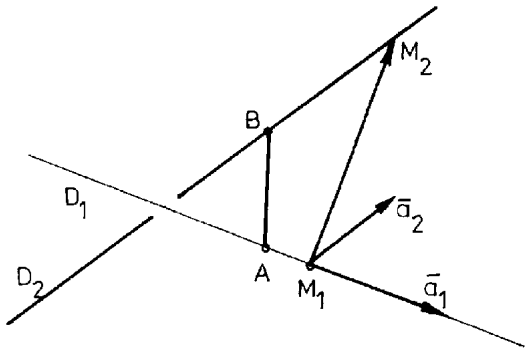


Fig. 38

Let AB be the distance between D_1 and D_2. For obtaining this distance, we take a plane through D_1 which is parallel to D_2; the distance from a point of D_2 to this plane is exactly AB. We also have the relation (see fig. 38)

$$d(D_1, D_2) = d(A, B) = \inf_{M_1 \in D_1, M_2 \in D_2} d(M_1, M_2).$$

2.5 Problems

1. Represent the points $A(5, 0, 0)$, $B(0, -2, 0)$, $C(0, 0, 3)$, $D(-3, 2, 0)$, $E(0, -1, -4)$, $F(2, 0, 4)$, $G(3, -5, 8)$, and find the position vector of G relative to the Cartesian system $\{O;\ \bar{i},\ \bar{j}; \bar{k}\}$.

2. Write the equations of a straight line which passes through the point $A(1, 1, -2)$ and is parallel to the straight line D, for the cases

1) $D : \dfrac{x-4}{2} = \dfrac{y+2}{3} = \dfrac{z+1}{4}$;

2) $D : \begin{cases} x - y - 3z + 2 = 0 \\ 2x - y + 2z - 3 = 0. \end{cases}$

3. Calculate the distance from the point $A(1, 1, 1)$ to the straight line D in each of the following cases

1) $D : \dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{3}$;

2) $D : \begin{cases} x - y + z = 0 \\ x + y - z + 2 = 0. \end{cases}$

4. Let be the straight lines

$$D_1:\ \frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{\alpha},\qquad D_2:\ \frac{x-2}{2}=\frac{y}{2}=\frac{z+1}{\alpha}.$$

1) determine α such that the straight lines D_1 and D_2 intersect and write the equation of the plane P which contains these straight lines;
2) compute $D(M_0,P)$, for $M_0(5,-4,1)$.

5. Let D_1 and D_2 two parallel straight lines, whose director vectors are respectively $(1,0,1)$ and $(-1,1,2)$. Write the parametric equations of the common perpendicular of D_1 and D_2, which contains the point $A(2,3,0)$.

6. Find the equations of the common perpendicular of the straight lines D_1 and D_2 and compute the distance between D_1 and D_2 in each of the following cases

1) $D_1:\ \frac{x-1}{1}=\frac{y+2}{4}=\frac{z}{1},\qquad D_2:\ \frac{x}{3}=\frac{y}{1}=\frac{z-1}{1},$

2) $D_1: x=1+2r,\ y=3+r,\ z=-2+r,\ \ r\in\mathbb{R},$

$D_2: x=1+s;\ y=-2-4s,\ z=9+2s,\ s\in\mathbb{R}.$

7. Consider the points

$$A(1,3,0),\ B(3,-2,1),\ C(\alpha,1,-3),\ D(7,-2,3).$$

Find α such that the points be coplanar; for this value of α, write the Cartesian equation of the plane which contains these points.

8. Write the equation of the plane which passes through the point $A(1,1,-1)$ and is perpendicular to D in each of the following cases

1) $D:\ \dfrac{x}{1}=\dfrac{y-1}{2}=\dfrac{z+1}{-1},$

2) $D:\ \begin{cases} x-y=0 \\ x+2y-z+1=0 \end{cases}$

9. Write the equation of the plane which passes through the point $M_0(-1,3,3)$ and contains the straight line D, where

1) $D:\ \begin{cases} x+2y+3z-1=0 \\ 2x-y-z-3=0 \end{cases},$

2) $D:\ \dfrac{x-4}{1}=\dfrac{y+2}{0}=\dfrac{z+1}{2}$,

3) $D:\ x=1+2t,\ y=-1+t,\ z=1+3t,\ t\in\mathbb{R}$.

10. Compute the angle between the straight lines

$$D_1:\ \frac{x-1}{1}=\frac{y+2}{4}=\frac{z}{1},\qquad D_2:\ \frac{x}{-1}=\frac{y}{8}=\frac{z}{1},$$

and write the equation of the plane which contains them.

11. Let be the planes

$$P: x-2y+2z-7=0,\quad Q: 2x-y-2z+1=0,\quad R: 2x+2y+z-2=0.$$

1) Show that the planes are perpendicular.
2) Find the common point of these three planes.
3) Find the distance from $A(2,4,7)$ to the plane P.

12. Consider the plane $P: x-y+2z+2=0$ and the points $A(2,3,-1)$, $B(1,2,4)$. Write the equations of the straight line passing through the symmetric points of A and B with respect to the plane P.

13. Find the projection of the straight line D on the plane

$$P: x+6y-z-2=0$$

in each of the following cases

1) $D:\begin{cases} x-y-3z+2=0\\ 2x-y+2z-3=0,\end{cases}$

2) $D:\ \dfrac{x-4}{2}=\dfrac{y+2}{3}=\dfrac{z+1}{4}$.

14. Let be the plane $P:\ 2x+y-z-1=0$ and the points

$$A(1,3,2),\ B(-1,2,1),\ C(0,1,-2),\ D(2,0,-1).$$

Find which of these points are on the same side as the origin $O(0,0,0)$ with respect to this plane.

Chapter 3

Transformations of coordinates

3.1 Transformations of Cartesian coordinates

Let be in E_3 two Cartesian frames $Oxyz$ and $O'x'y'z'$; let (x, y, z) and (x', y', z') be the coordinates of some point $M \in E_3$ relative to $Oxyz$ and $O'x'y'z'$, respectively. We want to express (x, y, z) as functions of (x', y', z'). We remark that the following relations hold true

$$\overline{OM} = \overline{OO'} + \overline{O'M}, \quad \begin{cases} \overline{OM} = x\bar{i} + y\bar{j} + z\bar{k}, \\ \overline{OO'} = a\bar{i} + b\bar{j} + c\bar{k}, \\ \overline{O'M} = x'\bar{i}' + y'\bar{j}' + z'\bar{k}'. \end{cases}$$

Also, the base of the second frame decomposes with respect to the first frame as

$$\begin{cases} \bar{i}' = (\bar{i}', \bar{i})\bar{i} + (\bar{i}', \bar{j})\bar{j} + (\bar{i}', \bar{k})\bar{k} \\ \bar{j}' = (\bar{j}', \bar{i})\bar{i} + (\bar{j}', \bar{j})\bar{j} + (\bar{j}', \bar{k})\bar{k} \\ \bar{k}' = (\bar{k}', \bar{i})\bar{i} + (\bar{k}', \bar{j})\bar{j} + (\bar{k}', \bar{k})\bar{k}. \end{cases}$$

These decompositions can be rewritten as

$$\begin{cases} \bar{i}' = a_{11}\bar{i} + a_{21}\bar{j} + a_{31}\bar{k} \\ \bar{j}' = a_{12}\bar{i} + a_{22}\bar{j} + a_{32}\bar{k} \\ \bar{k}' = a_{13}\bar{i} + a_{23}\bar{j} + a_{33}\bar{k}. \end{cases}$$

Since $\{\bar{i}, \bar{j}, \bar{k}\}$ and $\{\bar{i}', \bar{j}', \bar{k}'\}$ are orthonormal bases, it results that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is an orthogonal matrix, i.e., ${}^tAA = I_3$, (homework), and hence $detA = \pm 1$. We infer the required formula

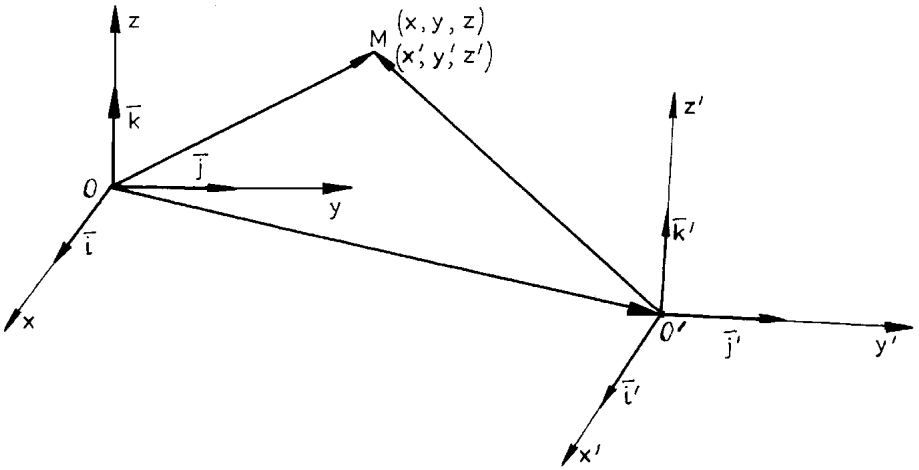


Fig. 39

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This matrix relation gives the passage from $Oxyz$ to $Ox'y'z'$. If $\bar{i}' = \bar{i},\ \bar{j}' = \bar{j},\ \bar{k}' = \bar{k}$, then the transformation above is a *translation* (see fig. 39).

If $O' = O$ and $detA = 1$, then the transformation above is a *rotation* (see fig. 40)

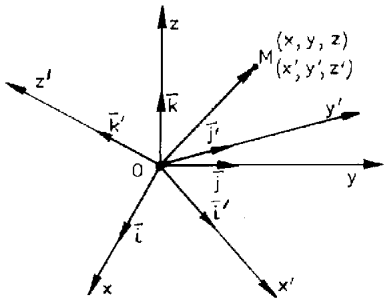


Fig. 40

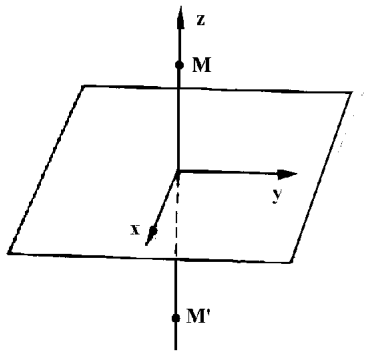


Fig. 41

If $O' = O$ and $detA = -1$, then the transformation above is a *rotation followed by a symmetry with respect to a plane* (see fig. 41).

Also, the rotation of angle θ in trigonometric sense around Oz is provided by the change of basis (see fig. 42)

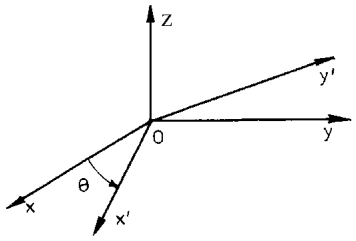


Fig. 42

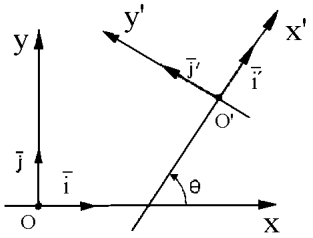


Fig. 43

$$\begin{cases} \bar{i}' = \bar{i}\cos\theta + \bar{j}\sin\theta \\ \bar{j}' = -\bar{i}\sin\theta + \bar{j}\cos\theta \\ \bar{k}' = \bar{k} \end{cases}$$

and has the equations

$$\begin{cases} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \\ z = z'. \end{cases}$$

As a particular case, one can obtain the roto-translation of angle θ in trigonometric sense in the plane, given by the equations (see fig. 43)

$$\begin{cases} x = x'\cos\theta - y'\sin\theta + a \\ y = x'\sin\theta + y'\cos\theta + b. \end{cases}$$

3.2 Polar, cylindrical and spherical coordinates

2.1 Polar coordinates. Let xOy be the Cartesian system of coordinates (see fig. 44). The pairs (ρ, θ) with the significance on the figure are called *polar coordinates* of the point M.

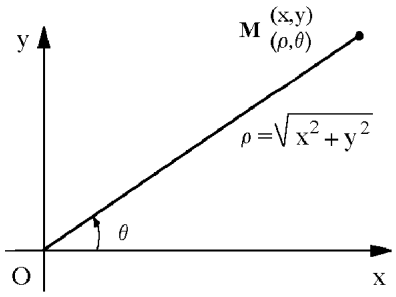


Fig. 44

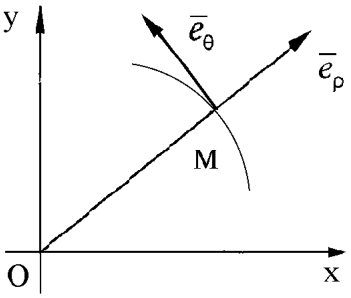


Fig. 45

Then the relations between the Cartesian coordinates (x, y) of some point M and the polar coordinates (ρ, θ) of the same point M, are

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}, \quad \rho \geq 0, \quad \theta \in [0, 2\pi].$$

This relation determines a bijection between $(0, \infty) \times [0, 2\pi)$ and $\mathbb{R}^2 \backslash \{O\}$.

Coordinate curves:

- the equation $\rho = \rho_0$ represents a circle;
- the equation $\theta = \theta_0$ represents a straight semiline.

One can consider *the polar frame field* $\{M; \bar{e}_\rho, \bar{e}_\theta\}$, whose orthonormal basis is (see fig. 45)

$$\begin{cases} \bar{e}_\rho = \cos\theta\ \bar{i} + \sin\theta\ \bar{j} \\ \bar{e}_\theta = -\sin\theta\ \bar{i} + \cos\theta\ \bar{j}. \end{cases}$$

2.2 Cylindrical coordinates. Let $Oxyz$ be the Cartesian system of coordinates. The triples (ρ, θ, z) with the significance on the figure are called *cylindrical coordinates* of the point M (see fig. 46).

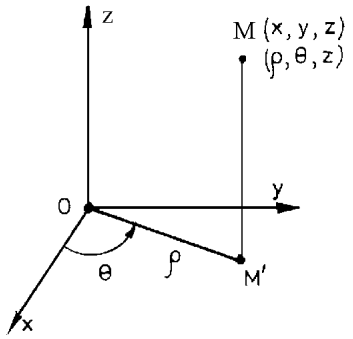


Fig. 46

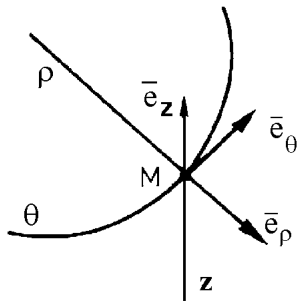


Fig. 47

Then the relations between the Cartesian coordinates (x, y, z) of some point M and the cylindrical coordinates (ρ, θ, z) of the same point M, are

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}, \quad \rho \geq 0, \quad \theta \in [0, 2\pi], \quad z \in \mathbb{R}.$$

These relations determine a bijection between $(0, \infty) \times [0, 2\pi) \times \mathbb{R}$ and $\mathbb{R}^3 \backslash Oz$.

Coordinate surfaces:

- the equation $\rho = \rho_0$ represents a circular cylinder with generators parallel to Oz;
- the equation $\theta = \theta_0$ describes a semiplane;
- the equation $z = z_0$ describes a plane parallel to xOy without the point $(0, 0, z_0)$.

Coordinate curves:

- the equations $\begin{cases} \rho = \rho_0 \\ \theta = \theta_0 \end{cases}$ describe a straight line orthogonal to xOy;
- the equations $\begin{cases} \theta = \theta_0 \\ z = z_0 \end{cases}$ describe a straight semiline parallel to xOy;
- the equations $\begin{cases} z = z_0 \\ \rho = \rho_0 \end{cases}$ describe a circle with the center on Oz and included in a plane parallel to xOy.

One can consider *the cylindrical frame field* (*moving frame*)

$$\{M;\ \bar{e}_\rho, \bar{e}_\theta, \bar{e}_z\},$$

which is defined on $\mathbb{R}^3\backslash Oz$, and whose orthonormal basis is (see fig. 47)

$$\begin{cases} \bar{e}_\rho = \cos\theta\ \bar{i} + \sin\theta\ \bar{j} \\ \bar{e}_\theta = -\sin\theta\ \bar{i} + \cos\theta\ \bar{j} \\ \bar{e}_z = \bar{k}. \end{cases}$$

2.3 Spherical coordinates. Let $Oxyz$ be the Cartesian system of coordinates. The triples (r, φ, θ) with the significance on the figure are called *spherical coordinates* of the point M (see fig. 48).

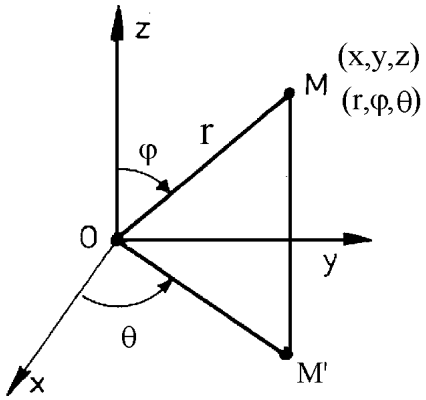


Fig. 48

Then the relations between the Cartesian coordinates (x, y, z) of some point M and the spherical coordinates (r, φ, θ) of the same point, are

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}, \quad r \geq 0, \quad \theta \in [0, 2\pi], \quad \varphi \in [0, \pi].$$

These relations determine a bijection between $(0, \infty) \times (0, \pi) \times [0, 2\pi)$ and $\mathbb{R}^3\backslash Oz$.

Coordinate surfaces:

- the equation $r = r_0$, describes a sphere with center in origin without the poles;
- the equation $\varphi = \varphi_0$ describes a cone without the vertex (origin);
- the equation $\theta = \theta_0$ describes a semiplane.

Coordinate curves:

- the equations $\begin{cases} r = r_0 \\ \varphi = \varphi_0 \end{cases}$ represent a circle with center on Oz which is contained

in a plane parallel to xOy;

- the equations $\begin{cases} \varphi = \varphi_0 \\ \theta = \theta_0 \end{cases}$ represent a semiline;
- the equations $\begin{cases} \theta = \theta_0 \\ r = r_0 \end{cases}$ represent a semicircle.

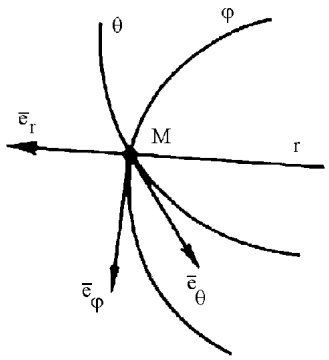


Fig. 49

One can consider *the spherical frame field* (*moving frame*) $\{M;\ \bar{e}_r, \bar{e}_\varphi, \bar{e}_\theta\}$ which is defined on $\mathbb{R}^3 \backslash Oz$, and whose orthonormal basis is (see fig. 49)

$$\begin{cases} \bar{e}_r = \sin\varphi\cos\theta\ \bar{i} + \sin\varphi\sin\theta\ \bar{j} + \cos\varphi\ \bar{k} \\ \bar{e}_\varphi = \cos\varphi\cos\theta\ \bar{i} + \cos\varphi\sin\theta\ \bar{j} - \sin\varphi\ \bar{k} \\ \bar{e}_\theta = -\sin\theta\ \bar{i} + \cos\theta\ \bar{j}. \end{cases}$$

3.3 Problems

1. Let be the Cartesian system $Oxyz$ and the points

$$A(3,0,0),\ B(0,2,0),\ C(0,0,6).$$

We build the rotated system $Ox'y'z'$ as follows: Oz' has the direction and the sense of the height OO' of the tetrahedron $OABC$; Oy' is parallel to $O'A'$, where A' is the foot of the height from A in the triangle ABC; the axis Ox' is chosen such that the system $Ox'y'z'$ to be positive oriented.

Write the matrix of the rotation and find the invariant direction (the real 1-dimensional eigen-space) with respect to this rotation.

2. Let be the straight line

$$D: \ \frac{z}{1} = \frac{y}{-2} = \frac{z}{2}.$$

Let $\bar{i}'$ be the director unit vector of the line D. Let $\bar{j}'$ be a unit vector which is orthogonal on D and which belongs to the plane yOz, and $\bar{k}'$ the unit vector chosen such that $\{\bar{i}', \bar{j}', \bar{k}'\}$ is an orthonormal basis.

Determine the formulas of change of basis and compare the orientations of the two bases.

3. 1) The polar coordinates of a point are $\rho = 2, \theta =$ atan 7. Find the Cartesian coordinates.

2) Find the Cartesian coordinates of the points $A(2; \pi/6; -2)$ and $B(1, 0, 1)$ given in cylindrical coordinates.

3) A point M on the sphere of radius $r = 4$ has the coordinates $\theta = 60^o, \varphi = 30^o$. Which are the cylindrical coordinates of M ?

4. 1) Rewrite in cylindrical coordinates the Cartesian equation of a cylinder of equation $x^2 + y^2 = a^2$.

2) Rewrite in spherical coordinates the Cartesian equation $x^2 + y^2 + z^2 = a^2$. Determine the equations expressed in spherical coordinates of the great circle of the sphere which contains the points $(a, \theta_1, \vec{\alpha}_1)$ and $(a, \theta_2, \vec{\alpha}_2)$.

3) Rewrite in spherical coordinates the following equations

$$(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2), \ \ (x^2 + y^2 + z^2)^2 \cdot (x^2 + y^2) = 4a^2x^2y^2.$$

5. Let be the points $A(5; \ \pi/3 \ , 4), \ \ B(7, \ 4\pi/3 \ , -2), \ \ C(2, \ 5\pi/6 \ , -1)$ in cylindric coordinates. Show that A and B belong to a plane which contains Oz. Find the Cartesian coordinates of the points A and C, and the distance between them.

6. Show that the angle of the straight lines D_1 and D_2 which are given in spherical coordinates by $D_1: \ \theta = \theta_1, \vec{\alpha} = \vec{\alpha}_1$ and $D_2: \ \theta = \theta_2, \vec{\alpha} = \vec{\alpha}_2$, is given by the relation

$$\cos\alpha = \sin\vec{\alpha}_1 \sin\vec{\alpha}_2 + \cos(\theta_1 - \theta_2)\cos\vec{\alpha}_1 \cos\vec{\alpha}_2.$$

7. Express the cylindrical coordinates as functions of the spherical ones and conversely.

8. Find the coordinates of the point $A(1, -2, 4)$ with respect to the roto-translated system of coordinates $x'O'y'z'$ which is obtained from the canonic system

$xOyz$ by a translation of vector $\overrightarrow{OO'}$ followed by a rotation, such that its axes are oriented by the vectors

$$\bar{i}+\bar{j},\ \bar{i}-\bar{j}+\bar{k},\ -2\bar{k}.$$

Chapter 4

Conics

Lesson 7

4.1 Conics

1. The definition of a conic. Let E_2 be the space of Euclidean plane geometry and $\{O, \bar{i}, \bar{j}\}$ a Cartesian frame which permits the identification $E_2 \equiv \mathbb{R}^2$. We shall call *affine quadratic form* a function $g : \mathbb{R}^2 \to \mathbb{R}$,

$$g(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

in which at least one of the real coefficients a_{11}, a_{12}, a_{22} is non-zero.

We consider the *level sets* of the affine form g,

$$\{M(x, y) |\ \ g(x, y) = c\},\ \text{for } c \in \mathbb{R},$$

and we shall call c the *height of the level set* and $g^{-1}(c)$ *the level set at height* c. Obviously, one can consider $c = 0$, using a translation in the range of g.

1.1 Definition. The set $\Gamma : g(x, y) = 0$ is called *conic* or *algebraic curve of order two.*

This definition is valid, since the degree of the equation is invariant relative to the choice of the Cartesian coordinate system, i.e., after performing a roto-translation

$$\begin{cases} x = x' \cos\theta - y' \sin\theta + a \\ y = x' \sin\theta + y' \cos\theta + b \end{cases}$$

the equation $g(x,y)=0$ of order two becomes $h(x',y')=0$, which is also of order two.

We also remark that, since $\Gamma = g^{-1}(\{0\})$ and g is continuous, Γ is a closed set in $\mathbb{R}^2$. In the sequel we prove that a conic is congruent with one of the following curves: circle, ellipse, hyperbola, parabola, pair of straight lines, a point or a void set (see fig. 50). In this sense, we shall use the change of Cartesian coordinates and elements of theory of quadratic forms.

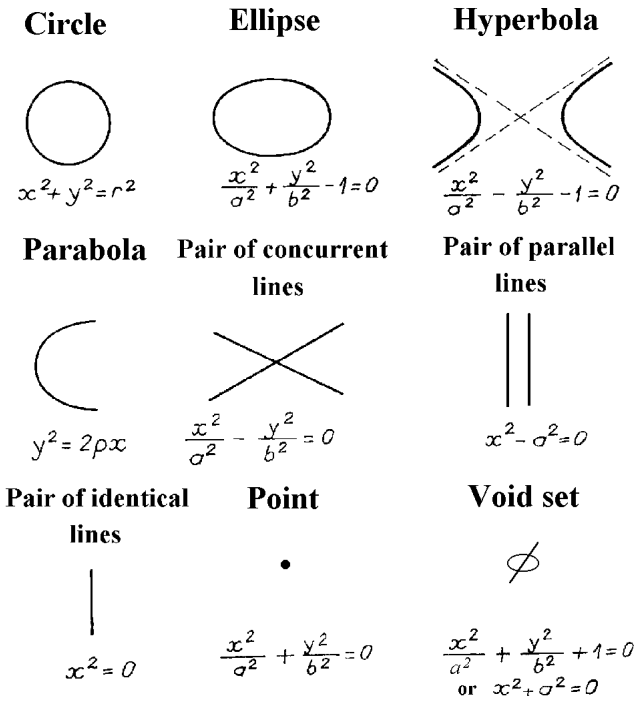


Fig. 50

1.2 Theorem. *The following numbers (called the invariants of the conic Γ)*

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{21} & a_{22} & a_{20} \\ a_{01} & a_{02} & a_{00} \end{vmatrix}, \quad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad I = a_{11}+a_{22},$$

where $a_{ij}=a_{ji},\ i,j=1,2,3$, are invariant under roto-translations.

Proof. (homework) After a roto-translation, the three quantities become respectively $\Delta',\ \delta',\ I'$ and we have

$$\Delta=\Delta',\ \delta=\delta',\ I=I'.$$

Considering the invariants of a conic, the corresponding types are obtained according to the following table

<table>
<tr><th colspan="3">Conditions</th><th>The conic</th></tr>
<tr><td rowspan="3">$\Delta = 0$</td><td colspan="2">$\delta < 0$</td><td>$\Gamma = \{(x_0, y_0)\}$</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>$\Gamma = D_1 \cup D_2$, where D_1 and D_2 are parallel or confounded lines, or $\Gamma = \emptyset$</td></tr>
<tr><td colspan="2">$\delta > 0$</td><td>$\Gamma = D_1 \cup D_2$, where D_1 and D_2 are concurrent lines. If $I = 0$, then $D_1 \perp D_2$</td></tr>
<tr><td rowspan="4">$\Delta \neq 0$</td><td rowspan="2">$\delta > 0$</td><td>$I\Delta > 0$</td><td>ellipse</td></tr>
<tr><td>$I\Delta < 0$</td><td>$\Gamma = \emptyset$</td></tr>
<tr><td colspan="2">$\delta = 0$</td><td>parabola</td></tr>
<tr><td colspan="2">$\delta < 0$</td><td>hyperbola. If $I = 0$, then equilateral hyperbola</td></tr>
</table>

4.2 Centre of symmetry for a conic

For an ellipse or hyperbola the *axes of the canonic frame* are their axes of symmetry, and the *origin of the canonic frame* is their center of symmetry.

For the parabola $y^2 = 2px$, Ox is the axis of symmetry.

Let be the conic $\Gamma : g(x, y) = 0$. We say that $C(x_0, y_0) \in E_2$ is a *critical point of* g, if its coordinates (x_0, y_0) satisfy the linear system

$$\frac{\partial g}{\partial x}(x, y) = 0, \quad \frac{\partial g}{\partial y}(x, y) = 0,$$

or, equivalently,

$$\begin{cases} a_{11}x + a_{12}y + a_{10} = 0 \\ a_{21}x + a_{22}y + a_{20} = 0. \end{cases}$$

We say that $C(x_0, y_0) \in E_2$ is a *center of symmetry* of the conic Γ if

$$\mathrm{sym}_C M \in \Gamma, \quad \forall M \in \Gamma.$$

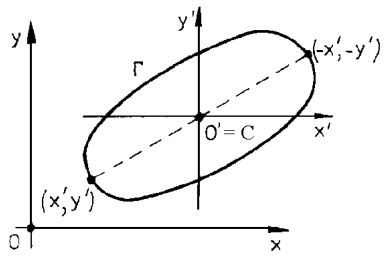


Fig. 51

2.1 Theorem. *The point $C(x_0, y_0)$ is a center of symmetry of the conic Γ : $g(x, y) = 0$ iff $C(x_0, y_0)$ is a critical point of the function g.*

Proof. Let the conic Γ admit the center of symmetry $C(x_0, y_0)$. We make the translation

$$\begin{cases} x = x_0 + x' \\ y = y_0 + y' \end{cases}$$

whence we obtain the equation of Γ relative to the new system of coordinates $x'Cy'$ (see fig. 51),

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + 2g_{x_0}x' + 2g_{y_0}y' + g(x_0, y_0) = 0,$$

where we denoted $g_{x_0} = \dfrac{\partial g}{\partial x}(x_0, y_0)$, $g_{y_0} = \dfrac{\partial g}{\partial y}(x_0, y_0)$. We can check that the condition

$$(x', y') \in \Gamma \to (-x', -y') \in \Gamma$$

is equivalent to fulfilling the relations $\begin{cases} \frac{\partial g}{\partial x}(x_0, y_0) = 0 \\ \frac{\partial g}{\partial y}(x_0, y_0) = 0, \end{cases}$ i.e., (x_0, y_0) have to satisfy the linear system

$$\begin{cases} a_{11}x + a_{12}y + a_{10} = 0 \\ a_{21}x + a_{22}y + a_{20} = 0, \end{cases}$$

whose determinant is exactly $\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. We have the following cases:

1) If $\delta \neq 0$, then we have a unique center. The equation of Γ reduced to the center is

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + g(x_0, y_0) = 0.$$

Such conics are: circle, ellipse, hyperbola, pair of concurrent lines, point, void set;

2) If $\delta = 0, \Delta \neq 0$, we have an incompatible system, and the conic is a parabola;

3) If $\delta = 0, \Delta = 0$, we have a straight line of centers. The corresponding conics are pairs of parallel or confounded lines and the void set.

4.3 Reduction of the equation of a conic to the canonic form

Let be the conic

$$\Gamma: \quad g(x,y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0.$$

To obtain the canonic equation of Γ, we consider two cases.
1) $a_{12} = 0$, when we make a translation;
2) $a_{12} \neq 0$, when first we make a rotation.

The method of eigenvalues

Suppose $a_{12} \neq 0$. The type of the conic is described by the quadratic form

$$\varphi(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2.$$

The associated matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, has the characteristic polynomial

$$P(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix},$$

whose roots λ_1 and λ_2 are real numbers (since A is a symmetric matrix).
We have the cases

1) λ_1, λ_2 have opposite sign; in this case $\delta < 0$ and we say that the conic Γ is *of hyperbolic type*;

2) λ_1, λ_2 have the same sign; in this case $\delta > 0$ and we say that the conic Γ is *of elliptic type*;

3) one root is zero; in this case $\delta = 0$ and we say that the conic Γ is *of parabolic type*.

For $i = 1, 2$, the linear systems

$$\begin{cases} (a_{11} - \lambda_i)u_i + a_{12}v_i = 0 \\ a_{21}u_i + (a_{22} - \lambda_i)v_i = 0 \end{cases}$$

give the eigenvectors $(u_1, v_1), (u_2, v_2)$ of the matrix A, which are orthogonal. We find the corresponding unit vectors $\bar{e}_1, \bar{e}_2$ and we construct the matrix $R = [\bar{e}_1, \bar{e}_2]$;

R is an orthogonal matrix, and hence $detR = \pm 1$. If $detR = 1$, we have a rotation. If $detR = -1$, then we have a rotation and a symmetry; in this case we change $\bar{e}_1$ to $-\bar{e}_1$, or interchange $\bar{e}_1$ with $\bar{e}_2$, to eliminate the symmetry. Then, the change of coordinates (rotation)

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x' \\ y' \end{pmatrix}$$

reduce the quadratic form φ to

$$\lambda_1 x'^2 + \lambda_2 y'^2,$$

and the unit vectors $\bar{e}_1, \bar{e}_2$ give the directions of Ox', Oy', respectively. Then follows a discussion and we make perfect squares in the equation of Γ.

If $\lambda_1, \lambda_2 \neq 0$, then we obtain an equation of the form

$$\lambda_1(x' + \ldots)^2 + \lambda_2(y' + \ldots)^2 + a = 0.$$

Denoting $x'' = x' + \ldots, \quad y'' = y' + \ldots$ respectively, i.e., making a translation, we infer the canonic form, which indicates the type of the conic Γ: ellipse, hyperbola, parabola, pair of straight lines (which may even coincide), etc.

Example.
Obtain the canonic form of the equation

$$3x^2 - 4xy - 2x + 4y - 3 = 0,$$

and draw the corresponding conic.

Solution. The matrix of the quadratic form $\varphi(x, y) = 3x^2 - 4xy$ is $\begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix}$. The characteristic equation of this matrix is

$$\lambda^2 - 3\lambda - 4 = 0,$$

and has the roots $\lambda_1 = -1, \ \lambda_2 = 4$.

The coordinates (u_1, v_1) of the eigenvector which corresponds to $\lambda_1 = -1$ is the solution of the system

$$\begin{cases} 4u_1 - 2v_1 = 0 \\ -2u_1 + v_1 = 0, \end{cases}$$

i.e., $(k, 2k), \ k \in \mathbb{R}\backslash\{0\}$. By normalization, we obtain the unit eigenvector $\bar{e}_1 = (1/\sqrt{5} \ , \ 2/\sqrt{5})$. Similarly, for $\lambda_2 = 4$ we find $\bar{e}_2 = (2/\sqrt{5} \ , \ -1/\sqrt{5})$. The matrix

$R = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}$ represents a rotation followed by a symmetry, because $\det R = -1$. For replacing R with a matrix of a rotation, we use one of the following procedures.

1) We renote $\lambda'_1 = \lambda_2,\ \lambda'_2 = \lambda_1$ and accordingly, $\bar{e}'_1 = \bar{e}_2,\ \bar{e}'_2 = \bar{e}_1$. The rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{cases} x = \frac{1}{\sqrt{5}}(2x' + y') \\ y = \frac{1}{\sqrt{5}}(-x' + 2y') \end{cases}$$

leads to

$$4x'^2 - y'^2 - \frac{8}{\sqrt{5}}x' + \frac{6}{\sqrt{5}}y' - 3 = 0.$$

We build perfect squares forcing the common multipliers 4 and -1, and rewrite the equation in the equivalent form

$$4\left(x' - \tfrac{1}{\sqrt{5}}\right)^2 - \left(y' - \tfrac{3}{\sqrt{5}}\right)^2 = 0.$$

Using the translation of the system $x'Oy'$ to the point $C_1\left(\frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$, given by

$$x' = x'' + \frac{1}{\sqrt{5}},\ y' = y'' + \frac{3}{\sqrt{5}},$$

we obtain the canonic equation of a hyperbola

$$\frac{x''^2}{1/2} - \frac{y''^2}{2} - 1 = 0.$$

The vertices of this hiperbola are located on the axis C_1x''. For drawing the conic, we perform a rotation of the system of axes xOy. The new coordinate axes Ox', Oy' have the directions of their unit vectors $\bar{e}'_1$ and $\bar{e}'_2$ respectively. The frame $x'Oy'$ is translated to the point C_1 (see fig. 52).

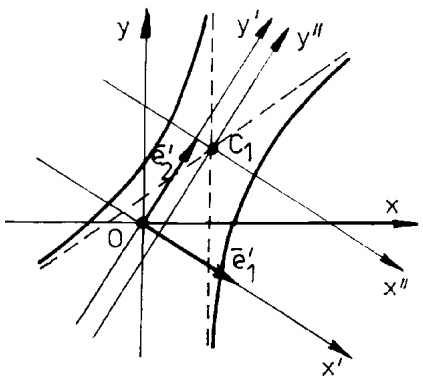


Fig. 52

2) We may also use, as second alternative, the unit vectors

$$\bar{e}''_1 = -\bar{e}_1,\ \bar{e}''_2 = \bar{e}_2.$$

The rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

leads to the equation

$$-x_1^2 + 4y_1^2 + \frac{6}{\sqrt{5}}x_1 + \frac{8}{\sqrt{5}}y_1 - 3 = 0.$$

The directions of the coordinate axes Ox_1, Oy_1 are determined by the unit vectors $\bar{e}''_1$ and $\bar{e}''_2$ respectively. The rotated system is translated to $C_2\left(\frac{-3}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, using the formulas

$$\begin{cases} x_1 = x_2 + \frac{3}{\sqrt{5}} \\ y_1 = y_2 - \frac{1}{\sqrt{5}}. \end{cases}$$

We obtain the canonic equation

$$\frac{x_2^2}{2} - \frac{y_2^2}{1/2} + 1 = 0$$

of the hyperbola, with respect to the canonic frame x_2Oy_2 (see fig. 52). This hyperbola has its vertices on C_2y_2.

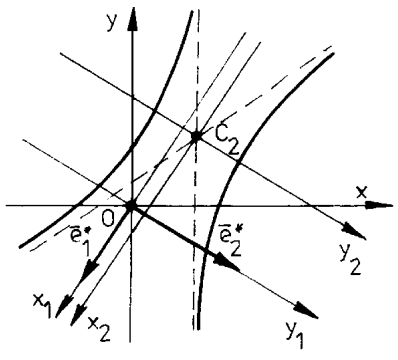


Fig. 53

The roto-translation method
If $a_{12} \neq 0$, then the equation

$$(a_{11} - a_{22}) \sin 2\theta = 2a_{12} \cos 2\theta$$

determines an angle θ and hence a rotation

$$\begin{cases} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{cases}$$

such that the equation $g(x, y) = h(x', y') = 0$ does not contain the term with $x'y'$. Afterwards, if it is necessary, we perform a translation.

Lesson 8

4.4 Intersection of a straight line and a conic

Let be the straight line

$$D: \quad x = x_0 + lt, y = y_0 + mt, \quad t \in \mathbb{R}$$

and the conic

$$\Gamma: \ g(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0.$$

Denote $\varphi(l, m) = a_{11}l^2 + 2a_{12}lm + a_{22}m^2$ and

$$\frac{\partial g}{\partial x_0} = \frac{\partial g}{\partial x}(x_0, y_0), \quad \frac{\partial g}{\partial y_0} = \frac{\partial g}{\partial y}(x_0, y_0).$$

Then the intersection $D \cap \Gamma$ is given by the values of the parameter t which satisfy the algebraic equation

$$t^2\varphi(l, m) + t(lg_{x_0} + mg_{y_0}) + g(x_0, y_0) = 0. \qquad (*)$$

Regarding its solutions, we have the following cases
I. Let $\varphi(l, m) \neq 0$. Then (*) is an equation of order two; in this case the discriminant is

$$q = (lg_{x_0} + mg_{y_0})^2 - 4\varphi(l, m)\ g(x_0, y_0);$$

I.1 If $q > 0$, then the equation (*) has two distinct roots $t_1 \neq t_2$ and for the corresponding points M_1, M_2, we have

$$D \cap \Gamma = \{M_1, M_2\}.$$

I.2 For $q = 0$ we infer $M_1 = M_2$, and D is a tangent line to Γ at $M_1 = M_2$.

I.3 For $q < 0$, D is exterior to Γ, i.e., $D \cap \Gamma = \emptyset$.

Assume that we have a given point $M_0 \in \Gamma$ and we want to determine the equation of the tangent at M_0 (see fig. 54). From $q = 0$ and $g(x_0, y_0) = 0$, we have

$$lg_{x_0} + mg_{y_0} = 0,$$

Imposing $g_{x_0}^2 + g_{y_0}^2 > 0$, the equation of the *tangent* to the conic Γ at its point M_0 is

$$(x - x_0)g_{x_0} + (y - y_0)g_{y_0} = 0.$$

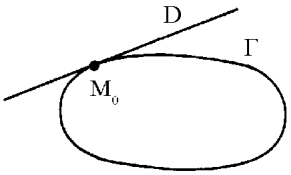


Fig. 54

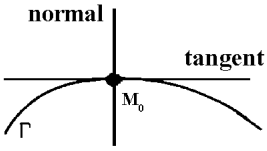


Fig. 55

Also, the *normal straight line to* Γ *at* M_0 has the equation (see fig. 55)

$$\frac{x - x_0}{g_{x_0}} = \frac{y - y_0}{g_{y_0}}.$$

II. If $\varphi(l, m) = 0$, then (l, m) is called *an asymptotic direction for* Γ.

II.1 We suppose that $lg_{x_0} + mg_{y_0} \neq 0$. If $D \cap \Gamma$ consists just of a point, then (*) is an equation of the first order.

II.2 If $lg_{x_0} + mg_{y_0} = 0$, then $D \cap \Gamma = \emptyset$, and the equation $\varphi(l, m) = 0$ rewrites

$$a_{11}l^2 + 2a_{12}lm + a_{22}m^2 = 0. \qquad (**)$$

We have three cases

- for $\delta < 0$, then there exist two different asymptotic directions;
- for $\delta = 0$, then there exists a double asymptotic direction;
- for $\delta > 0$, then we have no asymptotic direction.

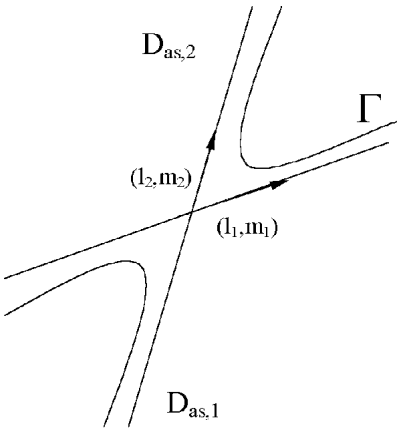


Fig. 56

4.1 Definition. Let Γ be a non-degenerate conic ($\Delta \neq 0$). If (l, m) satisfies (**) and $D \cap \Gamma = \emptyset$, then D is called an *asymptote* of Γ.

We have the following

4.2 Theorem. *An asymptote of* Γ *is described by the equation*

$$lg_x + mg_y = 0,$$

where (l, m) *is an asymptotic direction.*

Corollary. *1) The hyperbola has two asymptotes which pass through the center of the hyperbola (see fig. 56);*
2) The parabola has an asymptotic direction;
3) The ellipse has no asymptotic direction.

4.5 Polar and pole

Let $M_0(x_0, y_0) \in E_3$ and the conic

$$\Gamma: \quad g(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0.$$

The substitutions

$$\begin{cases} x^2 \to xx_0 \\ y^2 \to yy_0 \\ xy \to \frac{1}{2}(x_0y + xy_0), \end{cases} \quad \text{and} \quad \begin{cases} x \to \frac{1}{2}(x + x_0) \\ y \to \frac{1}{2}(y + y_0) \\ k \in \mathbb{R} \to k, \end{cases}$$

are called *halvings*. By the *halving of the equation* of Γ at $M_0(x_0, y_0)$, we understand the equation

$$a_{11}xx_0 + a_{12}(x_0y + xy_0) + a_{22}yy_0 + a_{10}(x + x_0) + a_{20}(y + y_0) + a_{00} = 0.$$

5.1 Theorem. *If $M(x_0, y_0)$ is not center of Γ, then the halving of the equation of Γ represents a straight line.*

Proof. This equation can be written in the form

$$(a_{11}x_0 + a_{12}y_0 + a_{10})x + (a_{12}x_0 + a_{22}y_0 + a_{20})y + a_{10}x_0 + a_{20}y_0 + a_{00} = 0.$$

It is an equation of first order in (x, y) if the coefficients of x and y do not vanish simultaneously.

Suppose that $M_0(x_0, y_0)$ is not center of Γ. The straight line D described by the halving of the equation of Γ at $M_0(x_0, y_0)$ is called *the polar* of Γ. The point M_0 is called *the pole* with reference to the polar.

5.2 Theorem. *Let Γ be a nondegenerate conic ($\Delta \neq 0$).*
If $D: ax + by + c = 0$ is a given straight line which:
- does not contain the center of Γ (for $\delta \neq 0$);
- is different from the axis of the parabola (for $\delta = 0$),
then there exists a point $M_0(x_0, y_0)$ whose polar is just D.

Proof. Identifying the polar with D, we find the linear system

$$\frac{a_{11}x_0 + a_{12}y_0 + a_{10}}{a} = \frac{a_{12}x_0 + a_{22}y_0 + a_{20}}{b} = \frac{a_{10}x_0 + a_{20}y_0 + a_{00}}{c},$$

which has a unique solution $M_0(x_0, y_0)$.

5.3 Theorem. *If $M_0(x_0, y_0) \in \Gamma$ is not the center of Γ, then the corresponding polar of Γ is tangent to Γ at M_0.*

Proof. The equation of the polar can be transcribed as

$$(x - x_0)g_{x_0} + (y - y_0)g_{y_0} + 2g(x_0, y_0) = 0.$$

Adding the condition $g(x_0, y_0) = 0$, we find the equation of the tangent.

5.4 Theorem. *1) If D is the polar relative to M_0 and $D \cap \Gamma = \{Q_1, Q_2\}$, then the intersection between the tangents to Γ at Q_1, Q_2 is exactly the pole M_0 (see fig. 57).*

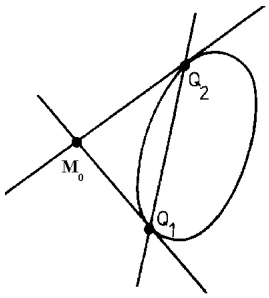


Fig. 57

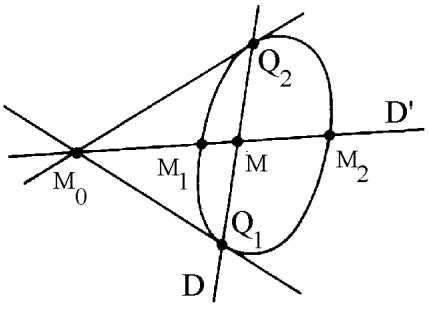


Fig. 58

2) A straight line D' through $M_0 \in E_2$ determines on Γ and on the associated polar line D the points M_1, M, M_2 such that M_0, M divides harmonically the pair M_1, M_2 (see fig. 58).

4.6 Diameters of a conic

Let be the conic $\Gamma:\ g(x,y)=0$ and g_x, g_y the partial derivatives of the function g.

If $\delta \neq 0$, then the solution of the linear system $\begin{cases} g_x = 0 \\ g_y = 0 \end{cases}$ is the center of the conic Γ.

If $\delta = 0, \Delta \neq 0$ (parabola), then there exist the numbers α, β such that $\beta g_y = g_x + \alpha$.

Let $\bar{d}(l,m) \neq \bar{0}$ be a vector which provides a given direction (see fig. 59).

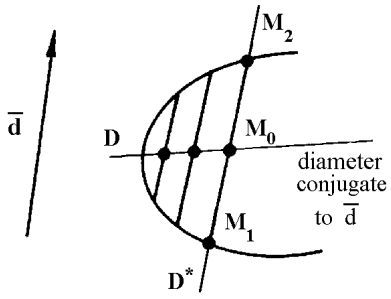


Fig. 59

6.1 Definition. The straight line

$$D: lg_x + mg_y = 0$$

is called a *diameter of the conic* Γ *conjugate to the direction* (l, m).

6.2 Theorem. *The mid-points of the set of chords of* Γ *which are parallel to* $\bar{d}(l, m)$*, lie on the diameter* $D: \ lg_x + mg_y = 0$.

Proof. We consider a straight line secant to Γ,

$$D^*: \quad \begin{cases} x = x_0 + lt \\ y = y_0 + mt \end{cases}, \quad t \in \mathbb{R}.$$

Assume that its intersection with Γ is $D^* \cap \Gamma = \{M_1, M_2\}$ (see fig. 59), and the two points correspond to the (distinct) values t_1, t_2, which are the solutions of the algebraic equation

$$t^2\varphi(l, m) + t(lg_{x_0} + mg_{y_0}) + g(x_0, y_0) = 0.$$

The mid-point of the segment $[M_1M_2]$ has the coordinates

$$\frac{x_1 + x_2}{2} = x_0 + \frac{t_1 + t_2}{2}, \ \frac{y_1 + y_2}{2} = y_0 + \frac{t_1 + t_2}{2}.$$

This point coincides with $M_0(x_0, y_0)$ iff

$$t_1 + t_2 = 0 \Leftrightarrow lg_{x_0} + mg_{y_0} = 0,$$

whence we get $M_0 \in D$. We have the following cases:

1) If $\delta \neq 0, \Delta \neq 0$ (circle, ellipse, hyperbola), then the diameter conjugate to an arbitrary direction determines a pencil of straight lines whose vertex is the center of the conic.

Let us consider two diameters

$$D: lg_x + mg_y = 0; \ \ D': l'g_x + m'g_y = 0.$$

D is said to be *conjugate to* D' if the direction of D is (l', m') and the direction of D' is (l, m); in this case we have

$$\frac{l'}{la_{12} + ma_{22}} = \frac{m'}{-(la_{11} + ma_{12})} \Leftrightarrow a_{11}ll' + a_{12}(lm' + l'm) + a_{22}mm' = 0.$$

Since this equation which describes directions of conjugate diameters is the halving of the equation of the asymptotic directions

$$a_{11}l^2 + 2a_{12}lm + a_{22}m^2 = 0,$$

we remark that for a hyperbola the asymptotes are self-conjugate diameters.

2) If $\delta = 0, \Delta \neq 0$ (parabola), the equation $lg_x + mg_y = 0$ can be rewritten $g_x + \gamma = 0, \gamma \in \mathbb{R}$, and determines a pencil of parallel lines of direction $(a_{12}, -a_{11}) || (a_{22}, -a_{12})$ which coincides with the direction of the axis of symmetry. The attempt of finding conjugate diameters to the direction of the axis of symmetry of the parabola, leads to the equation $a_{12}g_x - a_{11}g_y = 0$ or $a_{22}g_x - a_{12}g_y = 0$, which is an impossibility. Hence the direction of the axis of a parabola has no conjugate diameter.

4.7 Axes of a conic

The axes of the canonic Cartesian frame xOy are axes of symmetry for the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and for the hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0.$$

The Ox axis is axis of symmetry for the parabola $y^2 = 2px$.

Let us describe the axes of symmetry when the conic Γ is given by the general equation $g(x, y) = 0$.

A straight line D is called *axis of symmetry* of the conic Γ if

$$sym_D(M) \in \Gamma, \ \ \forall M \in \Gamma.$$

I. If Γ admits the center $C(x_0, y_0)$, i.e., $\delta \neq 0$, then C belongs to any axis of symmetry.

Suppose (l, m) is a direction orthogonal to the axis of symmetry D. Since the mid-points of the chords of the direction (l, m) lie on D, the axis is automatically a diameter conjugate to the direction (l, m), and consequently has the equation

$$lg_x + mg_y = 0.$$

II. If $\delta = 0$ (conics of parabolic type), then the last equation reduces to

$$g_x + \mu = 0 \quad \text{or} \quad g_y + \lambda = 0.$$

The directions of the axes can be found as follows. The equation $lg_x + mg_y = 0$ can be transcribed as

$$(a_{11}l + a_{12}m)x + (a_{12}l + a_{22}m)y + la_{10} + ma_{20} = 0.$$

The collinearity of normal vectors (l, m) and $(a_{11}l + a_{12}m, a_{12}l + a_{22}m)$ or of the director vectors $(-m, l)$ and $(-(a_{12}l + a_{22}m), a_{11}l + a_{12}m)$, i.e.,

$$\frac{a_{11}l + a_{12}m}{l} = \frac{a_{12}l + a_{22}m}{m}$$

produces the algebraic equation

$$(a_{11} - a_{22})ml + a_{12}(m^2 - l^2) = 0, \qquad (*)$$

which is known as the *equation of the directions of the axes.*

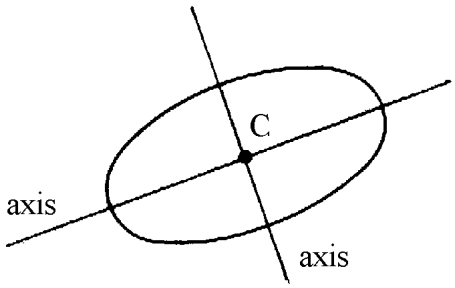


Fig. 60

Suppose $a_{11} \neq a_{22}$ or $a_{12} \neq 0$, i.e., the conic Γ is not a circle.

1) If $\delta \neq 0$ (conics of elliptic or hyperbolic type), then the equation (*) has two orthogonal solutions $(l_i, m_i), i = 1, 2$. The conic Γ has two orthogonal axes, which can be regarded as conjugate diameters (see fig. 60),

$$D_i : \quad l_i g_x + m_i g_y = 0, \quad i = 1, 2$$

or

$$D_i : \quad \frac{x - x_0}{l_i} = \frac{y - y_0}{m_i}.$$

The points $\Gamma \cap D_{1,2}$ are called *the vertices* of the conic.

2) If $\delta = 0$ (conics of parabolic type), then the equation has a double solution (a_{11}, a_{12}) or (a_{21}, a_{22}). Then we have just one axis which has the equation

$$a_{11}g_x + a_{12}g_y = 0, \quad \text{or} \quad a_{12}g_x + a_{22}g_y = 0.$$

These equations reduce to

$$g_x + \mu = 0 \quad \text{or } g_y + \lambda = 0.$$

The direction of the axis is $(a_{12}, -a_{11})$ or $(a_{22}, -a_{21})$.

4.8 Problems

1. Under the influence of some force, a point M moves on the circle

$$x^2 + y^2 - 6x + 4y + 9 = 0.$$

The action of the force ceases at the moment when the point reaches the position $(1, -2)$. Find the trajectory of the point afterwards.

2. Let be the conics

$$\begin{aligned} \Gamma_1 : &\quad x^2 - 2xy + 2y^2 - 4x - 6y + 3 = 0, \\ \Gamma_2 : &\quad x^2 - 2xy - y^2 - 4x - 6y + 3 = 0, \\ \Gamma_3 : &\quad x^2 + 2xy + y^2 + 2x + 2y - 4 = 0. \end{aligned}$$

For each of these conics compute the metric invariants and the coordinates of the center, and find the canonic equation.

3. Find the canonic equations and draw the conics

$$\begin{aligned} \Gamma_1 : &\quad 5x^2 - 4xy + 2y^2 - 16x + 4y - 22 = 0, \\ \Gamma_2 : &\quad 11x^2 - 24xy + 4y^2 + 2x + 16y + 11 = 0, \\ \Gamma_3 : &\quad x^2 - 2xy + y^2 - 4y + 6 = 0. \end{aligned}$$

4. Determine the position of the straight line D relative to the conic Γ in the following cases

1) $D:\ 5x - y - 5 = 0;\ \ \Gamma:\ x^2 - 2xy - 3y^2 - 4x - 6y + 3 = 0;$

2) $D:\ x = 3 + t,\ y = -1 + 2t;\ \ \Gamma:\ x^2 - 2xy - 2y^2 + 7x + 6y + 132 = 0;$

3) $D:\ x = 1 + 8t,\ y = 1 + 7t;\ \Gamma:\ x^2 + 2xy - 4y^2 + 3x - 2y = 0.$

5. Find the coordinates of the pole of the Ox axis relative to the conic

$$\Gamma:\ x^2 - 2y^2 - 3x + 7y + 1 = 0.$$

6. Let be the conic

$$\Gamma:\ 9x^2 + 6xy + y^2 - 5x - 7y - 4 = 0.$$

Find the equation of the diameter which is conjugated to

1) the axis Ox;
2) the axis Oy;
3) the direction $\bar{d}\,(1,-3)$.

Write the equation of the axis of symmetry of this conic.

7. Find the center, the axes and the vertices of the conic

$$\Gamma:\ 16x^2+4xy+19y^2+4x-22y-5=0.$$

Chapter 5

Quadrics

Lesson 9

5.1 Equations of a sphere

Let E_3 be the Euclidean space of elementary geometry. We consider the Cartesian frame $Oxyz$ which realizes a bijection between E_3 and $\mathbb{R}^3$. Then the distance between two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ is

$$d(M_1, M_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This formula generates a real function which is not everywhere differentiable. Its square gives a function of class $\mathcal{C}^\infty$.

We fix a point $C(x_0, y_0, z_0)$ and a real positive number $r > 0$. Then *the sphere S of center C and radius r* is the geometrical locus of the points $M(x, y, z)$ which satisfy the condition $d^2(C, M) = r^2$. Hence (see fig. 61)

$$S: \quad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2,$$

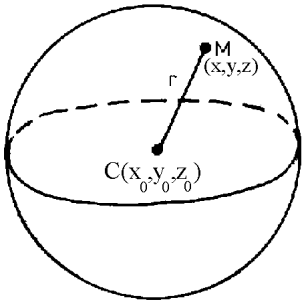


Fig. 61

and the equation is called *the Cartesian implicit equation of the sphere* S. This is equivalent to the three parametric equations, (see fig. 62)

$$S: \begin{cases} x = x_0 + r\cos u \sin v \\ y = y_0 + r \sin u \sin v \\ z = z_0 + r \cos v \end{cases}, \quad u \in [0, 2\pi], v \in [0, \pi],$$

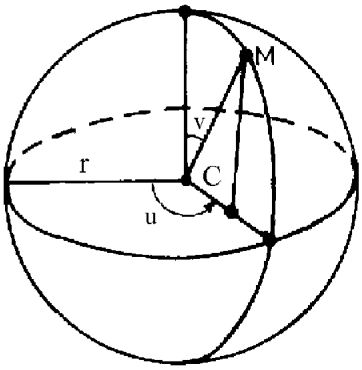


Fig. 62

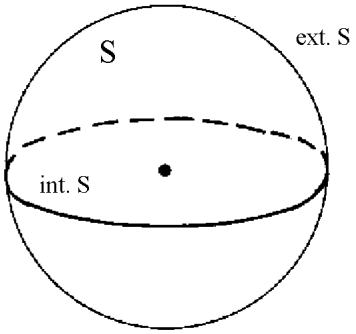


Fig. 63

whence we obtain *the vectorial equation of the sphere*

$$S: \quad \bar{r} = \bar{r}_0 + r(\cos u \sin v\bar{i} + \sin u \sin v\bar{j} + \cos v\bar{k}), \quad u \in [0, 2\pi],\ v \in [0, \pi].$$

On the other hand, we can consider the set Σ described by the Cartesian equation

$$\Sigma: \quad x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0.$$

Denoting $\rho = a^2 + b^2 + c^2 - d$, this equation rewrites

$$(x + a)^2 + (y + b)^2 + (z + c)^2 = \rho.$$

Then we remark that:

- if $\rho > 0$, then Σ is a sphere;
- if $\rho = 0$, then Σ is a point;
- if $\rho < 0$, then Σ is the void set.

Topologically, a sphere is a closed bounded set in $\mathbb{R}^3$, i.e., it is a compact set; it divides the space into two disjoint sets, $intS$ and $extS$ (the interior and the exterior of the sphere). We remark that $intS$ is a convex set (see fig. 63).

We consider a sphere S and a point $M_1(x_1, y_1, z_1) \in S$. A plane through M_1 which is orthogonal to the radius at M_1 is called *the tangent plane to S at M_1* (see fig. 64); its equation can be obtained by *halving* of the equation of the sphere at M_1.

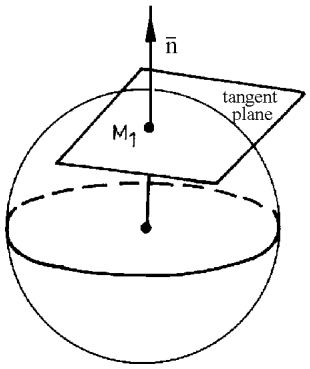


Fig. 64

5.2 Ellipsoid

The subset of $\mathbb{R}^3$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \quad a, b, c > 0$$

is called *an ellipsoid*. The numbers a, b, c are called its *semi - axes*. We remark that the coordinate planes are planes of symmetry for the ellipsoid, that the origin is center of symmetry and that the coordinate axes are axes of symmetry.

The intersections of the planes of coordinates and the ellipsoid are respectively the following ellipses

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \\ z = 0 \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} + \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}, \quad \begin{cases} \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}.$$

The intersection of the ellipsoid with planes parallel to xOy provides the ellipses

$$\begin{cases} \frac{x^2}{\frac{a^2}{c^2}(c^2-k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2-k^2)} - 1 = 0 \\ z = k, \quad k \in (-c, c), \end{cases}$$

which are similar to the first of the three ellipses above (see fig. 65).

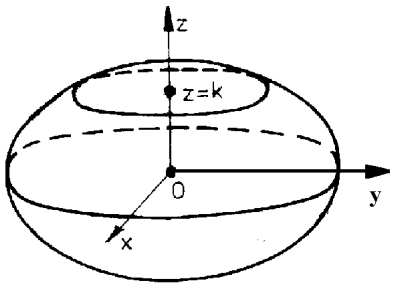


Fig. 65

9.1 Theorem. *The line of intersection of an ellipsoid with an arbitrary plane is an ellipse, a point or the void set.*

Proof. The intersection is an algebraic curve of the second order. The ellipsoid is a closed bounded set (compact set), hence this intersection is finite and cannot be a hyperbola, parabola or pair of straight lines but can be a conic of elliptic type. It follows that the ellipsoid has the shape in the figure (see fig. 66).

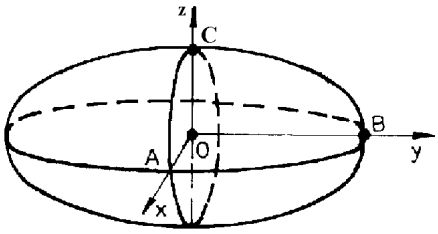


Fig. 66

9.2 Theorem. *The ellipsoid is a closed and bounded set (hence compact) in* $\mathbb{R}^3$.

Proof. From the equation of the ellipsoid it follows that

$$\frac{x^2}{a^2} \leq 1, \ \frac{y^2}{b^2} \leq 1, \ \frac{z^2}{c^2} \leq 1,$$

or, equivalently

$$-a \leq x \leq a,\ -b \leq y \leq b,\ -c \leq z \leq c,$$

and hence all the points of the ellipsoid are contained in the parallelepiped $[-a, a] \times [-b, b] \times [-c, c]$ which has finite sides.

The ellipsoid is a closed set of $\mathbb{R}^3$, since the set $\{1\} \subset \mathbb{R}$ is closed and the ellipsoid is the set $g^{-1}(\{1\})$, where

$$g : \mathbb{R}^3 \to \mathbb{R},\ \ g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

is a continuous function.

5.3 Hyperboloids

The subset of $\mathbb{R}^3$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$$

is called *the hyperboloid of one sheet* (see fig. 67).

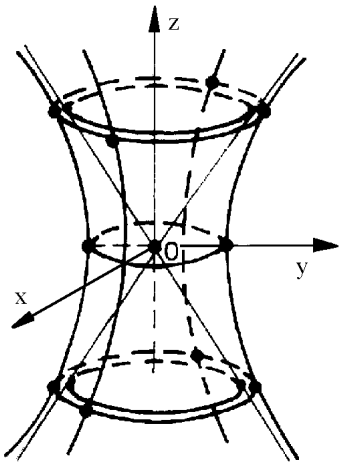


Fig. 67

This surface has the same symmetries as the ellipsoid; it has four vertices, and its intersections with the planes $x = 0$ and $y = 0$ are respectively the hyperbolas

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \\ x = 0 \end{cases}, \quad \text{and} \quad \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} - 1 = 0 \\ y = 0 \end{cases}.$$

The intersections of this surface with the planes $z = k$ are similar ellipses for all $k \in \mathbb{R}$ (see fig. 67), of equations

$$\begin{cases} \frac{x^2}{\frac{a^2}{c^2}(c^2+k^2)} + \frac{y^2}{\frac{b^2}{c^2}(c^2+k^2)} - 1 = 0 \\ z = k, \quad k \in \mathbb{R}. \end{cases}$$

The hyperboloid of one sheet is an unbounded set (since, e.g., it contains a hyperbola), and is closed in $\mathbb{R}^3$.

The hyperboloid of one sheet is used as a model for building factory chimneys, since it is a double ruled surface (see §6).

The subset of $\mathbb{R}^3$ of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \quad \text{with } a, b, c > 0$$

is called *the asymptotic cone* of the hyperboloid of one sheet (see fig. 67).

The subset of $\mathbb{R}^3$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0, \quad \text{with } a, b, c > 0$$

is called *the hyperboloid of two sheets* (see fig. 68).

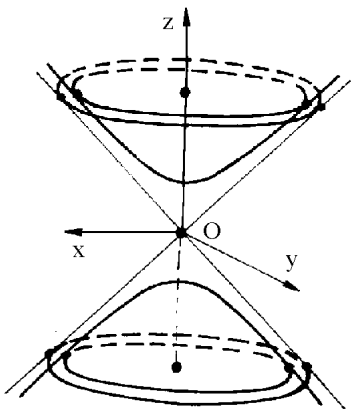


Fig. 68

This surface has the same symmetries as the hyperboloid of one sheet, and has only two vertices located on the Oz axis. The intersections of the hyperboloid of two sheets with the planes $x = 0$ and $y = 0$ are respectively the hyperbolas

$$\begin{cases} \frac{y^2}{b^2} - \frac{z^2}{c^2} + 1 = 0 \\ x = 0 \end{cases}, \quad \begin{cases} \frac{x^2}{a^2} - \frac{z^2}{c^2} + 1 = 0 \\ y = 0 \end{cases}.$$

We remark that the region $-c < z < c$ does not contain points of the hyperboloid of two sheets. The intersection of the surface with the planes $z = k,\ |k| > c$ provides the similar ellipses

$$\begin{cases} \frac{x^2}{\frac{a^2}{c^2}(k^2-c^2)} + \frac{y^2}{\frac{b^2}{c^2}(k^2-c^2)} - 1 = 0 \\ z = k, \quad |k| > c. \end{cases}$$

The hyperboloid of two sheets is a closed and unbounded set in $\mathbb{R}^3$. It admits the same asymptotic cone as the (previously described) hyperboloid of one sheet (see fig. 68).

We remark that for both the hyperboloids, the coordinate axes are axes of symmetry, the coordinate planes are planes of symmetry and the origin is center of symmetry.

A hyperboloid is cut by an arbitrary plane in various conic sections: ellipses, hyperbolas, pair of straight lines, etc.

Lesson 10

5.4 Paraboloids

The subset of $\mathbb{R}^3$ defined by the equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called *elliptic paraboloid* (see fig. 69).

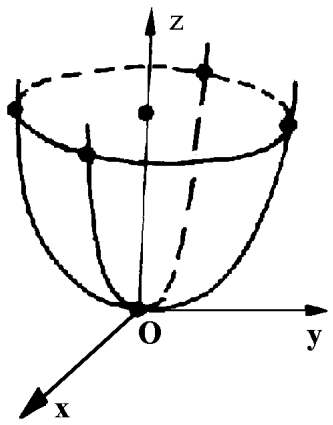


Fig. 69

Its planes of symmetry $x = 0$ and $y = 0$ are called *principal planes*; Oz is axis of symmetry (*principal axis*) and intersects the paraboloid at the origin; this point is called *the vertex* of the paraboloid (see fig. 69). The intersections of the paraboloid with the planes $x = 0$ and $y = 0$ are the parabolas

$$\begin{cases} z = \frac{y^2}{b^2} \\ x = 0 \end{cases}, \quad \begin{cases} z = \frac{x^2}{a^2} \\ y = 0 \end{cases},$$

and this shows that the elliptic paraboloid is an unbounded surface. By examining the Cartesian equation of the elliptic paraboloid, one can see that the points $M(x, y, z)$ of this surface satisfy $z \geq 0$. If we intersect the surface with the planes $z = k$, $k > 0$, we obtain the ellipses

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = k \\ z = k. \end{cases}$$

The elliptic paraboloid is a closed and unbounded (hence, non-compact) set in $\mathbb{R}^3$. It serves as model for fur caps.

The subset of $\mathbb{R}^3$ defined by the equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

is called *hyperbolic paraboloid* (or *saddle*) (see fig. 70).

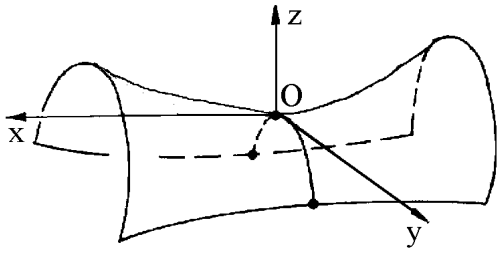


Fig. 70

This surface has the same symmetries as the elliptic paraboloid; the origin is the vertex of the surface. The intersection of the surface with the plane $x = 0$ is the parabola

$$\begin{cases} z = -\frac{y^2}{b^2} \\ x = 0 \end{cases},$$

which has its concavity directed to the negative sense of the Oz axis (since $z < 0$). The intersection of the surface with the plane $y = 0$ is the parabola

$$\begin{cases} z = \frac{x^2}{a^2} \\ y = 0 \end{cases},$$

which has its symmetry axis Oz and is oriented towards the positive sense of this axis.

Intersecting the saddle with the planes $z = k > 0$, we obtain the hyperbolas

$$\begin{cases} \frac{x^2}{a^2} - \frac{y^2}{b^2} = k \\ z = k \end{cases},$$

whose transverse axis is parallel to Ox.

Intersecting the saddle with the planes $z = k < 0$, we obtain the hyperbolas whose transverse axis is parallel to Oy.

The saddle is a closed and unbounded set in $\mathbb{R}^3$.

We remark that the planes xOz and yOz are symmetry planes for both the paraboloids described above; the axis Oz is a symmetry axis for them; the intersection of this axis with the surface is called the vertex. Both paraboloids are cut by planes parallel to xOz and yOz in parabolas. The xOy plane cuts an elliptic paraboloid in a point (the origin). The xOy plane cuts the saddle in two straight lines.

5.5 Cone and cylinders

The subset of $\mathbb{R}^3$ defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is a *cone* (see fig. 71).

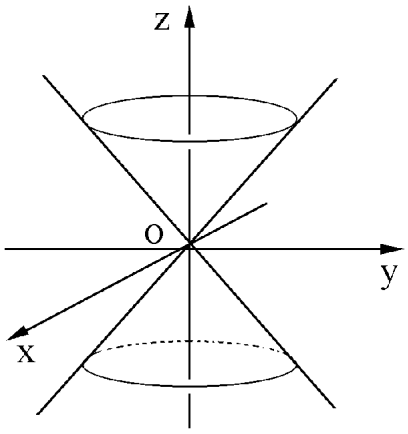


Fig. 71

The subset of $\mathbb{R}^3$ defined by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{1}$$

is an *elliptic cylinder* (see fig. 72);

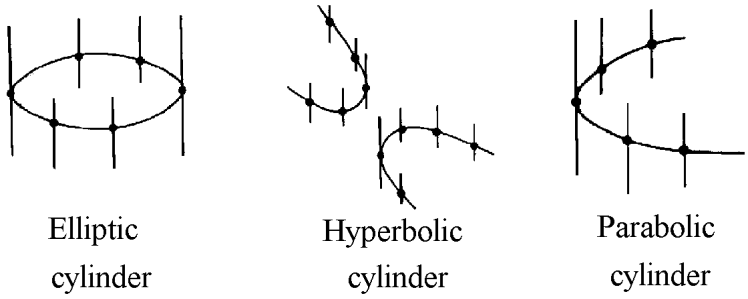


Fig. 72

the subset of $\mathbb{R}^3$ defined by the equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \tag{2}$$

is a *hyperbolic cylinder* (see fig. 72), and the one defined by

$$y^2 = 2px \tag{3}$$

is a *parabolic cylinder* (see fig. 72). The last one can be represented also by an equation of the form

$$y = ax^2 + bx + c.$$

Other quadrics. Besides the previously presented cuadrics, we have (see fig. 73):

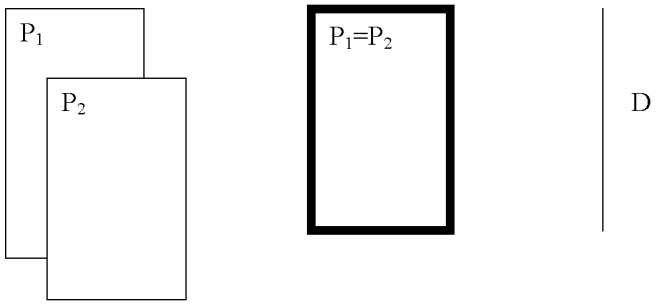


Fig. 73

the *pair of concurrent planes*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0;$$

pair of parallel planes,

$$x^2 - a^2 = 0;$$

a double plane,

$$x^2 = 0;$$

a straight line,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0;$$

a point

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0;$$

the void set

$$x^2+a^2=0, \text{ or } \quad \tfrac{x^2}{a^2}+\tfrac{y^2}{b^2}+1=0, \text{ or } \quad \tfrac{x^2}{a^2}+\tfrac{y^2}{b^2}+\tfrac{z^2}{c^2}+1=0.$$

5.6 Rectilinear generators

The hyperboloid of one sheet is a surface that contains (at each of its points) a pair of rectilinear generators. Because its equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$$

can be rewritten

$$\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=\left(-\frac{y}{b}+1\right)\left(\frac{y}{b}+1\right),$$

the two families of rectilinear generators (see fig. 74) have the equations

$$D_\lambda:\ \begin{cases}\frac{x}{a}-\frac{z}{c}=\lambda(1-\frac{y}{b})\\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}(1+\frac{y}{b})\end{cases}\ \text{and } D_\infty:\ \begin{cases}(1-\frac{y}{b})=0\\ \frac{x}{a}+\frac{z}{c}=0\end{cases}$$

$$D_\mu:\ \begin{cases}\frac{x}{a}-\frac{z}{c}=\mu(1+\frac{y}{b})\\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}(1-\frac{y}{b})\end{cases}\ \text{and } D_\infty:\ \begin{cases}(1+\frac{y}{b})=0\\ \frac{x}{a}+\frac{z}{c}=0\end{cases}\ .$$

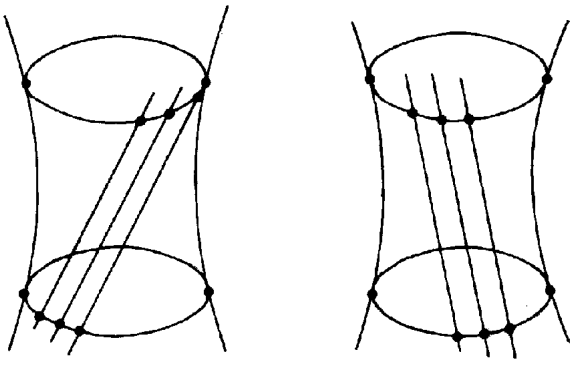


Fig. 74

The generators of the hyperboloid of one sheet satisfy the following properties.

1) the rectilinear generators belonging to one family do not intersect.

2) the rectilinear generators belonging to different families intersect. For that reason, the hyperboloid of one sheet is said to be *double ruled surface.*

The presence of rectilinear generators makes it possible to introduce a new method of generating the hyperboloid of one sheet. Namely, we take three rectilinear generators D_1, D_2, D_3 belonging to one family. Then each rectilinear generator D belonging to the second family intersects D_1, D_2, D_3 . Consequently, the surface is generated by moving a straight line D which intersects simultaneously the three given lines.

Also, the hyperbolic paraboloid (the saddle) contains (at each of its points) a pair of rectilinear generators which belong to two distinct families of generators. Namely, the equation of the saddle

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

can be rewritten

$$\left(\tfrac{x}{a} - \tfrac{y}{b} \right) \cdot \left(\tfrac{x}{a} + \tfrac{y}{b} \right) = 1 \cdot z,$$

whence the two families of rectilinear generators have the equations

$$D_\lambda : \begin{cases} \frac{x}{a} + \frac{y}{b} = \lambda z \\ \frac{x}{a} - \frac{y}{b} = \frac{1}{\lambda} \end{cases} \text{ and } D_\infty : \begin{cases} z = 0 \\ \frac{x}{a} - \frac{y}{b} = 0 \end{cases}$$

$$D_\mu : \begin{cases} \frac{x}{a} - \frac{y}{b} = \mu z \\ \frac{x}{a} + \frac{y}{b} = \frac{1}{\mu} \end{cases} \text{ and } D_\infty : \begin{cases} z = 0 \\ \frac{x}{a} + \frac{y}{b} = 0. \end{cases}$$

For that reason, the saddle is said to be *double ruled surface.*

The cone and the cylinders contain also rectilinear generators (are ruled surfaces).

Lesson 11

5.7 General equation of a quadric

Let E_3 be the 3-dimensional space of Euclidean geometry, $Oxyz$ a Cartesian frame which permits the identification $E_3 \equiv \mathbb{R}^3$. Let be the mapping $g : \mathbb{R}^3 \to \mathbb{R}$,

$$\begin{aligned} g(x,y,z) = \; & a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + \\ & + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + \\ & + 2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00}, \end{aligned}$$

in which at least one of the coefficients $a_{11}, \ldots, a_{23}$ is non-zero. We shall call such a mapping, *affine quadratic form*. Let

$$\{M(x,y,z)|g(x,y,z)=c\}$$

be the level sets of this form. Of course, one can consider only $c=0$, using a translation in the range of g.

7.1 Definition. The set

$$\Sigma: \quad g(x,y,z)=0$$

is called *quadric* or *algebraic surface of order two*.

This definition is correct, since the degree of the equation is invariant relative to the choice of the Cartesian coordinate system. For instance, a roto-translation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

with A orthogonal matrix ($A^tA = I_3$), leads also to an affine form

$$g(x,y,z) = h(x',y',z')$$

of order two. We also remark that Σ is a closed set.

In the sequel we shall prove that a quadric is congruent with one of the following surfaces: sphere, ellipsoid, hyperboloids, paraboloids, cone, cylinders, pair of planes (secant, parallel, identical), straight line, a point, the void set. Their equations with respect to the canonic frames were given in the preceding paragraphs.

7.2 Theorem. *The numbers*

$$\begin{aligned} \Delta &= \det \bar{A},\ \delta = detA, I = trA,\ \textit{and} \\ J &= \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} \end{aligned} \tag{5.1}$$

where

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{10} \\ a_{21} & a_{22} & a_{23} & a_{20} \\ a_{31} & a_{32} & a_{33} & a_{30} \\ a_{01} & a_{02} & a_{03} & a_{00} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

with $a_{ij} = a_{ji}$, for all $i, j \in \{0, 1, 2, 3\}$, are invariant under roto-translations, and are called the invariants of the quadric Σ.

Remarks. 1) For $\Delta = 0$, the quadric Σ is called *degenerate*; and for $\Delta \neq 0$, *nondegenerate*. The quadric Σ is degenerate iff $g(x, y, z)$ is the product of two functions of degree one in x, y, z, or iff Σ is the union of two planes.

2) The *center (of symmetry) of the quadric* Σ (if it exists) is given by the solution of the system of equations

$$\begin{cases} \frac{1}{2}g_x \equiv a_{11}x + a_{12}y + a_{13}z + a_{10} = 0 \\ \frac{1}{2}g_y \equiv a_{21}x + a_{22}y + a_{23}z + a_{20} = 0 \\ \frac{1}{2}g_z \equiv a_{31}x + a_{32}y + a_{33}z + a_{30} = 0. \end{cases}$$

We remark that for $\delta \neq 0$ we have an unique center of symmetry for the quadric Σ.

7.3 Reduction of the equation of a quadric to the canonic form

Let $\Sigma:\ g(x, y, z) = 0$ be a quadric. We have the following cases:

1) $a_{12} = a_{13} = a_{23} = 0$, then it is enough to perform a translation and (if necessary) a rotation.

2) if one of the coefficients a_{12}, a_{13}, a_{23} is different from zero, then we consider the quadratic form

$$\varphi(x, y, z) = (x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which determines the type of the quadric, and we look for the reduction of φ to the canonic expression. We solve the characteristic equation attached to A,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0,$$

find the roots and determine the orthogonal unit eigenvectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$ and the associated matrix $R = [\bar{e}_1, \bar{e}_2, \bar{e}_3]$ which is an orthogonal matrix; we arrange the vectors such that $\det R = 1$, i.e., to obtain the matrix of a *rotation*. The change of coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

transforms the equation $g(x,y,z)=0$ into

$$\lambda_1 x'^2+\lambda_2 y'^2+\lambda_3 z'^2+\ldots=0.$$

It follows a discussion and we look for perfect squares. If $\lambda_1,\lambda_2,\lambda_3\neq 0$, then we have an equation of the form

$$\lambda_1(x'+\ldots)^2+\lambda_2(y'+\ldots)^2+\lambda_3(z'+\ldots)^2+a=0.$$

Denoting $x''=x'+\ldots,\ y''=y'+\ldots,\ z''=z'+\ldots$, i.e., making a translation, we obtain the canonic equation

$$\lambda_1 x''^2+\lambda_2 y''^2+\lambda_3 z''^2+\frac{\delta}{\Delta}=0.$$

7.4 Intersection between a straight line and a quadric

Let $D:\begin{cases} x=x_0+lt \\ y=y_0+mt \\ z=z_0+nt \end{cases}$, $t\in\mathbb{R}$ be a straight line and let $\Sigma:\ g(x,y,z)=0$ be a quadric. Then the intersection $D\cap\Sigma$ (see fig. 75) corresponds to the roots t_1,t_2 of the equation

$$t^2\varphi(l,m,n)+t(lg_{x_0}+mg_{y_0}+ng_{z_0})+g(x_0,y_0,z_0)=0, \qquad (*)$$

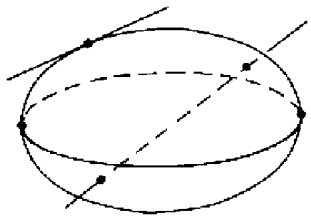


Fig. 75

where we denoted

$$\varphi(l,m,n)=a_{11}l^2+a_{22}m^2+a_{33}n^2+2a_{12}lm+2a_{23}mn+2a_{13}ln.$$

Discussion. I. If $\varphi(l,m,n)\neq 0$, then (*) is an algebraic equation of order two and denoting

$$q=(lg_{x_0}+mg_{y_0}+ng_{z_0})^2-4\varphi(l,m,n)g(x_0,y_0,z_0)$$

we have the following cases:

I.1 If $q > 0$, then there are two distinct (real) roots t_1, t_2 which provide two intersection points, $D \cap \Sigma = \{M_1, M_2\}$.

I.2 If $q = 0$, then the intersection points coincide (double point) $M_1 = M_2$ and D is tangent to the quadric Σ at M_1. Considering $M_0(x_0, y_0, z_0) \in \Sigma$, we have the tangent line D at M_0 iff

$$lg_{x_0} + mg_{y_0} + ng_{z_0} = 0,$$

with $g_{x_0}^2 + g_{y_0}^2 + g_{z_0}^2 > 0$.

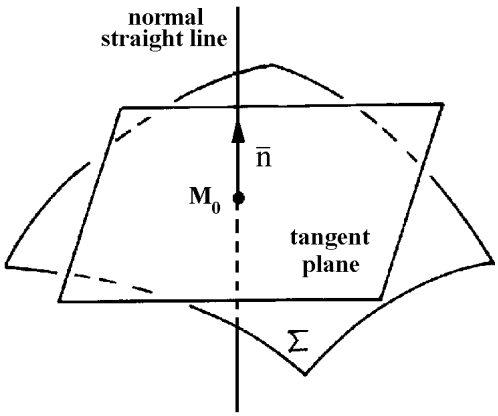


Fig. 76

The condition that a point lies on an arbitrary line tangent to Σ at M_0 would be "$\overline{M_0M}$ collinear to (l, m, n)", and produces

$$(x - x_0)g_{x_0} + (y - y_0)g_{y_0} + (z - z_0)g_{z_0} = 0,$$

which is the equation of a plane - the *tangent plane to* Σ *at* M_0 (see fig. 76). This equation can be also obtained by *halving* the equation $g(x, y, z) = 0$ of the quadric Σ at the point M_0.

The *normal straight line* to Σ at M_0 (see fig. 76), will then have the equations

$$\frac{x - x_0}{g_{x_0}} = \frac{y - y_0}{g_{y_0}} = \frac{z - z_0}{g_{z_0}}.$$

I.3 If $q < 0$, then we have $D \cap \Sigma = \emptyset$.

II. If $\varphi(l,m,n)=0$, then we have an equation of degree at most one, and the straight line D is secant or exterior to Σ.

7.5 Intersection between a plane and a quadric

We consider a plane $P: ax+by+cz+d=0$ and the quadric $\Sigma: g(x,y,z)=0$. Then we have (see fig. 77)

$$P\cap\Sigma:\ \begin{cases} ax+by+cz+d=0 \\ g(x,y,z)=0 \end{cases}.$$

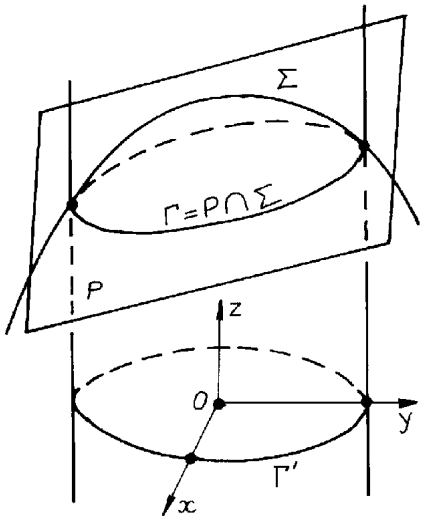


Fig. 77

For instance, assuming that $c\neq 0$, we obtain

$$P\cap\Sigma:\ \begin{cases} z=\alpha x+\beta y+\gamma \\ g(x,y,\alpha x+\beta y+\gamma)=0, \end{cases}$$

and $P\cap\Sigma$ can be regarded as the intersection between a plane and a cylinder of order two whose generators are parallel to Oz (see fig. 77). Consequently, $P\cap\Sigma$ is a conic in $\mathbb{R}^3$.

5.8 Problems

1. Find the equation of each of the following spheres:
1) the sphere with the center $C(2,0,3)$ which is tangent to the plane

$$3x+2y+4z-45=0;$$

2) the sphere which contains the points

$$A(-1,0,0),\ B(0,2,1),\ C(0,-1,-1),\ D(3,1,-1);$$

3) the sphere with the center $C(1,-2,1)$ which is interior tangent to the sphere

$$x^2+y^2+z^2-x-3y+3z-2=0.$$

2. Determine the coordinates of the center of the quadric

$$x^2+y^2+5z^2-6xy+2xz-2yz-4x+8y-12z+14=0,$$

then perform a translation of the Cartesian frame to the center of the quadric and write the equation of the quadric relative to the translated frame $O'x'y'z'$.

3. Find the canonic form of the equations of the following quadrics
$\Sigma_1:\ y^2+4xy-8xz-4yz+6x-5=0;$
$\Sigma_2:\ 2y^2-7z^2+112x-16y-14z-87=0;$
$\Sigma_3:\ xy+z^2-2=0.$

4. Compute the distance between the points of intersection of the straight line

$$D:\ x=t,\ y=2-t,\ z=-3+2t,\ \ t\in\mathbb{R}$$

with the quadric

$$x^2-y^2+z^2-4y+6z+9=0.$$

5. Let be the hyperboloid of one sheet

$$\Sigma:\ \frac{x^2}{9}+\frac{y^2}{4}-z^2-1=0$$

and the plane

$$P:\ 4x-3y-12z-6=0.$$

Describe the curve $P\cap\Sigma$, using its projections on the coordinate planes.

6. Let be the hyperbolic paraboloid

$$\Sigma:\ \frac{x^2}{9}-\frac{y^2}{4}=z$$

and the point $M_0(0, 2, -1)$. Write the equations of the rectilinear generators which contain M_0, and determine the angle between these generators.

7. Show that the intersection between the plane

$$P: \ 6x + 3y - 2z + 6 = 0$$

and the hyperboloid of one sheet

$$\Sigma: \ x^2 - \frac{y^2}{4} + \frac{z^2}{9} - 1 = 0$$

consists of two straight lines whose equations are required.

Part II

DIFFERENTIAL GEOMETRY

Chapter 6

Curves

Lesson 12

6.1 Introductory notions

Differentiable functions. A mapping $f : \mathbb{R}^n \to \mathbb{R}$ is called *differentiable function* if it is of class C^∞ (i.e., it has continuous partial derivatives of all orders). More general, a mapping

$$f : \mathbb{R}^n \to \mathbb{R}^m, \quad f = (f_1, \ldots, f_m)$$

is said to be *differentiable* if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable, for every $i \in \{1, \ldots, m\}$. In this case, the derivative of f is *the Jacobian matrix*

$$J(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial f_m}{\partial x_1} & \ldots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

We remark that for $m = n$, we can define *the Jacobian* of f, as the function

$$\frac{D(f_1, \ldots, f_n)}{D(x_1, \ldots, x_n)} = \det J(f).$$

Denominations. The mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is called
1) *immersion*, if rank $J(f) = n \leq m$;
2) *submersion*, if rank $J(f) = m \leq n$;

3) *regular*, if it is immersion or submersion;
4) *diffeomorphism* iff $m = n$ and it is a bijective immersion;
5) *one - to - one* iff it is injective;
6) *onto* iff it is surjective;
7) A point at which f is not regular is called *critical* or *singular point* of f.

We remark that if f is a diffeomorphism, then f and f^{-1} have the same class. For using the preceding mathematical notions, one should consider *the inverse function theorem* and *the theorem of implicit functions*.

Tangent vectors. Let $\mathbb{R}^n$ be the Euclidean real vector space and $P, Q \in \mathbb{R}^n$.

1.1 Definition. The ordered pair (P, Q) is called *a tangent vector to* $\mathbb{R}^n$ *at* P; in this case, we call P *the point of application*, Q *the extremity (the end point)* and the difference $\vec{v} = Q - P$, *the vector part* of the tangent vector.

The vector itself will be denoted as $\vec{v}_P$, and we shall always picture $\vec{v}_P$ as the arrow from the point P to the point $P + \vec{v}$ (see fig. 78).

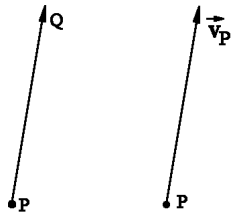


Fig. 78

We have the *equality* between two tangent vectors $\vec{v}_P = \vec{w}_Q$ iff $\vec{v} = \vec{w}$ and $P = Q$. Also, we shall say that the two vectors are *parallel*, and denote $\vec{v}_P || \vec{v}_Q$ iff the vector parts are collinear (have proportional components) (see fig. 79).

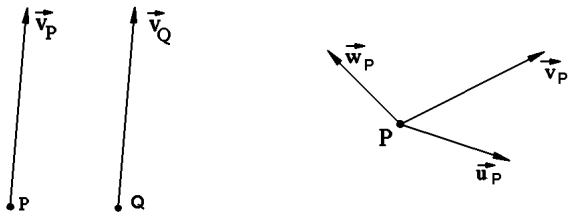


Fig. 79 **Fig. 80**

It is essential to recognize that $\vec{v}_P$ and $\vec{v}_Q$ are different tangent vectors if $P \neq Q$.

1.2 Definition. Let be P a point of $\mathbb{R}^n$. The set $T_P\,\mathbb{R}^3$ consisting of all tangent vectors that have P as point of application, is called *the tangent space of* $\mathbb{R}^n$ *at* P (see fig. 80).

We define $\vec{v}_P + \vec{w}_P$ to be $(\vec{v} + \vec{w})_P$, and $c(\vec{v}_P)$ to be $(c\vec{v})_P$. These operations on $T_P\,\mathbb{R}^n$ make it a vector space isomorphic to $\mathbb{R}^n$ itself, where the mapping $\varphi : R^n \to T_P\,\mathbb{R}^n$

$$\varphi(\vec{v}) = \vec{v}_p, \quad \forall \vec{v} \in \mathbb{R}^n$$

defines an isomorphism between these vector spaces.

We can define on $T_P\,\mathbb{R}^n$ the *scalar product*

$$(\vec{v}_P, \vec{w}_P) = (\vec{v}, \vec{w})_P,$$

and *the angle* φ between the tangent vectors $\vec{v}_P$ and $\vec{w}_P$ (see fig. 81) by the relation

$$\cos\varphi = \frac{(\vec{v}, \vec{w})}{||\vec{v}|| \cdot ||\vec{w}||}, \quad \varphi \in [0, \pi].$$

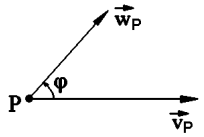


Fig. 81

The scalar product permits us to define the *norm* of a tangent vector $\vec{v}_P \in T_P\,\mathbb{R}^n$, as $||\vec{v}_P|| = ||\vec{v}||$. We shall also say that two vectors $\vec{v}_P$ and $\vec{w}_P$ are *orthogonal* iff $(\vec{v}_P, \vec{w}_P) = 0$; we shall call *unit vector* a vector $\vec{e}_P \in T_P\,\mathbb{R}^n$ of length one.

1.3 Definition. Let consider in $\mathbb{R}^n$ a basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$. Then it produces *the frame* at P, given by the vectors $\{\vec{e}_{1P}, \ldots, \vec{e}_{nP}\} \subset T_P\,\mathbb{R}^n$; if the basis in R^n is the canonic one, i.e.,

$$\vec{e}_1 = (1, \ldots, 0), \ldots, \vec{e}_n = (0, \ldots, 1),$$

then the frame will be called *the natural frame at* P.

We notice that any vector $\vec{v}_P \in T_P\,\mathbb{R}^n$ decomposes with respect to this frame as

$$\vec{v}_P = (\vec{v}_P, \vec{e}_{1P})\vec{e}_{1P} + \cdots + (\vec{v}_P, \vec{e}_{nP})\vec{e}_{nP}.$$

In the particular case $n = 3$, the *canonic frame* in $T_P\,\mathbb{R}^3$ will be (see fig. 82)

$$\{\vec{i}_P, \vec{j}_P, \vec{k}_P\} \subset T_P\,\mathbb{R}^3.$$

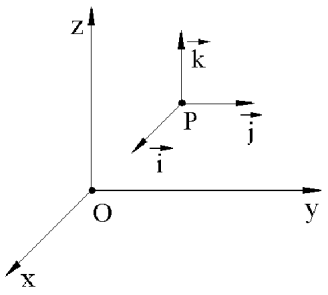


Fig. 82

Also, we can define in $T_P\,\mathbb{R}^3$ the *vector product* (see fig. 83), as

$$\vec{v}_P \times \vec{w}_P = \vec{v} \times \vec{w}.$$

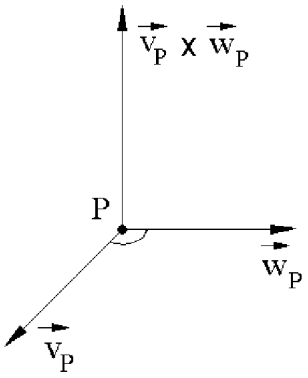


Fig. 83

Vector fields

1.4 Definitions. a) A *vector field* $\vec{V}$ on $\mathbb{R}^n$ is a function that assigns to each point P of $\mathbb{R}^n$ a tangent vector $\vec{V}(P)$ to $\mathbb{R}^n$ at P. Hence a vector field can be regarded as a collection of arrows.

Let $T\,\mathbb{R}^n = \bigcup_{P\in\mathbb{R}^n} T_P\,\mathbb{R}^n$ be the tangent bundle of $\mathbb{R}^n$. Then a vector field $\vec{V}$ on $\mathbb{R}^n$ is a function

$$\vec{V} : \mathbb{R}^n \to T\,\mathbb{R}^n, \quad \vec{V}(P) \in T_P\,\mathbb{R}^n.$$

b) For $\vec{V}$ constant we say that the vector field is a *parallel field*; as examples, we have the *fundamental vector fields*

$$\vec{U}_{1_P} = (1, 0, \ldots, 0)_P, \quad \ldots \quad , \vec{U}_{n_P} = (0, \ldots, 0, 1)_P.$$

This set of vector fields forms *the natural frame field* on $\mathbb{R}^n$.

1.5 Theorem. *If $\vec{V}$ is a vector field on $\mathbb{R}^n$, there are n unique determined real-valued functions $f_1, \ldots, f_n$ on $\mathbb{R}^n$ such that*

$$\vec{V} = f_1\vec{U}_1 + \ldots + f_n\vec{U}_n,$$

where f_i are called the Euclidean coordinate functions of $\vec{V}$.

We shall say that the vector field $\vec{V}$ is differentiable iff the functions f_i are differentiable.

As an example, we have the *electrostatic field*

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon} \cdot \frac{q_0}{||\vec{r}||^3}\vec{r}, \quad \forall P \in \mathbb{R}^3\backslash\{0\}.$$

Covariant derivative. We shall assume that all the functions considered below are differentiable of class C^∞.

I. Let D be an open set of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be a real function. Let $P \in D$ and $\vec{v}$ a tangent vector to D at the point P. We set the interval I such that $P + tV \in D$, $\forall t \in I$, where V is the point which corresponds to the vector $\vec{v}$. Obviously $t \to P + tV$ represents the restriction of a straight line and if f is differentiable, then the composed function $t \to f(P + tV)$ is also differentiable.

1.6 Definition. The number

$$D_{\vec{v}}f(P) = \frac{d}{dt}f(P + tV)\bigg|_{t=0}$$

is called *the derivative of f with respect to $\vec{v}$.*

The number $D_{\vec{v}}f(P)$ shows the change of the value $f(P)$ when P moves in the sense and direction of $\vec{v}$. If $\vec{v}$ is a unit vector, then $D_{\vec{v}}f$ is called *the derivative of f*

with respect to the direction $\vec{v}$.

1.7 Lemma. *If* $\vec{v}_P = (v_1, ..., v_n)_P$, *then*

$$D_{\vec{v}}f(P) = v_1\frac{\partial f}{\partial x_1}(P) + ... + v_n\frac{\partial f}{\partial x_n}(P) = (\vec{v}, \nabla f(P)) = df(P)(\vec{v}),$$

where ∇f *is the gradient of* f, *and* df *is the differential of* f.

The proof is straightforward, as a consequence of the theorem of derivation of composed functions.

Using the Cauchy-Schwarz inequality

$$|D_{\vec{v}}f(P)| = |(\vec{v},\ \nabla f(P))| \leq \|\vec{v}\|\|\nabla f(P)\|,$$

in which the equality holds iff $\vec{v}$ and $\nabla f(P)$ are collinear, we obtain that:

- the mapping $\vec{v} \to D_{\vec{v}}f(P),\ ||\vec{v}|| = 1$ has a minimum equal to $-\|\nabla f(P)\|$ if $\vec{v} = -\nabla f(P)/||\nabla f(P)||$. Hence, $-\nabla f(P)$ indicates locally the direction and the sense in which f decreases fastest.
- the mapping $\vec{v} \to D_{\vec{v}}f(P),\ ||\vec{v}|| = 1$ has a maximum equal to $\|\nabla f(P)\|$ if $\vec{v} = \nabla f(P)/||\nabla f(P)||$. Hence, $\nabla f(P)$ indicates locally the direction and the sense in which f increases fastest.

1.8 Theorem. *Let* $f, g : D \to \mathbb{R},\ \vec{v}, \vec{w} \in T_PD$ *and* $a, b \in \mathbb{R}$. *We have*

$$D_{a\vec{v}+b\vec{w}}f(P) = aD_{\vec{v}}f(P) + bD_{\vec{w}}f(P)$$
$$D_{\vec{v}}(af + bg)(P) = aD_{\vec{v}}f(P) + bD_{\vec{v}}g(P)$$
$$D_{\vec{v}}(fg)(P) = g(P)D_{\vec{v}}f(P) + f(P)D_{\vec{v}}g(P).$$

The proof relies on Lemma 1.7 and on the properties of the gradient.

Using this notion, we can define the action of a vector field $\vec{V}$ on a function f (both being defined on D), as the real-valued function denoted with $D_{\vec{V}}f$ and whose value at each point $P \in D$ is the number $D_{\vec{V}(P)}f(P)$. The function $D_{\vec{V}}f$ is called *the derivative of the function* f *relative to the field* $\vec{V}$. In particular, for the case $n = 3$, we have

$$D_{\vec{i}}f = \frac{\partial f}{\partial x}, \qquad D_{\vec{j}}f = \frac{\partial f}{\partial y}, \qquad D_{\vec{k}}f = \frac{\partial f}{\partial z}.$$

The derivative $D_{\vec{V}}f$ has the following properties

$$D_{f\vec{V}+g\vec{W}}h = fD_{\vec{V}}h + gD_{\vec{W}}h$$
$$D_{\vec{V}}(af + bg) = aD_{\vec{V}}f + bD_{\vec{V}}g$$
$$D_{\vec{V}}(fg) = fD_{\vec{V}}g + gD_{\vec{V}}f.$$

II. We shall define in the following a notion which generalizes the derivative $D_{\vec{v}}f(P)$ and represents an operation on vector fields. Let $\vec{W}$ be a vector field defined on the open set D in $\mathbb{R}^n$ and $\vec{v}$ a tangent vector to D at the point x. We consider the composed function $t \to \vec{W}(P+tV),\ t \in I$, where I is determined by the condition $P + tV \in D$.

1.9 Definition. The vector

$$D_{\vec{v}}\vec{W}(P) = \frac{d}{dt}\vec{W}(P+tV)\Big|_{t=0}$$

which is tangent to D at the point P is called *the covariant derivative of* $\vec{W}$ *with respect to* $\vec{v}$.

The covariant derivative $D_{\vec{v}}\vec{W}(P)$ measures the initial rate of change of $\vec{W}(P)$ when P moves in the sense of $\vec{v}$ (see fig. 84).

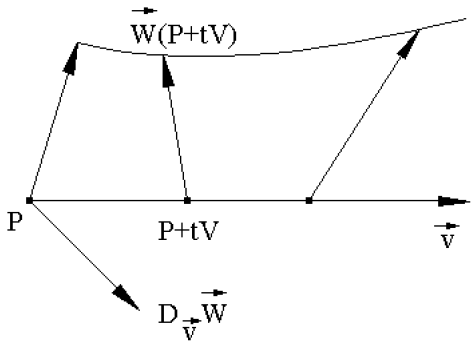


Fig. 84

1.10 Lemma. If $\vec{W} = w_1\vec{U}_1 + ... + w_n\vec{U}_n$ and $\vec{v}$ is a tangent vector to D at the point P, then

$$D_{\vec{v}}\vec{W}(P) = D_{\vec{v}}w_1(P)\vec{U}_1(P) + ... + D_{\vec{v}}w_n(P)\vec{U}_n(P).$$

Proof. We remark that

$$\vec{W}(P+tV) = w_1(P+tV)\vec{U}_1(P+tV) + ... + w_n(P+tV)\vec{U}_n(P+tV).$$

We derive this field at $t = 0$ by deriving its components. Having in view Definition 1.6, the conclusion follows.

The properties of the covariant derivative come out of Lemma 1.10 and from the properties of the derivative $D_{\vec{v}}f(P)$ stated in Theorem 1.8.

1.11 Theorem. *Let $\vec{V}$ and $\vec{W}$ two vector fields on D, let $\vec{v}, \vec{w} \in T_P D$ and $a, b \in \mathbb{R}$. We have*

$$\begin{aligned}
&D_{a\vec{v}+b\vec{w}}\vec{W} = aD_{\vec{v}}\vec{W} + bD_{\vec{w}}\vec{W}\\
&D_{\vec{v}}(a\vec{V} + b\vec{W}) = aD_{\vec{v}}\vec{V} + bD_{\vec{v}}\vec{W}\\
&D_{\vec{v}}(f\vec{W}) = (D_{\vec{v}}f)\vec{W} + fD_{\vec{v}}\vec{W}\\
&D_{\vec{v}}(\vec{V}, \vec{W}) = (D_{\vec{v}}\vec{V},\ \vec{W}) + (\vec{V}, D_{\vec{v}}\vec{W}).
\end{aligned}$$

The notion above can be extended considering the covariant derivative of a vector field $\vec{W}$ relative to the vector field $\vec{V}$. The result is also a vector field, which is denoted by $D_{\vec{V}}\vec{W}$ and whose value at P is $D_{\vec{V}(P)}\vec{W}(P)$. If $\vec{W} = w_1\vec{U}_1 + ... + w_n\vec{U}_n$, then

$$D_{\vec{V}}\vec{W} = (D_{\vec{V}}w_1)\vec{U}_1 + ... + (D_{\vec{V}}w_n)\vec{U}_n.$$

Based on the preceding considerations, it results that $D_{\vec{V}}$ has the following properties

$$\begin{aligned}
&D_{f\vec{V}+g\vec{W}}\vec{Y} = fD_{\vec{V}}\vec{Y} + gD_{\vec{W}}\vec{Y}\\
&D_{\vec{V}}(a\vec{Y} + b\vec{Z}) = aD_{\vec{V}}\vec{Y} + bD_{\vec{V}}\vec{Z}\\
&D_{\vec{V}}(f\vec{Y}) = (D_{\vec{V}}f)\vec{Y} + fD_{\vec{V}}\vec{Y}\\
&D_{\vec{V}}(\vec{Y}, \vec{Z}) = (D_{\vec{V}}\vec{Y}, \vec{Z}) + (\vec{Y}, D_{\vec{V}}\vec{Z}).
\end{aligned}$$

Remarks. 1) The role of $\vec{V}$ in the covariant derivative $D_{\vec{V}}\vec{W}$, is algebraic; $\vec{W}$ is derived.

2) The covariant derivatives of the fundamental fields $\vec{U}_i, i \in \{1, ..., n\}$, are null.

Lesson 13

6.2 Curves in $\mathbf{R}^n$

Let be the vector space $\mathbb{R}^n$ and the space $T_O\,\mathbb{R}^n$ tangent to $\mathbb{R}^n$ at its origin O. Sometimes we identify $\mathbb{R}^n$ to $T_O\,\mathbb{R}^n$.

2.1 Definition. A differentiable function

$$\alpha : I \to \mathbb{R}^n, \quad \alpha(t) = (x_1(t), \ldots, x_n(t)),$$

is called *curve in* $\mathbb{R}^n$; the functions $x_1, \ldots, x_n : I \to \mathbb{R}^n$ are called its *Euclidean coordinate functions.*

For a point $P \in \mathbb{R}^3$, we have that (see fig. 85)

$$P \in \alpha(I) \Leftrightarrow \exists t \in I, \alpha(t) = P.$$

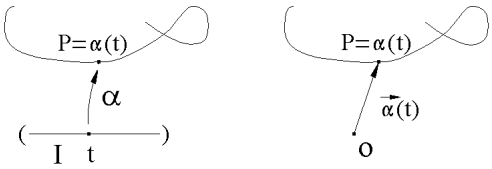


Fig. 85

Sometimes we name the set of points $\alpha(I)$ as *curve*, and in this case we say that t is *the parameter on the curve* $\alpha(I)$ and α, *the parametrization.* We call the equality

$$\vec{\alpha}(t) = x_1(t)\vec{e}_1 + \ldots + x_n(t)\vec{e}_n$$

which can be rewritten in condensed form $\vec{\alpha} = \vec{\alpha}(t)$ the *vectorial equation* of the curve (see fig. 85). The equation is equivalent to the *parametric equations of the curve*

$$\begin{cases} x_1 = x_1(t) \\ \ldots \\ x_n = x_n(t). \end{cases}$$

Example. Let be the real function $f : I \to \mathbb{R}$. Then *the graph of equation* $y = f(x)$, provides a parametrized curve

$$\alpha : \begin{cases} x = t \\ y = f(t) \end{cases}, \quad t \in I.$$

The associated vectorial equation is $\vec{\alpha} = \vec{\alpha}(t)$, where

$$\vec{\alpha}(t) = t\vec{i} + f(t)\vec{j}.$$

2.2 Definition. A point $P \in \alpha(I)$ is called a *simple point* if the equation $\alpha(t) = P$ has a unique solution $t \in I$; if exist more points $t \in I$ such that $\alpha(t) = P$, then P is called *multiple point*: double, triple, etc, (see fig. 86).

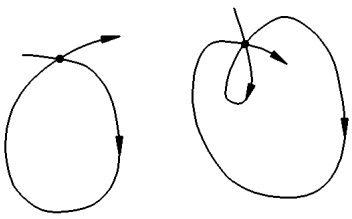


Fig. 86

2.3 Definitions. a) The curve α is called *simple curve* if all its points are simple.

b) If the curve $\alpha : [a, b] \to \mathbb{R}^n$ satisfies $\alpha(a) = \alpha(b)$, then it is called *closed curve* (see fig. 87).

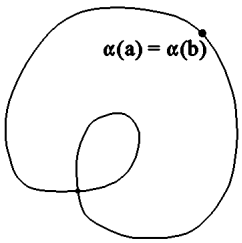


Fig. 87

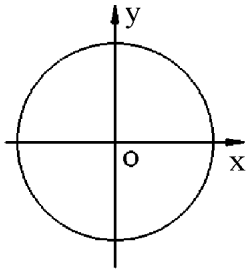


Fig. 88

The circle (see fig. 88)

$$\alpha : \begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad t \in [0, 2\pi]$$

is an example of closed and simple curve.

2.4 Definition. The curve $\alpha : I \to \mathbb{R}^n$ is called *periodic* if there exists $T > 0$ such that

$$\alpha(t) = \alpha(t + T), \quad \forall t \in I.$$

A periodic curve is closed and its domain I can be extended to $\mathbb{R}$.

The right circular helix. The curve $\alpha(t) = (a \cos t, a \sin t, 0)$ is a circle of radius $a > 0$ in the xOy plane of $\mathbb{R}^3$. If we allow this curve to rise (or fall) with a constant rate, we obtain a *helix* (see fig. 89) $\alpha : \mathbb{R} \to \mathbb{R}^3$ given by the formula

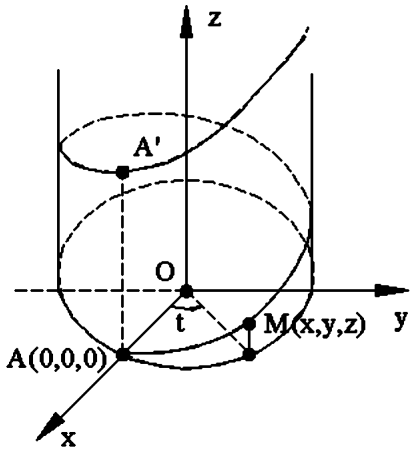


Fig. 89

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad t \in \mathbb{R}; \quad a > 0, b \neq 0.$$

Practically, we move a point $M(x, y, z)$ on a circular cylinder with axis Oz, the motion being composed of a rotation around Oz and a translation along Oz.

6.3 Tangent line and normal hyperplane of a curve

Let be the curve $\alpha : I \to \mathbb{R}^n$. Consider $t \in I$ and denote $\alpha(t) = P$, $\alpha(t+h) = Q$, for some $h \in (-\varepsilon, \varepsilon)$. The derivative

$$\bar{\alpha}'(t) = \lim_{h\to 0} \frac{\bar{\alpha}(t+h) - \bar{\alpha}(t)}{h} = \lim_{h\to 0} \frac{\overline{PQ}}{h},$$

making sense for free vectors, gives a vector $\vec{a}'(t)$ tangent to α at the point $P = \alpha(t)$, which is called *the velocity vector of* α *at* t (see fig. 90).

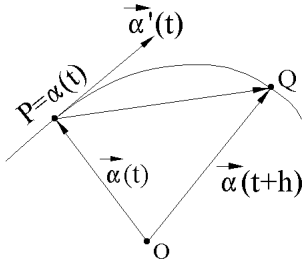


Fig. 90

We can decompose the velocity $\vec{\alpha}'(t)$ as

$$\vec{\alpha}'(t) = x_1'(t)\vec{e}_1 + \ldots + x_n'(t)\vec{e}_n.$$

3.1 Definitions. Let be the curve $\alpha : I \to \mathbb{R}^n$.

a) If $\vec{\alpha}'(t) \neq 0$, then $P = \alpha(t)$ is called *regular point of* α; if all the points of α are regular, we say that α is a *regular curve*.

b) Let $P = \alpha(t)$ be a regular point of α. The straight line defined by the point P and the director vector $\bar{\alpha}'(t)$ is called *the tangent of* α *at* P (see fig. 91); its equations are

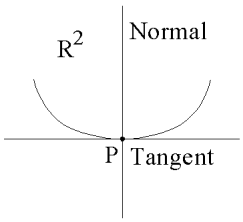


Fig. 91

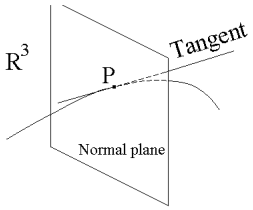


Fig. 92

$$\frac{x_1 - x_1(t)}{x_1'(t)} = \ldots = \frac{x_n - x_n(t)}{x_n'(t)}.$$

c) The hyperplane defined by P and the normal vector $\bar{\alpha}'(t)$ is called *the normal hyperplane of* α *at* P (see fig. 92); its equation is

$$(x_1 - x_1(t))\, x_1'(t) + \ldots + (x_n - x_n(t))\, x_n'(t) = 0.$$

Generally, the vectors $\vec{\alpha}''(t), \ldots, \vec{\alpha}^{(m)}(t), \ldots$ are tangent to $\mathbb{R}^n$ at $P = \alpha(t)$.

3.2 Definitions. a) A point $P = \alpha(t)$ such that $\vec{\alpha}'(t) = \vec{0}$ is called *critical* or *singular point* of the curve α.

b) If there exists $m > 1$, such that

$$\vec{\alpha}'(t) = \ldots = \vec{\alpha}^{(m-1)}(t) = \vec{0}, \quad \text{and} \quad \vec{\alpha}^{(m)}(t) \neq \vec{0},$$

then we say that the point $P = \alpha(t)$ is a *singular point of order* m of the curve α.

By the Taylor formula (which makes sense only for free vectors), at a singular point of order m, we have

$$\bar{\alpha}(t+h) = \bar{\alpha}(t) + \frac{h^m}{m!}[\bar{\alpha}^m(t) + \bar{\varepsilon}(h)]$$

with $\lim_{h\to 0} \bar{\varepsilon}(h) = \bar{0}$; hence we find

$$\bar{\alpha}^{(m)}(t) = m! \lim_{h\to 0} \frac{\bar{\alpha}(t+h) - \bar{\alpha}(t)}{h^m}.$$

It results that $\vec{\alpha}^{(m)}(t)$ is a vector tangent to α at $P = \alpha(t)$.

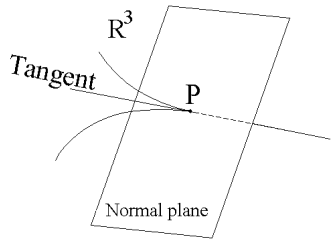


Fig. 93

3.3 Definition. Let $P = \alpha(t)$ be a singular point of order m; the straight line given by P and the director vector $\bar{\alpha}^{(m)}(t)$ is called *the tangent line to α at P*. Its equations are

$$\frac{x_1 - x_1(t)}{x_1^{(m)}(t)} = \ldots = \frac{x_n - x_n(t)}{x_n^{(m)}(t)}.$$

The hyperplane given by P having as normal vector $\bar{\alpha}^{(m)}(t)$ is called *the normal hyperplane to α at P* (see fig. 93), and its equation is

$$(x_1 - x_1(t))\, x_1^{(m)}(t) + \ldots + (x_n - x_n(t))\, x_n^{(m)}(t) = 0.$$

Remarks. 1) The *angle between two intersecting curves* $\alpha_1 : I_1 \to \mathbb{R}^3$ and $\alpha_2 : I_2 \to \mathbb{R}^3$ at their common point

$$P = \alpha_1(t_1) = \alpha_2(t_2), \;\; t_1 \in I_1, \; t_2 \in I_2,$$

is the angle between their velocities at P, given by the vectors $\vec{\alpha}_1'(t_1)$ and $\vec{\alpha}_2'(t_2)$ (see fig. 94).

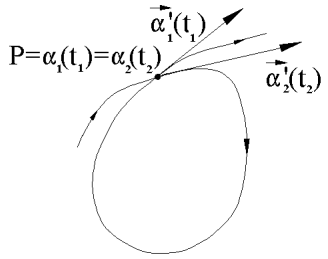


Fig. 94

2) A regular curve $\alpha : I \to \mathbb{R}^3$ on which the sense of moving is given by its velocity, is called *oriented curve*.

3) In kinematics, one can picture a curve in $\mathbb{R}^3$ as a trip taken by a moving point $\alpha(t)$ when t varies in I; in this case, the points of the image $\alpha(I)$ form *the trajectory*, the parameter t is called *the time* on α; $\vec{\alpha}'(t)$ is called *the velocity at the moment* t and the vector $\vec{\alpha}''(t)$ is called *the acceleration of* α *at* t. The length $v(t) = ||\vec{\alpha}'(t)||$ is called *the speed* of the curve α at t.

3.4 Definition. *A vector field* $\vec{Y}$ on a curve $\alpha : I \to \mathbb{R}^n$ is a function that assigns to each number t in I a tangent vector $\vec{Y}(t)$ to $\mathbb{R}^n$ at the point $\alpha(t)$ (see fig. 95).

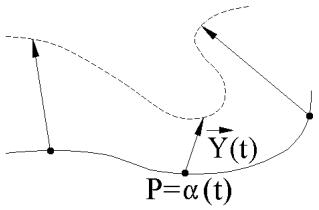


Fig. 95

For example, the velocity $\vec{\alpha}'$ and the acceleration $\vec{\alpha}''$ are vector fields on the curve α.

A vector field $\vec{Y}$ on α decomposes at each point $\alpha(t)$ of α as

$$\vec{Y}(t) = y_1(t)\vec{e}_1 + \ldots + y_n(t)\vec{e}_n.$$

The operations of addition, scalar multiplication, dot product, vector product (in three dimensions) of vector fields (all defined on the same curve), are defined in the usual pointwise fashion.

Remarks. 1) We can perform *the differentiation* of a vector field $\vec{Y}$ defined on a curve α. It provides a new vector field on α given by

$$\vec{Y}'(t) = y_1'(t)\vec{e}_1 + \ldots + y_n'(t)\vec{e}_n.$$

2) The derivative of the scalar product of two vector fields $\vec{Y}$ and $\vec{Z}$ defined on α has the property

$$(\vec{Y}, \vec{Z})' = (\vec{Y}', \vec{Z}) + (\vec{Y}, \vec{Z}').$$

3) If the vector field $\vec{Y}$ defined on α is a unit vector field, i.e.,

$$||\vec{Y}(t)|| = 1, \ \ \forall t \in I,$$

then we have

$$(\vec{Y}(t), \vec{Y}'(t)) = 0.$$

More generally, if the point on the curve moves with a constant speed, then

$$\vec{\alpha}'(t) \perp \vec{\alpha}''(t), \ \ \forall t \in I,$$

so we can say that the field $\vec{\alpha}\,''$ is normal to the curve at each point $\alpha(t)$.

Lesson 14

6.4 Infinite branches

Let $I = (a, b)$ be an open interval; we include the cases when $a = -\infty$ or $b = \infty$. Let $\alpha : I \to \mathbb{R}^n$ be a parametrized curve, such that $\alpha(I)$ is unbounded. We shall analyse if an infinite branch of $\alpha(I)$ can be assimilated to the infinite branch of a straight line.

4.1 Definition. Let t_0 be an extremity of I. If

$$\lim_{t \to t_0} ||\vec{\alpha}(t)|| = \infty,$$

then we say that α *has an infinite branch* for $t \to t_0$.

We remark that if $\alpha(t) = (x_1(t), \ldots, x_n(t))$, then

$$||\vec{\alpha}(t)|| = \sqrt{x_1^2(t) + \ldots + x_n^2(t)}$$

and the condition for having an infinite branch rewrites

$$\lim_{t \to t_0} x_k(t) = \pm\infty$$

for some $k \in \{1, \ldots, n\}$.

4.2 Definitions. Let $t_0 \in I$ provide an infinite branch for the curve $\alpha : I \to \mathbb{R}^n$.
a) The unit vector

$$\vec{u} = \lim_{t \to t_0} \frac{\vec{\alpha}(t)}{||\vec{\alpha}(t)||}$$

(if it exists) is called *an asymptotic direction* of α, for $t \to t_0$.
b) The straight line D is called *asymptote of the infinite branch* (see fig. 96) if

$$\lim_{t \to t_0} d(P; D) = 0,$$

where we denoted $P = \alpha(t)$.

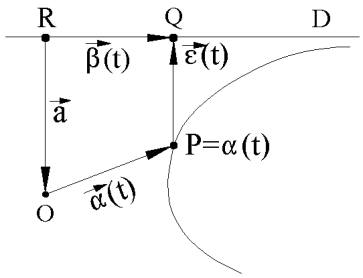


Fig. 96

This is equivalent to $\lim_{t \to t_0} \bar{\varepsilon}(t) = \bar{0}$, where $\bar{\varepsilon}(t)$ is orthogonal to D and R is a fixed point of D.

4.3. Theorem. *The direction of an asymptote of α is an asymptotic direction.*

Proof. We have the relation (see fig. 96)

$$\bar{\beta}(t) = \bar{a} + \bar{\alpha}(t) + \bar{\varepsilon}(t).$$

We try to estimate $\lim_{t \to t_0} \frac{\bar{\beta}(t)}{||\bar{\beta}(t)||}$. Using the properties of the norm, we have the inequality

$$||\bar{\beta}(t)|| - ||\bar{a} + \bar{\varepsilon}(t)|| \leq ||\bar{\alpha}(t)|| \leq ||\bar{\beta}(t)|| + ||\bar{a} + \bar{\varepsilon}(t)||.$$

Then for $||\bar{\alpha}|| \to \infty$ we get $||\bar{\beta}|| \to \infty$ and

$$\lim_{t \to t_0} \frac{||\bar{\alpha}(t)||}{||\bar{\beta}(t)||} = 1,$$

and

$$\lim_{t\to t_0}\frac{\bar\beta(t)}{||\bar\beta(t)||}=\lim_{t\to t_0}\frac{\bar\beta(t)}{||\bar\alpha(t)||}\frac{||\bar\alpha(t)||}{||\bar\beta(t)||}=\lim_{t\to t_0}\frac{\bar\alpha(t)}{||\bar\alpha(t)||}=\bar u.$$

Remarks. 1) The asymptote for $t\to t_0$ is unique, since the limit is unique.

2) If there exists a point $A\in\mathbb{R}^n$ such that $\lim\limits_{t\to t_0}\alpha(t)=A$, then A is called *asymptotic point.*

6.5 Normal representation of a curve

Let $\alpha: I\to\mathbb{R}^n$ be a curve; its velocity vector is $\vec\alpha'(t)$. We define *the speed of* α *at* t to be the length of $\vec\alpha'(t)$, i.e.,

$$v(t)=||\vec\alpha'(t)||.$$

In coordinates, we have $\alpha(t)=(x_1(t),\dots,x_n(t))$, and

$$v(t)=\sqrt{x_1'^2(t)+\dots+x_n'^2(t)}.$$

In physics, the distance traveled by a moving point is determined by integrating its speed with respect to time. Thus we define *arc length of* α *from* $t=a$ *to* $t=b$, to be the number

$$l=\int_a^b||\vec\alpha'(t)||dt,$$

and the *arc-length function*

$$s(t)=\int_a^t||\vec\alpha'(t)||dt.$$

Sometimes one is interested only in the route $\alpha(I)$ and not in the particular speed at which $\alpha(t)$ traverses its route. One way to ignore the speed of a curve is to reparametrize it to a curve β which has unit speed $||\vec\beta'||=1$.

5.1 Definition. Let $\alpha: I\to\mathbb{R}^n$ be a curve and $h: J\to I$ a differentiable function. The function $\beta=\alpha\circ h$ is called *the reparametrization of* α *by* h (see fig. 97).

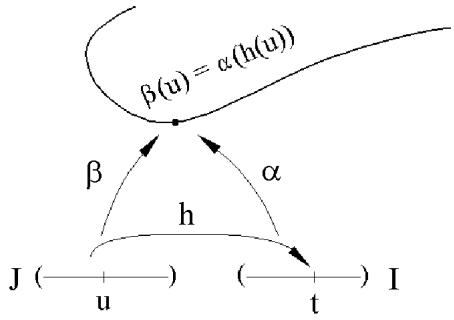


Fig. 97

We have the relations

$$\alpha(h(u)) = \beta(u), \qquad \frac{d\vec{\beta}}{du}(u) = \frac{d\vec{\alpha}}{dh}(h(u))\frac{dh}{du}(u).$$

In order to preserve the multiplicity of points is necessary that h be a bijection; for preserving the regularity we need the condition $h' \neq 0$; to preserve the orientation of the curve, we need $h' > 0$. If h is a diffeomorphism, then the curves α and β are called *equivalent curves*.

5.2 Theorem. *If is α is a regular curve in $\mathbb{R}^n$, then there exists a reparametrization β of α such that β has a unit speed.*

This reparametrization β is called *normal representation of α*.

Proof. Fix $a \in I$ and consider the arc-length function

$$s(t) = \int_a^t ||\vec{\alpha}'(u)||du.$$

Denoting briefly $s = s(t)$, we have $\frac{ds}{dt} = ||\vec{\alpha}'(t)|| > 0$; hence the function s has an inverse $t = t(s)$, whose derivative is $\frac{dt}{ds} = \frac{1}{ds/dt} > 0$. We define $\beta(s) = \alpha(t(s))$. The velocity of this curve is then $\frac{d\vec{\beta}}{ds} = \frac{d\vec{\alpha}}{dt}\frac{dt}{ds}$, whence we get $\left|\left|\frac{d\vec{\beta}}{ds}\right|\right| = 1$, so β is a unit-speed curve.

Notice that a curve is parametrized by arc-length iff its speed is 1.

Example. Let be the curve

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad t \in \mathbb{R}.$$

Then we have $\vec{\alpha}'(t) = (-asint, acost, b), ||\vec{\alpha}'|| = \sqrt{a^2+b^2}$ and the arc-length function

$$s = \int_0^t \sqrt{a^2+b^2}dt = t\sqrt{a^2+b^2}.$$

The inverse function is then $t(s) = \frac{s}{\sqrt{a^2+b^2}}$ and the normal representation of α is

$$\beta(s) = \alpha(t(s)) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, b\frac{s}{\sqrt{a^2+b^2}}\right), \quad t \in \mathbb{R}.$$

Curves in $\mathbf{R}^2$

Let $\alpha : I \to \mathbb{R}^2$, $\alpha(t) = (x(t)), y(t))$ be a curve. Then the derivative of the position vector $\vec{\alpha}(t) = x(t)\vec{i} + y(t)\vec{j}$ is the velocity

$$\vec{\alpha}'(t) = x'(t)\vec{i} + y'(t)\vec{j}.$$

For $t \in I$ fixed, let $\alpha(t)$ be a regular point of α. Then

$$\frac{x - x(t)}{x'(t)} = \frac{y - y(t)}{y'(t)}$$

is the equation of the *tangent to* α *at* $\alpha(t)$; also, the *normal to* α *at* $\alpha(t)$ has the equation

$$(x - x(t))x'(t) + (y - y(t))y'(t) = 0.$$

If $\alpha(t)$ is a singular point of order m of α, then the vector

$$\vec{\alpha}^{(m)}(t) = x^{(m)}(t)\vec{i} + y^{(m)}(t)\vec{j}$$

provides the tangent direction to α at its point $\alpha(t)$. The *tangent* at $\alpha(t)$ has the equation

$$\frac{x - x(t)}{x^{(m)}(t)} = \frac{y - y(t)}{y^{(m)}(t)},$$

and the *normal* at $\alpha(t)$,

$$(x - x(t))x^{(m)}(t) + (y - y(t))y^{(m)}(t) = 0.$$

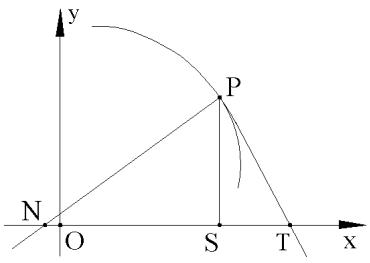


Fig. 98

Also, the segments TP and TS (see fig. 98) are called respectively *the tangent segment* and *the subtangent segment* of α at $\alpha(t)$. In a similar manner one can define *the normal segment* PN and *the subnormal* segment SN of α at $\alpha(t)$.

6.6 Shape of a plane curve in a neighborhood of its points

In the following, we shall provide some information about the shape of a curve $\alpha : I \to \mathbb{R}^2$ in the neighborhood of a point $\alpha(t) = P$. The set

$$V = \{Q \mid d(P, Q) < \varepsilon, Q \in \alpha(I)\}$$

for some $\varepsilon > 0$, is called *a neighborhood of* P *in* $\alpha(I)$. Let be a point $Q = \alpha(t+h) \in V$, where h belongs to a neighborhood of zero. Then we have the Taylor formula (which makes sense only for free vectors)

$$\bar{\alpha}(t+h) = \bar{\alpha}(t) + \frac{h}{1!}\bar{\alpha}'(t) + \frac{h^2}{2!}\bar{\alpha}''(t) + \ldots + \frac{h^n}{n!}[\bar{\alpha}^{(n)}(t) + \bar{\varepsilon}(h)],$$

where $\lim\limits_{h\to 0} \bar{\varepsilon}(h) = \bar{0}$. We can rewrite it as

$$\overline{PQ} = \bar{\alpha}(t+h) - \bar{\alpha}(t) \cong \frac{h}{1!}\bar{\alpha}'(t) + \ldots + \frac{h^n}{n!}\bar{\alpha}^{(n)}(t).$$

Consequently $\overline{PQ}$ can be approximated by a linear combination of n vectors. In the the following we shall approximate $\overline{PQ}$ like a linear combination of two linearly independent vectors.

If $P = \alpha(t)$ is a regular point, then we have $\vec{\alpha}'(t) \neq \vec{0}$ and the tangent at P is determined by P and $\vec{\alpha}'(t)$. If P is a regular point, then we have the following cases

1) Suppose $\vec{\alpha}'(t), \vec{\alpha}''(t)$ form a basis at P. Then the Taylor expansion of order 2 provides:

$$\overline{PQ} \cong \frac{h}{1!}\bar{\alpha}'(t) + \frac{h^2}{2!}\bar{\alpha}''(t).$$

The coordinates of the point Q relative to the considered frame (see fig. 99) are $(h, h^2/2)$; when h passes through zero, we pass from $(-,+)$ via $(0,0)$ to $(+,+)$.

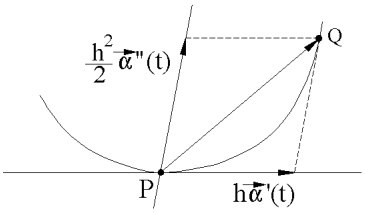


Fig. 99

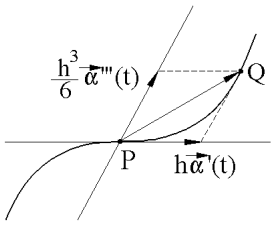


Fig. 100

2) If $\vec{\alpha}''(t)$ is collinear to $\vec{\alpha}'(t)$ and $\vec{\alpha}'(t), \vec{\alpha}'''(t)$ form a basis at $T_P\,\mathbb{R}^2$, then we have

$$\overline{PQ} \cong h\bar{\alpha}'(t) + \frac{h^3}{6}\bar{\alpha}'''(t)$$

and the point Q has the coordinates $(h, h^3/6)$; when h passes through zero, we pass from $(-,-)$ via $(0,0)$ to $(+,+)$, and the origin P of the frame is called *an inflection point* (see fig. 100).

Generalization. Let $\vec{\alpha}'(t) \neq \vec{0}$ and $\vec{\alpha}''(t), \ldots, \vec{\alpha}^{(m-1)}(t)$ be collinear to $\vec{\alpha}'(t)$; if $\vec{\alpha}^{(m)}(t)$ is non-collinear with $\vec{\alpha}'(t)$, then $\{\vec{\alpha}'(t), \vec{\alpha}^{(m)}(t)\}$ form a basis of $T_P\,\mathbb{R}^2$, the basis of a frame at P. Relative to this frame, the point Q has the coordinates $(h, h^m/m!)$. If m is odd, then the curve has an inflection point at P.

We say that $P = \alpha(t)$ is *a singular point of* α if $\vec{\alpha}'(t) = \vec{0}$.

3) Let $\vec{\alpha}''(t), \vec{\alpha}'''(t)$ form a basis of $T_P\,\mathbb{R}^2$. Then the direction of the tangent to α at P is given by $\vec{\alpha}''(t)$. In this case we have

$$\overline{PQ} \cong \frac{h^2}{2}\bar{\alpha}''(t) + \frac{h^3}{6}\bar{\alpha}'''(t)$$

so, with respect to the frame given by the basis above at P, a point Q has the coordinates $(h/2, h^3/6)$; when h passes through zero, Q describes a *cusp of the first type*. The image of α around P looks like a bifurcation of lines of railway (see fig. 101).

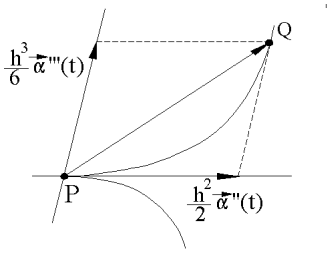


Fig. 101

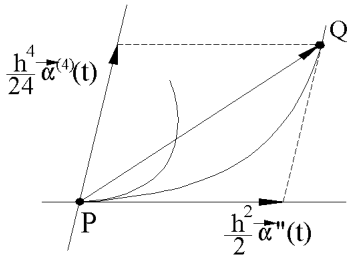


Fig. 102

4) Let $\vec{\alpha}''(t) \neq \bar{0}$, $\vec{\alpha}'''(t)$ collinear to $\vec{\alpha}''(t)$ and let $\vec{\alpha}''(t), \bar{\alpha}^{iv}(t)$ form a basis of $T_P\,\mathbb{R}^2$. Then we have

$$\overline{PQ} \cong \frac{h^2}{2}\bar{\alpha}''(t) + \frac{h^4}{24}\vec{\alpha}^{iv}(t);$$

relative to the frame at P provided by the basis above, the point Q has the coordinates $(h/2, h^4/24)$. When h passes through zero, Q describes a *cusp of second type*. The image of α around P looks like a bifurcation of lines of railway (see fig. 102).

Generalization. If $\vec{\alpha}'(t) = \ldots = \vec{\alpha}^{(m-1)}(t) = \vec{0}$, $\vec{\alpha}^{(m)}(t) \neq \vec{0}$ and $\{\vec{\alpha}^{(m)}(t), \vec{\alpha}^{(n)}(t)\}$ form a basis of $T_P\,\mathbb{R}^2$, then the coordinates of Q relative to the frame at P are $(h^m/m!, h^n/n!)$. We have the following cases:

- m is odd and n is even (see fig. 99);
- m is odd and n is odd (see fig. 100);
- m is even, n is odd (see fig. 101);
- m is even, n is even (see fig. 102).

Remark. If $\vec{\alpha}'(t) = \ldots = \vec{\alpha}^{(n)}(t) = \ldots = \vec{0}$ then it is not possible to introduce the tangent to the curve $\alpha(I)$ at $P = \alpha(t)$ using the derivatives of α.

Lesson 15

6.7 Curves in $\mathbf{R}^2$ defined by Cartesian implicit equations

We have already had examples of curves defined by Cartesian implicit equations, e.g., straight lines: $ax+by+c=0$, conics: $a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{10}x+2a_{20}y+a_{00}=0$, etc.

Generally, let us analyse the set described by an equation of the form

$$f(x, y) = 0,$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. We consider the Jacobian matrix

$$J(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \nabla f.$$

We say that a point $P(x, y) \in \mathbb{R}^2$ for which $\nabla f(P) = (0, 0)$, is a *critical point* of f. A point which is not critical, is called *regular point* of f. We call *level sets* of f, the sets of points

$$C = f^{-1}(c) = \{(x, y) | (x, y) \in \mathbb{R}^2, f(x, y) = c\},$$

where $c \in \mathbb{R}$ is called *the height of the level set*. We also say that C is the level set at height c. We call $z = f(x, y)$ *the explicit Cartesian equation of the graph of the function* f, and $f(x, y) = c$ *the implicit Cartesian equation of the set* C.

Usually, C contains critical and regular points as well. Let $M_0(x_0, y_0) \in C$; we call *neighborhood of* M_0 *in* C, the set of all points in C which satisfy the condition $(x - x_0) + (y - y_0)^2 < \epsilon$ for some $\epsilon > 0$.

7.1 Theorem. *If a solution* $M_0(x_0, y_0)$ *of the equation* $f(x, y) = c$ *is a regular point of* f*, then there exists a neighborhood* V *of* M_0 *in* C *such that* V *is a simple and regular curve* (see fig. 103).

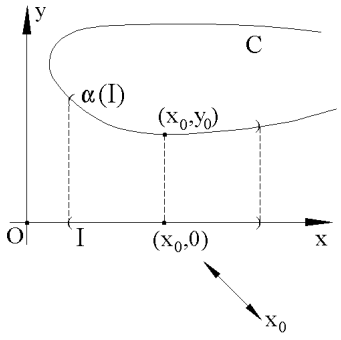


Fig. 103

Proof. Suppose $\frac{\partial f}{\partial y}(M_0) \neq 0$. According to the theorem of the implicit functions, there exists a neighborhood I_{x_0} in $\mathbb{R}$ and a diffentiable function $x \to y(x)$ such that

$$y_0 = y(x_0),\ f(x, y(x)) = c \quad \text{and} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

Generally C is made of critical points and regular arcs. If f is a polynomial of order n, then the curve C is called *algebraic curve of order n*. As examples are the straight lines, the conics, etc.

7.2 Theorem. *Let $M_0(x_0, y_0)$ be a regular point of C. Then, the tangent to C at M_0 has the equation*

$$(x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0) = 0,$$

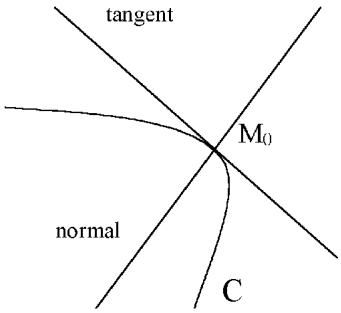


Fig. 104

and the normal line of C at M_0 (see fig. 104) has the equation

$$\frac{x - x_0}{\frac{\partial f}{\partial x}(x_0, y_0)} = \frac{y - y_0}{\frac{\partial f}{\partial y}(x_0, y_0)}.$$

Remark. The critical points of f are classified in the theory of catastrophes [44].

6.8 Sketch (drawing) of a plane curve

Let $\alpha : I \to \mathbb{R}^2$, $\alpha(t) = (x(t), y(t))$ be a plane curve. For drawing the image $\alpha(I) \subset \mathbb{R}^2$ relative to the Cartesian frame xOy we shall examine the following problems:

8.1 *The domain I; the accumulation points which do not belong to I, the limits of the functions $t \to x(t)$, $t \to y(t)$ to these points; the asymptotic points (if such points do exist).*

8.2 *The intersections of the graph with the axes.*

8.3 *The symmetries of* $\alpha(I)$. If $\forall t \in I,\ \exists t' \in I$ such that
(1) $x(t') = x(t),\ y(t') = -y(t)$,
(2) $x(t') = -x(t),\ y(t') = y(t)$,
(3) $x(t') = -x(t),\ y(t') = -y(t)$,
(4) $x(t') = y(t),\ y(t') = x(t)$,
etc, then the curve is symmetric respectively with respect to (1) the Ox axis, (2) the Oy axis, (3) the origin, (4) the first bisectrix, etc.

We remark that the systems (1), (2), and (3) contain as a particular case the study of the parity and imparity of the functions $t \to x(t)$ and $t \to y(t)$.

If there exists $r \in \mathbb{R}$ such that for all $t \in I$ the point $\alpha(r-t)$ is obtained from $\alpha(t)$ by a symmetry (relative to a point or a straight line), then it follows $t' = r - t$, which is equivalent to $\frac{t+t'}{2} = \frac{r}{2}$. Hence t and t' are symmetric in $\mathbb{R}$ with respect to $\frac{r}{2}$. In this case, we plot the part of $\alpha(I)$ which corresponds to $I \cap [r/2, \infty)$, and the rest of the curve is completed by symmetry.

If $\alpha\left(\frac{1}{t}\right)$ is obtained from $\alpha(t)$ by a symmetry, then it follows that $t' = \frac{1}{t}$ or $tt' = 1$. In this case we plot the part of $\alpha(I)$ which corresponds to $I \cap ([-1,0) \cup (0,1])$, and the rest of the graph is completed by symmetry.

8.4 *The singular points of* α. Among the regular points one should determine the inflexion points, and the points at which $\vec{\alpha}^{(n)},\ n = 2, 3, ...$ are collinear to $\vec{\alpha}'$.

Finding the singular points (when these exist) and of the tangents at these points. One should determine the inflexion points, the cusp points, the singular points of order n at which $\vec{\alpha}^{(m)},\ m = n+1,\ n+2, ...$ are collinear to $\vec{\alpha}^{(n)} \neq \vec{0}$ and the singular points at which $\vec{\alpha}^{(k)} = \vec{0},\ k = 1, 2, ...$

8.5 *The multiple points and the tangents at these points.* If the system

$$t_1 \neq t_2,\ x(t_1) = x(t_2),\ y(t_1) = y(t_2)$$

has solutions, then its solutions provide multiple points. If the system has no solution, then the curve has only simple points. If at least one of the components of the function α is injective, then α is injective.

8.6 *The periodicity of* α. We study if α is a periodic curve, i.e., if

$$\exists T > 0,\ \ \alpha(t+T) = \alpha(t), \quad \forall t \in I.$$

If α is periodic of period T, then it is sufficient to consider the restriction $\alpha : [0, T] \to R^2$. From α being a periodic curve, it follows that

$$t \to x(t),\ \ t \to y(t)$$

are periodic, having periods which do not necessarily coincide with the one of α. If $t \to x(t)$ is periodic, and has the period T_1, $t \to y(t)$ is periodic and has the period T_2, and $\frac{T_1}{T_2} = \frac{p}{q} \in \mathbb{Q}$, then α is periodic and its period is $T = qT_1 = pT_2$.

8.7 *The variation table of the functions $t \to x(t)$ and $t \to y(t)$.*

8.8 *The infinite branches and the asymptotes (if these exist).* The following situations can occur:

1) $\lim\limits_{t \to t_0} x(t) = \pm\infty$, $\lim\limits_{t \to t_0} y(t) = b$.

In this case the equation of the asymptote is $y = b$. To find the relative position of the branch with respect to the asymptote, we read from the table the sign of $y(t) - b$ in the neighborhood of t_0.

2) $\lim\limits_{t \to t_0} x(t) = a$, $\lim\limits_{t \to t_0} y(t) = \pm\infty$.

In this case the equation of the asymptote is $x = a$.

3) $\lim\limits_{t \to t_0} x(t) = \pm\infty$, $\lim\limits_{t \to t_0} y(t) = \pm\infty$.

- If $\lim\limits_{t \to t_0} \frac{y(t)}{x(t)} = 0$, then $(1, 0)$ is an asymptotic direction, but the curve does not admit an asymptote (it has a *parabolic branch*).
- If $\lim\limits_{t \to t_0} \frac{x(t)}{y(t)} = 0$, then $(0, 1)$ is an asymptotic direction. The curve does not admit an asymptote (it has a parabolic branch).
- If $\lim\limits_{t \to t_0} \frac{y(t)}{x(t)} = m$, then $(1, m)$ is an asymptotic direction. If

$$\lim_{t \to t_0} (y(t) - mx(t)) = n,$$

then the curve admits the asymptote $y = mx + n$. If $\lim\limits_{t \to t_0} (y(t) - mx(t)) = \pm\infty$, then curve does not admit an asymptote (it has a parabolic branch).

8.9 Example. The curve $x^3 + y^3 - 3axy = 0$, $a > 0$, is called *the folium of Descartes*.

a) Find a parametrization of the curve.

b) Sketch the curve.

Answer. a) The axis Oy intersects the curve at the origin. We intersect the curve with the straight line $y = tx$. Replacing y in the equation of the curve, we find $x^2(x + t^3x - 3at) = 0$. First we have $x^2 = 0$, which corresponds to the double

point $(0,0)$. Then, for $t \neq -1$, we find

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

b) For drawing the curve, we follow the steps:

1. *Domain and symmetries.* We can see that

$$t \in (-\infty, -1) \cup (-1, \infty).$$

At the accumulation points which do not belong to the domain we have to compute the limits. We find:

$$\lim_{t\to\pm\infty} x(t) = 0, \quad \lim_{t\nearrow -1} x(t) = \infty, \quad \lim_{t\searrow -1} x(t) = -\infty$$

$$\lim_{t\to\pm\infty} y(t) = 0, \quad \lim_{t\nearrow -1} y(t) = -\infty, \quad \lim_{t\searrow -1} y(t) = \infty.$$

We remark that $O(0,0)$ is an asymptotic point for $t \to \infty$ as well as for $t \to -\infty$. This point coincides with the (common) point $t = 0$.

The system

$$\begin{cases} \dfrac{3at'}{1+t'^3} = \dfrac{3at^2}{1+t^3} \\ \dfrac{3at'^2}{1+t'^3} = \dfrac{3at}{1+t^3} \end{cases}$$

has an infinity of solutions, because it is satisfied for any t and t' which are related by $tt' = 1$. Hence the curve is symmetric with respect to the first bisectrix. Therefore it is sufficient to draw just the branch $t \in (-1, 0) \cup (0, 1]$, and the rest is completed by symmetry.

2. *Regular and singular points.* We have

$$x' = \frac{3a - 6at^3}{(1+t^3)^2}, \quad y' = \frac{6at - 3at^4}{(1+t^3)^2}.$$

From $x' = 0$ we find $t = \frac{1}{\sqrt[3]{2}}$. From $y' = 0$ we get $t_1 = 0,\ t_2 = \sqrt[3]{2}$. This implies that the folium of Descartes is a regular curve.

3. *Multiple points.* Having in view the meaning of t - which is a slope of a straight line, $t \to -\infty$ and $t \to \infty$ provide the same point $(0,0)$ on the curve (asymptotic point), which corresponds to the intersection of the curve with the $Oy(x = 0)$ axis. On the other hand, $t = 0$ gives the point $(0,0)$. Thus, the origin is a double point.

The system

$$\begin{cases} t_1 \neq t_2 \\ \dfrac{3at_1}{1+t_1^3} = \dfrac{3at_2}{1+t_2^3} \\ \dfrac{3at_1^2}{1+t_1^3} = \dfrac{3at_2^2}{1+t_2^3} \end{cases}$$

shows that there are no more multiple points on the curve.
The tangents at $O(0,0)$ are the axes Ox and Oy.

4. *The variation table for $x(t)$ and $y(t)$.*

	$-\infty$			-1			0	$1/\sqrt[3]{2}$		$\sqrt[3]{2}$		$+\infty$
x'		$+$			$+$		$+$	$+$	0	$-$		
x	0	$\nearrow$	∞	$-\infty$		$\nearrow$	0	$\nearrow$	$a\sqrt[3]{4}$	$\searrow$	$a\sqrt[3]{2}$	$\searrow 0$
y'		$-$			$-$		0	$+$		0	$-$	
y	0	$\searrow$	$-\infty$	∞		$\searrow$	0	$\nearrow$	$a\sqrt[3]{2}$	$\nearrow$	$a\sqrt[3]{4}$	$\searrow 0$

5. *Infinite branches and asymptotes.*
For $t \nearrow -1$ and $t \searrow -1$ we have infinite branches.

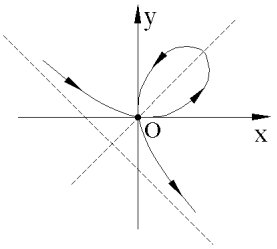


Fig. 105

Since

$$\lim_{t\to -1} \frac{y(t)}{x(t)} = -1,$$

it follows that both the infinite branches admit the asymptotic direction $(1,-1)$.
On the other hand

$$\lim_{t\to -1} (y(t)+x(t)) = \lim_{t\to -1} \frac{3at}{1+t^2-t} = -a$$

and hence we obtain the slant asymptote $y+x+a=0$.

6. *The sketch of the curve* (see fig. 105).

6.9 Frenet formulas for plane curves

We now derive the mathematical measurement of the turning of a curve in $\mathbb{R}^2$. Let

$$\beta : J \to \mathbb{R}^2,\ \beta(s) = (x(s), y(s))$$

be a unit speed curve , i.e., $||\vec{\beta}'(s)|| = 1, \forall s \in J$. Then $\vec{T} = \vec{\beta}'$ is called *the unit tangent vector field on* β; we define *the normal nit vector of* β as

$$\vec{N}(s) = (-y'(s), x'(s)),$$

which is obtained by rotating $\vec{T}$ through 90^o (see fig. 106). The orthonormal moving frame $\{\vec{T}, \vec{N}\}$ is called *the Frenet frame field.*

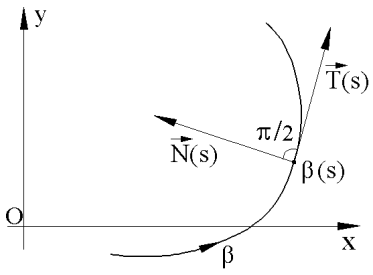


Fig. 106

We remark that $(\vec{T}, \vec{T}) = 1 \Rightarrow (\vec{T}', \vec{T}) = 0$. Hence, $\vec{T}'$ is collinear to $\vec{N}$. The function $k : J \to \mathbb{R}$, defined by

$$\vec{T}' = k\vec{N}$$

is called *the curvature function of* β. We can see that the larger $|k|$ is, the sharper the turning of β.

9.1 Theorem. *Let $\beta : J \to \mathbb{R}^2$ be a unit speed curve with curvature function k.*
1) (Frenet formulas) The decompositions of the vector fields $\vec{T}', \vec{N}'$ with respect to the Frenet frame field $\{\vec{T}, \vec{N}\}$ are

$$\begin{cases} \vec{T}' = k\vec{N} \\ \vec{N}' = -k\vec{T}. \end{cases}$$

2) We have the relation $k = \dfrac{d\varphi}{ds}$, where φ is the angle which gives the slope of the tangent (see fig. 107).

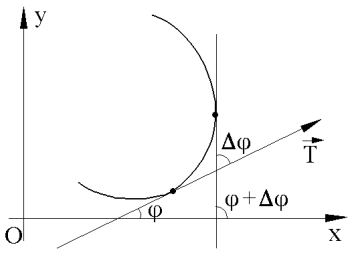


Fig. 107

Proof. 1) The first relation was justified above. For the second, we decompose $\vec{N}'$ with respect to the orthonormal frame $\{\vec{T}, \vec{N}\}$,

$$\vec{N}' = (\vec{N}', \vec{T})\vec{T} + (\vec{N}', \vec{N})\vec{N}.$$

Let us compute the coefficients. Since $(\vec{N}, \vec{N}) = 1$, we have $(\vec{N}', \vec{N}) = 0$. Using $(\vec{N}, \vec{T}) = 0$, we obtain by differentiation

$$(\vec{N}', \vec{T}) + (\vec{N}, \vec{T}') = 0 \Rightarrow (\vec{N}', \vec{T}) = -(\vec{N}, \vec{T}') = -(\vec{N}, k\vec{N}) = -k.$$

2) We can decompose $\vec{T}$ relative to the canonic basis $\{\vec{i}, \vec{j}\}$ as

$$\vec{T} = \vec{i}\cos\varphi + \vec{j}\sin\varphi,$$

and by deriving it, we have

$$\vec{T}' = (-\vec{i}\sin\varphi + \vec{j}\cos\varphi)\frac{d\varphi}{ds} = \frac{d\varphi}{ds}\vec{N}.$$

Now let us find the general formula for the curvature of a curve reported to an arbitrary parameter.

9.2 Theorem. *Let $\alpha : I \to \mathbb{R}^2$ be a regular curve. Then the curvature of α is given by the formula*

$$k = \frac{(\vec{\alpha}'', \mathcal{R}(\vec{\alpha}'))}{||\vec{\alpha}'||^3} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}},$$

where $\mathcal{R}$ is the operator of rotation with the angle $\pi/2$. We call $\frac{1}{|k|}$ *radius of curvature.*

Proof. First we express the velocity $\vec{\alpha}'$ and the acceleration $\vec{\alpha}''$ with respect to $\{\vec{T}, \vec{N}\}$. Since t is an arbitrary parameter, we have $\vec{\alpha}' = v\vec{T}$, whence, by deriving, we obtain

$$\vec{\alpha}'' = \frac{dv}{dt}\vec{T} + kv^2\vec{N}.$$

Also, we have

$$\mathcal{R}(\vec{\alpha}') = \mathcal{R}(v\vec{T}) = v\vec{N}.$$

Taking the scalar product between the two vectors above, we yield

$$(\vec{\alpha}'', \mathcal{R}(\vec{\alpha}')) = kv^3,$$

whence the relation in the statement.

The tangential component of $\vec{\alpha}''$ measures the rate of change of the length of $\vec{\alpha}'$; the normal component measures the rate of change of the direction of $\vec{\alpha}'$.

9.3 Definition. A regular curve $\alpha : I \to \mathbb{R}^2$ is said to be *convex* if, for each $t_0 \in I$, the curve lies entirely on one side of the tangent at $\alpha(t_0)$.

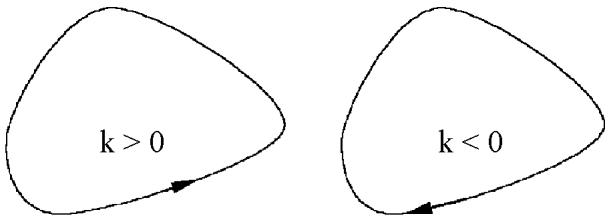


Fig. 108

9.4 Theorem. *Let $\alpha : I \to \mathbb{R}^2$ be a simple closed regular curve. Then α is convex iff it satisfies (see fig. 108)*

$$k(t) \geq 0, \ \forall t \in I \quad \text{or} \quad k(t) \leq 0, \ \forall t \in I.$$

Lesson 16

6.10 Theory of contact

Let $\alpha_1, \alpha_2 : I \to \mathbb{R}^2$ and let M_0 be a common regular point, i.e., there exists $t_0 \in I$ such that $M_0 = \alpha_1(t_0) = \alpha_2(t_0)$. Let us denote $P = \alpha_1(t), \ Q = \alpha_2(t)$. Then we have $\overline{PQ} = \bar{\alpha}_2 - \bar{\alpha}_1$ (see fig. 109).

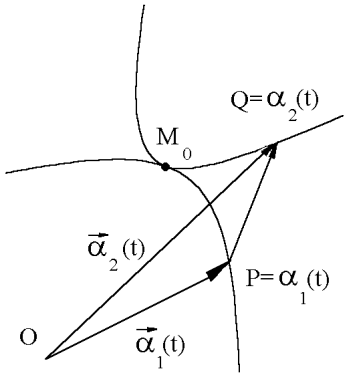


Fig. 109

10.1 Definition. We say that $\alpha_1(I)$ and $\alpha_2(I)$ have at M_0 *a contact of order* m if

$$\lim_{t\to t_0}\frac{\overline{PQ}}{(t-t_0)^k}=\begin{cases}\bar{0},\ \text{when } k=1,\dots,m\\ \neq\bar{0},\ \text{when } k=m+1.\end{cases}$$

Using the Taylor series for α_1 and α_2, we have

$$\bar{\alpha}_1(t)=\bar{\alpha}_1(t_0)+\frac{t-t_0}{1!}\bar{\alpha}_1'(t_0)+\ldots+\frac{(t-t_0)^m}{m!}[\bar{\alpha}_1^{(m)}(t)+\bar{\epsilon}_1(t-t_0)]$$

$$\bar{\alpha}_2(t)=\bar{\alpha}_2(t_0)+\frac{t-t_0}{1!}\bar{\alpha}_2'(t_0)+\ldots+\frac{(t-t_0)^m}{m!}[\bar{\alpha}_2^{(m)}(t)+\bar{\epsilon}_2(t-t_0)].$$

Hence we can see that a contact of order m means that the following conditions are fulfilled

$$\bar{\alpha}_1^{(k)}(t_0)=\bar{\alpha}_2^{(k)}(t_0),\ k=0,1,...,m\quad\text{ and }\quad\bar{\alpha}_1^{(m+1)}(t_0)\neq\bar{\alpha}_2^{(m+1)}(t_0).$$

We remark that M_0 is in fact a *singular point of order* $m+1$ *of the curve* $\alpha_2-\alpha_1$. Therefore, the contact of order m occurs when α_1 and α_2 have $m+1$ confounded common points at M_0; in particular, for contact of order 1, the curves α_1 and α_2 have the same tangent at M_0.

If $\alpha:\begin{cases}x=x(t)\\ y=y(t)\end{cases}$, $t\in I$ and $C: f(x,y)=0$ are two curves which have a common point $M_0(x_0,y_0)=\alpha(t_0)$, $t_0\in I$, then we can state the following

10.2 Theorem. *The curves* α *and* C *have at* M_0 *a contact of order* m *iff*

$$\Phi^{(k)}(t_0)=0,\ k=0,1,...,m\quad\textit{and}\quad\Phi^{(m+1)}(t_0)\neq 0,$$

where $\Phi(t) = f(x(t), y(t)),\ \forall t \in I$.

10.3 Osculating curves. Let $\alpha : \begin{cases} x = x(t) \\ y = y(t) \end{cases},\ t \in I$ be a curve and

$$C_a :\ f(x, y; a_1, ..., a_{m+1}) = 0,\ a = (a_1, ..., a_{m+1}) \in \mathbb{R}^{m+1}$$

be a family of curves with $m+1$ parameters, where f is a differentiable function of $m+3$ variables. Then a curve from the family C_a which has with α at a common point, a contact of order m (i.e., $m+1$ confounded common points) is called *osculating curve of* α; it is called *super-osculating* iff it has a contact at least of order $m+1$.

The theorem 10.2 determines the osculating curve by $m+1$ equations in $m+1$ unknowns.

An osculating curve appears as the limit position of a curve C_a which passes through $m+1$ points of α, when these points tend along α towards the initial point. We have the following cases:

a) If $m = 2k+1$, then the osculating curve does not traverse α at the regular point M_0 (see fig. 110);

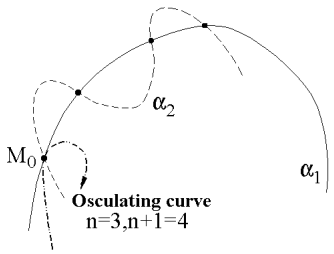


Fig. 110

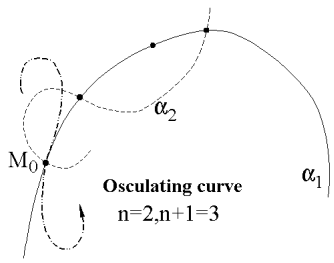


Fig. 111

b) If $m = 2k$, then the osculating curve traverses α at the regular point M_0 (see fig. 111).

The osculating curve is the best approximation of the curve α around the common point.

10.4 Osculating straight lines. The family of straight lines is characterized by the equation

$$ax + by + c = 0 \quad \text{with} \quad a^2 + b^2 > 0,$$

which has two essential parameters. Given a curve $\alpha : I \to \mathbb{R}^2$ and a regular point on it, we can determine an osculating straight line (a contact of order 1, i.e., 2 confounded common points) which is *the tangent of* α. Also, at a singular point of order m, the curve and the tangent are two superosculating curves, since the tangent is a superosculating straight line (a contact of order m, i.e., $m+1$ confounded points) (see fig. 112).

Fig. 112

For determining the tangent line at a regular point of the plane curve

$$\alpha : \ I \to \mathbb{R}^2, \quad \alpha(t) = (x(t), y(t)),$$

we use the theorem 10.2, imposing the conditions

$$\begin{cases} ax(t) + by(t) + c = 0 \\ ax'(t) + by'(t) + c = 0, \ t \text{ fixed}, \end{cases}$$

whence we obtain a non-trivial solution (a, b, c).

10.5 Osculating circle (circle of curvature). The family of circles is characterized by the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

which has three essential parameters, i.e., the coordinates of the center (x_0, y_0) and the radius r.

Given a curve $\alpha : \ I \to \mathbb{R}^2, \alpha(t) = (x(t), y(t))$, and a regular point on it, we can determine the essential parameters of the osculating circle which has a contact of order 2 (i.e., 3 confounded common points) with α, or a superosculating circle (having, e.g., a contact of order 3, i.e., 4 confounded common points).

If the circle has the equation $(\vec{r} - \vec{r}_0, \vec{r} - \vec{r}_0) = R^2$ and the curve has the vector equation $\vec{\alpha}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, then considering the function

$$\Phi = (\vec{\alpha} - \vec{r}_0, \vec{\alpha} - \vec{r}_0) - R^2,$$

the vanishing of Φ and its first two derivatives at a fixed point t provides the system

$$\begin{cases} (\vec{\alpha}-\vec{r}_0,\vec{\alpha}-\vec{r}_0)-R^2=0 \\ (\vec{\alpha}',\vec{\alpha}-\vec{r}_0)=0 \\ (\vec{\alpha}'',\vec{\alpha}-\vec{r}_0)+(\vec{\alpha}',\vec{\alpha}')=0. \end{cases}$$

From the second equation we obtain

$$\vec{r}_0=\vec{\alpha}+\lambda\vec{N},$$

where $\vec{N}$ is the second unit vector of the Frenet frame

$$\left\{\vec{T}=\frac{x'\vec{i}+y'\vec{j}}{\sqrt{x'^2+y'^2}},\vec{N}=\frac{-y'\vec{i}+x'\vec{j}}{\sqrt{x'^2+y'^2}}\right\}.$$

Then, from the third equation, we have

$$\lambda=\frac{1}{k},\quad k=\frac{x'y''-x''y'}{(x'^2+y'^2)^{3/2}},$$

and from the first equation, we find $R=\frac{1}{|k|}$ or, explicitly,

$$R=\frac{(x'^2+y'^2)^{3/2}}{|x'y''-x''y'|}.$$

We call $R=\frac{1}{|k|}$ *radius of curvature.* The point whose position vector is

$$\vec{r}_0=\vec{\alpha}+\frac{1}{k}\vec{N}$$

is called *center of curvature* (see fig. 113).

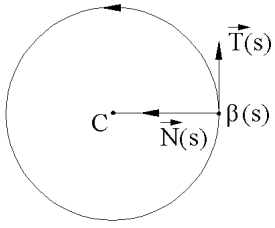


Fig. 113

Explicitly, the coordinates of the center of curvature are

$$\begin{cases} x_0 = x(t) - y'(t)\frac{x'^2+y'^2}{x'y''-x''y'}(t) \\ y_0 = y(t) + x'(t)\frac{x'^2+y'^2}{x'y''-x''y'}(t), \end{cases}$$

where t is fixed.

The curve

$$\vec{\gamma} = \vec{\alpha} + \frac{1}{k}\vec{N}$$

is called *the evolute of* α and it is the geometrical locus of centers of curvature of α (the envelope of the family of normals). We say that $\vec{\alpha}$ is *the involute of* $\vec{\gamma}$. Generally, involutes of a curve are orthogonal trajectories of its tangents.

10.6 Envelope of a family of curves. Let

$$C_a : f(x, y, a) = 0,\ a \in \mathbb{R}$$

be a family of curves, where f is a differentiable function in three variables. A curve

$$\alpha : I \to \mathbb{R}^2, \alpha(a) = (x(a), y(a))$$

is called *the envelope of the family* C_a, if it satisfies the following conditions:

1) for all $P \in \alpha(I)$ we can indicate a unique curve C_a which contains P as a regular point and the contact at P is at least of order 1;

2) for any curve of the family $C_a, a \in \mathbb{R}$, there exists a regular point $P \in \alpha(I)$ such that the contact at P is at least of order 1;

3) the curve α and any of the members of the family C_a have no common arcs.

So, the envelope is the curve which is tangent to each of the curves of the family C_a (see fig. 114).

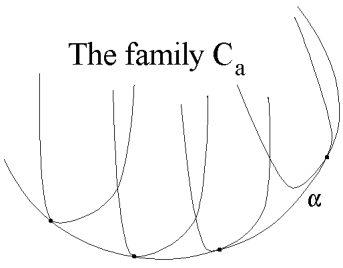


Fig. 114

10.7 Theorem. *The envelope of the family C_a is included in the curve*

$$\Gamma : \begin{cases} f(x,y,a)=0 \\ \dfrac{\partial f}{\partial a}(x,y,a)=0. \end{cases}$$

Proof. We consider

$$f(x,y,a)=0 \qquad (*)$$

as an equation in the unknown a and parameters x, y. The geometrical locus of $M(x,y)$ with the property that (*) has a root which is at least double, is described by the system $\begin{cases} f(x,y,a)=0 \\ \frac{\partial f}{\partial a}(x,y,a)=0. \end{cases}$. Each curve of C_a is tangent to Γ because $C_a \cap \Gamma$ corresponds to the double root a. Hence Γ contains the envelope of the family C_a; also, Γ might contain critical points of f.

10.8 Example. Find the envelope of the family of curves

$$C_a : f(x,y;a)=(y-a)^2-(x-a^2)^3=0.$$

Answer. We consider the system

$$\begin{cases} f(x,y;a)=(y-a)^2-(x-a^2)^3=0 \\ \frac{\partial f}{\partial a}(x,y;a)=-2(y-a)+6a(x-a^2)^2=0, \end{cases}$$

which leads to $y^2-x=0$ and to

$$\alpha(a)=\left(\frac{9a^4+1}{9a^2},\ \frac{27a^4+1}{27a^3}\right).$$

We remark that the points at which $y^2-x=0$ holds true are critical points for the family C_a, i.e., these are points at which we have

$$f(x,y;a)=0, \quad \frac{\partial f}{\partial x}(x,y;a)=0, \quad \frac{\partial f}{\partial y}(x,y;a)=0.$$

Therefore the curve α alone is the envelope of the family C_a.

6.11 Plane curves in polar coordinates

Assume that the plane xOy is endowed with a polar moving frame; then, to a Cartesian pair of coordinates (x,y) corresponds the polar pair (ρ,θ). Within this framework, a plane curve can be given by its *polar equation*,

$$\rho=f(\theta).$$

11.1 Let (ρ, θ) a point of the given curve, different from the pole. The angle V between the tangent at this point and the corresponding position vector is given by the relation

$$\mathrm{tg}V = \frac{\rho}{\rho'}.$$

We shall consider an adapted Cartesian frame XOY, in which OX is the supporting oriented line of the position vector of the point (ρ, θ), and OY is perpendicular on OX, such that the frame XOY to be positive oriented. Then *the tangent and the normal lines* at (ρ, θ) have, respectively, the equations (see fig. 115)

$$Y = \frac{\rho}{\rho'}(X - \rho) \quad \text{and} \quad \frac{\rho}{\rho'}Y + X - \rho = 0.$$

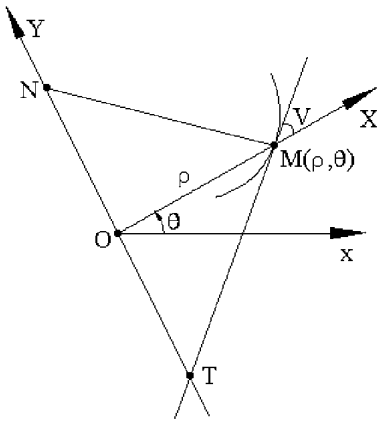


Fig. 115

The polar subtangent segment OT and *the polar subnormal segment ON* have respectively the lengths (see fig. 115)

$$\frac{\rho^2}{|\rho'|} \qquad \text{and} \qquad |\rho'|.$$

If the considered curve contains the pole O, then the tangent at the pole makes with Ox the angle θ_1, which is a root of the equation $f(\theta) = 0$.

11.2 The *multiple points* of a curve given by the polar equation $\rho = f(\theta)$ are determined by the conditions

$$f(\theta_1) = f(\theta_2),\ \theta_2 = 2k\pi + \theta_1,\ \text{for some } k \in \mathbb{Z}^*$$

and

$$f(\theta_1) = f(\theta_2),\ \theta_2 = 2l\pi + \pi + \theta_1,\ \text{for some } l \in \mathbb{Z}^*$$

11.3 The *shape of the curve* is determined by examining the sign of the curvature

$$k = \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2 + \rho'^2)^{3/2}}.$$

The following cases emerge

- for $k < 0$ we have the points near which the curve bends opposed to the pole;
- for $k > 0$ we have the points near which the curve bends towards the pole;
- for $k = 0$ we have, generally, inflexion points.

11.4 The *accumulation points* θ_0 for which the limit of $\rho = f(\theta)$ is infinite provide the asymptotic directions. Let be a given asymptotic direction θ_0. We denote

$$d = \lim_{\theta \to \theta_0} f(\theta)\sin(\theta - \theta_0).$$

If d is finite, then the curve admits one asymptote D whose ordonate at the origin is d (see fig. 116). If $d = \infty$, then the curve has just a parabolic branch.

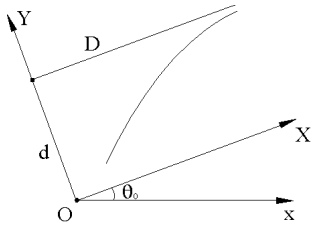


Fig. 116

Lesson 17

Curves in $\mathbf{R}^3$

Let $\alpha : I \to \mathbb{R}^3,\ \alpha(t) = (x(t), y(t), z(t))$ be a curve and $P = \alpha(t_0)$ a *regular* point on α. The *tangent to* α *at* P has the equations

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)};$$

the normal plane to α at P has the equation

$$(x - x(t_0))x'(t_0) + (y - y(t_0))y'(t_0) + (z - z(t_0))z'(t_0) = 0.$$

Let $\alpha : I \to \mathbb{R}^3,\ \alpha(t) = (x(t), y(t), z(t))$ be a curve and $P = \alpha(t_0)$ a *singular* point of order m on α. The *tangent to α at P* has the equations

$$\frac{x - x(t_0)}{x^m(t_0)} = \frac{y - y(t_0)}{y^m(t_0)} = \frac{z - z(t_0)}{z^m(t_0)};$$

the normal plane to α at P has the equation

$$(x - x(t_0))x^m(t_0) + (y - y(t_0))y^m(t_0) + (z - z(t_0))z^m(t_0) = 0.$$

We say that the curve α is *a plane curve* if there exists a plane which contains $\alpha(I)$.

6.12 Curves in $\mathbf{R}^3$ defined by Cartesian implicit equations

We have already had examples of curves in $\mathbb{R}^3$ defined by a system of Cartesian implicit equations, e.g., straight lines

$$D : \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0, \end{cases}$$

conics

$$\Gamma : \begin{cases} ax + by + cz + d = 0 \\ a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + \\ \qquad\qquad +2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0, \end{cases}$$

etc. We want to generalize the preceding cases. Let be the differentiable function

$$F = (f, g) : \mathbb{R}^3 \to \mathbb{R}^2,\ F(x, y, z) = (f(x, y, z), g(x, y, z))$$

with *the Jacobian matrix of F*,

$$J(F) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}.$$

If at some point $M(x, y, z)$ we have

$$\frac{D(f,g)}{D(y,z)} = \frac{D(f,g)}{D(z,x)} = \frac{D(f,g)}{D(x,y)} = 0,$$

then M is called *critical point of F*; a point which is not critical is called *regular point of F*.

The set C of points which satisfy the system

$$C: \begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$$

is the intersection of two level sets. If some point $M_0(x_0, y_0, z_0)$ belongs to C, then a neighborhood of M_0 in C is the set of all points in C which are inside a sphere with center at M_0.

12.1 Theorem. *If a solution M_0 of the system* $\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$ *is a regular point of F, then there exists a neighborhood V_{M_0} in C such that V is simple and regular curve (see fig. 117).*

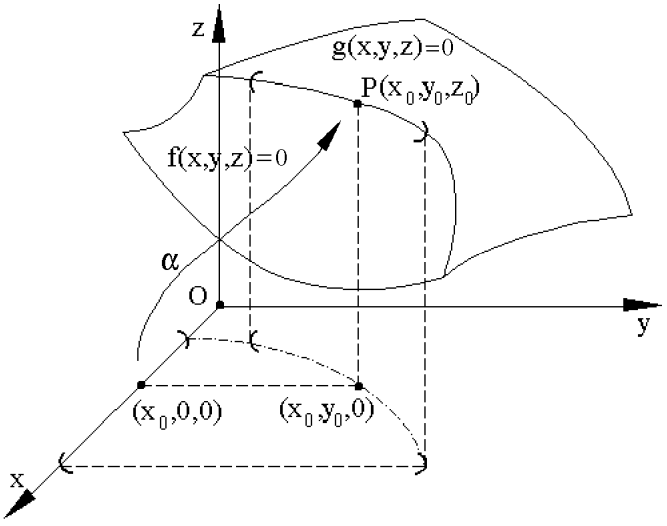


Fig. 117

Proof. Assume that $\frac{D(f,g)}{D(y_0,z_0)} \neq 0$. Then there exists a neighborhood I_{x_0} and a differentiable function $x \to (y(x), z(x))$, such that

$$\begin{cases} y_0 = y(x_0) \\ z_0 = z(x_0) \end{cases}, \quad \begin{cases} f(x, y(x), z(x)) \equiv 0 \\ g(x, y(x), z(x)) \equiv 0. \end{cases}$$

By straightforward partial derivation, we get the system

$$\begin{cases} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = 0 \\ \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} + \frac{\partial g}{\partial z}\frac{dz}{dx} = 0. \end{cases}$$

Then V is the graph of $\begin{cases} y = y(x) \\ z = z(x) \end{cases}$, and can be parametrized as being the image of

$$\alpha : I \to \mathbb{R}^3, \alpha(t) = (t, y(t), z(t)).$$

Remark that generally, C is formed of critical points and regular arcs.

12.2 Theorem. *Let $M_0(x_0, y_0, z_0)$ be a regular point of the curve*

$$C : \begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases}.$$

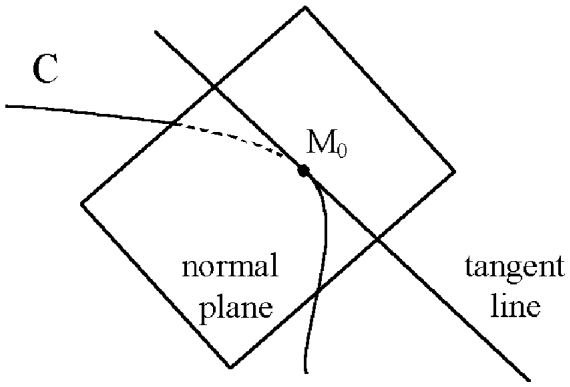


Fig. 118

Then the tangent line of C at M_0 (see fig. 118) has the equations

$$\frac{x - x_0}{\frac{D(f,g)}{D(y_0,z_0)}} = \frac{y - y_0}{\frac{D(f,g)}{D(z_0,x_0)}} = \frac{z - z_0}{\frac{D(f,g)}{D(x_0,y_0)}},$$

and the normal plane to C at M_0 (see fig. 118) has the equation

$$(x - x_0)\frac{D(f,g)}{D(y_0,z_0)} + (y - y_0)\frac{D(f,g)}{D(z_0,x_0)} + (z - z_0)\frac{D(f,g)}{D(x_0,y_0)} = 0.$$

6.13 Frenet formulas for unit-speed curves

We now derive mathematical measurements of the turning and twisting of a curve in $\mathbb{R}^3$.

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve and assume that the vectors $\vec{\alpha}', \vec{\alpha}'', \vec{\alpha}'''$ are non-coplanar. The Taylor formula of order three produces the approximation

$$\overline{PQ} \approx \frac{h}{1!}\bar{\alpha}'(t) + \frac{h^2}{2!}\bar{\alpha}''(t) + \frac{h^3}{3!}\bar{\alpha}'''(t);$$

the current point $Q = \alpha(t+h)$ on the curve α has the coordinates relative to the frame

$$\{\vec{\alpha}'(t), \vec{\alpha}''(t), \vec{\alpha}'''(t)\}$$

given by

$$(h, \frac{h^2}{2}, \frac{h^3}{6}).$$

When h passes through zero, the signs of the coordinates of Q can be $(-,+,-), (0,0,0), (+,+,+)$, providing hence an idea about the behavior of α around $P = \alpha(t)$. The arc described by Q belongs to the semispace of $\vec{\alpha}''$; it crosses the support of $\vec{\alpha}'$ and $\vec{\alpha}''$ and the plane determined by the point P and the vectors $\vec{\alpha}', \vec{\alpha}''$, which is called *osculating plane of α at P*.

The turning of α from the tangent line is called *bending*; the variation of the last component of Q is related to *the twisting of α from the osculating plane* (see fig. 119).

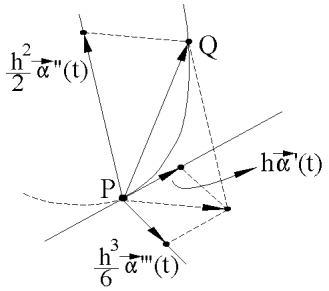


Fig. 119

Let $\beta : I \to \mathbb{R}^3$ be a unit-speed curve ($||\vec{\beta}'(s)|| = 1, \forall s \in I$).

- The vector field

$$\vec{T} = \vec{\beta}'$$

is called *the unit tangent vector field on β*. Since $||\vec{T}|| = 1$, $\vec{T}' = \vec{\beta}''$ describes the

way the curve is turning. The vector field $\vec{T}'$ is called *curvature vector field of* β; it is normal to $\vec{\beta}'$. The function

$$k(s) = ||\vec{T}'(s)||,\ s \in I$$

is called *the curvature function of* β; it gives a numerical measurement of the turning of β; we remark that $k \geq 0$.

Assuming that $k > 0$, the vector field

$$\vec{N} = \frac{1}{k}\vec{T}'$$

gives the direction in which β is turning at each point.

- The vector field $\vec{N}$ is called *the unit principal normal vector field.*
- The vector field

$$\vec{B} = \vec{T} \times \vec{N}$$

is called *the binormal vector field of* β (see fig. 120).

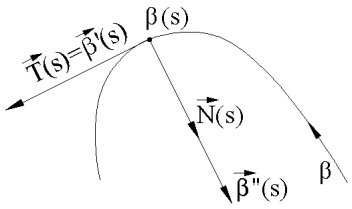


Fig. 120

So, we have the unit vector fields $\{\vec{T}, \vec{N}, \vec{B}\}$, which are mutually orthogonal at each point; they form *the Frenet frame field on* β; this is a *moving frame* since its origin $\beta(s)$ and its basis varies with s along the curve β.

We call *tangent, principal normal and binormal* the axes of the Frenet frame. These are the straight lines determined by $\beta(s)$ and the unit vectors $\vec{T}, \vec{N}, \vec{B}$ respectively; they are also called *edges of the Frenet frame.*

We call *normal plane, rectifying plane and osculating plane*, the planes which contain the origin $\beta(s)$ of the frame, and are normal respectively to the unit vectors $\vec{T}, \vec{N}, \vec{B}$ (see fig. 121); they are also called *the faces of the Frenet frame.*

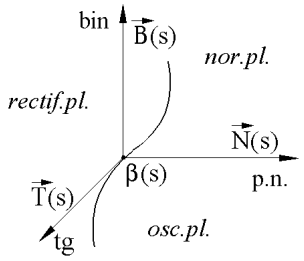


Fig. 121

Decomposing the vector field $\vec{B}'$ relative to the Frenet basis, we obtain

$$\vec{B}' = (\vec{B}', \vec{T})\,\vec{T} + (\vec{B}', \vec{N})\,\vec{N} + (\vec{B}', \vec{B})\,\vec{B}.$$

We remark that

$$1 = ||\vec{B}||^2 = (\vec{B}, \vec{B}) \Rightarrow (\vec{B}', \vec{B}) = 0$$

and that

$$(\vec{B}, \vec{T}) = 0 \Rightarrow (\vec{B}', \vec{T}) = -(\vec{B}, \vec{T}') = -(\vec{B}, k\vec{N}) = -k(\vec{B}, \vec{N}) = 0,$$

whence the decomposition above provides the relation

$$\vec{B}' = (\vec{B}', \vec{N})\,\vec{N},$$

i.e., $\vec{B}'$ is collinear to $\vec{N}$; hence there is a function $\tau : I \to \mathbb{R}$, called *the torsion function*, defined by the relation

$$\vec{B}' = -\tau\vec{N},$$

where the minus sign is traditional; τ does measure the *torsion* (or the *twisting*) of the curve β.

13.1 Theorem (Frenet formulas for unit-speed curves).
If $\beta : J \to \mathbb{R}^3$ is a unit-speed curve with curvature $k > 0$ and torsion τ, then

$$\vec{T}' = k\vec{N}, \quad \vec{N}' = -k\vec{T} + \tau\vec{B}, \quad \vec{B}' = -\tau\vec{N}.$$

Proof. The first and third relations were justified before. For the second, we decompose $\vec{N}'$ relative to the frame basis,

$$\vec{N} = (\vec{N}', \vec{T})\,\vec{T} + (\vec{N}', \vec{N})\,\vec{N} + (\vec{N}', \vec{B})\,\vec{B}.$$

Since $||\vec{N}|| = 1 \Rightarrow (\vec{N}', \vec{N}) = 0$ and the derivation of the relations

$$(\vec{N}, \vec{T}) = 0, \ \ (\vec{N}, \vec{B}) = 0$$

provide

$$(\vec{N}', \vec{T}) = -(\vec{N}, \vec{T}') = -(\vec{N}, k\vec{N}) = -k$$
$$(\vec{N}', \vec{B}) = -(\vec{N}, \vec{B}') = -(\vec{N}, -\tau\vec{N}) = \tau,$$

which finally yield $\vec{N}' = -k\vec{T} + \tau\vec{B}$.

13.2 The Frenet approximation. Applying the Taylor formula to β, we obtain the approximation

$$\bar{\beta}(s) \approx \bar{\beta}(0) + \frac{s}{1!}\bar{\beta}'(0) + \frac{s^2}{2!}\bar{\beta}''(0) + \frac{s^3}{3!}\bar{\beta}'''(0).$$

On the other hand

$$\bar{\beta}' = \bar{T}, \ \bar{\beta}'' = \bar{T}' = k\bar{N}; \ \bar{\beta}''' = k'\bar{N} + k\bar{N}'.$$

Hence the Taylor approximation is transformed into the Frenet approximation

$$\bar{\beta}(s) \approx \bar{\beta}(0) + s\bar{T}_0 + k_0\frac{s^2}{2}\bar{N}_0 + k_0\tau_0\frac{s^3}{6}\bar{B}_0,$$

where we denoted

$$k_0 = k(0), \ \tau_0 = \tau(0); \quad \vec{T}_0 = \vec{T}(0), \ \vec{N}_0 = \vec{N}(0), \ \vec{B}_0 = \vec{B}(0).$$

Examining the Frenet approximation, we have:

- for $s = 0$ (for keeping only the first right term of the approximation) we obtain the *point* $\bar{\beta}(0)$;
- the approximation of first order is obtained by keeping the first two right terms, and is *the tangent line* to β at $\beta(0)$;
- the approximation of second order is obtained by keeping the first three right terms, and is *a parabola*, which is the best quadratic approximation of β near $\beta(0)$.

The approximation of third order (provided by all the three terms) is called *Frenet approximation* (see fig. 122).

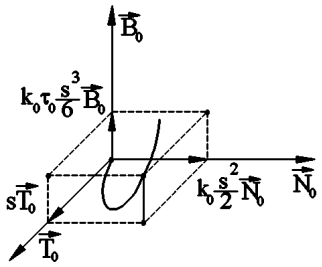


Fig. 122

Remarks. A *plane curve in* $\mathbb{R}^3$ is a curve that lies in a single plane of $\mathbb{R}^3$.

1) A curve β is a plane curve iff $\tau \equiv 0$.

2) If $k > 0$ is constant along β and $\tau \equiv 0$, then β is a part of a circle of radius $1/k$.

Lesson 18

6.14 Frenet formulas for arbitrary-speed curves

Let $\alpha : I \to \mathbb{R}^3$ be a regular curve and β its normal representation; we have $\alpha(t) = \beta(s(t))$.

If $k_\beta > 0$ and $\tau_\beta, \vec{T}_\beta, \vec{N}_\beta, \vec{B}_\beta$ are defined as in §13, then we define for α the following:

- the *curvature function* $k = k_\beta \circ s$,
- the *torsion function* $\tau = \tau_\beta \circ s$,
- the *unit tangent vector field* $\vec{T} = \vec{T}_\beta \circ s$,
- the *unit principal normal vector field* $\vec{N} = \vec{N}_\beta \circ s$,
- the *unit binormal vector field* $\vec{B} = \vec{B}_\beta \circ s$ (see fig. 123).

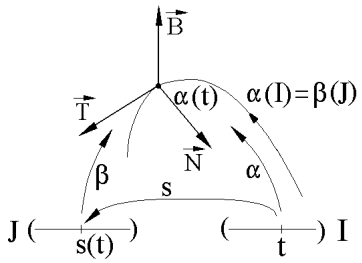


Fig. 123

We can see that k and k_β are different functions, but they give exactly the same description of the turning of the common route of α and β. For purely theoretical work this simple change of parameter is often all that is needed. However, for explicit numerical computations this transference is impractical, since it is rarely possible to find explicit formulas for β.

14.1 Theorem (Frenet formulas). *Let α be a regular curve with the speed v and curvature $k > 0$. Then we have the relations*

$$\begin{aligned}\vec{T}' &= kv\vec{N}\\ \vec{N}' &= -kv\vec{T} + \tau v\vec{B}\\ \vec{B}' &= -\tau v\vec{N}.\end{aligned}$$

Proof. We have, for instance,

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}_\beta}{ds}\frac{ds}{dt}.$$

Using then the corresponding relations for $\vec{N}, \vec{B}$ and the Frenet formulas for unit speed curves, we obtain the stated result.

We analyse the velocity and the acceleration by expressing them in terms of the Frenet frame field.

14.2 Lemma. *Let α be a regular curve with speed v. Then we have (see fig. 124)*

$$\vec{\alpha}' = v\vec{T} \quad \textit{and} \quad \vec{\alpha}'' = \frac{dv}{dt}\vec{T} + kv^2\vec{N}.$$

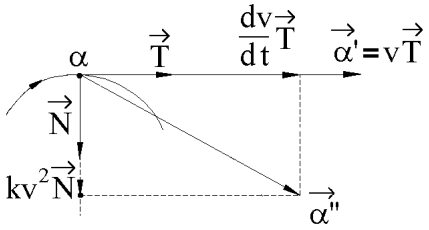


Fig. 124

Proof. We have $\vec{T} = \frac{\vec{\alpha}'}{||\vec{\alpha}'||}$ whence the first relation; deriving it, we obtain by straightforward computation and using the Frenet formulas, the second.

Having in view the second relation, we call $\frac{dv}{dt}$ the *tangential component* of $\bar{\alpha}''$; it measures the rate of change of the length of $\bar{\alpha}'$. *The normal component* kv^2 describes the rate of change of the direction of $\bar{\alpha}'$.

14.3 Computational formulas. Let α be a regular curve with $k > 0$. Then the unit vector fields of the Frenet frame are provided by

$$\begin{cases} \vec{T} = \frac{\vec{\alpha}'}{||\vec{\alpha}'||} \\ \vec{N} = \vec{B} \times \vec{T} \\ \vec{B} = \frac{\vec{\alpha}' \times \vec{\alpha}''}{||\vec{\alpha}' \times \vec{\alpha}''||} \end{cases}$$

Also, we have

$$\vec{\alpha}' \times \vec{\alpha}'' = v\vec{T} \times \left(\frac{dv}{dt}\vec{T} + kv^2\vec{N} \right) = kv^3\vec{B} \Rightarrow ||\vec{\alpha}' \times \vec{\alpha}''|| = kv^3,$$

whence the *curvature function* is given by

$$k = \frac{||\vec{\alpha}' \times \vec{\alpha}''||}{||\vec{\alpha}'||^3} \quad .$$

Regarding the torsion, we have

$$\vec{\alpha}''' = \frac{d^2v}{dt^2}\vec{T} + \frac{dv}{dt}\vec{T}' + \left(\frac{dk}{dt}v^2 + 2kv\frac{dv}{dt} \right) \vec{N} + kv^2\vec{N}' =$$

$$= \left(\frac{d^2v}{dt^2} - k^2v^3 \right) \vec{T} + \left(kv\frac{dv}{dt} + \frac{dk}{dt}v^2 + 2kv\frac{dv}{dt} \right) \vec{N} + kv^3\tau\vec{B}$$

and hence

$$(\vec{\alpha}' \times \vec{\alpha}'', \vec{\alpha}''') = k^2v^6\tau.$$

It follows the *torsion*

$$\tau = \frac{(\vec{\alpha}' \times \vec{\alpha}'', \vec{\alpha}''')}{||\vec{\alpha}' \times \vec{\alpha}''||^2}.$$

Exercise. Parametrize the curve $\begin{cases} xyz = 1 \\ x = y \end{cases}$ and find its Frenet frame; compute the curvature and torsion functions and write down the Frenet equations.

6.15 Problems

1. Let be the curve

$$\alpha : [0, \pi] \to \mathbb{R},\ \alpha(t) = (\sin t, \cos^2 t, 5\sin t, 1 - 3\cos^2 t).$$

Show that α is closed and its extension $\alpha : \mathbb{R} \to \mathbb{R}^4$ is periodic.

2. Let be the curve

$$\alpha : \mathbb{R} \to \mathbb{R}^4,\ \alpha(t) = (\sin t, 1 + \cos^2 t, \sin t + \cos^2 t, \sin t).$$

Show that the point $\alpha\left(\frac{\pi}{4}\right)$ is a regular one, and the tangent to α at this point is perpendicular on the straight line of direction

$$\vec{d} = (1, 1, -\sqrt{2} - 2, -1).$$

Determine the normal hyperplane to α at the point $\alpha\left(\frac{\pi}{4}\right)$.

3. Let $\vec{Y}$ be a vector field on the curve

$$\alpha(t) = (\sin t, 1 + \cos t, \sin t + \cos^2 t, \sin^2 t).$$

In each of the following cases, express $\vec{Y}$ in the form

$$\vec{Y} = y_1\vec{e}_1 + y_2\vec{e}_2 + y_3\vec{e}_3 + y_4\vec{e}_4.$$

1) $\vec{Y}(t)$ is the vector which has its application point at $\alpha(t)$ and the extremity at the origin of $\mathbb{R}^4$.
2) $\vec{Y}(t) = \vec{\alpha}'(t) - \vec{\alpha}''(t)$.
3) $\vec{Y}(t)$ has the length 1 and is orthogonal to $\vec{\alpha}'(t)$, $\vec{\alpha}''(t)$ and to $\vec{\alpha}'''(t)$.
4) $\vec{Y}$ is the vector of origin $\alpha(t)$ and of extremity at $\alpha(t + \pi)$.

4. Study the infinite branches and determine the asymptotes (if these exist) of the curves
1) $\alpha = (x, y) : \mathbb{R}\backslash\{1\} \to \mathbb{R}^2,\ x(t) = \dfrac{t^3 + 2}{t^3 - 1},\ y(t) = \dfrac{3t}{t^3 - 1}$;
2) $\alpha = (x, y) : \mathbb{R} \to \mathbb{R}^2,\ x(t) = \dfrac{2 + t^2}{1 + t^2},\ y(t) = \dfrac{t^3}{1 + t^2}$.

5. Let be the curve

$$\alpha = \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}^5,\ \alpha(t) = \left(\frac{1}{t-1}, \frac{2t}{t^2-1}, \frac{t^2+1}{t^2-1}, \frac{2}{t+1}, \frac{t}{t+1}\right).$$

1) Show that the point $(-1, 0, -1, 2, 0)$ belongs to α and find the tangent and the normal hyperplane of the curve at this point.

2) Study the infinite branches of the curve and determine its asymptotes (in case that they exist).

6. Let be the curve

$$\alpha : (0,4) \to \mathbb{R}^3,\ \alpha(t) = (\sqrt{t}, t\sqrt{t}, 1-t).$$

Reparametrize it by $h : (0,2) \to \mathbb{R},\ h(u) = u^2$.

7. Let be the curve

$$\alpha : [0, 2\pi) \to \mathbb{R}^2,\ \alpha(t) = (2\cos t - \cos 2t,\ 2\sin t - \sin 2t).$$

Find its normal parameter relative to the origin $t = 0$ and its normal parametrization.

8. Let be the curve

$$\alpha : \mathbb{R}\backslash\{0\} \to \mathbb{R}^2,\ \alpha(t) = \left(\int \frac{\sin t}{t} dt,\ \int \frac{\cos t}{t} dt \right).$$

Show that α is a regular curve and compute the length of the arc which corresponds to $t \in [2,3]$.

9. Let be the curves $\alpha_{1,2} :\ [0, 2\pi] \to \mathbb{R}^4$,

$$\begin{aligned} \alpha_1(t) &= (\cos t, (\sqrt{2}\sin t - 1)^2, 2 + \sin t, 1 - \cos t); \\ \alpha_2(t) &= (\sin t, 0, 2 + \cos t, 1 - \sin t)\,. \end{aligned}$$

1) Show that these are closed curves which have a common point M_0.
2) Show that the two curves have a contact of order one at M_0.

10. Show that

$$\alpha : \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = \left(\frac{2+t^2}{1+t^2},\ \frac{t^3}{1+t^2} \right)$$

is a simple curve. Find its singular points, the tangents and the normals at these points.

11. The curve described by a point M located on a circle of radius r which rolls (without friction) along a straight line is called *cycloid*. Find the parametric equations of the cycloid (see fig. 125).

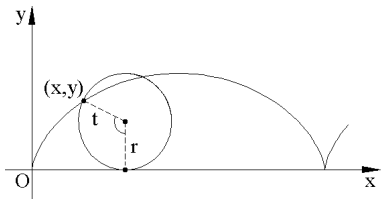


Fig. 125

Find its normal parameter and the length of its first arc. Compute the lengths of the tangent, subtangent, normal and subnormal segments at an arbitrary point of the cycloid.

Hint. The parametrization is

$$\alpha : \begin{cases} x = r(t - \sin t), \\ y = r(1 - \cos t), \end{cases} \quad t \in \mathbb{R}.$$

12. Determine the curves of constant level $f^{-1}(-1), f^{-1}(0), f^{-1}(1)$ for $f(x, y) = x^2 - y^2$. In each case find at which of its points the tangent space is $[\nabla f(P)]^{\perp}$.

13. Find a global parametrization for each of the following curves, oriented by $\dfrac{\nabla f}{\|\nabla f\|}$, where f is the function which is defined by the left side of each of the subsequent equations

1) $ax + by = c, \quad c \neq 0.$

2) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \quad a \neq 0, \quad b \neq 0.$

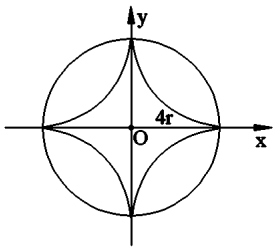
3) $y - ax^2 = c, \quad a \neq 0.$

4) $x^2 - y^2 = 1, \quad x > 0.$

14. Show that the *astroid* (see fig. 126)

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \ \alpha(t) = (4r\cos^3 t, 4r\sin^3 t).$$

has cusp points and determine the tangents at these points.



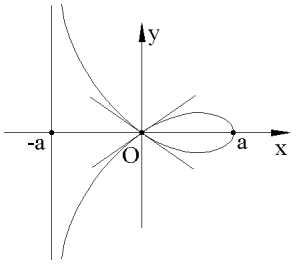


Fig. 126 **Fig. 127**

15. Consider the *strophoid* $C = f^{-1}(0)$ (see fig. 127), where

$$f(x, y) = y^2x + ay^2 + x^3 - ax^2.$$

Show that the origin is a double point and find the tangents at this point.

16. Let be the curve (see fig. 128)

$$C = f^{-1}(0),\ f(x, y) = x^3 - x^2 - y^2.$$

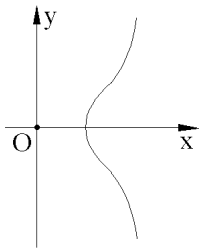
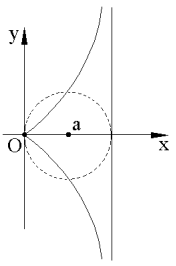


Fig. 128

Show that the origin is an isolated point of the curve C.

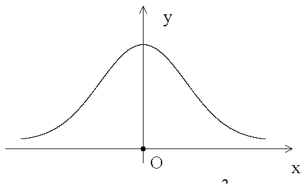
17. Show that the origin is a cusp point for *Diocles' cyssoid* (see fig. 129)

$$C = f^{-1}(0),\ f(x, y) = x^3 + xy^2 - 2ay^2.$$



Diocles' cyssoid

Fig. 129



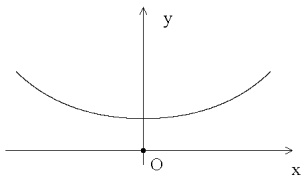
The Gauss curve : $y = e^{-x^2}$

Fig. 130

18. Draw the graph of the following functions

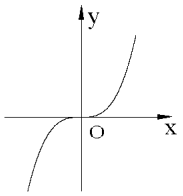
1) *Gauss' curve*: $y = e^{-x^2}$ (see fig. 130);

2) *The catenary*: $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$ (see fig. 131);



The catenary : $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$

Fig. 131

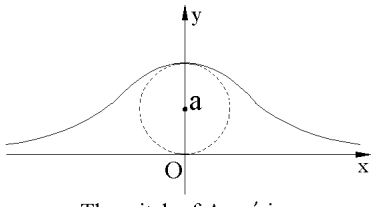


The cubic parabola: $y = ax^3$

Fig. 132

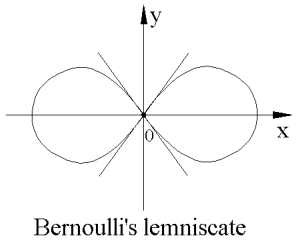
3) *The cubic parabola*: $y = ax^3$ (see fig. 132);

4) *The witch of Agnési*: $x^2y = 4a^2(2a - y)$ (see fig. 133).



The witch of Agnési

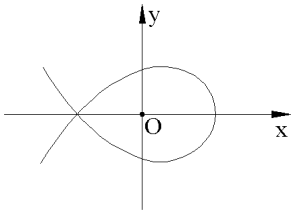
Fig. 133



Bernoulli's lemniscate

Fig. 134

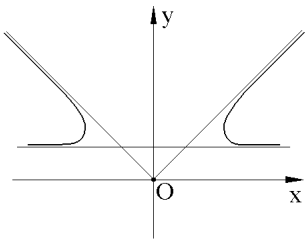
5) *Bernoulli's lemniscate*: $(x^2+y^2)^2+2a^2(y^2-x^2)=0$ (see fig. 134).
6) *The Lissajous' curve*: $x=\cos 2t,\ y=\sin 3t$ (see fig. 135).



Lissajous' curve

Fig. 135

7) $x=\frac{\mathrm{ch}t}{t},\quad y=\frac{\mathrm{sh}t}{t}$ (see fig. 136).



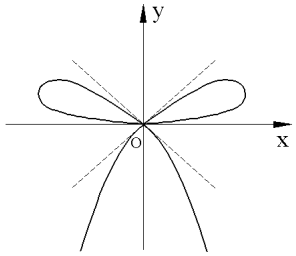


Fig. 136 **Fig. 137**

8) $x=t-t^3,\quad y=t^2-t^4$ (see fig. 137).

19. Find the parabola of second order whose axis is parallel to Oy and which has a contact of order two with the cubic parabola $y=x^3$ at the point $x=1$.

20. Find the envelopes of the following families of curves

1) $(x-a)^2+y^2-\frac{a^2}{2}=0$.
2) $\frac{x}{\alpha}+\frac{y}{\beta}-1=0$, when $\alpha^m-\beta^m-a^m=0$ ($a=\text{const.}$).
3) $x\cos a+y\sin a-1=0$.
4) $y=ax+\frac{p}{2a}$.

21. Find the evolute for each of the following curves

1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0.$
2) $x=a(t-\sin t), \quad y=a(1-\cos t).$

22. Draw *the three-folded roses* (see fig. 138)

$$\rho=a\sin 3\theta, \quad \rho=a\cos 3\theta.$$

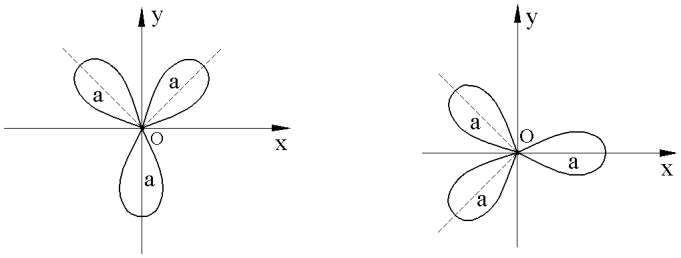


Fig. 138

23. Let be the curve

$$\beta(s)=\left(\frac{(1+s)^{3/2}}{3},\ \frac{(1-s)^{3/2}}{3},\ \frac{s}{\sqrt{2}}\right), \quad s\in(-1,1).$$

Determine the Frenet elements $\vec{T},\vec{N},\vec{B},k,\tau$.

24. Find the normal parameter for each of the following curves
1) $\alpha:\mathbb{R}\to\mathbb{R}^3, \quad \alpha(t)=(\cos t,\sin t,t^2/2),$
2) $\alpha:\mathbb{R}\to\mathbb{R}^3, \quad \alpha(t)=(a\mathrm{ch}t,a\mathrm{sh}t,at),$
3) $\alpha:\mathbb{R}\to\mathbb{R}^3, \quad \alpha(t)=(t\cos t,t\sin t,t^2/2),$
taking as origin the point at which the curve intersects the plane xOy. Find the Frenet elements $\vec{T},\vec{N},\vec{B},k,\tau$ and write the equations of the sides and faces of the Frenet frame at this point.

25. We call *Ţiţeica curve*, a curve for which

$$\frac{1}{\tau}\cdot d^2 = \text{ const.},$$

where τ is the torsion at an arbitrary point of the curve, and d is the distance from a fixed point to the osculating plane of the curve at that point. Show that the following is a Ţiţeica curve:

$$C : \begin{cases} xyz = 1 \\ y^2 = x. \end{cases}$$

Lesson 19

Chapter 7

Surfaces

7.1 Charts, surfaces, parametrizations

Let $\mathbb{R}^3$ be the space of triples of real numbers, identified with $T_O\,\mathbb{R}^3$. A *surface* is a subset of $\mathbb{R}^3$ which is smooth and two-dimensional. We shall describe this in mathematical terms by the next definitions.

1.1 Definition. Let D be an open set in $\mathbb{R}^2$. A function $r : D \to \mathbb{R}^3$ which is injective, differentiable and regular is called a *chart*.

Its image $r(D)$ is smooth and two-dimensional, where regularity means the smoothness condition. We require injectivity for r, in order to prevent $r(D)$ from cutting across itself (see fig. 139). The image $r(D)$ is called two-dimensional since the domain D is included in a two-dimensional vector space.

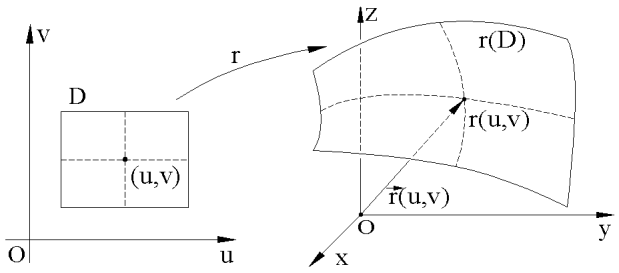


Fig. 139

The mapping r is characterized by its Euclidean components,

$$r(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in D.$$

The injectivity of r means $r(u_1, v_1) = r(u_2, v_2) \Rightarrow u_1 = u_2, v_1 = v_2$. The differentiability of r is reduced to the fact that $x(u, v), y(u, v), z(u, v)$ are differentiable functions. The regularity of r is represented by the fact that the transpose of the Jacobian matrix of r,

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

has rank 2. In vector notation, r writes

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

so $r(D)$ is described by the extremity of a variable vector which is applied to the origin (see fig. 139). In this case the regularity is expressed by the condition $\vec{r}_u \times \vec{r}_v \neq \vec{0}$. The vectors $\vec{r}_u$ and $\vec{r}_v$ are called *partial velocities* of r.

If we drop the injectivity condition, then the mapping r is called *parametrization* of the region $r(D)$, the variables u and v are called *parameters*, and $r(D)$ is described by the *parametric equations*

$$\begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \quad u, v \in D.$$

Let $r : D \to \mathbb{R}^3$, $r(u, v) = (x(u, v), y(u, v), z(u, v))$ be a chart. Holding v constant we obtain a curve

$$u \to r(u, v_0),$$

which is called the *u-parameter curve* or *the curve* $v = v_0$. Also, the curve

$$v \to r(u_0, v)$$

is called *v-parameter curve* or *the curve* $u = u_0$ (see fig. 140). These curves are called *coordinate curves* on $r(D)$.

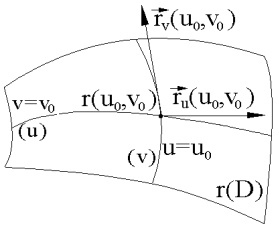


Fig. 140

We remark that both these curves pass through the point $r(u_0, v_0)$ and that $r(D)$ is covered by these two families of curves. The partial velocities $\vec{r}_u(u_0, v_0)$ and $\vec{r}_v(u_0, v_0)$ are tangent to the coordinate curves at $r(u_0, v_0)$; these vectors are linearly independent since r is a regular mapping.

Let $M \subset \mathbb{R}^3$ and P be a point in M. A *neighborhood U of P in M* is the set of all points in M whose Euclidean distance from P is less than some number $\varepsilon > 0$. If $r : D \to M$ is a chart, then $r(D)$ is open in M. Conversely, in order to assure that we associate to open sets in M open sets in $\mathbb{R}^2$, it is necessary to use proper charts, a chart $r : D \to r(D) \subset \mathbb{R}^3$ being called *proper* if the mapping $r^{-1} : r(D) \to D$ is continuous.

1.2 Definition. A subset M of $\mathbb{R}^3$ is called *a surface* if for each point P of M there exists a proper chart in M whose image contains a neighborhood of P in M (see fig. 141).

The image $r(D)$ of a chart is a surface, called *simple surface*. Any surface is a union of simple surfaces.

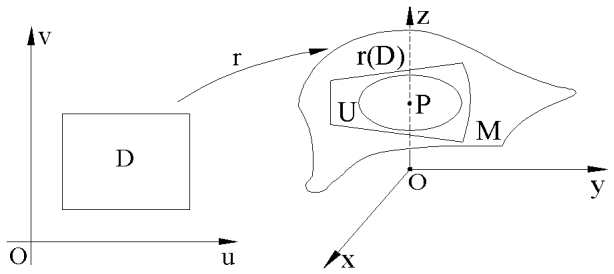


Fig. 141

1.3 Surface described by an explicit Cartesian equation. Let D be an open subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a differentiable function. Let

$$M : \ z = f(x, y)$$

where M is provided by the graph of f. Then M is a surface, being the image of the *Monge chart*

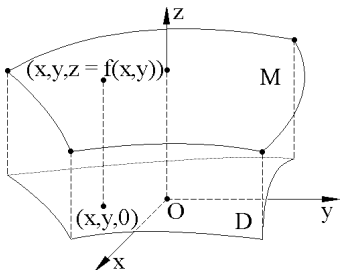


Fig. 142

$$\begin{cases} x = u \\ y = v \\ z = f(u, v), \end{cases}$$

which is automatically proper. The equation $z = f(x, y)$ is called *Cartesian explicit equation* of the surface M.

1.4 Sphere as a surface. The set of points $P(x, y, z) \in \mathbb{R}^3$ characterized by the equation

$$x^2 + y^2 + z^2 = 1$$

is the unit sphere centered at the origin (see fig. 143).

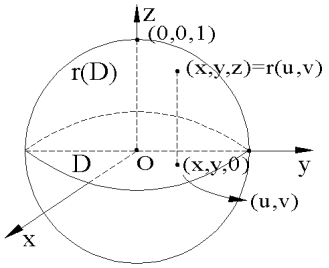


Fig. 143

The sphere can be provided by *two* explicit Cartesian equations,

$$z = \pm\sqrt{1 - x^2 - y^2}.$$

The Monge chart associated to the function

$$z = f(x, y) \equiv \sqrt{1 - x^2 - y^2}, \quad f : D = \{(x, y) | x^2 + y^2 < 1\} \subset \mathbb{R}^2 \to \mathbb{R}$$

is proper. Its image covers the superior semisphere. We can easily notice that images of six such charts (obtained by solving the implicit equation of the sphere with respect to x, y and z), are enough to cover the sphere, which hence becomes a surface in the sense of Definition 1.1.

1.5 Surface described by an implicit Cartesian equation.

We have already studied surfaces provided by algebraic equations of degree one (planes)

$$ax + by + cz + d = 0$$

and of degree two (quadrics)

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + \\ +2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0.$$

Generally, such surfaces will be provided by considering equations attached to a differentiable function $F : \mathbb{R}^3 \to \mathbb{R}$. The points at which the Jacobian matrix (the gradient) of F vanishes, will be called *critical points*; the points which are not critical, will be called *regular points*. We shall consider the *level sets of* F,

$$M = F^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 | F(x, y, z) = c\},\ c \in \mathbb{R}.$$

The constant c will be called *the height of the level set*. Generally, M contains critical and regular points of F.

1.6 Theorem. *If a solution $P(x_0, y_0, z_0)$ of the equation*

$$F(x, y, z) = c$$

is a regular point of F, then there exists a neighborhood V of P in M such that V is a simple surface (see fig. 144).

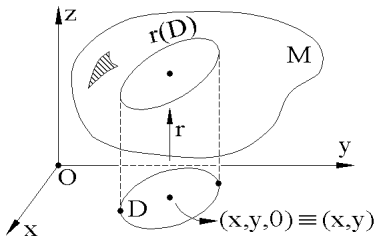


Fig. 144

Proof. Suppose $\frac{\partial f}{\partial z}(P) \neq 0$. Then the theorem of implicit functions asserts that there exists a neighborhood $U(x_0, y_0)$ in $\mathbb{R}^2$ and a differentiable function $(x, y) \to z(x, y)$ such that

$$z(x_0, y_0) = z_0,\ \ F(x, y, z(x, y)) \equiv c,$$

and hence, by partial derivation, we obtain the relations

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0,$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0.$$

The neighborhood V is the graph of the differentiable function $z = z(x, y)$, i.e., the image of

$$r : U \to M, \quad r(u, v) = (u, v, z(u, v)).$$

Generally M is a union of critical points and simple surfaces. The equation $F(x, y, z) = c$ is called *implicit Cartesian equation* of the surface M.

If f is a polynomial of order n, then the surface M is called *algebraic surface of order n*. As examples, we mention the planes, the quadrics, etc.

Remark. The critical points of F are classified in the theory of catastrophes [44].

7.2 Generated surfaces

A *generated surface* is a surface produced by moving a curve using certain rules.

Ruled surfaces. We call *ruled surface*, a surface swept out by a straight line L moving along a curve α. The various positions of the generating line L are called *rulings (generators) of the surface*. The curve $\alpha : I \to \mathbb{R}^3$ is called *director curve* (see fig. 145).

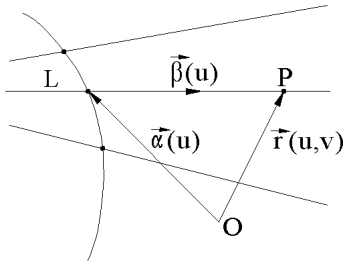


Fig. 145

The vectorial representation of such a surface is of the form

$$\bar{r}(u, v) = \bar{\alpha}(u) + v\bar{\beta}(u), \quad v \in \mathbb{R}, \ u \in I.$$

The condition of being a parametrization is equivalent to

$$\bar{r}_u \times \bar{r}_v \neq \bar{0}.$$

I. The *cylinders* are ruled surfaces (see fig. 146) whose vectorial representation is of the form

$$\bar{r}(u, v) = \bar{\alpha}(u) + v\bar{\beta}, \quad \bar{\beta} = \text{ const.}$$

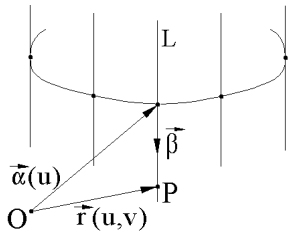


Fig. 146

In other words, a ruled surface is a cylinder iff the rulings are parallel.

2.1 Theorem. *Let P, Q be polynomial functions of order 1 on $\mathbb{R}^3$. A cylinder with the ruling $L: \ P = 0, \ Q = 0$ is described by an equation of the form $F(P, Q) = 0$.*

Proof. The straight lines which are parallel to L have the equations of the form

$$P = u, \ Q = w. \tag{1}$$

The condition that these meet the curve

$$x = x(t), \ y = y(t), \ z = z(t), \ \ t \in I \tag{2}$$

is obtained by eliminating x, y, z and t from the equations (1) and (2); we obtain an equation of the form

$$F(u, v) = 0. \tag{3}$$

Now we reinterpret the surface as the geometrical locus of straight lines (1) with condition (3), whence we obtain the equation $F(P, Q) = 0$. The converse is also true.

Remark. An equation of the form $f(x, y) = 0$ in the unknowns $(x, y, z) \in \mathbb{R}^3$ represents a cylinder whose rulings are parallel to Oz.

II. The *cones* are ruled surfaces (see fig. 147) with the vectorial representation of the form

$$\bar{r}(u, v) = \bar{\alpha} + v\bar{\beta}(u), \ \ \bar{\alpha} = \text{ const.}$$

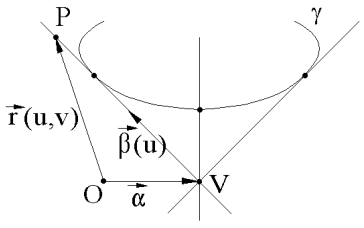


Fig. 147

In other words, a ruled surface is a cone iff all the rulings pass through a point V. The point V is called *vertex of the cone*. The curve $\gamma : I \to \mathbb{R}^3$ (see fig. 147) is called *director curve*.

2.2 Theorem. *Let P, Q, R be polynomials of order 1 on $\mathbb{R}^3$. A cone with the vertex*

$$V : \ P = 0, \ \ Q = 0, \ \ R = 0$$

is described by an equation of the form

$$F\left(\frac{P}{R}, \frac{Q}{R}\right) = 0.$$

Proof. The straight lines through V are members of the pencil

$$\begin{cases} P = uR \\ Q = wR \end{cases} . \tag{1}$$

The condition that these meet the curve

$$\alpha : \ x = x(t), \ y = y(t), \ z = z(t) \tag{2}$$

provides (after eliminating x, y, z, t) a relation of the form

$$F(u, v) = 0. \tag{3}$$

Now we reinterpret the surface as the geometrical locus of (1) with condition (3), whence we get the equation $F(\frac{P}{R}, \frac{Q}{R}) = 0$. The converse is also true.

Remark. If $G : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous function of degree zero, then $G(P, Q, R) = 0$ describes a cone.

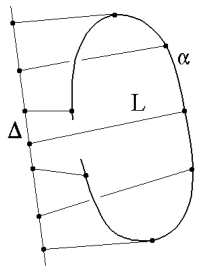


Fig. 148

III. A *conoid surface* is a ruled surface (see fig. 148) whose rulings are straight lines which all intersect a fixed axis Δ and a given curve α, which is called *director curve*. If we take for Δ the Oz axis, then the conoid admits the parametrization

$$r(u, v) = (u \cos\theta(v), \ u \sin\theta(v), \ h(v)).$$

A conoid whose rulings remain parallel to a fixed plane P, is called *conoid with director plane*. Such a conoid admits a vectorial representation of the form

$$\bar{r}(u,v) = \bar{\alpha}(u) + v\bar{\beta}(u),$$

where $\bar{\beta} \| P$ and $(\bar{\alpha} - \bar{\beta}) \| \Delta$.

2.3 Theorem. *Let P, Q, R be polynomials of order 1 on $\mathbb{R}^3$. Let M be the conoid with the director plane $P = 0$ and the axis $\Delta : Q = 0,\ R = 0$. The set*

$$M \setminus \{(x,y,z) \mid P(x,y,z) = u,\ \ R(x,y,z) = 0\}$$

is characterized by an equation of the form

$$F\left(P,\ \frac{Q}{R}\right) = 0.$$

Proof. Excepting the straight lines of equations $D : P = u,\ \ R = 0$, the straight lines which are parallel to P have the equations

$$\begin{cases} P = u \\ Q - wR = 0. \end{cases} \tag{1}$$

The condition that these lines intersect a curve

$$\alpha : x = x(t),\ y = y(t),\ z = z(t) \tag{2}$$

is obtained by eliminating x, y, z, t between the five considered equations (1) and (2). It results that $F(u,w) = 0$ and hence the conoid with director plane has the equation

$$F\left(P,\ \tfrac{Q}{R}\right) = 0. \tag{3}$$

Conversely, let N be the subset consisting of all the points in $\mathbb{R}^3$ which are characterized by an equation of the form (3). If $P = 0$ is the equation of a plane which intersects the line $\Delta : Q = 0,\ R = 0$ and if N does not contain critical points of F, then N is a conoid. Indeed, for any real solution (x,y,z) of (3), there exist two real numbers u, w such that $P = u,\ Q - wR = 0$ with $F(u,w) = 0$.

Definition. The *right conoid* is a ruled surface whose rulings are straight lines which all intersect a fixed axis and are parallel to a plane which is perpendicular on the axis (see fig. 149).

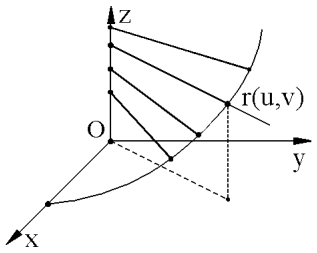


Fig. 149

Surfaces of revolution. A *surface of revolution* is a surface obtained by revolving a curve C around a fixed straight line L called *axis of rotation* (see fig. 150).

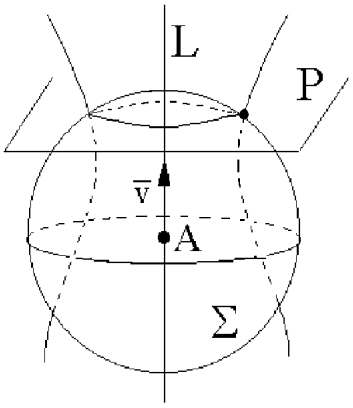


Fig. 150

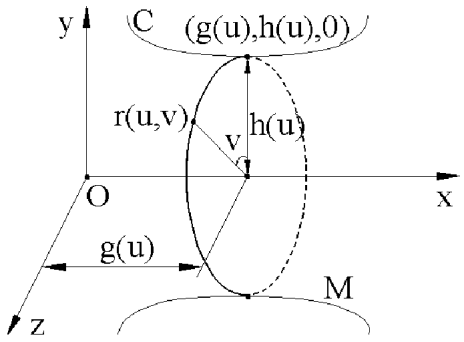


Fig. 151

Let be the straight line

$$L : \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

which passes through $A(x_0, y_0, z_0)$ and has the director vector $\bar{v} = l\bar{i} + m\bar{j} + n\bar{k}$. Let also

$$P = (x - x_0)l + (y - y_0)m + (z - z_0)n = 0,$$

be the plane through A which is normal to L at the same point. Let

$$\Sigma : \ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - r^2 = 0$$

be a sphere with center at $A \in L$.

2.3 Theorem. *A surface of revolution with axis L is described by a Cartesian equation of the form $F(P, \Sigma) = 0$.*

Proof. We think M as the geometrical locus of circles

$$\begin{cases} P = u \\ \Sigma = w \end{cases} \tag{1}$$

which meet the curve

$$x = x(t), y = y(t), z = z(t). \tag{2}$$

The compatibility condition is

$$F(u, w) = 0. \tag{3}$$

We reinterpret as geometrical locus of circles (1) with condition (3), obtaining the equation of the surface $F(P, \Sigma) = 0$. The converse argument is true.

Suppose the curve C is in the same plane with the axis L, and $C \cap L = \emptyset$. The circles determined by the revolving points of C are called *parallels*; the different positions of C obtained by revolving are called *meridians*.

For simplification we accept that C belongs to xOy, and L is just Ox.

1) If $C: \ f(x, y) = 0, \ z = 0, \ y > 0$, then $M: \ f(x, y^2 + z^2) = 0$.

2) If C is given by the parametrization

$$\alpha(u) = (g(u), h(u), 0), \ h(u) > 0, \ u \in I,$$

then the surface M admits the parametrization (see fig. 151)

$$r(u, v) = (g(u), \ h(u)\cos v, \ h(u)\sin v), \ u \in I, \ v \in \mathbb{R}.$$

Lesson 20

7.3 Tangent plane and normal line

Let M be a surface and $r: \ D \to M$ a chart. Consider a simple curve $\alpha: \ I \to M$.

3.1 Lemma. If $\alpha(I) \subset r(D)$, then there exists a unique pair of functions (u, v) such that $\alpha(t) = r(u(t), v(t)), \ \forall t \in I$ (see fig. 152).

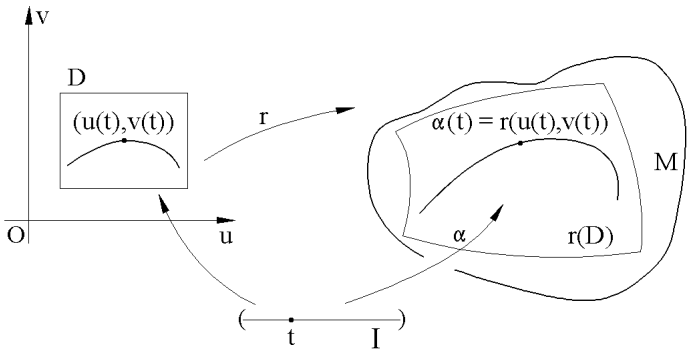


Fig. 152

Proof. The desired mapping is

$$(u(t), v(t)) = (r\mid_{Im\alpha})^{-1} \circ \alpha\ (t),\ \ (u, v):\ I \to D \subset \mathbb{R}^2.$$

3.2 Definition. Let M be a surface and $P \in M$. A vector $\vec{w}$ applied to P is called *tangent to M at P* provided that $\vec{w}$ is a velocity vector of some curve in M which passes through P. The set of all tangent vectors to M at P is called *the tangent plane of M at P*, and is denoted by T_PM.

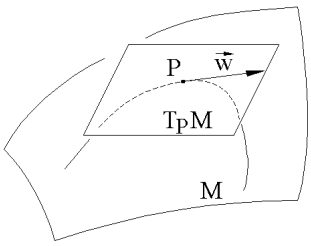


Fig. 153

We shall prove that T_PM can be canonically organized as a two-dimensional vector space (see fig. 153).

3.3 Theorem. *Let us consider a point $P \in M$, the chart r and let $r(u_0, v_0) = P$. A vector $\vec{w}_P$ is tangent to M at P iff it can be written as a linear combination of $\vec{r}_{u_0}$ and $\vec{r}_{v_0}$ (see fig. 154).*

Proof. If $\alpha:\ I \to M$ is a curve on M such that $\alpha(t_0) = P$ and $\vec{\alpha}'(t_0) = \vec{w}$, then the evaluation at t_0 yields $\vec{w} = \vec{\alpha}'(t_0) = u'(t_0)\vec{r}_{u_0} + v'(t_0)\vec{r}_{v_0}$. Conversely, assume that $\vec{w} = c_1\vec{r}_{u_0} + c_2\vec{r}_{v_0}$. Then the curve whose velocity at t_0 is $\vec{w}$ is

$$\alpha(t) = r(u_0 + (t - t_0)c_1, v_0 + (t - t_0)c_2).$$

So T_PM is generated by $\vec{r}_{u_0}$ and $\vec{r}_{v_0}$.

As plane in $\mathbb{R}^3$, the tangent space T_PM is determined by the point $P = r(u_0, v_0)$ and the vectors $\vec{r}_{u_0}, \vec{r}_{v_0}$ (see fig. 154), having the equation

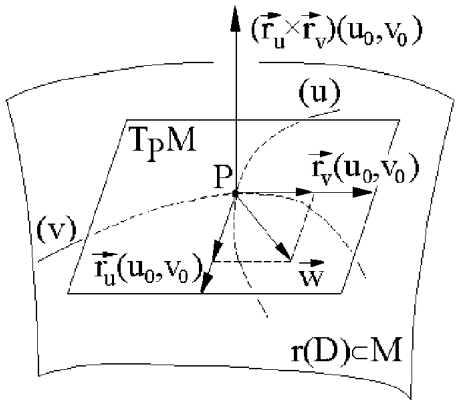


Fig. 154

$$T_PM: \begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_{u_0} & y_{u_0} & z_{u_0} \\ x_{v_0} & y_{v_0} & z_{v_0} \end{vmatrix} = 0;$$

The normal vector to this plane is $\vec{Z}(u_0, v_0) = \vec{r}_{u_0} \times \vec{r}_{v_0}$, and hence the equations of the normal line to M at P are

$$\frac{x-x_0}{\frac{D(y,z)}{D(u_0,v_0)}} = \frac{y-y_0}{\frac{D(z,x)}{D(u_0,v_0)}} = \frac{z-z_0}{\frac{D(x,y)}{D(u_0,v_0)}}.$$

3.4 Definition. Let M be a surface. A vector field $\vec{X}$ for which each vector $\vec{X}(P)$ is tangent to M at $P \in M$ is called *tangent vector field on M*. A *normal vector field on M* is a vector field for which each vector $\vec{X}(P)$ is normal to M at $P \in M$, i.e., it is perpendicular to T_PM at P.

3.5 Theorem. *Let be the surface $M: F(x, y, z) = 0$, where F is a submersion (i.e., the rank of its Jacobian matrix is 1). Then ∇F is a normal vector field on M.*

Proof. By hypothesis, the vector field ∇F is nowhere zero. Let us show that

$$(\nabla F(P), \vec{w}) = 0, \ \forall \vec{w} \in T_PM.$$

To this aim we remark that a curve $\alpha : I \to M$ produces the identity

$$F(x(t), y(t), z(t)) \equiv 0, \ \forall t \in I.$$

By deriving this relation, we obtain

$$\frac{\partial F}{\partial x}(\alpha)\frac{dx}{dt} + \frac{\partial F}{\partial y}(\alpha)\frac{dy}{dt} + \frac{\partial F}{\partial z}(\alpha)\frac{dz}{dt} = 0.$$

Choosing α such that the initial velocity at $\alpha(0) = P$ is

$$\vec{\alpha}'(0) = \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k},$$

we find

$$0 = F_x(P)w_1 + F_y(P)w_2 + F_z(P)w_3 = (\nabla F(P), \vec{w}).$$

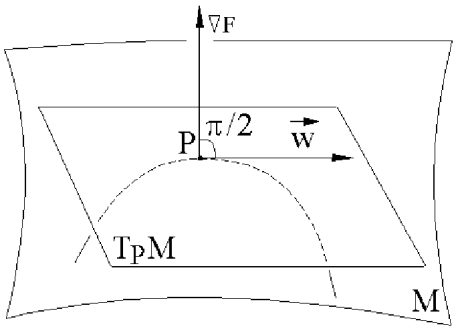


Fig. 155

From analytical point of view, the equation of the tangent plane to M at $P(x_0, y_0, z_0)$ is

$$(x - x_0)F_{x_0} + (y - y_0)F_{y_0} + (z - z_0)F_{z_0} = 0,$$

and the equations of the normal line are (see fig. 155)

$$\frac{x - x_0}{F_{x_0}} = \frac{y - y_0}{F_{y_0}} = \frac{z - z_0}{F_{z_0}}.$$

Particularly, for the explicit Cartesian representation $z = f(x, y)$ we have the equations for

- the tangent plane, $z - z_0 = p_0(x - x_0) + q_0(y - y_0)$;
- the normal line, $\dfrac{x - x_0}{p_0} = \dfrac{y - y_0}{q_0} = \dfrac{z - z_0}{-1}$,

where we denoted $p_0 = \frac{\partial f}{\partial x}(x_0, y_0), \ q_0 = \frac{\partial f}{\partial y}(x_0, y_0)$.

Remark. The tangent plane T_PM is the linear approximation of the surface M around the point $P \in M$.

7.4 Topological properties of surfaces

This section will introduce the notions of connected surfaces, simply connected surfaces, compact surfaces and orientable surfaces.

Connected surfaces. Let $\alpha : I \to M$ be a curve on the surface M and a compact interval $[a, b] \subset I$. The restriction $\alpha : [a, b] \to M$ is called *curve segment* or 1-*segment* on the surface M.

4.1 Definition. A surface M is called *connected* (one-piece surface) if for all $P, Q \in M$ there exists a curve segment $\alpha : [a, b] \to M$ which is at least continuous, such that $\alpha(a) = P$ and $\alpha(b) = Q$, i.e., the image $\alpha([a, b]) \subset M$ joins the points P and Q (see fig. 156).

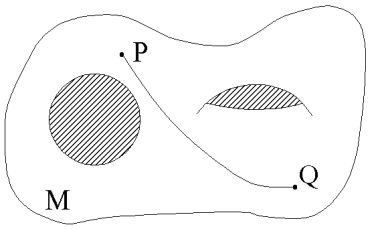


Fig. 156

A connected surface M is made "of one piece". E.g., excepting the hyperboloid of two sheets which is not connected, all the non-degenerate quadrics are connected.

4.2 Theorem. *Let M_1 and M_2 be two surfaces. If M_1 is connected and $F : M_1 \to M_2$ is a surjective and differentiable mapping, then M_2 is connected.*

Proof. Let R, S be two arbitrary points in M_2. Since F is surjective, there exist $P, Q \in M$ such that $F(P) = R,\ F(Q) = S$. But M_1 is connected, and hence there exists a curve segment α which joins P with Q. The image of α through the continuous function F is a curve segment which joins R with S, and hence M_2 is connected.

Corrolary. *If D is an open connected set in $\mathbb{R}^2$ and $r : D \to M$ is a parametrization, then the region $r(D) \subset M$ is connected.*

Example. The hyperboloid of one sheet is connected, since it is the image of the differentiable function

$$r : \mathbb{R}^2 \to \mathbb{R}^3, \quad r(u, v) = (x, y, z), \quad \begin{cases} x = a\ \mathrm{ch}u \cos v \\ y = b\ \mathrm{ch}u \sin v \\ z = c\ \mathrm{sh}u. \end{cases}$$

defined on the connected set $\mathbb{R}^2$.

Analogously we can see that each of the sheets of the hyperboloid of two sheets is connected. Since these sheets are disjoint sets, it follows that the hyperboloid of two sheets is not connected.

4.3 Theorem. *Let M be a connected surface and $f : M \to \mathbb{R}$ a differentiable mapping.*

1) If $df = 0$, then $f = const.$

2) If f does not vanish on M, then $f > 0$ or $f < 0$ on M.

Proof. Let $P, Q \in M$ be two arbitrary points. By hypothesis, they can be joined by a curve segment $\alpha : [a, b] \to M$. Using the differentiable function

$$f \circ \alpha : [a, b] \to [f(\alpha(c)),\ f(\alpha(d))],\quad c, d \in [a, b],$$

to which we apply the Lagrange theorem and the Darboux property, we infer $f =$ const, and respectively $f > 0$ or $f < 0$ everywhere.

Remark. A surface M can be regarded as a topologic space. In this sense, M is connected iff it satisfies one of the following two conditions:

a) M is not the union of two disjoint open nonempty sets;

b) the only subsets of M which are closed and open are just $\emptyset \subset M$ and M.

Simply connected surfaces. The surface M is said to be *simply connected* if any closed curve on the surface M can be deformed continuously in M to a point of M (see fig. 157).

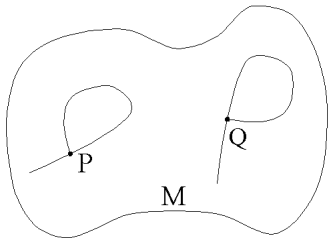


Fig. 157

As an example, the sphere is (connected and) simply connected; the circular cylinder (is connected but) is not simply connected, because the circles located inside planes which are orthogonal on the axis of the cylinder, cannot be deformed continuously to a point without exiting from the cylinder; the hyperboloid of two sheets (is not

connected but) is simply connected, etc.

Compact surfaces. Let $(u, v) \in \mathbb{R}^2$ and the closed *2-dimensional interval* (closed rectangle)

$$D_0 : a \leq u \leq b,\ c \leq v \leq d.$$

Let M be a surface. A differentiable function of type $r : D_0 \to M$ is called *2–segment* (plate) in M (see fig. 158). Here, the attribute *differentiable* is used in the sense that r can be extended differentiably to a 2-dimensional open interval which contains D_0.

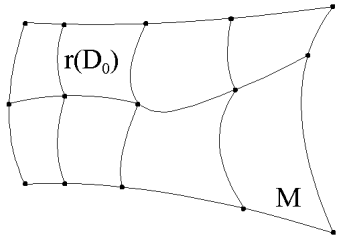


Fig. 158

4.4 Definition. The surface M is called *compact* if it can be covered by the images of a finite number of 2-segments.

For example, the sphere is a compact surface, because it is the image of the 2-segment

$$\begin{aligned} r(u, v) &= (x_0 + a\cos u \sin v,\ y_0 + a \sin u \sin v,\ z_0 + a \cos v), \\ &\forall (u, v) \in D_0 = [0, 2\pi] \times [0, \pi]. \end{aligned}$$

Also, the torus of revolution is a compact surface.

It is known that any real continuous function defined on a closed rectangle D_0 attends its maximum/minimum at a point of D_0. This is used to prove that if f is a continuous function defined on a compact surface M, then f attends its maximum/minimum at a point of M. This result proves to be efficient in showing that a surface is *not* compact. For example, the cylinder M with the generators parallel to Oz is not compact, because the (continuous) coordinate function $z : M \to \mathbb{R}$ is unbounded, its image is the whole (noncompact) set $\mathbb{R}$, whence M is noncompact.

• A subset $M \subset \mathbb{R}^3$ is said to be *closed* if its complementary set $\mathbb{R}^3 \backslash M$ is open in $\mathbb{R}^3$.

• A subset $M \subset \mathbb{R}^3$ is said to be *bounded* if the Euclidean distance between any two of its points is smaller than a given real number. In other words, a set

M is bounded iff it can be included inside a sphere of finite radius. Hence the problem of boundedness of M reduces to the study of the maximum of the function $f(x,y,z) = x^2 + y^2 + z^2$ on M or to the existence of the maxima/minima of the functions $f_1(x,y,z) = x, f_2(x,y,z) = y, f_3(x,y,z) = z$ on M.

4.5 The Heine - Borel Theorem. *A subset of* $\mathbb{R}^3$ *is compact iff it is bounded and closed.*

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentable function. A surface of type

$$M : F(x,y,z) = c$$

is closed, because it consists of the points $F^{-1}(\{c\})$, the set $\{c\}$ is a closed subset of $\mathbb{R}$ and F is a continuous function. Hence, a surface $M : F(x,y,z) = c$ is compact if it is bounded in $\mathbb{R}^3$. As an example, the ellipsoid is compact, but the hyperboloids are not compact.

4.6 Theorem. *If* $F(x,y,z)$ *is a homogeneous polynomial of degree* $n \geq 1$ *which takes at least one positive value, then*

$$M : F(x,y,z) = 0$$

is a surface. If $F(P) > 0$ *excepting* $P(0,0,0)$*, then* M *is compact, and conversely.*

Orientable surfaces. Globally, the surfaces can have two sides or just one side. This fact can be formalized using the concept of *orientability.*

4.7 Definitions. Let M be a surface.
a) A vector field $\vec{Z}$ is *a normal vector field to* M if

$$\vec{Z}(P) \perp T_P M, \ \forall P \in M.$$

b) The surface M is called *orientable* if there exists a normal vector field $\vec{Z}$ on M that is nonzero at each point of M.
c) If M is orientable, then a unit vector field $\vec{U}$ on M is called *an orientation on* M.
If $\vec{Z}$ is a nonzero normal vector field on M, then $\pm\vec{U} = \pm\frac{\vec{Z}}{||\vec{Z}||}$ define two orientations on M.

Example. Let $M : F(x,y,z) = 0$, where F is a submersion. Then $Z = \nabla F$ is a normal vector field on M without zeros and hence every surface that can be defined

implicitly is orientable (e.g., all cylinders, surfaces of revolution, quadric surfaces etc.). The same holds true for surfaces $M:\ z = f(x,y)$. For $M = r(D)$, the vector field $Z = \bar{r}_u \times \bar{r}_v$ is normal to M at each point $r(u,v) \in M$.

We have also the following results:

1) any simple surface is orientable,

2) any surface $M : F(x,y,z) = c$, (which does not have critical points of F) is orientable. E.g., the plane, the sphere, the non-degenerate quadrics are orientable surfaces; in particular, any surface defined by a Cartesian explicit equation is orientable.

3) any connected and simply connected surface is orientable;

4) any connected compact surface is orientable;

5) if M_2 is a orientable surface and $F : M_1 \to M_2$ is a regular mapping, then the surface M_1 is orientable.

4.8 Theorem. *Any connected orientable surface admits exactly two orientations.*

Proof. Let $\vec{Z}$ be an anywhere nonvanishing global normal vector field defined on M. It provides the fields

$$\pm\vec{U} = \pm\frac{\vec{Z}}{\|\vec{Z}\|}$$

which are unit vector fields. We shall prove that any other unit normal vector field has to be $+\vec{U}$ or $-\vec{U}$. Let $\vec{V}$ be an arbitrary unit normal vector field defined on M. Obviously, we have $\vec{V} = (\vec{V},\vec{U})\vec{U}$, where $f = (\vec{V},\vec{U})$ takes at each point the values -1 or $+1$. Using theorem 4.3, it follows that f takes everywhere either the value $+1$, or the value -1.

Intuitively, this result shows that the connected orientable surfaces have two faces. An orientable surface M together with a selected orientation $\vec{U}$, i.e., the pair $(M,\vec{U})$, is therefore called *oriented surface*. We shall denote the face which corresponds to the sense of $+\vec{U}$ with "+" (and call it *the positive face*), and the opposite one, with "$-$" (*the negative face*); as an example, (see fig. 159), we can orient a surface in accordance with the right hand rule.

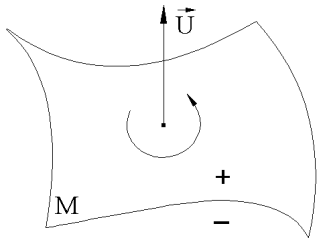


Fig. 159

For an oriented surface which encompass a finite volume, the positive sense is usually taken from interior towards exterior (see fig. 160).

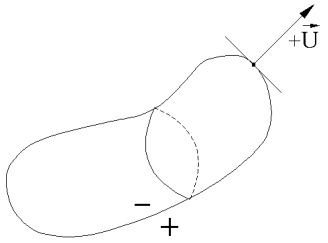


Fig. 160

If M is not connected, but is made of connected and orientable pieces (e.g., the hyperboloid of two sheets), then the orientation of its components determines an orientation of M.

4.10 Regarding the *non-orientable surfaces*, i.e., the surfaces which have only one face, the simplest example is *the Möbius band*.

One of its models can be obtained by twisting a rectangular piece of paper $ABCD$, and gluing its sides AB and DC such that A coincides with C and B with D (see fig. 161).

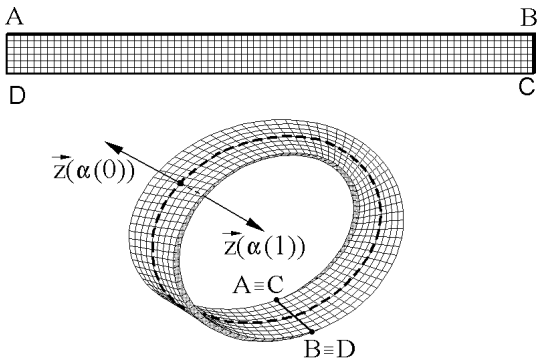


Fig. 161

The Möbius band is not globally orientable, because any normal vector field $\vec{Z}$ defined on M has to vanish at a point of M for changing its sense while passing through this point. For that, we consider the curve $\alpha : [0,1] \to M$ which is closed and satisfies $\alpha(0) = \alpha(1)$ (see fig. 161).

If we assume that $\vec{Z}$ does not vanish anywhere, then the twisting of M provides the contradiction $\vec{Z}(\alpha(0)) = -\vec{Z}(\alpha(1))$, since the function $\vec{Z}(\alpha(t))$ is by hypothesis differentiable and hence continuous.

Still, regarding the local orientability of non-orientable surfaces, one can prove the following

4.9 Theorem. *Any point P of a non-orientable surface is contained in a connected and orientable region.*

Proof. There exists a chart M, $r : D \to M$, $P \in r(D)$ such that D is open and connected. Via the Theorem 4.2 the image $r(D) \subset M$ is connected. It is oriented by the unit normal vector field

$$\vec{Z} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}, \quad \forall (u,v) \in D.$$

Consequently the non-orientable surfaces admit just *local* everywhere nonvanishing normal vector fields.

Lesson 21

7.5 Weingarten map

We described the shape of a curve in $\mathbb{R}^3$ by its curvature and torsion. The shape of a surface is described infinitesimally by the Weingarten map, a linear transformation of the tangent space into itself.

Let $\vec{Z}$ be a vector field on a surface M and let $\vec{w}$ be a vector tangent to M at P. To compute $D_{\vec{w}}\vec{Z}$, we keep account of the following ways:

1) If $\alpha : I \to M$ is a curve on M, such that $\vec{\alpha}'(t_0) = \vec{w}$ (see fig. 162), then

$$D_{\vec{w}}\vec{Z} = \frac{d}{dt}\vec{Z}(\alpha(t)) \mid_{t_0} .$$

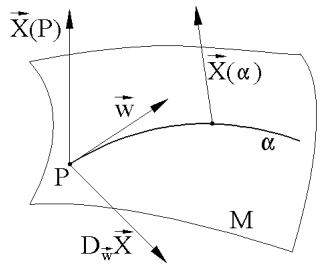


Fig. 162

2) If $\vec{Z} = f\vec{i} + g\vec{j} + h\vec{k}$, then

$$D_{\vec{w}}\vec{Z} = (D_{\vec{w}}f)\vec{i} + (D_{\vec{w}}g)\vec{j} + (D_{\vec{w}}h)\vec{k},$$

where the operator in the parantheses is the *directional derivative*.

For an orientable surface M, we shall consider the unit normal vector field $\vec{U}$, which provides the orientation of M.

5.1 Definition. Let $P \in M$, and let T_PM be the tangent space to M at P. The function

$$S_P(\vec{w}) = -D_{\vec{w}}\vec{U},\ \vec{w} \in T_PM$$

is called *the Weingarten map* (see fig. 163).

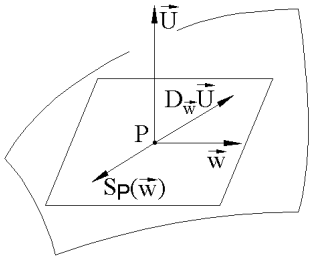


Fig. 163

The vector $D_{\vec{w}}\vec{U}$ tells how the tangent planes of M are varying in the $\vec{w}$ direction; this gives an infinitesimal description of the way M itself is curving in $\mathbb{R}^3$.

5.2 Theorem. *The Weingarten map S_P is a linear transformation $S_P : T_PM \to T_PM$.*

Proof. From $1 = (\vec{U}, \vec{U})$ we infer $0 = 2(D_{\vec{w}}\vec{U}, \vec{U}) = -2(S_P(\vec{w}), \vec{U})$ and hence $S_P(\vec{w}) \in T_PM$. On the other hand

$$S_P(a\vec{v} + b\vec{w}) = aS_P(\vec{v}) + bS_P(\vec{w}).$$

Therefore the function $P \to S_P$ associates to each point $P \in M$ a linear mapping, an endomorphism of the vector space T_PM. If $\alpha : I \to M$ and we denote also by $\vec{U}$ the restriction of $\vec{U}$ to $\alpha(I)$, then $S(\vec{\alpha}') = -\vec{U}'$.

5.3 Theorem. *The Weingarten map S_P is a symmetric operator, i.e.,*

$$(S_P(\vec{v}), \vec{w}) = (\vec{v}, S_P(\vec{w})), \quad \forall \vec{v}, \vec{w} \in T_PM.$$

The proof will be given in §8.

Let consider the Weingarten map S_P at $P \in M$, and let $\{\vec{v}, \vec{w}\}$ be a basis in T_PM; then decomposing the images $S_P(\vec{v}), S_P(\vec{w})$, i.e., writing

$$\begin{cases} S_P(\vec{v}) = a\vec{v} + c\vec{w} \\ S_P(\vec{w}) = b\vec{v} + d\vec{w} \end{cases}$$

we obtain that the matrix of S_P relative to the basis $\{\vec{v}, \vec{w}\}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\{\vec{v}, \vec{w}\}$ are orthonormal vectors, then this is a symmetric matrix.

We shall discuss in the following the geometric meaning of the eigenvalues, the eigenvectors, the trace, and the determinant of the Weingarten map.

7.6 Normal curvature

Let M be an orientable surface and let the unit normal vector field $\vec{U}$ provide the orientation. Let S be the Weingarten map. The shape of M influences the shape of the curves in M ; using the shape of curves we can obtain information on the shape of M. Usually the bending of a curve is indicated by the acceleration of the curve. To relate this bending with the shape of the surface, we project the acceleration on the unit normal vector field of the surface.

6.1 Lemma. *If $\alpha : I \to M$ is a curve on M, then*

$$(S(\vec{\alpha}'), \vec{\alpha}') = (\vec{\alpha}'', \vec{U}).$$

Proof. We have $(\vec{\alpha}', \vec{U}) = 0$; its differentiation yields $(\vec{\alpha}'', \vec{U}) + (\vec{\alpha}', \vec{U}') = 0$. Since $S(\vec{\alpha}') = -\vec{U}'$, we obtain the desired result.

The function $(\vec{\alpha}'', \vec{U})$ is called *the normal component of acceleration*, and depends only on the velocity $\vec{\alpha}'$ and S. Thus, all the curves in M with a given velocity at the point P will have the same normal component of $\vec{\alpha}''$ (see fig. 164).

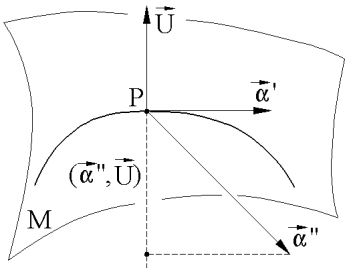


Fig. 164

We standardize $\vec{\alpha}'$ by reducing it to a unit vector $\vec{e}$.

6.2 Definition. Let $\vec{e}$ be a unit vector tangent to M at P. The number

$$k_n(\vec{e}) = (S(\vec{e}), \vec{e})$$

is called *the normal curvature of M in the $\vec{e}$ direction.*

The preceding name is justified since $k_n(-\vec{e}) = k_n(\vec{e})$, and we see that k_n generates a real function on the set of all tangent directions at the point $P \in M$.

Theorem (Meusnier). *Let $\vec{e}$ be a unit vector tangent to M at P, and $\alpha : I \to M$ a unit speed curve on M, such that $\alpha(0) = P$, $\vec{\alpha}'(0) = \vec{e}$. Then we have*

$$k_n(\vec{e}) = k_\alpha(0) \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between the principal normal $\vec{N}(0)$ of α and the unit normal vector $\vec{U}(P)$ of the surface M (see fig. 165), and $k_\alpha(0) \geq 0$ is the curvature of α at the point P.

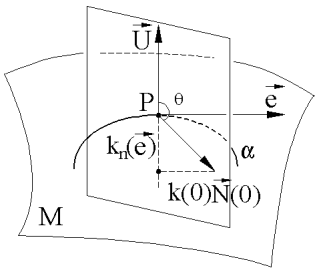


Fig. 165

Proof. Indeed, using the previous lemma and the Frenet formulas, we have

$$k_n(\vec{e}) = (S(\vec{e}), \vec{e}) = (S(\vec{\alpha}'(0)), \vec{\alpha}'(0)) = (\vec{\alpha}''(0), \vec{U}(P)) =$$

$$= k_\alpha(0)(\vec{N}(0), \vec{U}(P)) = k_\alpha(0)\cos\theta,$$

This formula shows that:

- if the vector $\vec{N}(0)$ is collinear to the vector $\vec{U}(P)$, then

$$k_n(\vec{e}) = \pm k_\alpha(0);$$

- among all the curves of M which are tangent to the unit vector $\vec{e}$ at P, the least curvature is accomplished by the curve located at the intersection between M and the plane determined by P, $\vec{U}(P)$, $\vec{e}$; this curve is called *the normal section of M in the direction of $\vec{e}$.*

For the normal section σ with unit-speed (see fig. 166), we have the relations

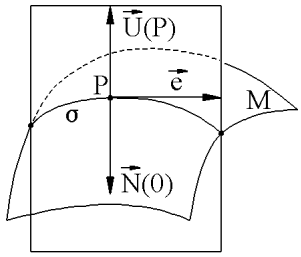


Fig. 166

$$\vec{\sigma}'(0) = \vec{e} \quad \text{and} \quad \vec{\sigma}''(0) = k_\sigma(0)\vec{N}(0),$$

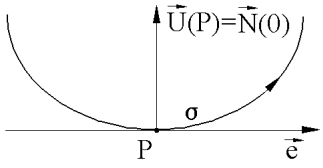
where the acceleration $\vec{\sigma}''(0)$ belongs to the sectional plane and it is perpendicular to the velocity $\vec{\sigma}'(0) = \vec{e}$, and $\vec{N}(0) = \pm\vec{U}(P)$. On this curve, we have at the point $P = \sigma(0)$ the relation

$$k_n(\vec{e}) = \pm k_\sigma(0).$$

Let P be a point of the surface M, $\vec{e}$ a tangent unit vector to M at P, and $k_n(\vec{e})$ the corresponding normal curvature. Using the normal section of M in the direction of $\vec{e}$ and taking account that the principal normal provides the sense in which the curve bends, we can give the following geometric interpretations of the sign of $k_n(\vec{e})$ with respect to the selection of the unit normal vector field $\vec{U}$.

Geometric meaning of the sign of normal curvature.

1) If $k_n(\vec{e}) > 0$, then $\vec{N}(0) = \vec{U}(P)$ and the normal section is bending towards $\vec{U}(P)$, whence the surface is bending towards $\vec{U}(P)$ (see fig. 167).



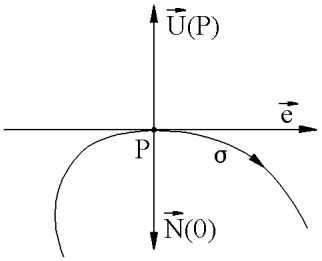


Fig. 167 **Fig. 168**

2) If $k_n(\vec{e}) < 0$, then σ is bending away from $\vec{U}(P)$ (see fig. 168).

3) If $k_n(\vec{e}) = 0$, then $k_\sigma(0) = 0$ and $\vec{N}(0)$ is undefined; therefore, it is not possible to decide the shape of the surface in the direction $\vec{e}$.

Let us now fix a point $P \in M$ and imagine that a unit tangent vector $\vec{e}$ at P revolves, sweeping out the unit circle in T_PM (see fig. 169).

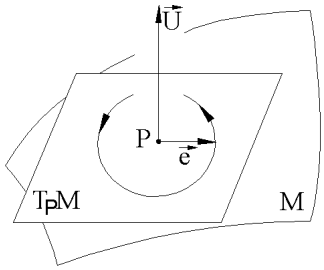


Fig. 169

We consider the restriction

$$k_n(\vec{e}) = (S(\vec{e}), \vec{e}),\ ||\vec{e}|| = 1.$$

Because the circle $||\vec{e}|| = 1$ is compact set, $k_n(\vec{e})$ has a minimum and a maximum, namely

$$k_1 = \max_{||\vec{e}||=1} k_n(\vec{e}),\quad k_2 = \min_{||\vec{e}||=1} k_n(\vec{e}).$$

6.3 Definition. The scalars $k_1,\ k_2$ are called *principal curvatures of M at P*; the directions in which these extreme values occur are called *principal directions of M at P*; the unit vectors in these directions are called *principal vectors.*

If $k_1 = k_2$, then $k_n(\vec{e})$ is constant and the point $P \in M$ is called *umbilic.*

6.4 Theorem.

If there exists a constant $k \in \mathbb{R}$ such that for all $\vec{e} \in T_P M$ with $||\vec{e}|| = 1$ we have $S(\vec{e}) = k\vec{e}$, then P is an umbilic.

Otherwise, the eigenvalues of S are the principal curvatures and the unit eigenvectors of S are the principal unit vectors (note that there are two principal orthogonal directions).

Proof. Let $\{\vec{f_1}, \vec{f_2}\}$ be an orthonormal basis in $T_P M$. Then we have

$$\begin{cases} S(\vec{f_1}) = S_{11}\vec{f_1} + S_{21}\vec{f_2} \\ S(\vec{f_2}) = S_{12}\vec{f_1} + S_{22}\vec{f_2}. \end{cases}$$

Since the matrix (S_{ij}) is symmetric, there exist the (real) eigenvalues λ_1, λ_2; we have that $\lambda_1 = \lambda_2$ iff $S_{11} = S_{22},\ S_{12} = 0$, whence the eigenvectors are undetermined and the point is an umbilic. For different eigenvalues, assuming $\lambda_1 > \lambda_2$ we get the orthogonal unit eigenvectors $\{\vec{e}_1, \vec{e}_2\},\ S(\vec{e}_1) = \lambda_1\vec{e}_1,\ S(\vec{e}_2) = \lambda_2\vec{e}_2$.

Let us show the relations

$$\lambda_1 = \max_{||\vec{e}||=1} k_n(\vec{e}), \quad \lambda_2 = \min_{||\vec{e}||=1} k_n(\vec{e}).$$

For that we decompose a tangent unit vector $\vec{e}$ as (see fig. 170)

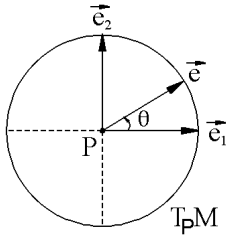


Fig. 170

$$\vec{e} = \cos\theta\vec{e}_1 + \sin\theta\vec{e}_2,$$

and we use the expression

$$k_n(\vec{e}) = (S(\vec{e}), \vec{e}) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$

Hence we get automatically

$$\lambda_2 \le k_n(\vec{e}) \le \lambda_1,\ \theta \in [0, 2\pi].$$

For $\theta = 0$ we have $k_n(\vec{e}) = \lambda_1$ and for $\theta = \pi/2$ we obtain $k_n(\vec{e}) = \lambda_2$.

Euler formula. Let k_1, k_2 be the principal curvatures and $\vec{e}_1, \vec{e}_2$ be the principal unit vectors of M at P. If $\vec{e} = \cos\theta\vec{e}_1 + \sin\theta\vec{e}_2$, then we infer *the Euler formula*

$$k_n(\vec{e}) = k_1\cos^2\theta + k_2\sin^2\theta, \quad \theta \in [0, 2\pi].$$

The quadratic approximation of M. We fix P and assume that
(1) P is the origin of $\mathbb{R}^3$;
(2) $T_PM = xOy$;
(3) $\vec{i},\ \vec{j}$ are principal unit vectors.
Then $M:\ z = f(x, y)$ has the unit normal vector field

$$\vec{U} = \frac{-f_x\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{1 + f_x^2 + f_y^2}},$$

and $f(0,0) = f_x(0,0) = f_y(0,0) = 0$. Let us consider the Taylor approximation (see fig. 171)

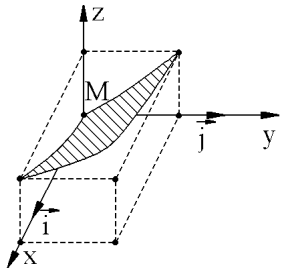


Fig. 171

$$z \cong \frac{1}{2}(x^2 f_{x^2}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{y^2}(0,0)).$$

The relations

$$\begin{cases} S(\vec{i}) = f_{x^2}(0,0)\vec{i} + f_{xy}(0,0)\vec{j} \\ S(\vec{j}) = f_{xy}(0,0)\vec{i} + f_{y^2}(0,0)\vec{j} \end{cases}$$

imply $f_{xy}(0,0) = 0$, and $k_1 = f_{x^2}(0,0)$ and $k_2 = f_{y^2}(0,0)$. Hence we obtain the quadric

$$z = \frac{1}{2}(k_1x^2 + k_2y^2),$$

which is called *the quadratic approximation of* M. We recall that T_PM is the linear approximation of M.

Lesson 22

7.7 Gauss curvature and mean curvature

Let M be an orientable surface and S its Weingarten map. Now we examine the geometrical meaning of the determinant and of the trace of S.

7.1 Definition.
1) The function $K: M \to \mathbb{R}$, $K = \det S$ is called *Gauss curvature*;
2) The function $H: M \to \mathbb{R}$, $H = \frac{1}{2} trace S$ is called *mean curvature.*
The Gauss curvature K and the mean curvature H can be expressed by the principal curvatures k_1, k_2 of the surface M.

7.2 Theorem. *The connection between* k_1, k_2, K, H *is described by*

$$K = k_1 \cdot k_2, \qquad H = \frac{1}{2}(k_1 + k_2).$$

Proof. Let $\{\vec{e}_1, \vec{e}_2\}$ be the orthonormal basis of T_PM determined by the principal unit vectors. Since we have the decompositions

$$S(\vec{e}_1) = k_1\vec{e}_1, \quad S(\vec{e}_2) = k_2\vec{e}_2,$$

with respect to the basis $\{\vec{e}_1, \vec{e}_2\}$, the matrix of S writes $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, whence the result.

Remark. If $\vec{U}$ is changed to $-\vec{U}$, then the signs of both k_1, k_2 change, but K remains unaffected.

Geometrical interpretation of the sign of $K(P)$. Let k_1, k_2 be the principal curvatures of the surface M. We use Euler formula $k_n(\vec{e}) = k_1 \cos^2\theta + k_2 \sin^2\theta$ and the quadratic approximation $z = \frac{1}{2}(k_1x^2 + k_2y^2)$.
1) If $K(P) > 0$, since $K = k_1 \cdot k_2$, it follows that k_1 and k_2 have the same sign and since $k_n(\vec{e}) = k_1 \cos^2\theta + k_2 \sin^2\theta$, either $k_n(\vec{e}) > 0$, or $k_n(\vec{e}) < 0$; in this case, the

quadratic approximation is an elliptic paraboloid and hence, in the neighborhood of P, the surface looks like an elliptic paraboloid (see fig. 172).

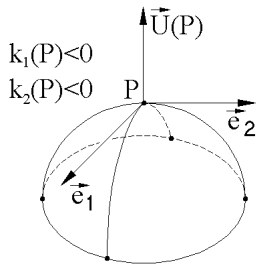


Fig. 172

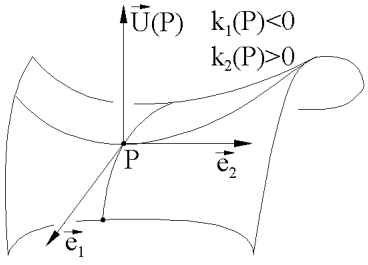


Fig. 173

2) If $K(P) < 0$, then the quadratic approximation is a hyperbolic parabolid (see fig. 173).

3) If $K(P) = 0$ and $k_1(P) \neq 0,\ k_2(P) = 0$ then the quadratic approximation is a parabolic cylinder (see fig. 174).

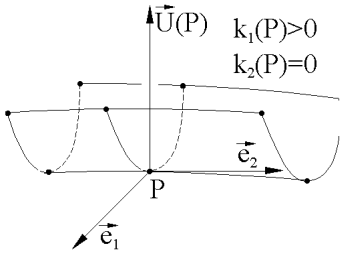


Fig. 174

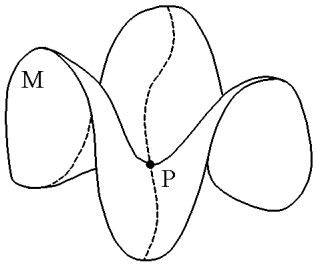


Fig. 175

4) If $k_1(P) = k_2(P) = 0$, then P is called *planar point* (see fig. 175), since the quadratic approximation is reduced to the tangent plane.

If $K(P)$ is constant for all $P \in M$, then M is called *surface of constant curvature*. If $K(P) = 0, \forall P \in M$, then M is called *flat surface*, or *locally Euclidean*. If $H(P) = 0, \forall P \in M$, then M is called *minimal surface*. Generally, we have the relation

$$k_{1,2} = H \pm \sqrt{H^2 - K},$$

which shows that the principal curvatures k_1, k_2 are differentiable functions excepting the points which satisfy $H^2 = K$, i.e., $k_1 = k_2$ (umbilic points).

7.8 Fundamental forms and computational techniques

Let M be a surface and let

$$r:\ D \to M,\ \ r(u,v) = (x(u,v), y(u,v), z(u,v))$$

be a chart in M; in vector notation, we have

$$\vec{r}(u,v) = x(u,v)\vec{i} + v(u,v)\vec{j} + z(u,v)\vec{k}.$$

We attach three real-valued functions:

$$E = (\vec{r}_u, \vec{r}_u), F = (\vec{r}_u, \vec{r}_v), G = (\vec{r}_v, \vec{r}_v)$$

on D; we can easily see that $E > 0, G > 0$ and $F = \sqrt{EG}cos\theta$, where θ is the angle between $\vec{r}_u$ and $\vec{r}_v$. These functions measure the way in which r distorts the flat region D in $\mathbb{R}^2$ in order to apply it in the curved region $r(D)$ in M.

8.1 The first fundamental form. The scalar product on $\mathbb{R}^3$ induces a symmetric bilinear form on T_PM by restriction. This form is called *the first fundamental form of the surface* M and will be denoted by I. Let us consider two tangent vectors

$$\vec{a} = a_1\vec{r}_u + a_2\vec{r}_v,\ \vec{b} = b_1\vec{r}_u + b_2\vec{r}_v \in T_PM.$$

Then

$$I(\vec{a}, \vec{b}) = (\vec{a}, \vec{b}) = Ea_1b_1 + F(a_1b_2 + a_2b_1) + Ga_2b_2.$$

The attached quadratic form of this bilinear form will have the same name.

Considering the differential $d\vec{r} = \vec{r}_u du + \vec{r}_v dv$, which is a tangent vector to $r(D)$, the first fundamental form reduces to *the square of the arc element* of a curve in $r(D)$,

$$ds^2 = (d\vec{r}, d\vec{r}) = Edu^2 + 2Fdudv + Fdv^2.$$

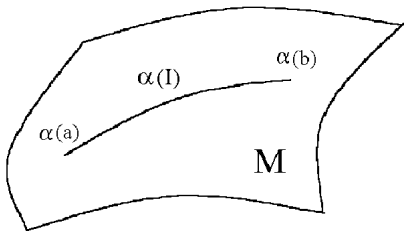


Fig. 176

Using ds is possible to define the length of a curve on M (see fig. 176). Namely, if $\alpha : [a, b] \to M$ is a regular curve, then its length is given by

$$l(\alpha) = \int_a^b \left\| \frac{d\vec{\alpha}}{dt} \right\| dt = \int_a^b \sqrt{I_{\alpha(t)}(\vec{\alpha}'(t), \vec{\alpha}'(t))} dt.$$

If M is a connected surface, we can define a *distance* (or *metric*) on M, as follows. Denoting by Ω the set of regular curves $\alpha : [a, b] \to M$ which join the point $P = \alpha(a)$ with the point $Q = \alpha(b)$, the distance from P to Q is defined by

$$d(P, Q) = \inf_{\alpha \in \Omega} l(\alpha).$$

Hence, any connected surface is a metric space, whose topology determined by the metric coincides with the one induced from $\mathbb{R}^3$ (the submanifold topology).

8.2 The second fundamental form.

The Weingarten map S at the point $r(u, v) \in r(D)$ is a symmetric transformation with respect to the scalar product defined by I. To introduce the Weingarten map S we need the normal vector field $\vec{Z} = \vec{r}_u \times \vec{r}_v$ whose length satisfies

$$||\vec{r}_u \times \vec{r}_v||^2 = EG - F^2,$$

(Lagrange identity), and the unit normal vector field

$$\vec{U} = \frac{\vec{r}_u \times \vec{r}_v}{||\vec{r}_u \times \vec{r}_v||}$$

The bilinear symmetric form

$$II(\vec{a}, \vec{b}) = (S(\vec{a}), \vec{b}) = la_1b_1 + m(a_1b_2 + a_2b_1) + na_2b_2,$$

$$l = (S(\vec{r}_u), \vec{r}_u), m = (S(\vec{r}_u), \vec{r}_v) = (\vec{r}_u, S(\vec{r}_v), n = (S(\vec{r}_v), \vec{r}_v),$$

is called *the second fundamental form of the surface*. The associated quadratic form will have the same name, and it can be described by

$$\sigma = (S(d\vec{r}), d\vec{r}) = ldu^2 + 2mdudv + ndv^2.$$

The real functions l, m, n are determined by the first fundamental form and the Weingarten map.

Because $\{\vec{r}_u,\ \vec{r}_v\}$ gives a basis for the tangent space T_PM, it is clear that l, m, n determine the Weingarten map. Consequently the second fundamental form of a

surface produces the same mathematical information as the Weingarten map. Now we want to express the Gauss curvature and the mean curvature as functions of E, F, G, l, m, n.

8.3 Lemma. *The following relations hold*

$$S(\vec{r}_u) \times S(\vec{r}_v) = K(P)\ \ \vec{r}_u \times \vec{r}_v,$$
$$S(\vec{r}_u) \times \vec{r}_v + \vec{r}_u \times S(\vec{r}_v) = 2H(P)\ \ \vec{r}_u \times \vec{r}_v.$$

Proof. We decompose the vectors $S(\vec{r}_u), S(\vec{r}_v)$ relative to the basis $\{\vec{r}_u, \vec{r}_v\}$, obtaining

$$S(\vec{r}_u) = a\vec{r}_u + b\vec{r}_v, \quad S(\vec{r}_v) = c\vec{r}_u + d\vec{r}_v.$$

By direct calculation, we infer

$$K(P) = ad - bc, \quad H(P) = \frac{1}{2}(a + d).$$

8.4 Theorem. *The Gauss curvature and mean curvature have the following expressions*

$$K(r) = \frac{ln - m^2}{EG - F^2}, \quad H(r) = \frac{Gl - 2Fm + En}{2(EG - F^2)}.$$

Proof. Using the lemma and the identity $(\vec{a} \times \vec{b}, \vec{c} \times \vec{d}) = \begin{vmatrix} (\vec{a}, \vec{c}) & (\vec{a}, \vec{d}) \\ (\vec{b}, \vec{c}) & (\vec{b}, \vec{d}) \end{vmatrix}$, we obtain

$$K = \frac{(S(\vec{r}_u) \times S(\vec{r}_v), \vec{r}_u \times \vec{r}_v)}{||\vec{r}_u \times \vec{r}_v||^2} = \frac{ln - m^2}{EG - F^2};$$

similarly, we yield the relation for H.

We remark that since $\{\vec{r}_u, \vec{r}_v\}$ is not an orthonormal basis, l, m, n do not lead to simple expressions for $S(\vec{r}_u)$ and $S(\vec{r}_v)$ in terms of $\vec{r}_u, \vec{r}_v$; however, they do provide simple expressions for K and H.

Let us find a simple way to compute l, m, n. Using $(\vec{U}, \vec{r}_u) = 0$, by partial derivation we have

$$0 = \frac{\partial}{\partial v}(\vec{U}, \vec{r}_u) = (\vec{U}_v, \vec{r}_u) + (\vec{U}, \vec{r}_{uv}).$$

Since $\vec{U}_v = -S(\vec{r}_v)$, we obtain

$$m = (S(\vec{r}_v), \vec{r}_u) = (\vec{U}, \vec{r}_{uv}) = (\vec{U}, \vec{r}_{vu}) = (\vec{r}_v, S(\vec{r}_u)),$$

and this is sufficient to prove that S is symmetric. Similarly we obtain new expressions for $l,\ n$ which are suitable for direct computations.

8.5 Theorem. *The coefficients l, m, n of the second fundamental form of the parametrized surface $M = r(D)$ are given by*

$$l = (\vec{U}, \vec{r}_{u^2}),\ m = (\vec{U}, \vec{r}_{uv}),\ n = (\vec{U}, \vec{r}_{v^2}),$$

or briefly

$$II(d\vec{r}, d\vec{r}) = l\ du^2 + 2m\ du\ dv + m\ dv^2 = (\vec{U}, d^2\vec{r}).$$

Remarks. 1) In the case when r is provided by a Monge chart, i.e.,

$$M = r(D),\ r(x, y) = (x, y, f(x, y)),$$

or briefly $M:\ z = f(x, y),\ \forall (x, y) \in D \subset \mathbb{R}^2$, we have

$$\bar{U} = \frac{-f_x\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{1 + f_x^2 + f_y^2}},\ S_P(\vec{a}) = -D_{\vec{a}}\vec{U},$$
$$k_n(\vec{a}) = (S_P(\vec{a}), \vec{a}) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\ d^2 f(\vec{a}, \vec{a}),$$
$$II(\vec{a}, \vec{b}) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\ d^2 f(\vec{a}, \vec{b}),$$

for $\vec{a}, \vec{b}$ tangent vectors to M.

2) Let M be given by a Cartesian implicit equation,

$$M:\ F(x, y, z) = 0.$$

Then we have

$$\bar{U} = \frac{F_x\vec{i} + F_y\vec{j} + F_z\vec{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}},\ S_P(\vec{a}) = -D_{\vec{a}}\bar{U},$$
$$k_n(\vec{a}) = (S_P(\vec{a}), \vec{a}) = -\frac{1}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\ d^2 F(\vec{a}, \vec{a}),$$
$$II(\vec{a}, \vec{b}) = \frac{1}{\sqrt{F_x^2 + F_y^2 + F_z^2}}\ d^2 F(\vec{a}, \vec{b}),$$

for $\vec{a}, \vec{b}$ tangent vectors to M.

Examples.
1) The *helicoid*

$$x = u\cos v,\ y = u\sin v,\ z = v$$

is a minimal surface with $-1 \leq K < 0$.

2) The saddle surface $z = xy$ is not minimal and has negative Gauss curvature.

7.9 Curves on surfaces

We shall describe three types of curves on a surface, which have geometrical meaning: principal curves, asymptotic curves and geodesics. Let M be a surface and $\vec{U}$ be the unit normal vector field on M.

Principal curves

9.1 Definition. A regular curve is called *principal curve* (or *line of curvature*) if $\vec{\alpha}'$ determines a principal direction at each point of the curve.

Hence, the principal curves are trajectories whose directions provide maximal/minimal normal curvature. Modulo reparametrizations, through any non-umbilical point of the surface M pass exactly two principal curves (which are orthogonal at the point).

Theorem. *Let α be a regular curve on M and $\vec{U}$ the restriction of the unit normal vector field to α.*

1) The curve α is principal iff the vector fields U' and $\vec{\alpha}'$ are collinear at each point of α.

2) If α is a principal curve, then

$$k_n\left(\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right) = \frac{(\vec{\alpha}'', \vec{U})}{(\vec{\alpha}', \vec{\alpha}')}.$$

Proof. 1) The curve α is principal iff $S(\vec{\alpha}')$ and $\vec{\alpha}'$ are collinear. On the other side we have $S(\vec{\alpha}') = -\vec{U}'$, whence the assertion.

2) Since $\dfrac{\vec{\alpha}'}{\|\vec{\alpha}'\|}$ is a principal vector, we have for $k_* \in \{k_1, k_2\}$,

$$k_* = k_n\left(\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right) = \left(S\left(\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right), \frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}\right) = \frac{(S(\vec{\alpha}'), \vec{\alpha}')}{\|\vec{\alpha}'\|^2} = \frac{(\vec{\alpha}'', \vec{U})}{(\vec{\alpha}', \vec{\alpha}')}.$$

Asymptotic curves

9.2 Definition. The directions tangent to M whose attached normal curvatures are zero, are called *asymptotic directions*. Thus a direction given by the vector $\vec{w} \in T_P M$ is asymptotic iff $(S(\vec{w}), \vec{w}) = 0$.

9.3 Theorem. *Let $P \in M$.*
1) If $K(P) > 0$, then there are no asymptotic directions at P;
2) If $K < 0$, then there are two asymptotic directions at P (see fig. 177); the angle θ between a principal direction and an asymptotic direction is given by

$$\tan\theta = -\frac{k_1(P)}{k_2(P)}.$$

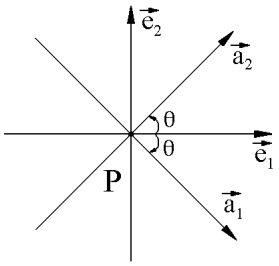


Fig. 177

3) If $K(P) = 0$, then every direction is asymptotic if P is a planar point; otherwise there exists just one asymptotic direction, which is also principal.

Hint. We use Euler's formula $k(\bar{e}) = k_1(P)\cos^2\theta + k_2(P)\sin^2\theta$.

9.4 Definition. A regular curve $\alpha : I \to M$ such that $\vec{\alpha}'$ gives an asymptotic direction at every point of $\alpha(I)$, is called *asymptotic curve*.

9.5 Theorem. *Let $\alpha : I \to M$ be a regular curve and $\vec{U}$ the restriction of the unit normal vector field of M to α. The following statements are equivalent:*
1) α is an asymptotic curve;
2) $\vec{U}'$ and $\vec{\alpha}'$ are orthogonal at all the points of $\alpha(I)$;
3) $\vec{\alpha}''$ is tangent to M.

Proof. 1) $\Leftrightarrow$ 2). The curve α is an asymptotic line iff

$$k_n(\vec{\alpha}') = 0 \Leftrightarrow (S(\vec{\alpha}'), \vec{\alpha}') = 0,$$

and considering that $S(\vec{\alpha}') = -\vec{U}'$, this becomes equivalent to

$$(\vec{U}', \vec{\alpha}') = 0.$$

1) $\Leftrightarrow$ 3). Deriving the relation $(\vec{U}, \vec{\alpha}') = 0$, we infer $(\vec{U}', \vec{\alpha}') + (\vec{U}, \vec{\alpha}'') = 0$, whence

$$(\vec{U}', \vec{\alpha}') = 0 \Leftrightarrow (\vec{U}, \vec{\alpha}'') = 0.$$

9.6 Theorem. *A surface M is minimal iff at each of its points there exist two orthogonal asymptotic directions.*

Proof. We have

$$H(P) = 0 \Leftrightarrow k_1(P) = -k_2(P)$$

which is equivalent to the fact that either

(1) P is planar (and the criterion is fulfilled trivially), or

(2) $K(P) < 0$ and $\theta \in \left\{\frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{2}\right\}$, i.e., the two asymptotic directions are orthogonal.

9.7 Theorem. *If M is a ruled surface, then $K \leq 0$. The ruled surface M is locally Euclidean ($K = 0$) iff $\vec{U}$ is parallel along each ruling of M (along the ruling, we have also the same tangent plane).*

Proof. A straight line $\vec{\alpha} = \vec{a} + t\vec{b},\ t \in \mathbb{R}$ included in the surface M is an asymptotic curve, because $\vec{\alpha}'' = \vec{0}$ is a vector tangent to M. By definition, through each point of a ruled surface passes a straight line which is contained in the surface. Hence, at each point we have at least one asymptotic direction. Then, applying theorem 9.3, it results $K \leq 0$.

Let α be an arbitrary ruling of M. If $\vec{U}$ is parallel along α, then $S(\vec{\alpha}') = -\vec{U}' = \vec{0}$. Hence α is a principal curve with the normal curvature $k_n(\vec{\alpha}') = 0$, which infers $K = k_1 k_2 = 0$.

Conversely, if $K = 0$, from the case (3) of the theorem 9.3 it results that the asymptotic directions of M are principal curves. Hence each generator α is simultaneously principal curve, i.e.,

$$S(\vec{\alpha}') = k_n(\vec{\alpha}')\vec{\alpha}',$$

and asymptotic curve, i.e., $k_n(\vec{\alpha}') = 0$. It follows that

$$\vec{U}' = -S(\vec{\alpha}') = \vec{0}.$$

Geodesics

9.8 Definition. A regular curve $\alpha : I \to M$ whose acceleration $\vec{\alpha}''$ is always normal to M is called a *geodesic*.

We remark that on a geodesic we have $(\vec{\alpha}'', \vec{\alpha}') = 0$, whence $(\vec{\alpha}', \vec{\alpha}') \equiv ||\vec{\alpha}'||^2$ is constant. So, the geodesics have constant speed. As examples, the straight lines regarded as curves in a plane are geodesics. Also, on cylinders the helices and on surfaces of revolution the parallels (circles) are also geodesics. E.g., on the circular cylinder

$$M : \ x^2 + y^2 = 1, \ M = r(\mathbb{R}^2), \ r(u, v) = (\cos u, \sin u, v),$$

the curves

$$\alpha(t) = (a \cos t, a \sin t, bt), \quad t \in \mathbb{R}$$

are geodesics (helices, circles, straight lines).

Remark. Though the definition of a geodesic on a surface is independent of the Weingarten map, still there exists a strong relation between the Frenet elements of a geodesic and the Weingarten map. For showing this, we consider a unit-speed geodesic α. Since $\vec{N} = \frac{1}{k}\vec{\alpha}''$ is a normal vector to M, we have

$$-\vec{N}' = S(\vec{\alpha}') = S(\vec{T})$$

and applying the Frenet formulas, we obtain

$$S(\bar{T}) = k\bar{T} - \tau\bar{B}.$$

7.10 Area of a surface

Let σ be a piece of a surface $M \subset \mathbb{R}^3$, regarded as the image of the function $r : D_0 \to M$, i.e., $\sigma = r(D_0)$, where D_0 is a closed rectangle. Let $D = int\ D_0$ and $r : D \to M$ be a chart.

10.1 In the following we shall determine the area of $\sigma = r(D_0)$.

Let $\vec{r} = \vec{r}(u, v), \ (u, v) \in D_0$ be the vector equation of σ. The coordinate lines divide σ into curvilinear quadrilaterals. From these, we choose the one which is bounded by the curves (see fig. 178)

$$(u), (v), \ (u + \Delta u), \ (v + \Delta v),$$

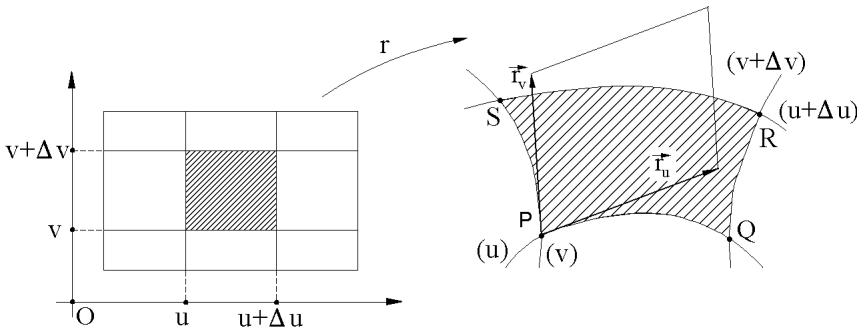


Fig. 178

The area of the curvilinear quadrilateral $PQRS$ is approximated by the area of the parallelogram built in the tangent plane T_PM on the vectors $\vec{r}_u\Delta u$ and $\vec{r}_v\Delta v$. Therefore, we have

$$\text{Area } (PQRS) = \|\vec{r}_u\Delta u \times \vec{r}_v\Delta v\| = \|\vec{r}_u \times \vec{r}_v\| \, \Delta u\Delta v$$

and

$$\text{Area } (\sigma) = \sum \text{ Area } (PQRS) = \sum \|\vec{r}_u \times \vec{r}_v\| \, \Delta u\Delta v.$$

These remarks let us say that the symbol

$$d\sigma = \|\vec{r}_u \times \vec{r}_v\| \, dudv = \sqrt{\vec{r}_u^2\vec{r}_v^2 - (\,\vec{r}_u, \vec{r}_v\,)^2} \, dudv = \sqrt{EG - F^2} \, dudv$$

is *the area element* of the surface σ. The area of σ is defined as the integral

$$A = \iint_\sigma d\sigma = \iint_{D_0} \sqrt{EG - F^2} \, dudv.$$

If σ is given by a Cartesian explicit equation $z = f(x, y)$, then we have

$$d\sigma = \sqrt{1 + p^2 + q^2} \, dxdy$$

and

$$A = \iint_{D_0} \sqrt{1 + p^2 + q^2} \, dxdy.$$

Let M be a surface and $\sigma_i \subset M$ a finite number of images of the form $r_i(D_{0i})$, $i = 1, \ldots, k$ which have in common just their edges $\partial\sigma_i$.

Then we define the area of the union as

$$\text{Area } (\sigma_1 \cup \sigma_2 ... \cup \sigma_k) = \sum_{j=1}^{k} \iint_{\sigma_j} d\sigma.$$

10.2 Examples

1) Let σ be the sphere of radius R. It can be regarded as the image of the domain

$$D : 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi$$

through the mapping

$$\begin{cases} x &= R \sin\varphi \cos\theta \\ y &= R \sin\varphi \sin\theta \\ z &= R \cos\varphi. \end{cases}$$

By straightforward calculation, we infer $\sqrt{EG - F^2} = R^2$. Hence

$$A = \int_0^{\pi} \int_0^{2\pi} R^2 \sin\varphi \, d\varphi d\theta = 4\pi R^2.$$

2) Let be the torus of radii $R > r > 0$ and parametric equations

$$\begin{cases} x = (R + r\cos u)\sin v \\ y = (R + r\cos u)\sin v \\ z = r \sin u \end{cases}, \qquad (u, v) \in [-\pi, \pi] \times [-\pi, \pi].$$

By direct calculation, we find the area of the torus $A = 4\pi^2 Rr$.

7.11 Problems

1. Let be the function

$$r : D \to \mathbb{R}^3, \; r(u, v) = (\cos 2\pi u, \sin 2\pi u, v), \;\; u \in \left(0, \frac{1}{2}\right), \; v \in \mathbb{R}.$$

Show that r is a proper chart and that $r(D)$ is the surface

$$M : x^2 + y^2 = 1, \; -1 < x < 1, \quad y > 0, \quad z \in \mathbb{R}.$$

2. Let be the immersion

$$r:\begin{cases} x=u\cos v \\ y=\sin v \\ z=a\ln\left(u+\sqrt{u^2-a^2}\right) \end{cases}, \quad a>0,\ u\in[a,\infty),\ v\in\mathbb{R}.$$

Determine the curves on the surface $r([a,\infty)\times\mathbb{R})$ which intersect the curves $v=v_0$ by a constant angle.

3. Determine the Cartesian implicit equations of the cylindric surfaces whose director curves are respectively

1) $x^2+y^2-3x+2y-1=0,\ z=0$;

2) $x^2-x+2y-1=0,\ z=0$,

and whose generators are orthogonal respectively to the planes of the director curves.

4. Let be the function

$$r:\mathbb{R}^2\to\mathbb{R}^3,\ r(u,v)=(1+uv,\ u+u^2v,\ u^2+u^3v).$$

1) Show that $r(\mathbb{R}^2)$ is a "conic surface";

2) Find the Cartesian equation of $r(\mathbb{R}^2)$.

5. A surface which admits simultaneously the parametrizations

$$\bar{r}(u,v)=\bar{\alpha}(u)+v\bar{\beta}(u) \ \text{ and } \ \bar{r}(u,v)=\bar{\gamma}(v)+u\bar{\delta}(v)$$

is called *double ruled surface*.

1) Show that any double ruled surface is a plane or a quadric.

2) Let be the map r defined by

$$x=a\left(u+\frac{1}{v}\right), y=b\left(u-\frac{1}{v}\right),\ z=\frac{2u}{v}, \quad u\in\mathbb{R}, v\in\mathbb{R}\backslash\{0\}.$$

Show that the image $r(\mathbb{R}\times(\mathbb{R}\backslash\{0\}))$ is a double ruled surface. Find the coordinate curves of the surface and the Cartesian equation of the surface.

6. The surface generated by the rotation of a circle around an axis which is contained in the plane of the circle (which does not intersect the circle) is called *torus*.

By a proper choice of axes, find a parametrization of the torus, then determine its Cartesian implicit equation.

7. Let $a > b > 0$ and the function $r : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$r(u, v) = ((a + b\cos v)\cos u,\ (a + b\cos v)\sin u,\ b\sin v).$$

1) Show that r is a double periodic parametrization of the torus.
2) Show that the torus is a connected and compact surface.

8. Let be the sphere $M : x^2 + y^2 + z^2 = a^2$ and the chart

$$r(u, v) = (a\cos u\sin v,\ a\sin u\sin v,\ a\cos v), \quad (u, v) \in (0, 2\pi) \times (0, \pi).$$

Find the coordinate expressions for the following functions defined on M:
1) $f(x, y, z) = x^2 + z^2$;
2) $f(x, y, z) = (y - z)^2 + x^2$.

9. Recognize the level sets $M = f^{-1}(c)$, for the function

$$f(x, y, z) = x^2 + y^2 - z^2, \qquad c \in \{c_1 = -1; c_2 = 0,\ c_3 = 1\}.$$

For each of them, determine at which point $P \in M$ the tangent space at P is exactly $[\nabla f(P)]^{\perp}$.

10. Let be the surface

$$M : x^{\alpha}y^{\beta}z^{\gamma} = 1,\ \alpha, \beta, \gamma \in \mathbb{R}.$$

The tangent plane to M at the point (x_0, y_0, z_0) intersects the coordinate axes respectively at the points A, B, C. Show that the point (x_0, y_0, z_0) is the baricenter of the "masses" α, β, γ, applied respectively in A, B, C.

11. Let $D \subset \mathbb{R}^2$ be a connected open set, and $r : D \to \mathbb{R}^3$ a parametrization. Show that if the normal to the surface $r(D)$ has a fixed direction, then the surface is a subset of a plane. Check this result for
1) $r : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}^3, \quad r(u, v) = (u^2 + v^2,\ uv,\ (u + v)^2)$,
2) $r : \mathbb{R}^2\backslash\{(u, v)|u > 0,\ v > 0,\ u - v > 0,\ u - 3v > 0\} \to \mathbb{R}^3$,

$$r(u, v) = \left((u - v)^2,\ u^2 - 3v^2,\ \frac{v}{2}(u - 2v)\right).$$

12. Let M be an oriented surface and let $\{\vec{a}, \vec{b}\}$ be a basis for the tangent plane T_PM. Show that the orientation of the basis $\{\vec{a}, \vec{b}\}$ is coherent with the orientation

of M given by $\vec{U}$, iff is fulfilled one of the following conditions

1) $(\vec{U}(P),\ \vec{a}\times\vec{b})>0$;

2) $\dfrac{\vec{b}}{\|\vec{b}\|}=\mathcal{R}_\theta\left(\dfrac{\vec{\dashv}}{\|\vec{\dashv}\|}\right)$ for $0<\theta<\pi$,

where $\mathcal{R}_\theta$ is the positive rotation of angle θ în T_PM.

13. Let be the surface $M: z=f(x,y)$, oriented by the choice of the unit normal vector field

$$\vec{U}=\frac{-f_x\vec{i}-f_y\vec{j}+\vec{k}}{\sqrt{1+f_x^2+f_y^2}}.$$

We assume that $f(0,0)=f_x(0,0)=f_y(0,0)=0$.

1) Show that $\vec{i}$ and $\vec{j}$ are vectors tangent to M at $(0,0,0)$.

2) Show that

$$S(\vec{i})=f_{x^2}(0,0)\vec{i}+f_{xy}(0,0)\vec{j}$$
$$S(\vec{j})=f_{xy}(0,0)\vec{i}+f_{y^2}(0,0)\vec{j}.$$

14. Let be the surface $M=f^{-1}(1)$, where $f(x,y,z)=-x^2+y^2+z^2$. Provide an orientation for the surface M. Find the normal curvature of the surface at its point $P(0,0,1)$, in the direction $\vec{a}$, where $\vec{a}$ is a unit vector, $\vec{a}\in T_PM$. Consider the particular cases:

1) $\vec{a}=\vec{i}=(1,0,0)$;

2) $\vec{a}=\vec{j}=(0,1,0)$;

3) $\vec{a}=\left(\dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}},\ 0\right)$.

15. Let k_1,k_2 be the principal curvatures of the surface S which separates a liquid. The normal pressure p, on the surface element at an arbitrary point is given by the *Laplace equation*

$$\sigma(k_1+k_2)=p,$$

where σ is the superficial tension, considered to be constant. Find the pressure p, when S is:

1) *Enneper's surface*

$$x=\mathrm{Re}\int(1-t^2)dt,\quad y=\mathrm{Im}\int(1+t^2)dt,\quad z=\mathrm{Re}\int 2tdt,$$

$$t = u + iv, \quad i^2 = -1.$$

2) *The saddle* $z = xy$.

16. Let be the surfaces
1) $M : z = e^{x+y} - 1$;
2) $M : z = \ln \cos x - \ln \cos y$;
3) $M : z = (x + 3y)^3$.

Find the principal curvatures and the quadratic approximation of the surface M, around its point $(0, 0, 0)$.

17. Let be the Monge chart $r : x = u,\ y = v,\ z = f(u, v)$. Show that

$$E = 1 + f_u^2,\ F = f_u f_v,\ G = 1 + f_v^2,$$

$$l = \frac{f_{uu}}{W}, \quad m = \frac{f_{uv}}{W}, \quad n = \frac{f_{vv}}{W},$$

where $W = \sqrt{1 + f_u^2 + f_v^2}$.

18. Show that the following immersion (the *Enneper surface*)

$$r : \mathbb{R}^2 \to \mathbb{R}^3,\ r(u, v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2 \right),$$

is a minimal surface.

19. Let be the curve

$$\alpha(u) = (x(u),\ y(u)), \qquad \begin{cases} x(u) = \int\limits_0^u \sqrt{1 - e^{-2t}}\, dt \\ y(u) = e^{-u},\ u > 0. \end{cases}$$

and let M be the surface of revolution obtained by rotating α around the Ox axis, called *the pseudosphere* (see fig. 179). Show that M has the Gauss curvature $K = -1$.

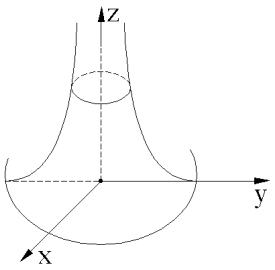


Fig. 179

20. A surface for which $K/d^4 = \text{const}$, where K is the Gauss curvature of the surface and d is the distance from a fixed point to the tangent plane of the surface at an arbitrary point of the surface, is called *Ţiţeica surface*. Show that $M : xyz - 1 = 0$ is a Ţiţeica surface.

21. Show that the Ţiţeica surface $M : xyz - 1 = 0$ is not connected.

22. Let β be a unit-speed curve having the curvature $k > 0$. The ruled surface

$$\bar{r}(u, v) = \bar{\beta}(u) + v\bar{T}(u),\ v > 0,$$

is called *the tangent surface* of β. Show that $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ and that the tangent surface of β is locally Euclidean.

23. Let be the immersion defined by

$$\begin{cases} x = \cos u \cos v \\ y = \cos u \sin v, \\ z = \sin u - \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{u}{2}\right) \end{cases}, \quad u \in \left(0, \frac{\pi}{2}\right),\ v \in \mathbb{R}.$$

Find the curvature lines and asymptotic lines of the surface

$$r\,(\,(0, \pi/2) \times \mathbb{R}\,)\,.$$

24. Let P be the center of the dot-shaped mass m, which moves on a surface with the acceleration $\vec{a}$ and let $\vec{N}$ the reaction (the normal vector at P to the surface). If the particle P is not subject to action of exterior forces, then show that P moves along a curve α, which is a geodesic of the surface.

25. Find the area of the surfaces:
1) $M : (x^2 + y^2 - 4)^2 + (z - 4)^2 = 1$.

2) $M : y^2 + z^2 - x^2 = 0,\ x^2 + y^2 \leq 1.$

26. Find the area of the ruled Möbius band:

$$\begin{cases} x = \cos u + v \cos u \cos \frac{u}{2} \\ y = \sin u + v \sin u \cos \frac{u}{2} \\ z = \sin \frac{u}{2} \end{cases}, \qquad u \in [0, 2\pi],\ v \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Chapter 8

MODELS OF SUBJECTS FOR EXAMS

1. 1) Paraboloids.
2) Compute the Gauss curvature of the surface $\Sigma : xyz = 1$.
3) Find the curvature and torsion of the curve

$$\alpha(t) = (t, t^2, \ln t),\ t > 0,$$

at its point $A(1, 1, 0)$.

2. 1) Conics (center, diameters).
2) Compute the Gauss curvature of the surface

$$r(u, v) = \left(u, v, \ln \frac{\cos u}{\cos v}\right),\ (u, v) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

at its point $A(1, 1, 0)$.
3) Determine the curvature and the torsion of the curve

$$\alpha(t) = (a \cos t, a \sin t, bt),\ t \in \mathbb{R}.$$

3. 1) Equations of a sphere.
2) Determine the points of the curve

$$\alpha(t) = (\ln t, t, t^2),\ t > 0,$$

at which the normal plane is parallel to the straight line

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$$

3) Establish the equation of the cylindric surface obtained by moving the straight line D on the director curve Γ, where

$$D : x = 1 - y = 2z \qquad \Gamma : \begin{cases} z = x \\ xyz = 1 \end{cases} .$$

4. 1) Tangent line and normal hyperplane.

2) Determine the canonic equation of the conic

$$\Gamma : 4xy + 3y^2 + 16x = 0.$$

3) Determine the fundamental forms of the surface

$$r(u, v) = (\cos u, \sin u, 2v),\ (u, v) \in \mathbb{R}^2.$$

5. 1) Frenet formulas for plane curves.

2) Find the canonic equation of the quadric $\Sigma : z = xy$.

3) Determine the equations of the normal straight line and of the tangent plane to the surface

$$r(u, v) = (u + v, u - v, uv),\ (u, v) \in \mathbb{R}^2$$

at its point $A(3, -1, 2)$. Find the Cartesian equation of the surface.

6. 1) Infinite branches of curves.

2) Find the canonic equation of the conic $\Gamma : x^2 - 4xy + 4y^2 - 6x = 0$.

3) Compute the mean curvature of the surface

$$r(u, v) = (\cos 2u, \sin 2u, 3v),\ (u, v) \in \mathbb{R}^2.$$

7. 1) Reduction of the equation of a quadric to the canonic form.

2) Determine the equations of the tangent plane and normal line to the surface $\Sigma : x^2 + y + e^z = 2$ at its point $A(1, 1, 0)$.

3) Find the osculating circle of the curve

$$\alpha(t) = (t, \ln t),\ t > 0,$$

at its point $A(1, 0)$.

8. 1) Surfaces of revolution.

2) We consider the family of planes

$$P : x + y + z + \lambda = 0,\ \lambda \in \mathbb{R}$$

and the quadric $\Sigma : x^2 + y^2 + z^2 + 2x = 0$. Determine $\lambda \in \mathbb{R}$ in each of the following cases:
a) P is tangent to Σ;
b) $P \cap \Sigma$ is a circle of radius $1/2$.

3) Determine the curvature and the torsion of the curve

$$\alpha(t) = (t^2 - t, t^2 + t, t^3),\ t \in \mathbb{R},$$

at its point $A(2, 0, -1)$.

9. 1) Conics: generalities, center, axes.

2) Determine the point of the curve

$$\alpha(t) = (2t - 1, t^3, 1 - t^2),\ t \in \mathbb{R},$$

at which the osculating plane is orthogonal to $P : x - y + z = 0$.

3) Compute the mean curvature of the surface $\Sigma : z = x^2 + y^2$.

10. 1) Ellipse and ellipsoid.

2) Compute the mean curvature of the surface

$$r(u, v) = \left(u - \frac{u^3}{3}, v - \frac{v^3}{3}, u^2 - v^2\right),\ (u, v) \in \mathbb{R}^2.$$

3) Determine the points of the curve

$$\alpha(t) = (\ln t, t, t^2),\ t > 0,$$

at which the osculating plane is orthogonal to the plane $P : x - y + z = 0$.

11. 1) Tangent plane and normal line.

2) Let be the plane $P : x - y = z$ and the sphere

$$\Sigma : x^2 + (y - 2)^2 + (z - 2)^2 = 9.$$

Does the plane P intersect the sphere Σ ? In the affirmative case, determine the center and the radius of the intersection circle.

3) Determine the equations of the edges and faces of the Frenet frame to the curve

$$\alpha(t) = (t, t^2, t^3),\ t \in \mathbb{R},$$

at its point $A(1, 1, 1)$.

12. 1) Shape of a plane curve in a neighborhood of its points.

2) Determine the canonic form of the equation of the quadric

$$\Sigma:\ 2y^2 - 7z^2 + 112x - 16y = 0.$$

3) Compute the Gauss curvature of the surface $\Sigma : z = xy$.

13. 1) Weingarten map.

2) Find the canonic equation of the conic

$$\Gamma : x^2 - 6xy + 9y^2 + 4x = 0.$$

3) Compute the curvature and the torsion of the curve

$$\alpha(t) = (t, t^2, t^3),\ t \in \mathbb{R},$$

at its point $A(1, 1, 1)$.

14. 1) Frenet formulas for spatial unit speed curves.

2) Find the equation of the surface obtained by rotating the circle

$$\Gamma : \begin{cases} (x-2)^2 + y^2 = 1 \\ z = 0 \end{cases}$$

about the axis Oy.

3) Determine the fundamental forms of the surface

$$r(u, v) = (\cos u, \sin u, 2v),\ (u, v) \in \mathbb{R}^2.$$

15. 1) Normal curvature.

2) Find the parametric equations of the surface of revolution obtained by rotating the curve $\Gamma : \begin{cases} x = 0 \\ z = y^2 \end{cases}$ about the axis Oy.

3) Determine the points of the curve

$$\alpha(t) = (\ln t, 2t, t^2),\ t > 0,$$

at which the principal normal is parallel to the plane $P : x + y + z = 0$.

16. 1) Theory of contact. Osculating straight line. Osculating circle.

2) Determine the equations of the tangent plane and of the normal line to the surface

$$r(u, v) = (u + \sin v, u - \sin v, v),\ (u, v) \in \mathbb{R}^2,$$

at the point $u = 1,\ v = \dfrac{\pi}{2}$.

3) Find the canonic equation of the conic

$$\Gamma : x^2 - 2xy + y^2 - 6x = 0.$$

17. 1) Pole and polar for a conic.

2) Find the torsion and the curvature of the curve

$$\alpha(t) = (t, t^2, t^3),\ t \in \mathbb{R},$$

at its point $A(-1, 1, -1)$.

3) Compute the Gauss curvature of the surface $\Sigma : z = xy + 1$.

18. 1) Cylinders as ruled surfaces.

2) Find the Frenet frame for the curve

$$\alpha(t) = (\cos t, \sin t, t),\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

at its point $A(1, 0, 0)$.

3) Find the sphere Σ_1 having the center $C(0, 0, 1)$ and which is tangent to the sphere

$$\Sigma_2 : (x - 1)^2 + y^2 + z^2 = 1.$$

Determine the equation of the common tangent plane.

19. 1) Hyperboloids.

2) Determine the fundamental forms of the surface

$$r(u, v) = (u \cos v, u \sin v, v),\ (u, v) \in \mathbb{R}^2.$$

3) Find the points of the curve

$$\alpha(t) = (\ln t, t, t^2),\ t > 0,$$

at which the Frenet frame reduces to $\{\vec{i}, \vec{i}+\vec{j}, \vec{k}\}$.

20. 1) Charts and surfaces.

2) Find the canonic equation of the conic

$$\Gamma : x^2 + 2xy + x + y = 0.$$

3) Find a parametrization of the curve $C : \begin{cases} xyz = 2 \\ z = x \end{cases}$. Determine the equations of the tangent line and of osculating plane to C at its point $A(-2, 1, -1)$.

21. 1) Curves.

2) Determine the fundamental forms of the surface

$$r(u, v) = (\cos 2u, \sin 2u, v),\ (u, v) \in \mathbb{R}^2.$$

3) Determine the center and the axes of the conic

$$\Gamma : x^2 - 4xy + 2x + 2y = 0.$$

22. 1) Normal representation of a curve.

2) Find the equation of the surface of revolution determined by rotating the straight line $D : x = y = z$ about the axis Oz.

3) Determine the canonic equation of the quadric $\Sigma :\ xz + y^2 - 1 = 0$. What does it represent ? Determine the equations of the normal at its point $A(1, 0, 1)$.

23. 1) Cones as ruled surfaces.

2) Find the canonic equation of the quadric $\Sigma :\ xy + z^2 - 2 = 0$. What does it represent ? Determine the equation of the tangent plane to Σ at its point $A(1, 2, 0)$.

3) Determine the shape of the curve

$$\alpha(t) = (t^3, t^4 - 2t^8),\ t \in \mathbb{R},$$

around the point $A(0, 0)$. Find the equations of the tangent and of the normal straight lines at the point A.

24. 1) Surfaces described by implicit Cartesian equations.

2) Find out if there exists a point M on the curve

$$\Gamma : \begin{cases} x = y^2 \\ z = y^3, \end{cases}$$

at which the normal plane is orthogonal to the straight line

$$D : \frac{x}{-2} = \frac{y}{1} = \frac{z}{3}.$$

3) Find the center and the asymptotes of the conic

$$\Gamma : 4xy - 3y^2 - 4x + 2y - 7 = 0.$$

25. 1) Reduction of the equation of a conic to the canonic form.

2) Find the parametric equations of the cone which passes through the curve

$$\Gamma : \begin{cases} x^2 + y^2 = 1 \\ z = 1, \end{cases}$$

and has the vertex at $A(0, 0, 2)$.

3) Determine the equations of the tangent plane and normal line to the surface

$$r(u, v) = (uv, u + v, v \ln u),\ (u, v) \in \mathbb{R}_+^* \times \mathbb{R},$$

at its point $A(1, 2, 0)$.

26. 1) Frenet formulas for spatial regular curves.

2) Find the canonic equation of the quadric

$$\Sigma : x^2 - y^2 + z^2 - 2xy + 5x = 0.$$

What does Σ represent ?

3) Determine the equations of the tangent plane and of the normal line to the surface

$$\Sigma : x^2 + y^2 + e^z = 1$$

at its point $M_0(1, 0, 0)$.

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