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S. Vacaru, P. Stavrinos, E. Gaburov and D. Gonţa

Clifford and RiemannFinsler Structures in Geometric Mechanics and Gravity

Selected Works

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Preface

The researches resulting in this massive book have been initiated by S. Vacaru fifteen years ago when he prepared a second Ph. Thesis in Mathematical Physics. Studying Finsler–Lagrange geometries he became aware of the potential applications of these geometries in exploring nonlinear aspects and nontrivial symmetries arising in various models of gravity, classical and quantum field theory and geometric mechanics.

Along years he convinced many people to enroll in solving some of his open problems and especially he attracted young students to specialize in this field. Some of his collaborators are among the co–authors of this book.

The book contains a collection of works on Riemann–Cartan and metric-affine manifolds provided with nonlinear connection structure and on generalized Finsler–Lagrange and Cartan–Hamilton geometries and Clifford structures modelled on such manifolds.

The authors develop and use the method of anholonomic frames with associated nonlinear connection structure and apply it to a great number of concrete problems: constructing of generic off–diagonal exact solution, in general, with nontrivial torsion and nonmetricity, possessing noncommutative symmetries and describing black ellipsoid/ torus configurations, locally anisotropic wormholes, gravitational solitons and warped factors and investigation of stability of such solutions; classification of Lagrange/Finsler affine spaces; definition of nonholonomic Dirac operators and their applications in commutative and noncommutative Finsler geometry.

This collection of works enriches very much the literature on generalized Finsler spaces and opens new ways toward applications by proposing new geometric approaches in gravity, string theory, quantum deformations and noncommutative models.

The book is extremely useful for the researchers in Differential Geometry and Mathematical Physics.

February, 2006

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Foreword

The general aim of this Selection of Works is to outline the methods of Riemann–Finsler geometry and generalizations as an aid in exploring certain less known nonlinear aspects and nontrivial symmetries of field equations defined by nonholonomic and noncommutative structures arising in various models of gravity, classical and quantum field theory and geometric mechanics. Accordingly, we move primarily in the realm of the geometry of nonholonomic manifolds for which the tangent bundles are provided with nonintegrable (anholonomic) distributions defining nonlinear connection (in brief, N–connection) structures. Such N–connections may be naturally associated to certain general off–diagonal metric terms and distinguish some preferred classes of adapted local frames and linear connections. This amounts to a program of unification when the Riemann–Cartan, Finsler–Lagrange spaces and various generalizations are commonly described by the corresponding geometric objects on N–anholonomic manifolds.

Our purposes and main concern are to illuminate common aspects in spinor differential geometry, gravity and geometric mechanics from the viewpoint of N–connection geometry and methods elaborated in investigating Finsler–Lagrange and related metric–affine spaces (in general, with nontrivial torsion and nonmetricity), to elaborate a corresponding language and techniques of nonholonomic deformations of geometric structures with various types of commutative and noncommutative symmetries and to benefit physicists interested in more application of advanced commutative and noncommutative geometric methods.

The guiding principle of the selected here works has been to show that the concept of N–anholonomic space seeks in roots when different type of geometries can be modelled by certain parametrizations of the N–connection structure and correspondingly adapted linear connection and metric structures. For instance, a class of such objects results in (pseudo) Riemann spaces but with preferred systems of reference, other classes of objects give rise into models of Finsler (Lagrange) geometries with metric compatible, or noncompatible, linear connections, all defined by the fundamental Finsler (regular Lagrange) functions and corresponding parametrizations and prescribed symmetries.

Despite a number of last decade works on research and applications of Finsler–Lagrange geometry by leading schools and prominent scholars in Romania (R. Miron, M. Anastasiei, A. Bejancu, ...), Japan (K. Matsumoto, S. Ikeda, H. Shimada,...), USA (S. S. Chern, S. Bao, Z. Shen, J. Vargas, R. G. Beil,...), Russia (G. Asanov, G. Yu. Bogoslovsky, ...), Germany (H. F. Goenner, K. Buchner, H. B. Rademacher,...), Canada (P. Antonelli, D. Hrimiuc,...), Hungary (L. Tamassy, S. Basco,...) and other Countries, there have not been yet obtained explicit results related to the phenomenology of Standard Model of particle physics, string theory, standard cosmological scenaria and astrophysics. The problem is that the main approaches and constructions in Finsler geometry and generalizations were elaborated in the bulk on tangent/ vector bundles and their higher generalizations. In this case all type of such locally anisotropic models are related to violations of the local Lorentz symmetry which is a fashion, for instance, in brane physics but, nevertheless, is subjected to substantial theoretical and experimental restrictions (J. D. Beckenstein and C. Will).

Our idea[1] was to define and work with Finsler (Lagrange) like geometric objects and structures not only on the tangent/vector bundles (and their higher order generalizations) but to model them on usual manifolds enabled with certain classes of nonholonomic distributions defined by exact sequences of subspaces of the tangent space to such manifolds. This way, for instance, we can model a Finsler geometry as a Riemann–Cartan manifold provided with certain types of N–connection and adapted linear connection and metric structures. The constructions have to be generalized for the metric–affine spaces provided with N–connection structure if there are considered the so–called Berwald–Moor or Chern connections for Finsler geometry, or, in an alternative way, one can be imposed such nonholonomic constrains on the frame structure when some subclasses of Finsler metrics are equivalently modelled on (pseudo) Riemann spaces provided with corresponding preferred systems of reference. This is possible for such configurations when the Ricci tensor for the so–called canonical distinguished connection in generalized Lagrange (or Finsler) space is constrained to be equal to the Ricci tensor for the Levi–Civita connection even the curvature tensors are different.

It was a very surprising result when a number of exact solutions modelling Finsler

[1]hereafter, in this Preface, we shall briefly outline the main ideas, concepts and results obtained during the last decade by a team of young researches in the Republic of Moldova (S. Vacaru, S. Ostaf, Yu. Goncharenko, E. Gaburov, D. Gonţa, N. Vicol, I. Chiosa ...) in collaboration (or having certain support) with some scientific groups and scholars in Romania, USA, Germany, Greece, Portugal and Spain (D. Singleton, H. Dehnen, P. Stavrinos, F. Etayo, B. Fauser, O. Ţintăreanu–Mircea, F. C. Popa, J. F. Gonzales–Hernandez, R. R. Santamaría and hosting by R. Miron, M. de Leon, M. Vişinescu, M. Anastasiei, T. Wolf, S. Anco, I. Gottlieb, C. Mociuţchi, B. Fauser, J. P. S. Lemos, L. Boya, P. Almeida, R. Picken, M. E. Gomez, M. Piso, M. Mars, L. Alías, C. Udrişte, D. Balan, V. Blanuţă, G. Zet, ...)

like structures were constructed in the Einstein and string gravity. Such solutions are defined by generic off–diagonal metrics, nonholonomic frames and linear connections (in general, with nontrivial torsion; examples of solutions with nontrivial nonmetricity were also constructed), when a subset of variables are holonomic and the subset of the rest ones are nonholonomic. That was an explicit proof that effective local anisotropies can be induced by off–diagonal metric terms and/or from extra dimensions. In a particular case, the locally anisotropic configurations can be modelled as exact solutions of the vacuum or nonvacuum Einstein equations. Sure, such Einstein–Finsler/ generalized Lagrange metrics and related nonholnomic frame structures are not subjected to the existing experimental restrictions and theoretical considerations formulated for the Finsler models on tangent/vector bundles.

The new classes of exact solutions describe three, four, or five dimensional space–times (there are possibilities for extensions to higher dimensions) with generalized symmetries when the metric, connection and frame coefficients depend on certain integration functions on two/ three / four variables. They may possess noncommutative symmetries even for commutative gravity models, or any generalizations to Lie/ Clifford algebroid structures, and can be extended to stable configurations in complex gravity. [2] Here it is appropriate to emphasize that the proposed 'anholnomic frame method' of constructing exact solutions was derived by using explicit methods from the Finsler–Lagrange geometry. Perhaps, this is the most general method of constructing exact solutions in gravity: it was elaborated as a geometric method by using the N–connection formalism.

The above mentioned results derived by using moving frames and nonholonomic structures feature several fundamental constructions: 1) Any Finsler–Lagrange geometry can be equivalently realized as an effective Riemann–Cartan nonholonomic manifold and, inversely, 2) any space-time with generic off–diagonal metric and nonholonomic frame and affine connection structures can be equivalently nonholonomically deformed into various types of Finsler/ Lagrange geometries. 3) As a matter of principle, realizing a Finsler configuration as a Riemannian nonholonomic manifold (with nonholonomically induced torsion), we may assemble this construction from the ingredients of noncommutative spin geometry, in the A. Connes approach, or we can formulate a noncommutative gauge–Finsler geometry via the Seiberg–Witten transform.

The differential geometry of N–anholonomic spinors and related Clifford structures provided with N–connections predated the results on 'nonholonomic' gravity and related classes of exact solutions. The first and second important results were, respectively, the

[2] The bibliography presented for the Introduction and at the end of Chapters contains exact citations of our works on anholonomic black ellipsoid/torus and disk solutions, locally anisotropic wormholes and Taub NUT spaces, nonholonomic Einstein–Dirac wave solitons, locally anisotropic cosmological solutions, warped configurations or with Lie/Clifford algebroid and/or noncommutative symmetries, ...

possibility to give a rigorous definition for spinors in Finsler spaces and elaboration of the concept of Finsler superspaces. The third result was the classification of such spaces in terms of nearly autoparallel maps (generalizing the classes of geodesic maps and conformal transforms) and their basic equations and invariants suggesting variants of definition of conservation lows for such (super) spaces. The forth such a fundamental result was a nontrivial proof that Finsler like structures can be derived in low energy limits of (super) string theory if the (super) frames with associated N–connection structure are introduced into consideration. There were obtained a set of results in the theory of locally anisotropic stochastic, kinetic and thermodynamic processes in generalized curved spacetimes. Finally, we mention here the constructions when from a regular Lagrange (Finsler) fundamental functions one derived canonically a corresponding Clifford/spinor structure which in its turn induces canonical noncommutative Lagrange (Finsler) geometries, nonholonomic Fedosov manifolds and generalized Lagrange (Finsler) Lie/Clifford algebroid structures.

The ideas that we can deal in a unified form, by applying the N–connection formalism, with various types of nonholnomic Riemann–Cartan–Weyl and generalized Finsler–Lagrange or Cartan–Hamilton spaces scan several new directions in modern geometry and physics: We hope that they will appeal researches (we also try to contribute explicitly in our works) in investigating nonholonomic Hopf strutures, N–anholonomic gerbes and noncommutative/ algebroid extensions, Atyiah–Singer theorems for Clifford–Lagrange spaces and in constructing exact solutions with nontrivial topological structure modelling Finsler gerbes, Ricci and Finsler–Lagrange fluxes, and applications in gravity and string theory, analogous modelling of gravity and gauge interactions and geometric mechanics.

Acknowledgments

The selection of works reflects by 15 years of authors' researches on generalized Finsler geometry and applications in modern physics. The most pleasant aspect in finishing this Foreword is having the opportunity to thank the many people (their names are given in footnote 1) who helped to perform this work, provided substantial support, collaborated and improved the results and suggested new very important ideas.

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Introduction

This collection of works grew out from explicit constructions proving that the Finsler and Lagrange geometries can be modelled as certain type nonintegrable distributions on Riemann–Cartan manifolds if the metric and connection structures on such spaces are compatible[3]. This is a rather surprising fact because the standard approaches were based on the idea that the Finsler geometry is more **general** then the Riemannian one when, roughly speaking, the metric anisotropically depends on "velocity" and the geometrical and physical models are elaborated on the tangent bundle. Much confusing may be made from such a generalization if one does not pay a due attention to the second fundamental geometric structure for the Finsler spaces called the nonlinear connection (N–connection), being defined by a nonholonomic distribution on the tangent bundle and related to a corresponding class of preferred systems of reference. There is the third fundamental geometric object, the linear connection, which for the Finsler like geometries is usually adapted to the N–connection structure.

In the former (let us say standard) approach, the Finsler and Lagrange spaces (the second class of spaces are derived similarly to the Finsler ones but for regular Lagrangians) are assembled from the mentioned three fundamental objects (the metric, N–connection and linear connection) defined in certain adapted forms on the tangent bundles provided with a canonical nonholonomic splitting into horizontal and vertical subspaces (stated by the exact sequence just defining the N–connection).

[3]one has to consider metric–affine spaces with nontrivial torsion and nonmetricity fields defined by the N–connection and adapted linear connection and metric structures if we work, for instance, with the metric noncompatible Berwald and Chern connections (they can be defined both in Finsler and Lagrange geometries)

As a matter of principle, we may consider that certain exact sequences and related nonintegrable distributions, also defining a N–connection structure, are prescribed, for instance, for a class of Riemann–Cartan manifolds. In this case, we work with sets of mixed holonomic coordinates (corresponding to the horizontal coordinates on the tangent bundle) and anholonomic[4] coordinates (corresponding to the vertical coordinates). The splitting into holonomic–anholonomic local coordinates and the corresponding conventional horizontal–vertical decomposition are globally stated by the N–connection structure as in the standard approaches to Finsler geometry. We are free to consider that a fibered structure is some way established as a generalized symmetry by a prescribed nonholonomic distribution defined for a usual manifold and not for a tangent or vector bundle.

There is a proof that for any vector bundles over paracompact manifolds the N–connection structure always exists, see Ref. [38]. On general manifolds this does not hold true but we can restrict our considerations to such Riemann–Cartan (or metric–affine) spaces when the metric structure[5] is someway related via nontrivial off–diagonal metric coefficients to the coefficients of the N–connection and associated nonholonomic frame structure. This way we can model Finsler like geometries on nonholonomic manifolds when the anisotropies depend on anholonomic coordinates (playing the role of "velocities" if to compare with the standard approaches to the Finsler geometry and generalizations). It is possible the case when such generic off–diagonal metric and N–connection and the linear connection structures are subjected to the condition to satisfy a variant of gravitational filed equations in Einstein–Cartan or string gravity. Any such solution is described by a Finsler like gravitational configuration which for corresponding constraints on the frame structure and distributions of matter defines a nonholonomic Einstein space.

0.1 For Experts in Differential Geometry and Gravity

The aim of this section is to present a self–contained treatment of generalized Finsler strucures modelled on Riemann–Cartan spaces provided with N–connections (we follow Chapters 2 and 3 in Ref. [80], see also details on modelling Lagrange and Finsler geometries on metric–affine spaces presented in Part I of this book).

[4]in literature, one introduced two equivalent terms: nonholonomic or anholonomic; we shall use both terms

[5]the corresponding metric tensor can not be diagonalized by any coordinate transforms

0.1.1 N–anholonomic manifolds

We formulate a coordinate free introduction into the geometry of nonholonomic manifolds. The reader may consult details in Refs. [62, 83, 78, 23]. Some important component/coordinate formulas are given in the next section.

Nonlinear connection structures

Let $\mathbf{V}$ be a smooth manifold of dimension $(n+m)$ with a local splitting in any point $u \in \mathbf{V}$ of type $\mathbf{V}_u = M_u \oplus V_u$, where M is a n–dimensional subspace and V is a m–dimensional subspace. It is supposed that one exists such a local decomposition when $\mathbf{V} \rightarrow M$ is a surjective submersion. Two important particular cases are that of a vector bundle, when we shall write $\mathbf{V} = \mathbf{E}$ (with $\mathbf{E}$ being the total space of a vector bundle $\pi : \mathbf{E} \rightarrow M$ with the base space M) and that of tangent bundle when we shall consider $\mathbf{V} = \mathbf{TM}$. The differential of a map $\pi : \mathbf{V} \rightarrow M$ defined by fiber preserving morphisms of the tangent bundles $T\mathbf{V}$ and TM is denoted by $\pi^\top : T\mathbf{V} \rightarrow TM$. The kernel of $\pi^\top$ defines the vertical subspace $v\mathbf{V}$ with a related inclusion mapping $i : v\mathbf{V} \rightarrow T\mathbf{V}$.

Definition 0.1.1. *A nonlinear connection (N–connection)* $\mathbf{N}$ *on a manifold* $\mathbf{V}$ *is defined by the splitting on the left of an exact sequence*

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0, \qquad (1)$$

i. e. by a morphism of submanifolds $\mathbf{N} : \ T\mathbf{V} \rightarrow v\mathbf{V}$ *such that* $\mathbf{N} \circ \mathbf{i}$ *is the unity in* $v\mathbf{V}$.

The exact sequence (1) states a nonintegrable (nonholonomic, equivalently, anholonomic) distribution on $\mathbf{V}$, i.e. this manifold is nonholonomic. We can say that a N–connection is defined by a global splitting into conventional horizontal (h) subspace, $(h\mathbf{V})$, and vertical (v) subspace, $(v\mathbf{V})$, corresponding to the Whitney sum

$$T\mathbf{V} = h\mathbf{V} \oplus_N v\mathbf{V} \qquad (2)$$

where $h\mathbf{V}$ is isomorphic to M. We put the label N to the symbol $\oplus$ in order to emphasize that such a splitting is associated to a N–connection structure.

For convenience, in the next Section, we give some important local formulas (see, for instance, the local representation for a N–connection (13)) for the basic geometric objects and formulas on spaces provided with N–connection structure. Here, we note that the concept of N–connection came from E. Cartan's works on Finsler geometry [17] (see a detailed historical study in Refs. [38, 23, 67] and alternative approaches developed by using the Ehressmann connection [22, 29]). Any manifold admitting an exact sequence of type (1) admits a N–connection structure. If $\mathbf{V} = \mathbf{E}$, a N–connection exists for any vector bundle $\mathbf{E}$ over a paracompact manifold M, see proof in Ref. [38].

The geometric objects on spaces provided with N–connection structure are denoted by "bolfaced" symbols. Such objects may be defined in "N–adapted" form by considering h– and v–decompositions (2). Following the conventions from [38, 54, 87, 67], one call such objects to be d–objects (i. e. they are distinguished by the N–connection; one considers d–vectors, d–forms, d–tensors, d–spinors, d–connections, ...). For instance, a d–vector is an element $\mathbf{X}$ of the module of the vector fields $\chi(\mathbf{V})$ on $\mathbf{V}$, which in N–adapted form may be written

$$\mathbf{X} = h\mathbf{X} + v\mathbf{X} \text{ or } \mathbf{X} = X \oplus_N \ {}^{\bullet}X,$$

where $h\mathbf{X}$ (equivalently, X) is the h–component and $v\mathbf{X}$ (equivalently, ${}^{\bullet}X$) is the v–component of $\mathbf{X}$.

A N–connection is characterized by its **N–connection curvature** (the Nijenhuis tensor)

$$\Omega(\mathbf{X}, \mathbf{Y}) \doteqdot [\ {}^{\bullet}X,\ {}^{\bullet}Y] + \ {}^{\bullet}[\mathbf{X}, \mathbf{Y}] - \ {}^{\bullet}[\ {}^{\bullet}X, \mathbf{Y}] - \ {}^{\bullet}[\mathbf{X}, {}^{\bullet}Y] \tag{3}$$

for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$, where $[\mathbf{X}, \mathbf{Y}] \doteqdot \mathbf{XY} - \mathbf{YX}$ and ${}^{\bullet}[,]$ is the v–projection of $[,]$, see also the coordinate formula (14) in section 0.2. This d–object Ω was introduced in Ref. [27] in order to define the curvature of a nonlinear connection in the tangent bundle over a smooth manifold. But this can be extended for any nonholonomic manifold, nonholnomic Clifford structure and any noncommutative / supersymmetric versions of bundle spaces provided with N–connection structure, i. e. with nonintegrable distributions of type (2), see [23, 67, 89].

Proposition 0.1.1. *A N–connection structure on* $\mathbf{V}$ *defines a nonholonomic N–adapted frame (vielbein) structure* $\mathbf{e} = (e, {}^{\bullet}e)$ *and its dual* $\widetilde{\mathbf{e}} = (\widetilde{e},\ {}^{\bullet}\widetilde{e})$ *with* e *and* ${}^{\bullet}\widetilde{e}$ *linearly depending on N–connection coefficients.*

Proof. It follows from explicit local constructions, see formulas (16), (15) and (17). □

Definition 0.1.2. *A manifold* $\mathbf{V}$ *is called N–anholonomic if it is defined a local (in general, nonintegrable) distribution (2) on its tangent space* $T\mathbf{V}$, *i.e.* $\mathbf{V}$ *is N–anholonomic if it is enabled with a N–connection structure (1).*

Curvatures and torsions of N–anholonomic manifolds

One can be defined N–adapted linear connection and metric structures on $\mathbf{V}$:

Definition 0.1.3. *A distinguished connection (d–connection)* $\mathbf{D}$ *on a N–anholonomic manifold* $\mathbf{V}$ *is a linear connection conserving under parallelism the Whitney sum (2). For any* $\mathbf{X} \in \chi(\mathbf{V})$, *one have a decomposition into h– and v–covariant derivatives,*

$$\mathbf{D}_{\mathbf{X}} \doteqdot \mathbf{X} \rfloor \mathbf{D} = X \rfloor \mathbf{D} + {}^{\bullet}X \rfloor \mathbf{D} = D_X + {}^{\bullet}D_X. \quad (4)$$

The symbol "$\rfloor$" in (4) denotes the interior product. We shall write conventionally that $\mathbf{D} = (D, {}^{\bullet}D)$.

For any d–connection $\mathbf{D}$ on a N–anholonomic manifold $\mathbf{V}$, it is possible to define the curvature and torsion tensor in usual form but adapted to the Whitney sum (2):

Definition 0.1.4. *The torsion*

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteqdot \mathbf{D}_{\mathbf{X}}\mathbf{Y} - \mathbf{D}_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] \quad (5)$$

of a d–connection $\mathbf{D} = (D, {}^{\bullet}D)$, *for any* $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$, *has a N–adapted decomposition*

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) = \mathbf{T}(X, Y) + \mathbf{T}(X, {}^{\bullet}Y) + \mathbf{T}({}^{\bullet}X, Y) + \mathbf{T}({}^{\bullet}X, {}^{\bullet}Y). \quad (6)$$

By further h- and v–projections of (6), denoting $h\mathbf{T} \doteqdot T$ and $v\mathbf{T} \doteqdot {}^{\bullet}T$, taking in the account that $h[{}^{\bullet}X, {}^{\bullet}Y] = 0$, one proves

Theorem 0.1.1. *The torsion of a d–connection* $\mathbf{D} = (D, {}^{\bullet}D)$ *is defined by five nontrivial d–torsion fields adapted to the h– and v–splitting by the N–connection structure*

$$\begin{aligned} T(X,Y) &\doteqdot D_X Y - D_Y X - h[X,Y], \quad {}^{\bullet}T(X,Y) \doteqdot {}^{\bullet}[Y,X], \\ T(X, {}^{\bullet}Y) &\doteqdot -{}^{\bullet}D_Y X - h[X, {}^{\bullet}Y], \; {}^{\bullet}T(X, {}^{\bullet}Y) \doteqdot {}^{\bullet}D_X Y - {}^{\bullet}[X, {}^{\bullet}Y], \\ {}^{\bullet}T({}^{\bullet}X, {}^{\bullet}Y) &\doteqdot {}^{\bullet}D_X {}^{\bullet}Y - {}^{\bullet}D_Y {}^{\bullet}X - {}^{\bullet}[{}^{\bullet}X, {}^{\bullet}Y]. \end{aligned}$$

The d–torsions $T(X,Y)$, ${}^{\bullet}T({}^{\bullet}X, {}^{\bullet}Y)$ are called respectively the $h(hh)$–torsion, $v(vv)$–torsion and so on. The formulas (26) present a local proof of this Theorem.

Definition 0.1.5. *The curvature of a d–connection* $\mathbf{D} = (D, {}^{\bullet}D)$ *is defined*

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \doteqdot \mathbf{D}_{\mathbf{X}}\mathbf{D}_{\mathbf{Y}} - \mathbf{D}_{\mathbf{Y}}\mathbf{D}_{\mathbf{X}} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]} \quad (7)$$

for any $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$.

Denoting $h\mathbf{R} = R$ and $v\mathbf{R} = {}^{\bullet}R$, by straightforward calculations, one check the properties

$$\begin{aligned} R(\mathbf{X}, \mathbf{Y}) {}^{\bullet}Z &= 0, \; {}^{\bullet}R(\mathbf{X}, \mathbf{Y}) Z = 0, \\ \mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} &= R(\mathbf{X}, \mathbf{Y}) Z + {}^{\bullet}R(\mathbf{X}, \mathbf{Y}) {}^{\bullet}Z \end{aligned}$$

for any for any $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \chi(\mathbf{V})$.

Theorem 0.1.2. *The curvature* $\mathbf{R}$ *of a d–connection* $\mathbf{D} =(D,\ {}^{\bullet}D)$ *is completely defined by six d–curvatures*

$$\begin{aligned}
\mathbf{R}(X,Y)Z &= \left(D_X D_Y - D_Y D_X - D_{[X,Y]} - {}^{\bullet}D_{[X,Y]}\right) Z, \\
\mathbf{R}(X,Y)\ {}^{\bullet}Z &= \left(D_X D_Y - D_Y D_X - D_{[X,Y]} - {}^{\bullet}D_{[X,Y]}\right)\ {}^{\bullet}Z, \\
\mathbf{R}(\ {}^{\bullet}X,Y)Z &= \left(\ {}^{\bullet}D_X D_Y - D_Y\ {}^{\bullet}D_X - D_{[\ {}^{\bullet}X,Y]} - {}^{\bullet}D_{[\ {}^{\bullet}X,Y]}\right) Z, \\
\mathbf{R}(\ {}^{\bullet}X,Y)\ {}^{\bullet}Z &= \left(\ {}^{\bullet}D_X\ {}^{\bullet}D_Y - {}^{\bullet}D_Y\ {}^{\bullet}D_X - D_{[\ {}^{\bullet}X,Y]} - {}^{\bullet}D_{[\ {}^{\bullet}X,Y]}\right)\ {}^{\bullet}Z, \\
\mathbf{R}(\ {}^{\bullet}X,\ {}^{\bullet}Y)Z &= \left(\ {}^{\bullet}D_X D_Y - D_Y\ {}^{\bullet}D_X - {}^{\bullet}D_{[\ {}^{\bullet}X,\ {}^{\bullet}Y]}\right) Z, \\
\mathbf{R}(\ {}^{\bullet}X,\ {}^{\bullet}Y)\ {}^{\bullet}Z &= \left(\ {}^{\bullet}D_X D_Y - D_Y\ {}^{\bullet}D_X - {}^{\bullet}D_{[\ {}^{\bullet}X,\ {}^{\bullet}Y]}\right)\ {}^{\bullet}Z.
\end{aligned}$$

The proof of Theorems 0.1.1 and 0.1.2 is given for vector bundles provided with N–connection structure in Ref. [38]. Similar Theorems and respective proofs hold true for superbundles [57], for noncommutative projective modules [67] and for N–anholonomic metric–affine spaces [78], where there are also give the main formulas in abstract coordinate form. The formulas (31) consist a coordinate proof of Theorem 0.1.2.

Definition 0.1.6. *A metric structure* $\breve{g}$ *on a N–anholonomic space* $\mathbf{V}$ *is a symmetric covariant second rank tensor field which is not degenerated and of constant signature in any point* $\mathbf{u} \in \mathbf{V}$.

In general, a metric structure is not adapted to a N–connection structure.

Definition 0.1.7. *A d–metric* $\mathbf{g} = g \oplus_N\ {}^{\bullet}g$ *is a usual metric tensor which contracted to a d–vector results in a dual d–vector, d–covector (the duality being defined by the inverse of this metric tensor).*

The relation between arbitrary metric structures and d–metrics is established by

Theorem 0.1.3. *Any metric* $\breve{g}$ *can be equivalently transformed into a d–metric*

$$\mathbf{g} = g(X,Y) + \ {}^{\bullet}g(\ {}^{\bullet}X,\ {}^{\bullet}Y) \tag{8}$$

for a corresponding N–connection structure.

Proof. We introduce denotations $h\breve{g}(X,Y) \doteqdot g(X,Y)$ and $v\breve{g}(\ {}^{\bullet}X,\ {}^{\bullet}Y) = \ {}^{\bullet}g(\ {}^{\bullet}X,\ {}^{\bullet}Y)$ and try to find a N–connection when

$$\breve{g}(X,\ {}^{\bullet}Y) = 0 \tag{9}$$

for any $\mathbf{X},\mathbf{Y} \in \chi(\mathbf{V})$. In local form, the equation (9) is just an algebraic equation for $\mathbf{N} = \{N_i^a\}$, see formulas (18), (19) and (20) and related explanations in section 0.2. □

Definition 0.1.8. *A d–connection* $\mathbf{D}$ *on* $\mathbf{V}$ *is said to be metric, i.e. it satisfies the metric compatibility (equivalently, metricity) conditions with a metric* $\breve{g}$ *and its equivalent d–metric* $\mathbf{g}$, *if there are satisfied the conditions*

$$\mathbf{D_X g} = \mathbf{0}. \tag{10}$$

Considering explicit h– and v–projecting of (10), one proves

Proposition 0.1.2. *A d–connection* $\mathbf{D}$ *on* $\mathbf{V}$ *is metric if and only if*

$$D_X g = 0,\ D_X\ {}^{\bullet}g = 0,\ {}^{\bullet}D_X g = 0,\ {}^{\bullet}D_X\ {}^{\bullet}g = 0.$$

One holds this important

Conclusion 0.1.1. *Following Propositions 0.1.1 and 0.1.2 and Theorem 0.1.3, we can elaborate the geometric constructions on a N–anholonomic manifold* $\mathbf{V}$ *in N–adapted form by considering N–adapted frames* $\mathbf{e} = (e, {}^{\bullet}e)$ *and co–frames* $\widetilde{\mathbf{e}} = (\widetilde{e},\ {}^{\bullet}\widetilde{e})$, *d–connection* $\mathbf{D}$ *and d–metric* $\mathbf{g} = [g,\ {}^{\bullet}g]$ *fields.*

In Riemannian geometry, there is a preferred linear Levi–Civita connection $\bigtriangledown$ which is metric compatible and torsionless, i.e.

$${}^{\bigtriangledown}\mathbf{T}(\mathbf{X},\mathbf{Y}) \doteqdot \bigtriangledown_{\mathbf{X}} \mathbf{Y} - \bigtriangledown_{\mathbf{Y}} \mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0,$$

and defined by the metric structure. On a general N–anholonomic manifold $\mathbf{V}$ provided with a d–metric structure $\mathbf{g} = [g,\ {}^{\bullet}g]$, the Levi–Civita connection defined by this metric is not adapted to the N–connection, i. e. to the splitting (2). The h– and v–distributions are nonintegrable ones and any d–connection adapted to a such splitting contains nontrivial d–torsion coefficients. Nevertheless, one exists a minimal extension of the Levi–Civita connection to a canonical d–connection which is defined only by a metric $\breve{g}$.

Theorem 0.1.4. *For any d–metric* $\mathbf{g} = [g,\ {}^{\bullet}g]$ *on a N–anholonomic manifold* $\mathbf{V}$, *there is a unique metric canonical d–connection* $\widehat{\mathbf{D}}$ *satisfying the conditions* $\widehat{\mathbf{D}}\mathbf{g} = 0$ *and with vanishing h(hh)–torsion, v(vv)–torsion, i. e.* $\widehat{T}(X,Y) = 0$ *and* ${}^{\bullet}\widehat{T}({}^{\bullet}X,\ {}^{\bullet}Y) = 0.$

Proof. The formulas (27) and (29) and related discussions state a proof, in component form, of this Theorem. □

The following Corollary gathers some basic information about N–anholonomic manifolds.

Corollary 0.1.1. *A N–connection structure defines three important geometric objects:*

1. *a (pseudo) Euclidean N–metric structure* $^{\eta}\mathbf{g} = \eta \oplus_N \ {}^{\bullet}\eta$, *i.e. a d–metric with (pseudo) Euclidean metric coefficients with respect to* $\widetilde{\mathbf{e}}$ *defined only by* $\mathbf{N}$;
2. *a N–metric canonical d–connection* $\widehat{\mathbf{D}}^N$ *defined only by* $^{\eta}\mathbf{g}$ *and* $\mathbf{N}$;
3. *a nonmetric Berwald type linear connection* $\mathbf{D}^B$.

Proof. Fixing a signature for the metric, $sign\ ^{\eta}\mathbf{g} = (\pm, \pm, ..., \pm)$, we introduce these values in (20) and get $^{\eta}\mathbf{g} = \eta \oplus_N \ {}^{\bullet}\eta$ of type (8), i.e. we prove the point 1. The point 2 is to be proved by an explicit construction by considering the coefficients of $^{\eta}\mathbf{g}$ into (29). This way, we get a canonical d–connection induced by the N–connection coefficients and satisfying the metricity conditions (10). In an approach to Finsler geometry [9], one emphasizes the constructions derived for the so–called Berwald type d–connection $\mathbf{D}^B$, considered to be the "most" minimal (linear on Ω) extension of the Levi–Civita connection, see formulas (30). Such d–connections can be defined for an arbitrary d–metric $\mathbf{g} = [g,\ {}^{\bullet}g]$, or for any $^{\eta}\mathbf{g} = \eta \oplus_N \ {}^{\bullet}\eta$. They are only "partially" metric because, for instance, $D^B g = 0$ and ${}^{\bullet}D^B\ {}^{\bullet}g = 0$ but, in general, $D^B\ {}^{\bullet}g \neq 0$ and ${}^{\bullet}D^B g \neq 0$, i. e. $\mathbf{D}^B\mathbf{g} \neq 0$, see Proposition 0.1.2. It is a more sophisticate problem to define spinors and supersymmetric physically valued models for such Finsler spaces, see discussions in [67, 71, 78]. □

Remark 0.1.1. *The d–connection* $\widehat{\mathbf{D}}^N$ *or* $\mathbf{D}^B$, *for* $^{\eta}\mathbf{g}$, *nonholonomic bases* $\mathbf{e} = (e, {}^{\bullet}e)$ *and* $\widetilde{\mathbf{e}} = (\widetilde{e},\ {}^{\bullet}\widetilde{e})$, *see Proposition 0.1.1 and the N–connection curvature* $\mathbf{\Omega}$ *(3), define completely the main properties of a N–anholonomic manifold* $\mathbf{V}$.

It is possible to extend the constructions for any additional d–metric and canonical d–connection structures. For our considerations on nonholnomic Clifford/ spinor structures, the class of metric d–connections plays a preferred role. That why we emphasize the physical importance of d–connections $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{D}}^N$ instead of $\mathbf{D}^B$ or any other nonmetric d–connections.

Finally, in this section, we note that the d–torsions and d–curvatures on N–anholonomic manifolds can be computed for any type of d–connection structure, see Theorems 0.1.1 and 0.1.2 and the component formulas (26) and (31).

0.1.2 Examples of N–anholonomic spaces:

For corresponding parametrizations of the N–connection, d–metric and d–connection coefficients of a N–anholonomic space, it is possible to model various classes of (generalized) Lagrange, Finsler and Riemann–Cartan spaces. We briefly analyze three such nonholonomic geometric structures.

Lagrange–Finsler geometry

This class of geometries is usually defined on tangent bundles [38] but it is possible to model such structures on general N–anholonomic manifolds, in particular in (pseudo) Riemannian and Riemann–Cartan geometry if nonholonomic frames are introduced into consideration [62, 83, 78, 73]. Let us outline the first approach when the N–anholonomic manifold $\mathbf{V}$ is taken to be just a tangent bundle (TM,π,M), where M is a n–dimensional base manifold, π is a surjective projection and TM is the total space. One denotes by $\widetilde{TM} = TM\backslash\{0\}$ where $\{0\}$ means the null section of map π.

We consider a differentiable fundamental Lagrange function $L(x,y)$ defined by a map $L : (x,y) \in TM \rightarrow L(x,y) \in \mathbb{R}$ of class $\mathcal{C}^\infty$ on $\widetilde{TM}$ and continuous on the null section $0 : M \rightarrow TM$ of π. The values $x = \{x^i\}$ are local coordinates on M and $(x,y) = (x^i, y^k)$ are local coordinates on TM. For simplicity, we consider this Lagrangian to be regular, i.e. with nondegenerated Hessian

$$ {}^L g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^i \partial y^j} \tag{11} $$

when $rank\,|g_{ij}| = n$ on $\widetilde{TM}$ and the left up "L" is an abstract label pointing that the values are defined by the Lagrangian L.

Definition 0.1.9. *A Lagrange space is a pair $L^n = [M, L(x,y)]$ with ${}^L g_{ij}(x,y)$ being of constant signature over $\widetilde{TM}$.*

The notion of Lagrange space was introduced by J. Kern [28] and elaborated in details in Ref. [38] as a natural extension of Finsler geometry.

Theorem 0.1.5. *There are canonical N–connection ${}^L\mathbf{N}$, almost complex ${}^L\mathbf{F}$, d–metric ${}^L\mathbf{g}$ and d–connection ${}^L\widehat{\mathbf{D}}$ structures defined by a regular Lagrangian $L(x,y)$ and its Hessian ${}^L g_{ij}(x,y)$ (11).*

Proof. The simplest way to prove this theorem is to take do this in local form (using formulas (35) and (37)) and then to globalize the constructions. The canonical ${}^L\mathbf{N}$ is defined by certain nonlinear spray configurations related to the solutions of Euler–Lagrange equations, see formula (35). It is given there the explicit matrix representation of ${}^L\mathbf{F}$ (36) which is a usual definition of almost complex structure, after ${}^L\mathbf{N}$ and N–adapted bases have been constructed. The d–metric (37) is a local formula for ${}^L\mathbf{g}$. Finally, the canonical d–connection ${}^L\widehat{\mathbf{D}}$ is a usual one but for ${}^L\mathbf{g}$ and ${}^L\mathbf{N}$ on $\widetilde{TM}$. □

A similar Theorem can be formulated and proved for the Finsler geometry:

Remark 0.1.2. *A Finsler space defined by a fundamental Finsler function* $F(x,y)$, *being homogeneous of type* $F(x,\lambda y) = \lambda F(x,y)$, *for nonzero* $\lambda \in \mathbb{R}$, *may be considered as a particular case of Lagrange geometry when* $L = F^2$.

From the Theorem 0.1.5 and Remark 0.1.2, one follows:

Result 0.1.1. *Any Lagrange mechanics with regular Lagrangian* $L(x,y)$ *(any Finsler geometry with fundamental function* $F(x,y)$*) can be modelled as a nonhlonomic Riemann–Cartan geometry with canonical structures* ${}^L\mathbf{N}$, ${}^L\mathbf{g}$ *and* ${}^L\widehat{\mathbf{D}}$ *(* ${}^F\mathbf{N}$, ${}^F\mathbf{g}$ *and* ${}^F\widehat{\mathbf{D}}$*) defined on a corresponding N–anholonomic manifold* $\mathbf{V}$.

It was concluded that any regular Lagrange mechanics/Finsler geometry can be geometrized/modelled as an almost Kähler space with canonical N–connection distribution, see [38] and, for N–anholonomic Fedosov manifolds, [23]. Such approaches based on almost complex structures are related with standard sympletic geometrizations of classical mechanics and field theory, for a review of results see Ref. [29].

For applications in optics of nonhomogeneous media [38] and gravity (see, for instance, Refs. [62, 83, 78, 71, 73]), one considers metrics of type $g_{ij} \sim e^{\lambda(x,y)}\ {}^Lg_{ij}(x,y)$ which can not be derived from a mechanical Lagrangian but from an effective "energy" function. In the so–called generalized Lagrange geometry, one introduced Sasaki type metrics (37), see section 0.2, with any general coefficients both for the metric and N–connection.

N–connections and gravity

Now we show how N–anholonomic configurations can defined in gravity theories. In this case, it is convenient to work on a general manifold $\mathbf{V}, \dim \mathbf{V} = n + m$ enabled with a global N–connection structure, instead of the tangent bundle $\widetilde{TM}$.

For the N–connection splitting of (pseudo) Riemannian–Cartan spaces of dimension $(n+m)$ (there were also considered (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modelled by N–anholonomic structures as exact solutions of gravitational field equations [78, 83, 69]. Inversely, all approaches to (super) string gravity theories deal with nontrivial torsion and (super) vielbein fields which under corresponding parametrizations model N–anholonomic spaces [57, 59, 89]. We summarize here some geometric properties of gravitational models with nontrivial N–anholonomic structure.

Definition 0.1.10. *A N–anholonomic Riemann–Cartan manifold* $^{RC}\mathbf{V}$ *is defined by a d–metric* $\mathbf{g}$ *and a metric d–connection* $\mathbf{D}$ *structures adapted to an exact sequence splitting (1) defined on this manifold.*

The d–metric structure $\mathbf{g}$ on $^{RC}\mathbf{V}$ is of type (8) and satisfies the metricity conditions (10). With respect to a local coordinate basis, the metric $\mathbf{g}$ is parametrized by a generic off–diagonal metric ansatz (19), see section 0.2. In a particular case, we can take $\mathbf{D} = \widehat{\mathbf{D}}$ and treat the torsion $\widehat{\mathbf{T}}$ as a nonholonomic frame effect induced by nonintegrable N–splitting. For more general applications, we have to consider additional torsion components, for instance, by the so–called H–field in string gravity.

Let us denote by $Ric(\mathbf{D})$ and $Sc(\mathbf{D})$, respectively, the Ricci tensor and curvature scalar defined by any metric d–connection $\mathbf{D}$ and d–metric $\mathbf{g}$ on $^{RC}\mathbf{V}$, see also the component formulas (32), (33) and (34) in Section 0.2. The Einstein equations are

$$En(\mathbf{D}) \doteqdot Ric(\mathbf{D}) - \frac{1}{2}\mathbf{g}Sc(\mathbf{D}) = \mathbf{\Upsilon} \tag{12}$$

where the source $\mathbf{\Upsilon}$ reflects any contributions of matter fields and corrections from, for instance, string/brane theories of gravity. In a closed physical model, the equation (12) have to be completed with equations for the matter fields, torsion contributions and so on (for instance, in the Einstein–Cartan theory one considers algebraic equations for the torsion and its source)... It should be noted here that because of nonholonomic structure of $^{RC}\mathbf{V}$, the tensor $Ric(\mathbf{D})$ is not symmetric and that $\mathbf{D}\left[En(\mathbf{D})\right] \neq 0$ which imposes a more sophisticate form of conservation laws on such spaces with generic "local anisotropy", see discussion in [78, 87] (this is similar with the case when the nonholonomic constraints in Lagrange mechanics modifies the definition of conservation laws). A very important class of models can be elaborated when $\mathbf{\Upsilon} = diag\left[\lambda^h(\mathbf{u})g, \lambda^v(\mathbf{u})\ ^{\bullet}g\right]$, which defines the so–called N–anholonomic Einstein spaces.

Result 0.1.2. *Various classes of vacuum and nonvacuum exact solutions of (12) parametrized by generic off–diagonal metrics, nonholonomic vielbeins and Levi–Civita or non–Riemannian connections in Einstein and extra dimension gravity models define explicit examples of N–anholonomic Einstein–Cartan (in particular, Einstein) spaces.*

Such exact solutions (for instance, with noncommutative, algebroid, toroidal, ellipsoid, ... symmetries) have been constructed in Refs. [62, 83, 23, 67, 69, 73, 71, 78, 87]. We note that a subclass of N–anholonomic Einstein spaces was related to generic off–diagonal solutions in general relativity by such nonholonomic constraints when $Ric(\widehat{\mathbf{D}}) = Ric(\bigtriangledown)$ even $\widehat{\mathbf{D}} \neq \bigtriangledown$, where $\widehat{\mathbf{D}}$ is the canonical d–connection and $\bigtriangledown$ is the Levi–Civita connection, see formulas (15.25) and (28) in section 0.2 and details in Ref. [73].

A direction in modern gravity is connected to analogous gravity models when certain gravitational effects and, for instance, black hole configurations are modelled by optical and acoustic media, see a recent review or results in [8]. Following our approach on geometric unification of gravity and Lagrange regular mechanics in terms of N–anholonomic spaces, one holds

Theorem 0.1.6. *A Lagrange (Finsler) space can be canonically modelled as an exact solution of the Einstein equations (12) on a N–anholonomic Riemann–Cartan space if and only if the canonical N–connection* ${}^{L}\mathbf{N}$ *(* ${}^{F}\mathbf{N}$*), d–metric* ${}^{L}\mathbf{g}$ *(* ${}^{F}\mathbf{g}$) *and d–connection* ${}^{L}\widehat{\mathbf{D}}$ *(* ${}^{F}\widehat{\mathbf{D}}$) *structures defined by the corresponding fundamental Lagrange function* $L(\mathbf{x},\mathbf{y})$ *(Finsler function* $F(\mathbf{x},\mathbf{y})$) *satisfy the gravitational field equations for certain physically reasonable sources* $\mathbf{\Upsilon}$.

Proof. We sketch the idea: It can be performed in local form by considering the Einstein tensor (34) defined by the ${}^{L}\mathbf{N}$ (${}^{F}\mathbf{N}$) in the form (35) and ${}^{L}\mathbf{g}$ (${}^{F}\mathbf{g}$) in the form (37) inducing the canonical d–connection ${}^{L}\widehat{\mathbf{D}}$ (${}^{F}\widehat{\mathbf{D}}$). For certain zero or nonzero $\mathbf{\Upsilon}$, such N–anholonomic configurations may be defined by exact solutions of the Einstein equations for a d–connection structure. A number of explicit examples were constructed for N–anholonomic Einstein spaces [62, 83, 23, 67, 69, 73, 71, 78, 87]. □

It should be noted that Theorem 0.1.6 states the explicit conditions when the Result 0.1.1 holds for N–anholonomic Einstein spaces.

Conclusion 0.1.2. *Generic off–diagonal metric and vielbein structures in gravity and regular Lagrange mechanics models can be geometrized in a unified form on N–anholonomic manifolds. In general, such spaces are not spin and this presents a strong motivation for elaborating the theory of nonholonomic gerbes and related Clifford/ spinor structures.*

Following this Conclusion, it is not surprising that a lot of gravitational effects (black hole configurations, collapse scenaria, cosmological anisotropies ....) can be modelled in nonlinear fluid, acoustic or optic media.

0.2 For Beginners in Riemann–Finsler Geometry

In this section, we outline some component formulas and equations defining the local geometry of N–anholonomic spaces, see details in Refs. [78, 67, 73, 38]. Elementary introductions on Riemann and Finsler geometry are contained in [41, 45].

Locally, a N–connection, see Definition 0.1.1, is stated by its coefficients $N_i^a(u)$,

$$\mathbf{N} = N_i^a(u)dx^i \otimes \partial_a \tag{13}$$

where the local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) are split in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where $i, j, k, \ldots = 1, 2, \ldots, n$ and $a, b, c, \ldots = n+1, n+2, \ldots, n+m$ when $\partial_i = \partial/\partial x^i$ and $\partial_a = \partial/\partial y^a$. The well known class of linear connections consists on a particular subclass with the coefficients being linear on y^a, i.e., $N_i^a(u) = \Gamma^a_{bj}(x) y^b$.

An explicit local calculus allows us to write the N–connection curvature (3) in the form

$$\mathbf{\Omega} = \frac{1}{2}\Omega^a_{ij} dx^i \wedge dx^j \otimes \partial_a,$$

with the N–connection curvature coefficients

$$\Omega^a_{ij} = \delta_{[j} N^a_{i]} = \delta_j N^a_i - \delta_i N^a_j = \partial_j N^a_i - \partial_i N^a_j + N^b_i \partial_b N^a_j - N^b_j \partial_b N^a_i. \quad (14)$$

Any N–connection $\mathbf{N} = N_i^a(u)$ induces a N–adapted frame (vielbein) structure

$$\mathbf{e}_\nu = \left(e_i = \partial_i - N^a_i(u)\partial_a, e_a = \partial_a\right), \quad (15)$$

and the dual frame (coframe) structure

$$\mathbf{e}^\mu = \left(e^i = dx^i, e^a = dy^a + N^a_i(u) dx^i\right). \quad (16)$$

The vielbeins (16) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W^\gamma_{\alpha\beta} \mathbf{e}_\gamma \quad (17)$$

with (antisymmetric) nontrivial anholonomy coefficients $W^b_{ia} = \partial_a N^b_i$ and $W^a_{ji} = \Omega^a_{ij}$.[6] These formulas present a local proof of Proposition 0.1.1 when

$$\mathbf{e} = \{\mathbf{e}_\nu\} = (\ e = \{e_i\}, {}^\bullet e = \{e_a\})$$

and

$$\widetilde{\mathbf{e}} = \{\mathbf{e}^\mu\} = \left(\widetilde{e} = \{e^i\},\ {}^\bullet\widetilde{e} = \{e^a\}\right).$$

Let us consider metric structure

$$\breve{g} = \underline{g}_{\alpha\beta}(u)\, du^\alpha \otimes du^\beta \quad (18)$$

[6]One preserves a relation to our previous denotations [54, 59] if we consider that $\mathbf{e}_\nu = (e_i, e_a)$ and $\mathbf{e}^\mu = (e^i, e^a)$ are, respectively, the former $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$ and $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$ when emphasize that operators (15) and (16) define, correspondingly, the "N–elongated" partial derivatives and differentials which are convenient for calculations on N–anholonomic manifolds.

defined with respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$ by coefficients

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \qquad (19)$$

Such a metric (19) is generic off–diagonal, i.e. it can not be diagonalized by any coordinate transforms if $N_i^a(u)$ are any general functions. The condition (9), for $X \rightarrow e_i$ and ${}^\bullet Y \rightarrow {}^\bullet e_a$, transform into

$$\breve{g}(e_i, {}^\bullet e_a) = 0, \text{ equivalently } \underline{g}_{ia} - N_i^b h_{ab} = 0$$

where $\underline{g}_{ia} \doteqdot g(\partial/\partial x^i, \partial/\partial y^a)$, which allows us to define in a unique form the coefficients $N_i^b = h^{ab}\underline{g}_{ia}$ where h^{ab} is inverse to h_{ab}. We can write the metric $\breve{g}$ with ansatz (19) in equivalent form, as a d–metric adapted to a N–connection structure, see Definition 0.1.7,

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\, \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u)\, e^i \otimes e^j + h_{ab}(u)\ {}^\bullet e^a \otimes {}^\bullet e^b, \qquad (20)$$

where $g_{ij} \doteqdot \mathbf{g}(e_i, e_j)$ and $h_{ab} \doteqdot \mathbf{g}({}^\bullet e_a, {}^\bullet e_b)$ and the vielbeins $\mathbf{e}_\alpha$ and $\mathbf{e}^\alpha$ are respectively of type (15) and (16).

We can say that the metric $\breve{g}$ (18) is equivalently transformed into (20) by performing a frame (vielbein) transform

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\ \underline{\alpha}} \partial_{\underline{\alpha}} \text{ and } \mathbf{e}^\beta = \mathbf{e}^\beta_{\ \underline{\beta}} du^{\underline{\beta}}.$$

with coefficients

$$\mathbf{e}_\alpha^{\ \underline{\alpha}}(u) = \begin{bmatrix} e_i^{\ \underline{i}}(u) & N_i^b(u) e_b^{\ \underline{a}}(u) \\ 0 & e_a^{\ \underline{a}}(u) \end{bmatrix}, \qquad (21)$$

$$\mathbf{e}^\beta_{\ \underline{\beta}}(u) = \begin{bmatrix} e^i_{\ \underline{i}}(u) & -N_k^b(u) e^k_{\ \underline{i}}(u) \\ 0 & e^a_{\ \underline{a}}(u) \end{bmatrix}, \qquad (22)$$

being linear on N_i^a. We can consider that a N–anholonomic manifold $\mathbf{V}$ provided with metric structure $\breve{g}$ (18) (equivalently, with d–metric (20)) is a special type of a manifold provided with a global splitting into conventional "horizontal" and "vertical" subspaces (2) induced by the "off–diagonal" terms $N_i^b(u)$ and a prescribed type of nonholonomic frame structure (17).

A d–connection, see Definition 0.1.3, splits into h– and v–covariant derivatives, $\mathbf{D} = D + {}^\bullet D$, where $D_k = \left(L^i_{jk}, L^a_{bk}\right)$ and ${}^\bullet D_c = \left(C^i_{jk}, C^a_{bc}\right)$ are correspondingly introduced as h- and v–parametrizations of (23),

$$L^i_{jk} = (\mathbf{D}_k e_j) \rfloor e^i, \quad L^a_{bk} = (\mathbf{D}_k e_b) \rfloor e^a, \; C^i_{jc} = (\mathbf{D}_c e_j) \rfloor e^i, \quad C^a_{bc} = (\mathbf{D}_c e_b) \rfloor e^a.$$

The components $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right)$ completely define a d–connection $\mathbf{D}$ on a N–anholonomic manifold $\mathbf{V}$.

The simplest way to perform a local covariant calculus by applying d–connections is to use N–adapted differential forms like $\mathbf{\Gamma}^{\alpha}_{\beta} = \mathbf{\Gamma}^{\alpha}_{\beta\gamma}\mathbf{e}^{\gamma}$ with the coefficients defined with respect to (16) and (15). One introduces the d–connection 1–form

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} = \mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}\mathbf{e}^{\gamma},$$

when the N–adapted components of d-connection $\mathbf{D}_{\alpha} = (\mathbf{e}_{\alpha}\rfloor\mathbf{D})$ are computed following formulas

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}(u) = (\mathbf{D}_{\alpha}\mathbf{e}_{\beta})\rfloor\mathbf{e}^{\gamma}, \tag{23}$$

where "$\rfloor$" denotes the interior product. This allows us to define in local form the torsion (5) $\mathbf{T} = \{\mathcal{T}^{\alpha}\}$, where

$$\mathcal{T}^{\alpha} \doteqdot \mathbf{D}\mathbf{e}^{\alpha} = d\mathbf{e}^{\alpha} + \Gamma^{\alpha}{}_{\beta} \wedge \mathbf{e}^{\beta} \tag{24}$$

and curvature (7) $\mathbf{R} = \{\mathcal{R}^{\alpha}{}_{\beta}\}$, where

$$\mathcal{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}_{\beta} = d\mathbf{\Gamma}^{\alpha}_{\beta} - \Gamma^{\gamma}{}_{\beta} \wedge \mathbf{\Gamma}^{\alpha}{}_{\gamma}. \tag{25}$$

The d–torsions components of a d–connection $\mathbf{D}$, see Theorem 0.1.1, are computed

$$\begin{aligned} T^{i}{}_{jk} &= L^{i}{}_{jk} - L^{i}{}_{kj},\ T^{i}{}_{ja} = -T^{i}{}_{aj} = C^{i}{}_{ja},\ T^{a}{}_{ji} = \Omega^{a}{}_{ji}, \\ T^{a}{}_{bi} &= T^{a}{}_{ib} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - L^{a}{}_{bi},\ T^{a}{}_{bc} = C^{a}{}_{bc} - C^{a}{}_{cb}. \end{aligned} \tag{26}$$

For instance, $T^{i}{}_{jk}$ and $T^{a}{}_{bc}$ are respectively the coefficients of the $h(hh)$–torsion $T(X,Y)$ and $v(vv)$–torsion $^{\bullet}T(^{\bullet}X, {}^{\bullet}Y)$.

The Levi–Civita linear connection $\bigtriangledown = \{^{\bigtriangledown}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}\}$, with vanishing both torsion and nonmetricity, $\bigtriangledown\breve{g} = 0$, is not adapted to the global splitting (2). There is a preferred, canonical d–connection structure, $\widehat{\mathbf{D}}$, on N–aholonomic manifold $\mathbf{V}$ constructed only from the metric and N–connection coefficients $[g_{ij}, h_{ab}, N^{a}_{i}]$ and satisfying the conditions $\widehat{\mathbf{D}}\mathbf{g} = 0$ and $\widehat{T}^{i}{}_{jk} = 0$ and $\widehat{T}^{a}{}_{bc} = 0$, see Theorem 0.1.4. By straightforward calculations with respect to the N–adapted bases (16) and (15), we can verify that the connection

$$\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = {}^{\bigtriangledown}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} + \widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} \tag{27}$$

with the deformation d–tensor [7]

$$\widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} = (P^{i}_{jk} = 0, P^{a}_{bk} = e_{b}(N^{a}_{k}), P^{i}_{jc} = -\frac{1}{2}g^{ik}\Omega^{a}{}_{kj}h_{ca}, P^{a}_{bc} = 0) \tag{28}$$

[7] $\widehat{\mathbf{P}}^{\alpha}_{\beta\gamma}$ is a tensor field of type (1,2). As is well known, the sum of a linear connection and a tensor field of type (1,2) is a new linear connection.

satisfies the conditions of the mentioned Theorem. It should be noted that, in general, the components $\widehat{T}^{i}{}_{ja}$, $\widehat{T}^{a}{}_{ji}$ and $\widehat{T}^{a}{}_{bi}$ are not zero. This is an anholonomic frame (or, equivalently, off–diagonal metric) effect. With respect to the N–adapted frames, the coefficients $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}=\left(\widehat{L}^{i}_{jk},\widehat{L}^{a}_{bk},\widehat{C}^{i}_{jc},\widehat{C}^{a}_{bc}\right)$ are computed:

$$\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2}g^{ir}\left(e_k g_{jr}+e_j g_{kr}-e_r g_{jk}\right), \\
\widehat{L}^{a}_{bk} &= e_b(N^a_k)+\frac{1}{2}h^{ac}\left(e_k h_{bc}-h_{dc}\ e_b N^d_k-h_{db}\ e_c N^d_k\right), \\
\widehat{C}^{i}_{jc} &= \frac{1}{2}g^{ik}e_c g_{jk},\ \widehat{C}^{a}_{bc}=\frac{1}{2}h^{ad}\left(e_c h_{bd}+e_c h_{cd}-e_d h_{bc}\right).
\end{aligned} \tag{29}$$

In some approaches to Finsler geometry [9], one uses the so–called Berwald d–connection $\mathbf{D}^{B}$ with the coefficients

$${}^{B}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}=\left(\ {}^{B}L^{i}_{jk}=\widehat{L}^{i}_{jk},\ {}^{B}L^{a}_{bk}=e_b(N^a_k),\ {}^{B}C^{i}_{jc}=0,\ {}^{B}C^{a}_{bc}=\widehat{C}^{a}_{bc}\right). \tag{30}$$

This d–connection minimally extends the Levi–Civita connection (it is just the Levi–Civita connection if the integrability conditions are satisfied, i.e. $\Omega^{a}{}_{kj}=0$, see (28)). But, in general, for this d–connection, the metricity conditions are not satisfied, for instance $D_a g_{ij}\neq 0$ and $D_i h_{ab}\neq 0$.

By a straightforward d–form calculus in (25), we can find the N–adapted components $\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}$ of the curvature $\mathbf{R}=\{\mathcal{R}^{\alpha}{}_{\beta}\}$ of a d–connection $\mathbf{D}$, i.e. the d–curvatures from Theorem 0.1.2:

$$\begin{aligned}
R^{i}{}_{hjk} &= e_k L^{i}{}_{hj}-e_j L^{i}{}_{hk}+L^{m}{}_{hj}L^{i}{}_{mk}-L^{m}{}_{hk}L^{i}{}_{mj}-C^{i}{}_{ha}\Omega^{a}{}_{kj}, \\
R^{a}{}_{bjk} &= e_k L^{a}{}_{bj}-e_j L^{a}{}_{bk}+L^{c}{}_{bj}L^{a}{}_{ck}-L^{c}{}_{bk}L^{a}{}_{cj}-C^{a}{}_{bc}\Omega^{c}{}_{kj}, \\
R^{i}{}_{jka} &= e_a L^{i}{}_{jk}-D_k C^{i}{}_{ja}+C^{i}{}_{jb}T^{b}{}_{ka}, \\
R^{c}{}_{bka} &= e_a L^{c}{}_{bk}-D_k C^{c}{}_{ba}+C^{c}{}_{bd}T^{c}{}_{ka}, \\
R^{i}{}_{jbc} &= e_c C^{i}{}_{jb}-e_b C^{i}{}_{jc}+C^{h}{}_{jb}C^{i}{}_{hc}-C^{h}{}_{jc}C^{i}{}_{hb}, \\
R^{a}{}_{bcd} &= e_d C^{a}{}_{bc}-e_c C^{a}{}_{bd}+C^{e}{}_{bc}C^{a}{}_{ed}-C^{e}{}_{bd}C^{a}{}_{ec}.
\end{aligned} \tag{31}$$

Contracting respectively the components of (31), one proves: The Ricci tensor $\mathbf{R}_{\alpha\beta}\doteqdot\mathbf{R}^{\tau}{}_{\alpha\beta\tau}$ is characterized by h- v–components, i.e. d–tensors,

$$R_{ij}\doteqdot R^{k}{}_{ijk},\quad R_{ia}\doteqdot -R^{k}{}_{ika},\ R_{ai}\doteqdot R^{b}{}_{aib},\ R_{ab}\doteqdot R^{c}{}_{abc}. \tag{32}$$

It should be noted that this tensor is not symmetric for arbitrary d–connections $\mathbf{D}$.

The scalar curvature of a d–connection is

$$ {}^{s}\mathbf{R} \doteqdot \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = g^{ij}R_{ij} + h^{ab}R_{ab}, \tag{33} $$

defined by a sum the h– and v–components of (32) and d–metric (20).

The Einstein tensor is defined and computed in standard form

$$ \mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ {}^{s}\mathbf{R} \tag{34} $$

For a Lagrange geometry, see Definition 0.1.9, by straightforward component calculations, one can be proved the fundamental results:

1. The Euler–Lagrange equations

$$ \frac{d}{d\tau}\left(\frac{\partial L}{\partial y^{i}}\right) - \frac{\partial L}{\partial x^{i}} = 0 $$

where $y^{i} = \frac{dx^{i}}{d\tau}$ for $x^{i}(\tau)$ depending on parameter τ, are equivalent to the "nonlinear" geodesic equations

$$ \frac{d^{2}x^{i}}{d\tau^{2}} + 2G^{i}(x^{k}, \frac{dx^{j}}{d\tau}) = 0 $$

defining paths of the canonical semispray

$$ S = y^{i}\frac{\partial}{\partial x^{i}} - 2G^{i}(x,y)\frac{\partial}{\partial y^{i}} $$

where

$$ 2G^{i}(x,y) = \frac{1}{2}\ {}^{L}g^{ij}\left(\frac{\partial^{2}L}{\partial y^{i}\partial x^{k}}y^{k} - \frac{\partial L}{\partial x^{i}}\right) $$

with ${}^{L}g^{ij}$ being inverse to (11).

2. There exists on $\widetilde{TM}$ a canonical N–connection

$$ {}^{L}N_{j}^{i} = \frac{\partial G^{i}(x,y)}{\partial y^{i}} \tag{35} $$

defined by the fundamental Lagrange function $L(x,y)$, which prescribes nonholonomic frame structures of type (15) and (16), ${}^{L}\mathbf{e}_{\nu} = (e_{i},\ {}^{\bullet}e_{k})$ and ${}^{L}\mathbf{e}^{\mu} = (e^{i},\ {}^{\bullet}e^{k})$.

3. The canonical N–connection (35), defining $\ ^{\bullet}e_i$, induces naturally an almost complex structure $\mathbf{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, where $\chi(\widetilde{TM})$ denotes the module of vector fields on $\widetilde{TM}$,
$$\mathbf{F}(e_i) = \ ^{\bullet}e_i \text{ and } \mathbf{F}(\ ^{\bullet}e_i) = -e_i,$$
when
$$\mathbf{F} = \ ^{\bullet}e_i \otimes e^i - e_i \otimes \ ^{\bullet}e^i \quad (36)$$
satisfies the condition $\mathbf{F}\rfloor\ \mathbf{F} = -\mathbf{I}$, i. e. $F^{\alpha}{}_{\beta}F^{\beta}{}_{\gamma} = -\delta^{\alpha}_{\gamma}$, where δ^{α}_{γ} is the Kronecker symbol and "$\rfloor$" denotes the interior product.

4. On $\widetilde{TM}$, there is a canonical metric structure
$$^{L}\mathbf{g} = \ ^{L}g_{ij}(x,y)\ e^i \otimes e^j + \ ^{L}g_{ij}(x,y)\ \ ^{\bullet}e^i \otimes \ ^{\bullet}e^j \quad (37)$$
constructed as a Sasaki type lift from M.

5. There is also a canonical d–connection structure $^{L}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ defined only by the components of $^{L}N^i_j$ and $^{L}g_{ij}$, i.e. by the coefficients of metric (37) which in its turn is induced by a regular Lagrangian. The values $^{L}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\ ^{L}\widehat{L}^i_{jk},\ ^{L}\widehat{C}^a_{bc})$ are computed just as similar values from (29). We note that on $\widetilde{TM}$ there are couples of distinguished sets of h- and v–components.

0.3 The Layout of the Book

This book is organized in three Parts comprising fifteen Chapters. Every Chapter represents a research paper, begins with an Abstract and ends with a Bibliography. We try to follow the original variants of the selected works but subject the text to some minimal grammar and style modifications if it is necessary.

The Foreword outlines the main results on modelling locally anisotropic and/or noncommutative structures in modern gravity and geometric mechanics. Chapter 0 presents an Introduction to the book: There are stated the main principles and concepts both for the experts in differential geometry and applications and for the beginners on Finsler and Lagrange geometry. We discuss the main references and results in such directions and present the corresponding list of references.

The Part I consists of three Chapters.

Chapter 1 is devoted to modelling Finsler–Lagrange and Hamilton–Cartan geometries on metric–affine spaces provided with N–connection structure. There are defined the

Finsler-, Lagrange– and Hamilton–affine spaces and elaborated complete scheme of their classification in terms of N–adapted geometric structures. The corresponding Tables are presented in the Appendix section.

Chapter 2 describes how Finsler–Lagrange metrics and connections can be extracted from the metric–affine gravity by introducing nonhlonomic distributions and extending the results for N–anholonomic manifolds. There are formulated and proved the main theorems on constructing exact solutions modelling such spacetimes in string gravity and models with nontrivial torsion and nonmetricity. Some examples of such solutions describing configurations with variable cosmological constants and three dimensional gravitational solitons propagating self–consistently in locally anisotropic spaces are constructed.

In Chapter 3 we construct exact solutions in metric–affine and string gravity with noncommutative symmetries defined by nontrivial N–connection structures. We generalize the methods of generating such noncommutative solutions in the commutative gauge and Einstein gravity theories [69, 64, 75, 68] and show that they can be performed in a form generalizing the solutions for the black ellipsoids [65, 66, 77]. We prove the stability of such locally anisotropic black holes objects and prove that stability can be preserved for extensions to solutions in complex gravity.

Part II provides an almost complect relief how the so–called "anholonomic frame method" of constructing exact solutions in gravity was proposed and developed. It reflects a set of nine electronic preprints [8] and communications at International Conferences concerning developments of a series of works [58, 63, 82, 88, 86, 83, 84, 85, 65, 66, 77, 21, 21, 69, 68, 70, 72, 74, 80].

Chapter 4 reflects the results of the first work where, in four dimensional gravity, an exact generic off–diagonal solution with ellipsoidal symmetry was constructed. It develops the the results of [58] and announces certain preliminary results published latter in [59, 63, 65, 66].

Chapter 5 is devoted to three dimensional (3D) black holes solutions with generic local anisotropy. It is well known that the vacuum 3D (pseudo) Riemannian gravity is trivial because the curvature vanishes if the Ricci tensor is zero. The firsts nontrivial solutions were obtained by adding a nonzero cosmological constant, a torsion field or other contributions, for instance, from string gravity. Our idea was to generate 3D nonholonomic configurations by deforming the frame structure for nonholonomic distributions (with anholonomically induced torsion). The results correct the errors from a previous preprint (S. Vacaru, gr–qc/ 9811048) caused by testing a Maple program on generating

[8] the exact references for these preprints are given as footnotes to Abstracts at the beginnings of Chapters 4–12

off–diagonal exact solutions. In collaboration with E. Gaburov and D. Gontţa, one were obtained all formulas in analytic form.

Chapter 6 shows a possible application of 3D nonholonomic exact solutions in applications of geometric thermodynamics to black hole physics. It revises a former preprint (S. Vacaru, gr–qc/ 9905053) to the case of elliptic local anisotropies (a common work together with P. Stavrinos and D. Gonţa). The results were further developed in Refs. [61, 63] and partially published in monograph [87].

Chapter 7 contains a research of warped configurations with generic local anisotropy. Such solutions prove that the running hierarchies became anisotropic if generic off–diagonal metric terms are included into consideration.

Chapter 8 introduces some classes of exact black ellipsoid solutions in brane gravity constructed by using the N–connection formalism. This is a common work together with E. Gaburov. Further developments were published in [85].

Chapter 9 elaborates a locally anisotropic models of inflational cosmology (a work together with D. Gonţa). Such models are defined by generic off–diagonal cosmological metrics and can be, for instance, with ellipsoid or toroidal symmetry [76, 21].

Chapter 10 reviews in detail the "anholonomic frame method" elaborated on the base of N–connection formalism and various methods from Finsler and Lagrange geometry. There are given the bulk of technical results used in Refs. [59, 63, 82, 88, 86, 83, 84, 85] and emphasized the cases of four and five dimensional ellipsoid configurations. The method was developed for the metric–affine spaces with N–connection structure (see Chapter 2) and revised for the solutions with noncommutative and Lie/Clifford algebroid symmetries (see Chapter 3 and Ref. [69]).

Chapter 11 develops the "anholonomic frame method" to the cases of toroidal configurations. Such solutions are not subjected to the restrictions of cosmic censorship criteria because, in general, the nonholonomic structures contain nontrivial torsion coefficients and additional sources induced by the off–diagonal metric terms. Such black tori solutions exist in five dimensional gravity and for nonholonomic configurations they are not restricted by black hole uniqueness theorems.

Chapter 12 extends the results of Chapters 10 and 11 when superpositions of ellipsoid and toroidal locally anisotropic configurations are constructed in explicit form. There are discussed possible applications in modern astrophysics as possible topological tests of the Einstein and extra dimension gravity.

In Part III, we mop to several foundations of noncommutative Finsler geometry and generalizations. In three Chapters, there are elaborated such models following possible realizations of Finsler and Lagrange structures as gauge models, Riemann–Cartan or string models of gravity provided with N–connection structure.

Chapter 13 extends the theory of N–connections to the case of projective finite modules (i.e. for noncommutative vector bundles). With respect to the noncommutative geometry there are outlined the necessary well known results from [18, 30, 26, 31] but with the aim to introduce nonholonomic structures. Noncommutative Finsler–gauge theories are investigated. There are developed the results elaborated in [64, 75].

Chapter 14 features several fundamental constructions when noncommutative Finsler configurations are derived in (super) string gravity. We show how locally anisotropic supergravity theories are derived in the low energy limit and via anisotropic topological compactification. There are analyzed noncommutative locally anisotropic field interactions. A model of anisotropic gravity is elaborated on noncommutative D–branes. Such constructions are provided with explicit examples of exact solutions: 1) black ellipsoids with noncommutative variables derived from string gravity; 2) 2D Finsler structures imbedded noncommutatively in string gravity; 3) moving soliton–black string configuration; 4) noncommutative anisotropic wormholes and strings.

Chapter 15 deals with the construction of nonholonomic spin geometry from the noncommutative point of view. We define noncommutative nonholonomic spaces and investigate the Clifford–Lagrange (–Finsler) structures. We prove that any regular fundamental Lagrange (Finsler) function induces a corresponding N–anholonomic spinor geometry and related nonholonomic Dirac operators. There are defined distinguished by N–connection spectral triples and proved the main theorems on extracting Finsler–Lagrange structures from noncommutative geometry.

Finally, we note that Chapters 13–15 scan some directions for further developments. For instance, the nonholonmic distributions can be considered on Hopf structures [32], Lie and Clifford algebroids [70, 72] and in relation to exact solutions with noncommutative symmetries [69]. We hope that such results will appeal to people both interested in noncommutative/ quantum developments of Finsler–Lagrange–Hamilton geometries and nonholonomic structures in gravity and string theory.

0.4 Sources on Finsler Geometry and Applications

We refer to the most important monographs, original articles and survey papers. Some of them sit in the junctions between different approaches and new applications. The bibliography is not exhaustive and reflects the authors interests and activity. The intend is to orient the nonspecialists on Finsler geometry, to emphasize some new perspectives and make a bridge to modern gravity and string theories and geometric mechanics. More specific details and discussions can be found in the references presented at the end of Chapters.

The first Finsler metric was considered by B. Riemann in his famous hability thesis in 1854 [42]. The geometric approach starts with the P. Finsler thesis work [24] in 1918 and the fundamental contributions by L. Berwald [12], a few years latter (see historical remarks and detailed bibliography in Refs. [43, 33, 37, 38, 60, 87]). The first monograph in the subject was due to E. Cartan [17].

The book [43] by H. Rund was for a long time the most comprehensive monograph on Finsler geometry.

In the middle of 80ths of the previous century, three new fundamental monographs stated renewed approaches and developments of Finsler geometry and applications: 1) The monograph by R. Miron and M. Anastasiei [37] elaborated a common approach to Finsler and Lagrange spaces following the geometry of nonlinear connections. Together with a set of further monographs [38, 34, 35, 39, 36], it reflects the results of the famous Romanian school on Finsler geometry and, in general, higher order generalizations of the Finsler–Lagrange and Cartan–Hamilton spaces. The monograph by G. Asanov [4] developed an approach related to new type of Finsler gauge symmetries and applications in relativity and field theories (the further work of his school [6, 5] is related to jet extensions and generalized nonlinear gauge symmetries). The monograph by M. Matsumoto [33] reflected the style and achievements on Finsler geometry in Japan.

Two monographs by A. Bejancu [10] and G. Yu. Bogoslovsky [13] complete the "80ths wave" on generalizations of Finsler geometry related respectively to the geometry of fiber bundles and certain bimetric theories of gravity.

During the last 15 years, the developments on Finsler geometry and applications can be conventionally distinguished into 5 main directions and applications (with interrelations of various sub–directions; we shall cite the works considered to be of key importance and discuss the items to which we contributed with our publications):

1. Generalized Finsler geometries with applications in geometric mechanics and optimal control theory. On higher order generalizations, there were published the monographs [34, 35, 39, 36] and, related results in optimal control theory, [47, 48].

2. Fisler methods in biology, ecology, diffusion and physics. It was published a series of monographs and collections of selected works like [2, 3, 1] (see there the main results and detailed references).

3. Nonmetric Finsler geometry, generalizations, violation of local Lorentz symmetry and applications. The direction originates from the L. Berwald and S. S. Chern works on Finsler geometry, see details in monographs [9, 44]. It was a fashion in the 20-30th years of the previous century to consider possible applications of the Riemann–Cartan–Weyl geometry (with nontrivial torsion and nonmetricity fields) in physics.

The Berwald and Chern connections (with various re–discovering and modifications by Rund, Moor and others) are typical ones which do not satisfy the compatibility conditions with the Finsler metric. Sure, they present certain interest in differential geometry but with more sophisticate applications in physics [16, 15, 25, 49] (for instance, it is a quite difficult problem to define spinors on spaces with nonmetricity and to construct supersymmetric and noncommutative extensions of such Finsler spaces). Here one should be noted that the E. Cartan [17] approach to Finsler geometry and a number of further developments [38, 60, 87] are based on canonical metric compatible Finsler conections and generalizations. In such cases, various real and complex spinor generalizations, supersymmetric models and noncommutative extensions are similar to those for the Riemann–Cartan geometry but with nonholonomic structures. The problem is discussed in details in Chapters 1–3 of this book.

There are investigated physical models with violation of the local Lorentz symmetry [4, 5, 6, 13, 14] being of special interest in modern gravity [19, 20]. Some authors [92, 11] consider that such Finsler spacetime and field interaction theories are subjected to substantial experimental restrictions but one should be noted that their conclusions are with respect to a restricted class of theories with nonmetricity and local broken Lorentz symmetry without a deep analysis of the N–connection structure. Such experimental restrictions do not hold in the metric compatible models when the Finsler like structures are defined, for instance, as exact solutions in general relativity and string gravity theories (see below, in point 5, some additional considerations and related references).

4. Super–Finsler spaces, Finsler–gauge gravity, locally anisotropic spinors and noncommutative geometry, geometric kinetics and stochastic processes and conservation laws.

A new classification of curved spaces in terms of chains of nearly autoparallel maps (generalizing the classes of conformal transforms and geodesic maps) and their invariants is possible both for the (pseudo) Riemannian and generalized Lagrange spaces and their supersymmetric generalizations [81, 60]. There were formulated the conservation laws on Riemann–Cartan–Weyl and Finsler–Lagrange spaces defined by the basic equations and invariant conditions for nearly autoparallel maps. It was also proven that the field equations of the Finsler–Lagrange (super) gravity can be formulated as Yang–Mills equations in affine (super) bundles provided with Cartan type connections and N–adapted frame structures [79, 60].

In monograph [10], there are summarized the A. Bejancu's results on gauge theories on Finsler spaces and supersymmetric models on usual manifolds but with supervector fibers and corresponding nonlinear connection structures. The author followed the approach to superspaces from [93] but had not used any global definition of superbundles and related nonintegrable super–distributions. It was not possible to do that in rigorous form without definition of spinors in Finsler spaces. The references [57, 60, 75] contains a comprehensive formulation of the geometry of generalized super–Finsler spaces. The approach is developed, for instance, by the authors of Ref. [7].

The idea to use spinor variables in Finsler spaces is due to Y. Takano (1983) [46] (see the monograph [87] for details, discussions, and references related to further contributions by T. Ono and Y. Takano, P. Stavrinos and V. Balan, who used 2–spinor variables but do not defined and do not proved the existence of general Clifford structures induced by Finsler metrics and connections; the spinor variables constructions, in that monograph, are compared with the S. Vacaru's approach to Finsler–spinors). The rigorous definition of locally anisotropic spinors on Finsler and Lagrange spaces was given in Refs. [53, 54] which was a nontrivial task because on Finsler like spaces there are not even local rotation symmetries. The differential geometry of spinors for Finsler, Lagrange and Hamilton spaces and their higher order generalizations, noncommutative extensions, and their applications in modern physics is elaborated in Refs. [59, 60, 87, 89, 82, 88, 75, 67, 71, 72, 74, 80]. The geometry of generalized Clifford–Finsler spaces, the further supersymmetric and noncommutative extensions, as well the proof that Finsler like structures appear in low energy limits of string theory [56, 57, 60] demonstrate that there are not conceptual problems for elaborating Finsler like theories in the framework of standard models in physics.

We refer also to applications of Finsler geometry in the theory of stochastic processes and kinetics and thermodynamics in curved spaces [61, 63]. Perhaps, the original idea on Finsler structures on phase spaces came from the A. A. Vlasov monograph [91]. The locally anisotropic processes in the language of Finsler geometry and generalizations were investigated in parallel by S. Vacaru [50, 51, 52, 55, 61, 60] and by P. Antonelli, T. Zastavniak and D. Hrimiuc (see details and a number of applications in Refs. [3, 2, 1]).

5. Generalized Finsler–Lagrange structures in gravity and string theory, anholonomic noncommutative and algebroid configurations, gravitational gerbes and exact solutions.

   There is a fundamental result for certain generic off–diagonal metric ansatz in four and five dimensional gravity: The Einstein equations for the so–called canonical distinguished connections are exactly integrable and such very general solutions depends on classes of functions depending on 2,3 and 4 variables (see details in Chapters 2, 10 and 11 and Refs. [68, 69, 73]). Perhaps, this is the most general method of constructing exact solutions in gravity by using geometric methods and the N–connection formalism. It was applied in different models of gravity, in general, with nontrivial torsion and nonmetricity.

Additionally to the formulas and references considered in Sections 0.1 and 0.2, we note that we analyzed the Finsler structures in explicit form in Refs. [21, 76] and that there are formulated explicit criteria when the Finsler–Lagrange geometries can be modelled in metric–affine, Riemann–Cartan, string and Einstein gravity models by corresponding nonholononmic frame and N–connection structures (see Chapters 1 and 2 and Refs. [73, 67]).

Section 0.3 outlines the main results and our publications related to modelling locally anisotropic configurations as exact solutions in gravity and generalization to noncommutative geometry, Lie/ Clifford algebroids and gerbes. Finally, we cite the works [90, 40] where 'hidden connections between general relativity and Finsler geometry' are discussed following alternative methods.

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Part I

Lagrange Geometry and Finsler–Affine Gravity

Chapter 1

Generalized Finsler Geometry in Einstein, String and Metric–Affine Gravity

Abstract [1]

We develop the method of anholonomic frames with associated nonlinear connection (in brief, N–connection) structure and show explicitly how geometries with local anisotropy (various type of Finsler–Lagrange–Cartan–Hamilton spaces) can be modelled on the metric–affine spaces. There are formulated the criteria when such generalized Finsler metrics are effectively defined in the Einstein, teleparallel, Riemann–Cartan and metric–affine gravity. We argue that every generic off–diagonal metric (which can not be diagonalized by coordinate transforms) is related to specific N–connection configurations. We elaborate the concept of generalized Finsler–affine geometry for spaces provided with arbitrary N–connection, metric and linear connection structures and characterized by gravitational field strengths, i. e. by nontrivial N–connection curvature, Riemannian curvature, torsion and nonmetricity. We apply an irreducible decomposition techniques (in our case with additional N–connection splitting) and study the dynamics of metric–affine gravity fields generating Finsler like configurations. The classification of basic eleven classes of metric–affine spaces with generic local anisotropy is presented.

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1.1 Introduction

Brane worlds and related string and gauge theories define the paradigm of modern physics and have generated enormous interest in higher–dimensional spacetimes amongst particle and astrophysics theorists (see recent advances in Refs. [1, 2, 3] and an outline of the gauge idea and gravity in Refs. [4, 5]). The unification scheme in the framework of string/ brane theory indicates that the classical (pseudo) Riemannian description is not valid on all scales of interactions. It turns out that low–energy dilaton and axi–dilaton interactions are tractable in terms of non–Riemannian mathematical structures possessing in particular anholonomic (super) frame [equivalently, (super) vielbein] fields [6], noncommutative geometry [7], quantum group structures [8] all containing, in general, nontrivial torsion and nonmetricity fields. For instance, in the closest alternatives to general relativity theory, the teleparallel gravity models [9], the spacetime is of Witzenbock type with trivial curvature but nontrivial torsion. The frame or co–frame filed (tetrad, vierbein, in four dimensions, 4D) is the basic dynamical variable treated as the gauge potential corresponding to the group of local translations.

Nowadays, it was established a standard point of view that a number of low energy (super) string and particle physics interactions, at least the nongravitational ones, are described by (super) gauge potentials interpreted as linear connections in suitable (super) bundle spaces. The formal identity between the geometry of fiber bundles [10] and gauge theory is recognized since the works [11] (see a recent discussing in connection to a unified description in of interactions in terms of composite fiber bundles in Ref. [12]).

The geometry of fiber bundles and the moving frame method originating from the E. Cartan works [6] constitute a modern approach to the Finsler geometry and generalizations (also suggested by E. Cartan [13] but finally elaborated in R. Miron and M. Anastasiei works [14]), see some earlier and recent developments in Refs. [15, 16, 17, 18, 19]. Various type of geometries with local anisotropy (Finsler, Lagrange, Hamilton, Cartan and their generalizations, according to the terminology proposed in [14]), are modelled on (co) vector / tangent bundles and their higher order generalizations [21, 20] with different applications in Lagrange and Hamilton mechanics or in generalized Finsler gravity. Such constructions were defined in low energy limits of (super) string theory and supergravity [22, 23] and generalized for spinor bundles [24] and affine– de Sitter frame bundles [25] provided with nonlinear connection (in brief, N–connection) structure of first and higher order anisotropy.

The gauge and moving frame geometric background is also presented in the metric–affine gravity (MAG) [4]. The geometry of this theory is very general being described by the two–forms of curvature and of torsion and the one–form of nonmetricity treated respectively as the gravitational field strengths for the linear connection, coframe and metric. The kinematic scheme of MAG is well understood at present time as well certain dynamical aspects of the vacuum configurations when the theory can be reduced to an effective Einstein–Proca model with nontrivial torsion and nonmetricity [26, 27, 28, 29]. There were constructed a number of exact solutions in MAG connecting the theory to modern string gravity and another extra dimension generalizations [30, 31, 32]. Nevertheless, one very important aspect has not been yet considered. As a gauge theory, the MAG can be expressed with respect to arbitrary frames and/or coframes. So, if we introduce frames with associated N–connection structure, the MAG should incorporate models with generic local anisotropy (Finsler like ones and their generalizations) which are distinguished by certain prescriptions for anholonomic frame transforms, N–connection coefficients and metric and linear connection structures adapted to such anholonomic configurations. Roughly speaking, the MAG contains the bulk of known generalized Finsler geometries which can be modelled on metric–affine spaces by defining splitting on subspaces like on (co) vector/ tangent bundles and considering certain anholonomically constrained moving frame dynamics and associated N–connection geometry.

Such metric–affine spaces with local anisotropy are enabled with generic off–diagonal metrics which can not be diagonalized by any coordinate transforms. The off–diagonal coefficients can be mapped into the components of a specific class of anholonomic frames, defining also the coefficients of the N–connection structure. It is possible to redefine equivalently all geometrical values like tensors, spinors and connections with respect to N–adapted anholonomic bases. If the N–connection, metric and linear connections are chosen for an explicit type of Finsler geometry, such a geometric structure is modelled on a metric–affine space (we claim that a Finsler–affine geometry is constructed). The point is to find explicitly by what type of frames and connections a locally anisotropic structure can be modelled by exact solutions in the framework of MAG. Such constructions can be performed in the Einstein–Proca sector of the MAG gravity and they can be defined even in general relativity theory (see the partners of this paper with field equations and exact solutions in MAG modelling Finsler like metrics and generalizations [33]).

Within the framework of moving frame method [6], we investigated in a series of works [34, 35, 36, 37] the conditions when various type of metrics with noncommutative symmetry and/or local anisotropy can be effectively modelled by anholonomic frames on (pseudo) Riemannian and Riemann–Cartan spaces [38]. We constructed explicit classes of such exact solutions in general relativity theory and extra dimension gravity models. They are parametrized by generic off–diagonal metrics which can not diagonalized by any coordinate transforms but only by anholonomic frame transforms. The new classes of solutions describe static black ellipsoid objects, locally anistoropic configurations with toroidal and/ or ellipsoidal symmetries, wormholes/ flux tubes and Taub-NUT metrics with polarizied constants and warped spinor–soliton–dilaton configurations. For certain conditions, some classes of such solutions preserve the four dimensional (4D) local Lorentz symmetry.

Our ongoing effort is to model different classes of geometries following a general approach to the geometry of (co) vector/tangent bundles and affine–de Sitter frame bundles [25] and superbundles [23] and or anisotropic spinor spaces [24] provided with N–connection structures. The basic geometric objects on such spaces are defined by proper classes of anholonomic frames and associated N–connections and correspondingly adapted metric and linear connections. There are examples when certain Finsler like configurations are modelled by some exact solutions in Einstein or Einstein–Cartan gravity and, inversely (the outgoing effort), by using the almost Hermitian formulation [14, 20, 24] of Lagrange/Hamilton and Finsler/Cartan geometry, we can consider Einstein and gauge gravity models defined on tangen/cotangent and vector/covector bundles. Recently, there were also obtained some explicit results demonstrating that the anholonomic frames geometry has a natural connection to noncommutative geometry in string/M–theory and noncommutative gauge models of gravity [36, 37] (on existing approaches to noncommutative geometry and gravity we cite Refs. [7]).

We consider torsion fields induced by anholonomic vielbein transforms when the theory can be extended to a gauge [5], metric–affine [4], a more particular Riemann–Cartan case [38], or to string gravity with B–field [2]. We are also interested to define the conditions when an exact solution possesses hidden noncommutative symmetries, induced torsion and/or locally anisotropic configurations constructed, for instance, in the framework of the Einstein theory. This direction of investigation develops the results obtained in Refs. [35] and should be distinguished from our previous works on the geometry of Clifford and spinor structures in generalized Finsler and Lagrange–Hamilton spacetimes [24]. Here we emphasize that the works [34, 35, 36, 37, 24] were elaborated following general methods of the geometry of anholonomic frames with associated N–connections in vector (super) bundles [14, 20, 23]. The concept of N–connection was proposed in Finsler geometry [15, 17, 18, 19, 16, 13]. As a set of coefficients it was firstly present the E. Cartan's monograph[13] and then was elaborated in a more explicit form by A. Kawaguchi [39]. It was proven that the N–connection structures can be defined also on (pseudo) Riemannian spaces and certain methods work effectively in constructing exact solutions in Einstein gravity [24, 34, 35].

In order to avoid possible terminology ambiguities, we note that for us the definition of N–connection is that proposed in global form by W. Barthel in 1963 [40] when a N–connection is defined as an exact sequence related to a corresponding Whitney sum of the vertical and horizontal subbundles, for instance, in a tangent vector bundle. [2] This

[2]Instead of a vector bundle we can consider a tangent bundle, or cotangent/covector ones, or even

concept is different from that accepted in Ref. [42] were the term 'nonlinear connection' is used for tetrads as N–connections which do not transform inhomogeneously under local frame rotations. That approach invokes nonlinear realizations of the local spacetime group (see also an early model of gauge gravity with nonlinear gauge group realizations [43] and its extensions to Finsler like [25] or noncommutative gauge gravity theories [36]).

In summary, the aim of the present work is to develop a unified scheme of anholonomic frames with associated N–connection structure for a large number of gauge and gravity models (in general, with locally isotropic and anisotropic interactions and various torsion and nonmetricity contributions) and effective generalized Finsler–Weyl–Riemann–Cartan geometries derived from MAG. We elaborate a detailed classification of such spaces with nontrivial N–connection geometry. The unified scheme and classification were inspired by a number of exact solutions parametrized by generic off–diagonal metrics and anholonomic frames in Einstein, Einstein–Cartan and string gravity. The resulting formalism admits inclusion of locally anisotropic spinor interactions and extensions to noncommutative geometry and string/ brane gravity [22, 23, 34, 35, 36, 37]. Thus, the geometry of metric–affine spaces enabled with an additional N–connection structure is sufficient not only to model the bulk of physically important non–Riemannian geometries on (pseudo) Riemannian spaces but also states the conditions when effective spaces with generic anisotropy can be derived as exact solutions of gravitational and matter field equations. In the present work we pay attention to the geometrical (pre–dynamical) aspects of the generalized Finsler–affine gravity which constitute a theoretical background for constructing a number of exact solutions in MAG in the partner papers [33].

The article is organized as follows. We begin, in Sec. 2, with a review of the main concepts from the metric–affine geometry and the geometry of anholonomic frames with associated N–connections. We introduce the basic definitions and formulate and prove the main theorems for the N–connection, linear connection and metric structures on metric–affine spaces and derive the formulas for torsion and curvature distinguished by N–connections. Next, in Sec. 3, we state the main properties of the linear and nonlinear connections modelling Finsler spaces and their generalizations and consider how the N–connection structure can be derived from a generic off–diagonal metric in a metric–affine space. Section 4 is devoted to the definition and investigation of generalized Finsler–affine spaces. We illustrate how by corresponding parametrizations of the off–diagonal metrics, anholonomic frames, N–connections and distinguished connections every type of generalized Finsler–Lagrange–Cartan–Hamilton geometry can be modelled in the metric–

general manifolds of necessary smooth class with adapted definitions of global sums of horizonal and vertical subspaces. The geometry of N–connections is investigated in details in Refs. [41, 14, 23, 24, 25] for various type of spaces.

affine gravity or any its restrictions to the Einstein–Cartan and general relativity theory. In Sec. 5, we conclude the results and point out how the synthesis of the Einstein, MAG and generalized Finsler gravity models can be realized and connected to the modern string gravity. In Appendix we elaborate a detailed classification of eleven classes of spaces with generic local anisotropy (i. e. possessing nontrivial N–connection structure) and various types of curvature, torsion and nonmetricity distinguished by N–connections.

Our basic notations and conventions combine those from Refs. [4, 14, 34, 35] and contain an interference of traditions from MAG and generalized Finsler geometry. The spacetime is modelled as a manifold V^{n+m} of necessary smoothly class of dimension $n+m$. The Greek indices $\alpha, \beta, ...$ can split into subclasses like $\alpha = (i, a)\,,\ \beta = (j, b)\,...$ where the Latin indices from the middle of the alphabet, $i, j, k, ...$ run values $1, 2, ...n$ and the Latin indices from the beginning of the alphabet, $a, b, c, ...$ run values $n+1, n+2,$ ..., $n+m$. We follow the Penrose convention on abstract indices [44] and use underlined indices like $\underline{\alpha} = (\underline{i}, \underline{a})$, for decompositions with respect to coordinate frames. The notations for connections $\Gamma^{\alpha}{}_{\beta\gamma}$, metrics $g_{\alpha\beta}$ and frames e_α and coframes ϑ^β, or other geometrical and physical objects, are the standard ones from MAG if a nonlinear connection (N–connection) structure is not emphasized on the spacetime. If a N–connection and corresponding anholonomic frame structure are prescribed, we use "boldfaced" symbols with possible splitting of the objects and indices like $\mathbf{V}^{n+m}$, $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right)$, $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})\,, \mathbf{e}_\alpha = (e_i, e_a)\,,$ ...being distinguished by N–connection (in brief, we use the terms d–objects, d–tensor, d–connection in order to say that they are for a metric–affine space modelling a generalized Finsler, or another type, anholonomic frame geometry). The symbol " $\doteqdot$" will be used is some formulas which state that the relation is introduced "by definition" and the end of proofs will be stated by symbol $\blacksquare$.

1.2 Metric–Affine Spaces and Nonlinear Connections

We outline the geometry of anholonomic frames with associated nonlinear connections (in brief, N–connections) in metric–affine spaces which in this work are necessary smooth class manifolds, or (co) vector/ tangent bundles provided with, in general, independent nonlinear and linear connections and metrics, and correspondingly derived strengths like N–connection curvature, Riemannian curvature, torsion and nonmetricity. The geometric formalism will be applied in the next sections where we shall prove that every class of (pseudo) Riemannian, Kaluza–Klein, Einstein–Cartan, metric–affine and generalized Lagrange–Finsler and Hamilton–Cartan spaces is characterized by corresponding N–connection, metric and linear connection structures.

1.2.1 Linear connections, metrics and anholonomic frames

We briefly review the standard results on linear connections and metrics (and related formulas for torsions, curvatures, Ricci and Einstein tensors and Bianchi identities) defined with respect to arbitrary anholonomic bases in order to fix a necessary reference which will be compared with generalized Finsler–affine structures we are going to propose in the next sections for spaces provided with N–connection. The results are outlined in a form with conventional splitting into horizontal and vertical subspaces and sub–indices. We follow the Ref. [45] but we use Greek indices and denote a covariant derivative by D preserving the symbol $\bigtriangledown$ for the Levi–Civita (metric and torsionless) connection. Similar formulas can be found, for instance. in Ref. [46].

Let V^{n+m} be a $(n+m)$–dimensional underlying manifold of necessary smooth class and denote by TV^{n+m} the corresponding tangent bundle. The local coordinates on $V^{n+m}, u=\{u^{\underline{\alpha}}=(x^{\underline{i}},y^{\underline{a}})\}$ conventionally split into two respective subgroups of "horizontal" coordinates (in brief, h–coordinates), $x=(x^{\underline{i}})$, and "vertical" coordinates (v–coordinates), $y=(y^{\underline{a}})$, with respective indices running the values $\underline{i},\underline{j},...=1,2,...,n$ and $\underline{a},\underline{b},...=n{+}1,n{+}2,...,n{+}m$. The splitting of coordinates is treated as a formal labelling if any fiber and/or the N–connection structures are not defined. Such a splitting of abstract coordinates $u^{\alpha}=(x^{i},y^{a})$ may be considered, for instance, for a general (pseudo) Riemannian manifold with x^{i} being some 'holonomic" variables (unconstrained) and y^{a} being "anholonomic" variables (subjected to some constraints), or in order to parametrize locally a vector bundle (E,μ,F,M) defined by an injective surjection $\mu:E\rightarrow M$ from the total space E to the base space M of dimension $\dim M=n$, with F being the typical vector space of dimension $\dim F=m$. For our purposes, we consider that both M and F can be, in general, provided with metric structures of arbitrary signatures. On vector bundles, the values $x=(x^{i})$ are coordinates on the base and $y=(y^{a})$ are coordinates in the fiber. If $\dim M=\dim F$, the vector bundle E transforms into the tangent bundle TM. The same conventional coordinate notation $u^{\alpha}=(x^{i},y^{a}\rightarrow p_{a})$ can be used for a dual vector bundle (E,μ,F^{*},M) with the typical fiber F^{*} being a covector space (of 1-forms) dual to F, where p_{a} are local (dual) coordinates. For simplicity, we shall label y^{a} as general coordinates even for dual spaces if this will not result in ambiguities. In general, our geometric constructions will be elaborated for a manifold V^{n+m} (a general metric–affine spaces) with some additional geometric structures and fibrations to be stated or modelled latter (for generalized Finsler geometries) on spacetimes under consideration.

At each point $p\in V^{n+m}$, there are defined basis vectors (local frames, vielbeins) $e_{\alpha}=$

$A_\alpha^{\ \underline{\alpha}}(u)\partial_{\underline{\alpha}} \in TV^{n+m}$, with $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ being tangent vectors to the local coordinate lines $u^{\underline{\alpha}} = u^{\underline{\alpha}}(\tau)$ with parameter τ. In every point p, there is also a dual basis $\vartheta^\beta = A^\beta_{\ \underline{\beta}}(u)du^{\underline{\beta}}$ with $du^{\underline{\beta}}$ considered as coordinate one forms. The duality conditions can be written in abstract form by using the interior product $\rfloor$, $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$, or in coordinate form $A_\alpha^{\ \underline{\alpha}} A^\beta_{\ \underline{\alpha}} = \delta^\beta_\alpha$, where the Einstein rule of summation on index $\underline{\alpha}$ is considered, δ^β_α is the Kronecker symbol. The "not underlined" indices $\alpha, \beta, ...$, or $i, j, ...$ and $a, b, ...$ are treated as abstract labels (as suggested by R. Penrose). We shall underline the coordinate indices only in the cases when it will be necessary to distinguish them from the abstract ones.

Any vector and 1–form fields, for instance, X and, respectively, $\widetilde{Y}$ on V^{n+m} are decomposed in h– and v–irreducible components,

$$X = X^\alpha e_\alpha = X^i e_i + X^a e_a = X^{\underline{\alpha}} \partial_{\underline{\alpha}} = X^{\underline{i}} \partial_{\underline{i}} + X^{\underline{a}} \partial_{\underline{a}}$$

and

$$\widetilde{Y} = \widetilde{Y}_\alpha \vartheta^\alpha = \widetilde{Y}_i \vartheta^i + \widetilde{Y}_a \vartheta^a = \widetilde{Y}_{\underline{\alpha}} du^{\underline{\alpha}} = \widetilde{Y}_{\underline{i}} dx^{\underline{i}} + \widetilde{Y}_{\underline{a}} dy^{\underline{a}}.$$

We shall omit labels like "$\sim$" for forms if this will not result in ambiguities.

Definition 1.2.1. *A linear (affine) connection D on V^{n+m} is a linear map (operator) sending every pair of smooth vector fields (X, Y) to a vector field $D_X Y$ such that*

$$D_X (sY + Z) = sD_X Y + D_X Z$$

for any scalar $s = const$ and for any scalar function $f(u^\alpha)$,

$$D_X (fY) = fD_X Y + (Xf)\, Y \text{ and } D_X f = Xf.$$

$D_X Y$ is called the covariant derivative of Y with respect to X (this is not a tensor). But we can always define a tensor $DY : X \to D_X Y$. The value DY is a $(1,1)$ tensor field and called the covariant derivative of Y.

With respect to a local basis e_α, we can define the scalars $\Gamma^\alpha_{\ \beta\gamma}$, called the components of the linear connection D, such that

$$D_\alpha e_\beta = \Gamma^\gamma_{\ \beta\alpha} e_\gamma \text{ and } D_\alpha \vartheta^\beta = -\Gamma^\beta_{\ \gamma\alpha} \vartheta^\gamma$$

were, by definition, $D_\alpha \doteqdot D_{e_\alpha}$ and because $e_\beta \vartheta^\beta = const$.

We can decompose

$$D_X Y = (D_X Y)^\beta e_\beta = \left[e_\alpha (Y^\beta) + \Gamma^\beta_{\ \gamma\alpha} \vartheta^\gamma \right] e_\beta \doteqdot Y^\beta_{\ ;\alpha} X^\alpha \tag{1.1}$$

where $Y^\beta_{\ ;\alpha}$ are the components of the tensor DY.

It is a trivial proof that any change of basis (vielbein transform), $e_{\alpha'} = B_{\alpha'}^{\ \alpha} e_\alpha$, with inverse $B^{\alpha'}_{\ \alpha}$, results in a corresponding (nontensor) rule of transformation of the components of the linear connection,

$$\Gamma^{\alpha'}_{\ \beta'\gamma'} = B^{\alpha'}_{\ \alpha}\left[B_{\beta'}^{\ \beta} B_{\gamma'}^{\ \gamma} \Gamma^{\alpha}_{\ \beta\gamma} + B_{\gamma'}^{\ \gamma} e_\gamma \left(B_{\beta'}^{\ \alpha}\right)\right]. \tag{1.2}$$

Definition 1.2.2. *A local basis e_β is anhlonomic (nonholonomic) if there are satisfied the conditions*

$$e_\alpha e_\beta - e_\beta e_\alpha = w^{\gamma}_{\alpha\beta} e_\gamma \tag{1.3}$$

for certain nontrivial anholonomy coefficients $w^{\gamma}_{\alpha\beta} = w^{\gamma}_{\alpha\beta}(u^\tau)$. A such basis is holonomic if $w^{\gamma}_{\alpha\beta} \doteqdot 0$.

For instance, any coordinate basis ∂_α is holonomic. Any holonomic basis can be transformed into a coordinate one by certain coordinate transforms.

Definition 1.2.3. *The torsion tensor is a tensor field $\mathcal{T}$ defined by*

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y], \tag{1.4}$$

where $[X,Y] = XY - YX$, for any smooth vector fields X and Y.

The components $T^{\gamma}_{\ \alpha\beta}$ of a torsion $\mathcal{T}$ with respect to a basis e_α are computed by introducing $X = e_\alpha$ and $Y = e_\beta$ in (1.4),

$$\mathcal{T}\ (e_\alpha, e_\beta) = D_\alpha e_\beta - D_\beta e_\alpha - [e_\alpha, e_\beta] = T^{\gamma}_{\ \alpha\beta} e_\gamma$$

where

$$T^{\gamma}_{\ \alpha\beta} = \Gamma^{\gamma}_{\ \beta\alpha} - \Gamma^{\gamma}_{\ \alpha\beta} - w^{\gamma}_{\alpha\beta}. \tag{1.5}$$

We note that with respect to anholonomic frames the coefficients of anholonomy $w^{\gamma}_{\alpha\beta}$ are contained in the formula for the torsion coefficients (so any anholonomy induces a specific torsion).

Definition 1.2.4. *The Riemann curvature tensor $\mathcal{R}$ is defined as a tensor field*

$$\mathcal{R}(X,Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z. \tag{1.6}$$

We can compute the components $R^{\alpha}_{\ \beta\gamma\tau}$of curvature $\mathcal{R}$, with respect to a basis e_α are computed by introducing $X = e_\gamma, Y = e_\tau,\ Z = e_\beta$ in (1.6). One obtains

$$\mathcal{R}(e_\gamma, e_\tau)\, e_\beta = R^{\alpha}_{\ \beta\gamma\tau} e_\alpha$$

where

$$R^{\alpha}_{\ \beta\gamma\tau} = e_{\tau}\left(\Gamma^{\alpha}_{\ \beta\gamma}\right) - e_{\gamma}\left(\Gamma^{\alpha}_{\ \beta\tau}\right) + \Gamma^{\nu}_{\ \beta\gamma}\Gamma^{\alpha}_{\ \nu\tau} - \Gamma^{\nu}_{\ \beta\tau}\Gamma^{\alpha}_{\ \nu\gamma} + w^{\nu}_{\gamma\tau}\Gamma^{\alpha}_{\ \beta\nu}. \tag{1.7}$$

We emphasize that the anholonomy and vielbein coefficients are contained in the formula for the curvature components (1.6). With respect to coordinate frames, $e_{\tau} = \partial_{\tau}$, with $w^{\nu}_{\gamma\tau} = 0$, we have the usual coordinate formula.

Definition 1.2.5. *The Ricci tensor $\mathcal{R}i$ is a tensor field obtained by contracting the Riemann tensor,*

$$R_{\ \beta\tau} = R^{\alpha}_{\ \beta\tau\alpha}. \tag{1.8}$$

We note that for a general affine (linear) connection the Ricci tensor is not symmetric $R_{\ \beta\tau} \doteqdot R_{\ \tau\beta}$.

Definition 1.2.6. *A metric tensor is a $(0,2)$ symmetric tensor field*

$$g = g_{\alpha\beta}(u^{\gamma})\vartheta^{\alpha} \otimes \vartheta^{\beta}$$

defining the quadratic (length) linear element,

$$ds^2 = g_{\alpha\beta}(u^{\gamma})\vartheta^{\alpha}\vartheta^{\beta} = g_{\underline{\alpha}\underline{\beta}}(u^{\underline{\gamma}})du^{\underline{\alpha}}du^{\underline{\beta}}.$$

For physical applications, we consider spaces with local Minkowski signature, when locally, in a point u_0^{γ}, the diagonalized metric is $g_{\underline{\alpha}\underline{\beta}}(u_0^{\gamma}) = \eta_{\underline{\alpha}\underline{\beta}} = (1, -1, -1, ...)$ or, for our further convenience, we shall use metrics with the local diagonal ansatz being defined by any permutation of this order.

Theorem 1.2.1. *If a manifold V^{n+m} is enabled with a metric structure g, then there is a unique torsionless connection, the Levi–Civita connection $D = \bigtriangledown$, satisfying the metricity condition*

$$\bigtriangledown g = 0. \tag{1.9}$$

The proof, as an explicit construction, is given in Ref. [45]. Here we present the formulas for the components $\Gamma^{\alpha}_{\bigtriangledown\ \beta\tau}$ of the connection $\bigtriangledown$, computed with respect to a basis e_{τ},

$$\begin{aligned} \Gamma_{\bigtriangledown\ \alpha\beta\gamma} &= g\left(e_{\alpha}, \bigtriangledown_{\gamma}e_{\beta}\right) = g_{\alpha\tau}\Gamma^{\tau}_{\bigtriangledown\ \alpha\beta} \\ &= \frac{1}{2}\left[e_{\beta}\left(g_{\alpha\gamma}\right) + e_{\gamma}\left(g_{\beta\alpha}\right) - e_{\alpha}\left(g_{\gamma\beta}\right) + w^{\tau}_{\ \gamma\beta}g_{\alpha\tau} + w^{\tau}_{\ \alpha\gamma}g_{\beta\tau} - w^{\tau}_{\ \beta\gamma}g_{\alpha\tau}\right]. \end{aligned} \tag{1.10}$$

By straightforward calculations , we can check that

$$\bigtriangledown_{\alpha}g_{\beta\gamma} = e_{\alpha}\left(g_{\beta_{\gamma}}\right) - \Gamma^{\tau}_{\bigtriangledown\ \beta\alpha}g_{\tau_{\gamma}} - \Gamma^{\tau}_{\bigtriangledown\ \gamma\alpha}g_{\beta\tau} \equiv 0$$

and, using the formula (1.5),

$$T^{\gamma}_{\bigtriangledown\ \alpha\beta} = \Gamma^{\gamma}_{\bigtriangledown\ \beta\alpha} - \Gamma^{\gamma}{}_{\bigtriangledown\alpha\beta} - w^{\gamma}_{\alpha\beta} \equiv 0.$$

We emphasize that the vielbein and anholonomy coefficients are contained in the formulas for the components of the Levi–Civita connection $\Gamma^{\tau}_{\bigtriangledown\ \alpha\beta}$ (1.10) given with respect to an anholonomic basis e_α. The torsion of this connection, by definition, vanishes with respect to all bases, anholonomic or holonomic ones. With respect to a coordinate base ∂_α, the components $\Gamma_{\bigtriangledown\ \alpha\beta\gamma}$ (1.10) transforms into the so–called 1-st type Christoffel symbols

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \Gamma^{\{\}}_{\alpha\beta\gamma} = \{\alpha\beta\gamma\} = \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\beta\alpha} - \partial_\alpha g_{\gamma\beta}\right). \tag{1.11}$$

If a space V^{n+m} posses a metric tensor, we can use $g_{\alpha\beta}$ and the inverse values $g^{\alpha\beta}$ for lowering and upping indices as well to contract tensor objects.

Definition 1.2.7.
a) The Ricci scalar R is defined

$$R \doteqdot g^{\alpha\beta} R_{\alpha\beta},$$

where $R_{\alpha\beta}$ is the Ricci tensor (1.8).
b) The Einstein tensor $\mathcal{G}$ has the coefficients

$$G_{\alpha\beta} \doteqdot R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta},$$

with respect to any anholonomic or anholonomic frame e_α.

We note that $G_{\alpha\beta}$ and $R_{\alpha\beta}$ are symmetric only for the Levi–Civita connection $\bigtriangledown$ and that $\bigtriangledown_\alpha G^{\alpha\beta} = 0$.

It should be emphasized that for any general affine connection D and metric g structures the metric compatibility conditions (1.9) are not satisfied.

Definition 1.2.8. *The nonmetricity field*

$$\mathcal{Q} = Q_{\alpha\beta}\ \vartheta^\alpha \otimes \vartheta^\beta$$

on a space V^{n+m} is defined by a tensor field with the coefficients

$$Q_{\gamma\alpha\beta} \doteqdot -D_\gamma g_{\alpha\beta} \tag{1.12}$$

where the covariant derivative D is defined by a linear connection 1–form $\Gamma^{\gamma}{}_{\alpha} = \Gamma^{\gamma}{}_{\alpha\beta}\vartheta^\beta$.

In result, we can generalize the concept of (pseudo) Riemann space [defined only by a locally (pseudo) Euclidean metric inducing the Levi–Civita connection with vanishing torsion] and Riemann–Cartan space [defined by any independent metric and linear connection with nontrivial torsion but with vanishing nonmetricity] (see details in Refs. [4, 38]):

Definition 1.2.9. *A metric–affine space is a manifold of necessary smooth class provided with independent linear connection and metric structures. In general, such spaces posses nontrivial curvature, torsion and nonmetricity (called strength fields).*

We can extend the geometric formalism in order to include into consideration the Finsler spaces and their generalizations. This is possible by introducing an additional fundamental geometric object called the N–connection.

1.2.2 Anholonomic frames and associated N–connections

Let us define the concept of nonlinear connection on a manifold V^{n+m}. [3] We denote by $\pi^T : TV^{n+m} \rightarrow TV^n$ the differential of the map $\pi : V^{n+m} \rightarrow V^n$ defined as a fiber–preserving morphism of the tangent bundle (TV^{n+m}, τ_E, V^n) to V^{n+m} and of tangent bundle (TV^n, τ, V^n). The kernel of the morphism π^T is a vector subbundle of the vector bundle $(TV^{n+m}, \tau_E, V^{n+m})$. This kernel is denoted $(vV^{n+m}, \tau_V, V^{n+m})$ and called the vertical subbundle over V^{n+m}. We denote the inclusion mapping by $i : vV^{n+m} \rightarrow TV^{n+m}$ when the local coordinates of a point $u \in V^{n+m}$ are written $u^\alpha = (x^i, y^a)$, where the values of indices are $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = n+1, n+2, ..., n+m$.

A vector $X_u \in TV^{n+m}$, tangent in the point $u \in V^{n+m}$, is locally represented as $(x, y, X, \widetilde{X}) = (x^i, y^a, X^i, X^a)$,where $(X^i) \in \mathbb{R}^n$ and $(X^a) \in \mathbb{R}^m$ are defined by the equality $X_u = X^i \partial_i + X^a \partial_a$ [$\partial_\alpha = (\partial_i, \partial_a)$ are usual partial derivatives on respective coordinates x^i and y^a]. For instance, $\pi^T \left(x, y, X, \widetilde{X} \right) = (x, X)$ and the submanifold vV^{n+m} contains elements of type $\left(x, y, 0, \widetilde{X} \right)$ and the local fibers of the vertical subbundle are isomorphic to $\mathbb{R}^m$. Having $\pi^T (\partial_a) = 0$, one comes out that ∂_a is a local basis of the vertical distribution $u \rightarrow v_u V^{n+m}$ on V^{n+m}, which is an integrable distribution.

Definition 1.2.10. *A nonlinear connection (N–connection)* $\mathbf{N}$ *in a space* (V^{n+m}, π, V^n) *is defined by the splitting on the left of the exact sequence*

$$0 \rightarrow vV^{n+m} \rightarrow TV^{n+m}/vV^{n+m} \rightarrow 0, \quad (1.13)$$

[3] see Refs. [40, 14] for original results and constructions on vector and tangent bundles.

i. e. a morphism of manifolds $N : TV^{n+m} \to vV^{n+m}$ *such that* $C \circ i$ *is the identity on* vV^{n+m}.

The kernel of the morphism $\mathbf{N}$ is a subbundle of $(TV^{n+m}, \tau_E, V^{n+m})$, it is called the horizontal subspace (being a subbundle for vector bundle constructions) and denoted by $(hV^{n+m}, \tau_H, V^{n+m})$. Every tangent bundle $(TV^{n+m},\ \tau_E,\ V^{n+m})$ provided with a N–connection structure is a Whitney sum of the vertical and horizontal subspaces (in brief, h- and v– subspaces), i. e.

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \tag{1.14}$$

It is proven that for every vector bundle (V^{n+m}, π, V^n) over a compact manifold V^n there exists a nonlinear connection [14] (the proof is similar if the bundle structure is modelled on a manifold).[4]

A N–connection $\mathbf{N}$ is defined locally by a set of coefficients $N_i^a(u^\alpha) = N_i^a(x^j, y^b)$ transforming as

$$N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a \tag{1.15}$$

under coordinate transforms on the space (V^{n+m}, μ, M) when $x^{i'} = x^{i'}(x^i)$ and $y^{a'} = M_a^{a'}(x) y^a$. The well known class of linear connections consists a particular parametization of its coefficients N_i^a to be linear on variables y^b,

$$N_i^a(x^j, y^b) = \Gamma_{bi}^a(x^j) y^b.$$

A N–connection structure can be associated to a prescribed ansatz of vielbein transforms

$$A_\alpha^{\underline{\alpha}}(u) = \mathbf{e}_\alpha^{\underline{\alpha}} = \begin{bmatrix} e_i^{\underline{i}}(u) & N_i^b(u) e_b^{\underline{a}}(u) \\ 0 & e_a^{\underline{a}}(u) \end{bmatrix}, \tag{1.16}$$

$$A^\beta{}_{\underline{\beta}}(u) = \mathbf{e}^\beta{}_{\underline{\beta}} = \begin{bmatrix} e^i{}_{\underline{i}}(u) & -N_k^b(u) e^k{}_{\underline{i}}(u) \\ 0 & e^a{}_{\underline{a}}(u) \end{bmatrix}, \tag{1.17}$$

in particular case $e_i^{\underline{i}} = \delta_i^{\underline{i}}$ and $e_a^{\underline{a}} = \delta_a^{\underline{a}}$ with $\delta_i^{\underline{i}}$ and $\delta_a^{\underline{a}}$ being the Kronecker symbols, defining a global splitting of $\mathbf{V}^{n+m}$ into "horizontal" and "vertical" subspaces with the N–vielbein structure

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\underline{\alpha}} \partial_{\underline{\alpha}} \text{ and } \vartheta^\beta = \mathbf{e}^\beta{}_{\underline{\beta}} du^{\underline{\beta}}.$$

[4] We note that the exact sequence (1.13) defines the N–connection in a global coordinate free form. In a similar form, the N–connection can be defined for covector bundles or, as particular cases for (co) tangent bundles. Generalizations for superspaces and noncommutative spaces are considered respectively in Refs. [23] and [36, 37].

In this work, we adopt the convention that for the spaces provided with N–connection structure the geometrical objects can be denoted by "boldfaced" symbols if it would be necessary to distinguish such objects from similar ones for spaces without N–connection. The results from subsection 1.2.1 can be redefined in order to be compatible with the N–connection structure and rewritten in terms of "boldfaced" values.

A N–connection $\mathbf{N}$ in a space $\mathbf{V}^{n+m}$ is parametrized, with respect to a local coordinate base,

$$\partial_\alpha = (\partial_i, \partial_a) \equiv \frac{\partial}{\partial u^\alpha} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right), \tag{1.18}$$

and dual base (cobase),

$$d^\alpha = (d^i, d^a) \equiv du^\alpha = (dx^i, dy^a), \tag{1.19}$$

by its components $N_i^a(u) = N_i^a(x, y)$,

$$\mathbf{N} = N_i^a(u) d^i \otimes \partial_a.$$

It is characterized by the N–connection curvature $\mathbf{\Omega} = \{\Omega_{ij}^a\}$ as a Nijenhuis tensor field $N_v(X, Y)$ associated to $\mathbf{N}$,

$$\mathbf{\Omega} = N_v = [vX, vY] + v[X, Y] - v[vX, Y] - v[X, vY],$$

for $X, Y \in \mathcal{X}(V^{n+m})$ [41] and $[,]$ denoting commutators. In local form one has

$$\mathbf{\Omega} = \frac{1}{2}\Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \tag{1.20}$$

The 'N–elongated' operators δ_j from (1.20) are defined from a certain vielbein configuration induced by the N–connection, the N–elongated partial derivatives (in brief, N–derivatives)

$$\mathbf{e}_\alpha \doteqdot \delta_\alpha = (\delta_i, \partial_a) \equiv \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(u)\partial_a, \frac{\partial}{\partial y^a} \right) \tag{1.21}$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^\beta \doteqdot \delta^{\ \beta} = \left(d^i, \delta^a\right) \equiv \delta u^\alpha = \left(\delta x^i = dx^i, \delta y^a = dy^a + N_i^a(u)\, dx^i\right) \tag{1.22}$$

called also, respectively, the N–frame and N–coframe. [5]

The N–coframe (1.22) is anholonomic because there are satisfied the anholonomy relations (1.3),

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \mathbf{w}^{\gamma}{}_{\alpha\beta}(u)\, \delta_\gamma \tag{1.23}$$

for which the anholonomy coefficients $\mathbf{w}^{\alpha}_{\beta\gamma}(u)$ are computed to have certain nontrivial values

$$\mathbf{w}^{a}{}_{ji} = -\mathbf{w}^{a}{}_{ij} = \Omega^a_{ij},\ \mathbf{w}^{b}{}_{ia} = -\mathbf{w}^{b}{}_{ai} = \partial_a N^b_i. \tag{1.24}$$

We emphasize that the N–connection formalism is a natural one for investigating physical systems with mixed sets of holonomic–anholonomic variables. The imposed anholonomic constraints (anisotropies) are characterized by the coefficients of N–connection which defines a global splitting of the components of geometrical objects with respect to some 'horizontal' (holonomic) and 'vertical' (anisotropic) directions. In brief, we shall use respectively the terms h- and/or v–components, h- and/or v–indices, and h- and/or v–subspaces

A N–connection structure on $\mathbf{V}^{n+m}$ defines the algebra of tensorial distinguished (by N–connection structure) fields $dT\,(T\mathbf{V}^{n+m})$ (d–fields, d–tensors, d–objects, if to follow the terminology from [14]) on $\mathbf{V}^{n+m}$ introduced as the tensor algebra $\mathcal{T} = \{\mathcal{T}^{pr}_{qs}\}$ of the distinguished tangent bundle $\mathcal{V}_{(d)}$, $p_d:\ h\mathbf{V}^{n+m} \oplus v\mathbf{V}^{n+m} \rightarrow \mathbf{V}^{n+m}$. An element $\mathbf{t} \in \mathcal{T}^{pr}_{qs}$, a d–tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, can be written in local form as

$$\mathbf{t} = t^{i_1 \ldots i_p a_1 \ldots a_r}_{j_1 \ldots j_q b_1 \ldots b_r}(u)\, \delta_{i_1} \otimes \ldots \otimes \delta_{i_p} \otimes \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes d^{j_1} \otimes \ldots \otimes d^{j_q} \otimes \delta^{b_1} \ldots \otimes \delta^{b_r}.$$

There are used the denotations $\mathcal{X}\left(\mathcal{V}_{(d)}\right)$ (or $\mathcal{X}(\mathbf{V}^{n+m}), \wedge^p\left(\mathcal{V}_{(d)}\right)$ (or $\wedge^p(\mathbf{V}^{n+m})$ and $\mathcal{F}\left(\mathcal{V}_{(d)}\right)$ (or $\mathcal{F}\,(\mathbf{V}^{n+m})$) for the module of d–vector fields on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$), the exterior algebra of p–forms on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$) and the set of real functions on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$).

1.2.3 Distinguished linear connection and metric structures

The d–objects on $\mathcal{V}_{(d)}$ are introduced in a coordinate free form as geometric objects adapted to the N–connection structure. In coordinate form, we can characterize such objects (linear connections, metrics or any tensor field) by certain group and coordinate

[5] We shall use both type of denotations $\mathbf{e}_\alpha \doteqdot \delta_\alpha$ and $\vartheta^\beta \doteqdot \delta^{\ \alpha}$ in order to preserve a connection to denotations from Refs. [14, 23, 24, 34, 35, 25, 37]. The 'boldfaced' symbols $\mathbf{e}_\alpha$ and ϑ^β are written in order to emphasize that they define N–adapted vielbeins and the symbols δ_α and $\delta^{\ \beta}$ will be used for the N–elongated partial derivatives and, respectively, differentials.

transforms adapted to the N-connection structure on $\mathbf{V}^{n+m}$, i. e. to the global space splitting (1.14) into h- and v–subspaces.

d–connections

We analyze the general properties of a class of linear connections being adapted to the N–connection structure (called d–connections).

Definition 1.2.11. *A d–connection* $\mathbf{D}$ *on* $\mathcal{V}_{(d)}$ *is defined as a linear connection* D*, see Definition 1.2.1, on* $\mathcal{V}_{(d)}$ *conserving under a parallelism the global decomposition of* $T\mathbf{V}^{n+m}$ *(1.14) into the horizontal subbundle,* $h\mathbf{V}^{n+m}$*, and vertical subbundle,* $v\mathbf{V}^{n+m}$*, of* $\mathcal{V}_{(d)}$*.*

A N-connection induces decompositions of d–tensor indices into sums of horizontal and vertical parts, for example, for every d–vector $\mathbf{X} \in \mathcal{X}\left(\mathcal{V}_{(d)}\right)$ and 1-form $\widetilde{\mathbf{X}} \in \Lambda^1\left(\mathcal{V}_{(d)}\right)$ we have respectively

$$X = hX + vX \text{ and } \quad \widetilde{X} = h\widetilde{X} + v\widetilde{X}.$$

For simplicity, we shall not use boldface symbols for d–vectors and d–forms if this will not result in ambiguities. In consequence, we can associate to every d–covariant derivation $\mathbf{D}_X = X\rfloor\mathbf{D}$ two new operators of h- and v–covariant derivations, $\mathbf{D}_X = D_X^{[h]} + D_X^{[v]}$, defined respectively

$$D_X^{[h]}Y = \mathbf{D}_{hX}Y \quad \text{and} \quad D_X^{[v]}Y = \mathbf{D}_{vX}Y,$$

for which the following conditions hold:

$$\begin{aligned} \mathbf{D}_X Y &= D_X^{[h]}Y + D_X^{[v]}Y, \\ D_X^{[h]}f &= (hX)f \quad \text{and} \quad D_X^{[v]}f = (vX)f, \end{aligned} \tag{1.25}$$

for any $X, Y \in \mathcal{X}(E), f \in \mathcal{F}(V^{n+m})$.

The N–adapted components $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ of a d-connection $\mathbf{D}_\alpha = (\delta_\alpha\rfloor\mathbf{D})$ are defined by the equations

$$\mathbf{D}_\alpha \delta_\beta = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\delta_\gamma,$$

from which one immediately follows

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}(u) = (\mathbf{D}_\alpha\delta_\beta)\rfloor\delta^\gamma. \tag{1.26}$$

The operations of h- and v-covariant derivations, $D_k^{[h]} = \{L^i_{jk}, L^a_{bk}\}$ and $D_c^{[v]} = \{C^i_{jk}, C^a_{bc}\}$ (see (1.25)) are introduced as corresponding h- and v–parametrizations of (1.26),

$$L^i_{jk} = (\mathbf{D}_k\delta_j)\rfloor d^i, \quad L^a_{bk} = (\mathbf{D}_k\partial_b)\rfloor\delta^a \tag{1.27}$$

$$C^i_{jc} = (\mathbf{D}_c\delta_j)\rfloor d^i, \quad C^a_{bc} = (\mathbf{D}_c\partial_b)\rfloor\delta^a. \tag{1.28}$$

A set of h–components (1.27) and v–components (1.28), distinguished in the form $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$, completely defines the local action of a d–connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$. For instance, having taken a d–tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{t} = t^{ia}_{jb}\delta_i \otimes \partial_a \otimes \partial^j \otimes \delta^b$, and a d–vector $\mathbf{X} = X^i\delta_i + X^a\partial_a$ we can write

$$\mathbf{D}_X\mathbf{t} = D^{[h]}_X\mathbf{t} + D^{[v]}_X\mathbf{t} = \left(X^k t^{ia}_{jb|k} + X^c t^{ia}_{jb\perp c}\right)\delta_i \otimes \partial_a \otimes d^j \otimes \delta^b,$$

where the h–covariant derivative is

$$t^{ia}_{jb|k} = \frac{\delta t^{ia}_{jb}}{\delta x^k} + L^i_{hk}t^{ha}_{jb} + L^a_{ck}t^{ic}_{jb} - L^h_{jk}t^{ia}_{hb} - L^c_{bk}t^{ia}_{jc}$$

and the v–covariant derivative is

$$t^{ia}_{jb\perp c} = \frac{\partial t^{ia}_{jb}}{\partial y^c} + C^i_{hc}t^{ha}_{jb} + C^a_{dc}t^{id}_{jb} - C^h_{jc}t^{ia}_{hb} - C^d_{bc}t^{ia}_{jd}.$$

For a scalar function $f \in \mathcal{F}\left(V^{n+m}\right)$ we have

$$D^{[h]}_k = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N^a_k\frac{\partial f}{\partial y^a} \text{ and } D^{[v]}_c f = \frac{\partial f}{\partial y^c}.$$

We note that these formulas are written in abstract index form and specify for d–connections the covariant derivation rule (1.1).

Metric structures and d–metrics

We introduce arbitrary metric structures on a space $\mathbf{V}^{n+m}$ and consider the possibility to adapt them to N–connection structures.

Definition 1.2.12. *A metric structure* $\mathbf{g}$ *on a space* $\mathbf{V}^{n+m}$ *is defined as a symmetric covariant tensor field of type* $(0,2)$, $g_{\alpha\beta}$, *being nondegenerate and of constant signature on* $\mathbf{V}^{n+m}$.

This Definition is completely similar to Definition 1.2.6 but in our case it is adapted to the N–connection structure. A N–connection $\mathbf{N} = \{N^{\underline{b}}_{\underline{i}}(u)\}$ and a metric structure

$$\mathbf{g} = g_{\underline{\alpha}\underline{\beta}}du^{\underline{\alpha}} \otimes du^{\underline{\beta}} \tag{1.29}$$

on $\mathbf{V}^{n+m}$ are mutually compatible if there are satisfied the conditions

$$\mathbf{g}\left(\delta_{\underline{i}}, \partial_{\underline{a}}\right) = 0, \text{ or equivalently, } g_{\underline{ia}}(u) - N^{\underline{b}}_{\underline{i}}(u)\, h_{\underline{ab}}(u) = 0, \tag{1.30}$$

where $h_{\underline{ab}} \doteqdot \mathbf{g}\left(\partial_{\underline{a}}, \partial_{\underline{b}}\right)$ and $g_{\underline{ia}} \doteqdot \mathbf{g}\left(\partial_{\underline{i}}, \partial_{\underline{a}}\right)$ resulting in

$$N_i^b\left(u\right) = h^{ab}\left(u\right) g_{ia}\left(u\right) \quad (1.31)$$

(the matrix h^{ab} is inverse to h_{ab}; for simplicity, we do not underly the indices in the last formula). In consequence, we obtain a h–v–decomposition of metric (in brief, d–metric)

$$\mathbf{g}(X,Y){=}h\mathbf{g}(X,Y) + v\mathbf{g}(X,Y), \quad (1.32)$$

where the d-tensor $h\mathbf{g}(X,Y) = \mathbf{g}(hX,hY)$ is of type $\left(\begin{array}{cc} 0 & 0 \\ 2 & 0 \end{array}\right)$ and the d-tensor $v\mathbf{g}(X,Y){=}\ \mathbf{h}(vX,vY)$ is of type $\left(\begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array}\right)$. With respect to a N–coframe (1.22), the d–metric (1.32) is written

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}\left(u\right)\delta^{\alpha}\otimes\delta^{\beta} = g_{ij}\left(u\right)d^{i}\otimes d^{j} + h_{ab}\left(u\right)\delta^{a}\otimes\delta^{b}, \quad (1.33)$$

where $g_{ij} \doteqdot \mathbf{g}\left(\delta_i, \delta_j\right)$. The d–metric (1.33) can be equivalently written in "off–diagonal" form if the basis of dual vectors consists from the coordinate differentials (1.19),

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array}\right]. \quad (1.34)$$

It is easy to check that one holds the relations

$$\mathbf{g}_{\alpha\beta} = \mathbf{e}_{\alpha}^{\ \underline{\alpha}}\mathbf{e}_{\beta}^{\ \underline{\beta}}\underline{g}_{\underline{\alpha}\underline{\beta}}$$

or, inversely,

$$\underline{g}_{\underline{\alpha}\underline{\beta}} = \mathbf{e}^{\alpha}_{\ \underline{\alpha}}\mathbf{e}^{\beta}_{\ \underline{\beta}}\mathbf{g}_{\alpha\beta}$$

as it is stated by respective vielbein transforms (1.16) and (1.17).

Remark 1.2.1. *A metric, for instance, parametrized in the form (1.34) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. If the anholonomy coefficients (1.24) vanish for a such parametrization, we can define certain coordinate transforms to diagonalize both the off–diagonal form (1.34) and the equivalent d–metric (1.33).*

Definition 1.2.13. *The nonmetricity d–field*

$$\mathcal{Q} = \mathbf{Q}_{\alpha\beta}\vartheta^{\alpha}\otimes\vartheta^{\beta} = \mathbf{Q}_{\alpha\beta}\delta^{\ \alpha}\otimes\delta^{\beta}$$

on a space $\mathbf{V}^{n+m}$ *provided with N–connection structure is defined by a d–tensor field with the coefficients*

$$\mathbf{Q}_{\alpha\beta} \doteqdot -\mathbf{D}\mathbf{g}_{\alpha\beta} \tag{1.35}$$

where the covariant derivative $\mathbf{D}$ *is for a d–connection* $\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, *see (1.26) with the respective splitting* $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right)$, *as to be adapted to the N–connection structure.*

This definition is similar to that given for metric–affine spaces (see definition 1.2.8) and Refs. [4], but in our case the N–connection establishes some 'preferred' N–adapted local frames (1.21) and (1.22) splitting all geometric objects into irreducible h- and v–components. A linear connection D_X is compatible with a d–metric $\mathbf{g}$ if

$$D_X\mathbf{g} = 0, \tag{1.36}$$

$\forall X \in \mathcal{X}\left(V^{n+m}\right)$, i. e. if $Q_{\alpha\beta} \equiv 0$. In a space provided with N–connection structure, the metricity condition (1.36) may split into a set of compatibility conditions on h- and v–subspaces. We should consider separately which of the conditions

$$D^{[h]}(h\mathbf{g}) = 0, D^{[v]}(h\mathbf{g}) = 0, D^{[h]}(v\mathbf{g}) = 0, D^{[v]}(v\mathbf{g}) = 0 \tag{1.37}$$

are satisfied, or not, for a given d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$. For instance, if $D^{[v]}(h\mathbf{g}) = 0$ and $D^{[h]}(v\mathbf{g}) = 0$, but, in general, $D^{[h]}(h\mathbf{g}) \neq 0$ and $D^{[v]}(v\mathbf{g}) \neq 0$ we can consider a nonmetricity d–field (d–nonmetricity) $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ with irreducible h–v–components (with respect to the N–connection decompositions), $\mathbf{Q}_{\gamma\alpha\beta} = \left(Q_{ijk}, Q_{abc}\right)$.

By acting on forms with the covariant derivative D, in a metric–affine space, we can also define another very important geometric objects (the 'gravitational field potentials', see [4]):

$$\text{torsion } \ \mathcal{T}^{\alpha} \doteqdot D\vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma^{\gamma}{}_{\beta} \wedge \vartheta^{\beta}, \text{ see Definition 1.2.3} \tag{1.38}$$

and

$$\text{curvature } \ \mathcal{R}^{\alpha}{}_{\beta} \doteqdot D\Gamma^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} - \Gamma^{\gamma}{}_{\beta} \wedge \Gamma^{\alpha}{}_{\gamma}, \text{ see Definition 1.2.4.} \tag{1.39}$$

The Bianchi identities are

$$DQ_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} + \mathcal{R}_{\beta\alpha}, \ D\mathcal{T}^{\alpha} \equiv \mathcal{R}_{\gamma}{}^{\alpha} \wedge \vartheta^{\gamma} \text{ and } D\mathcal{R}_{\gamma}{}^{\alpha} \equiv 0, \tag{1.40}$$

where we stress the fact that $Q_{\alpha\beta}, T^{\alpha}$ and $R_{\beta\alpha}$ are called also the strength fields of a metric–affine theory.

For spaces provided with N–connections, we write the corresponding formulas by using "boldfaced" symbols and change the usual differential d into N-adapted operator δ.

$$\mathbf{T}^{\alpha} \doteqdot \mathbf{D}\vartheta^{\alpha} = \delta\vartheta^{\alpha} + \mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \vartheta^{\beta} \quad (1.41)$$

and

$$\mathbf{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \delta\mathbf{\Gamma}^{\alpha}{}_{\beta} - \mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \mathbf{\Gamma}^{\alpha}{}_{\gamma} \quad (1.42)$$

where the Bianchi identities written in 'boldfaced' symbols split into h- and v–irreducible decompositions induced by the N–connection. [6]. We shall examine and compute the general form of torsion and curvature d–tensors in spaces provided with N–connection structure in section 1.2.4.

We note that the bulk of works on Finsler geometry and generalizations [15, 14, 20, 16, 17, 19, 13, 23, 24, 37] consider very general linear connection and metric fields being adapted to the N–connection structure. In another turn, the researches on metric–affine gravity [4, 38] concern generalizations to nonmetricity but not N–connections. In this work, we elaborate a unified moving frame geometric approach to both Finlser like and metric–affine geometries.

1.2.4 Torsions and curvatures of d–connections

We define and calculate the irreducible components of torsion and curvature in a space $\mathbf{V}^{n+m}$ provided with additional N–connection structure (these could be any metric–affine spaces [4], or their particular, like Riemann–Cartan [38], cases with vanishing nonmetricity and/or torsion, or any (co) vector / tangent bundles like in Finsler geometry and generalizations).

d–torsions and N–connections

We give a definition being equivalent to (1.41) but in d–operator form (the Definition 1.2.3 was for the spaces not possessing N–connection structure):

Definition 1.2.14. *The torsion* $\mathbf{T}$ *of a d–connection* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in space* $\mathbf{V}^{n+m}$ *is defined as an operator (d–tensor field) adapted to the N–connection structure*

$$\mathbf{T}(X,Y) = \mathbf{D}_X Y - \mathbf{D}_Y X - [X,Y]. \quad (1.43)$$

[6] see similar details in Ref. [14] for the case of vector/tangent bundles provided with mutually compatible N–connection, d–connection and d–metric structure

One holds the following h- and v–decompositions

$$\mathbf{T}(X,Y) = \mathbf{T}(hX,hY) + \mathbf{T}(hX,vY) + \mathbf{T}(vX,hY) + \mathbf{T}(vX,vY). \tag{1.44}$$

We consider the projections: $h\mathbf{T}(X,Y)$, $v\mathbf{T}(hX,hY)$, $h\mathbf{T}(hX,hY)$, ... and say that, for instance, $h\mathbf{T}(hX,hY)$ is the h(hh)-torsion of D , $vT(hX,hY)$ is the v(hh)-torsion of $\mathbf{D}$ and so on.

The torsion (1.43) is locally determined by five d–tensor fields, d–torsions (irreducible N–adapted h–v–decompositions) defined as

$$\begin{aligned} T^i_{jk} &= h\mathbf{T}(\delta_k,\delta_j)\rfloor d^i, \quad T^a_{jk} = v\mathbf{T}(\delta_k,\delta_j)\rfloor \delta^a, \quad P^i_{jb} = h\mathbf{T}(\partial_b,\delta_j)\rfloor d^i, \\ P^a_{jb} &= v\mathbf{T}(\partial_b,\delta_j)\rfloor \delta^a, \quad S^a_{bc} = v\mathbf{T}(\partial_c,\partial_b)\rfloor \delta^a. \end{aligned}$$

Using the formulas (1.21), (1.22), and (1.20), we can calculate the h–v–components of torsion (1.44) for a d–connection, i. e. we can prove [7]

Theorem 1.2.2. *The torsion* $\mathbf{T}^{\alpha}_{.\beta\gamma} = (T^i_{.jk}, T^i_{ja}, T^a_{.ij}, T^a_{.bi}, T^a_{.bc})$ *of a d–connection* $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ *(1.26) has irreducible h- v–components (d–torsions)*

$$\begin{aligned} T^i_{.jk} &= -T^i_{kj} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{.ja}, \ T^a_{.ji} = -T^a_{.ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i} = \Omega^a_{.ji}, \\ T^a_{.bi} &= -T^a_{.ib} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj}, \ T^a_{.bc} = -T^a_{.cb} = S^a_{.bc} = C^a_{bc} - C^a_{cb}. \end{aligned} \tag{1.45}$$

We note that on (pseudo) Riemanian spacetimes the d–torsions can be induced by the N–connection coefficients and reflect an anholonomic frame structures. Such objects vanishes when we transfer our considerations with respect to holonomic bases for a trivial N–connection and zero "vertical" dimension.

d–curvatures and N–connections

In operator form, the curvature (1.42) is stated from the

Definition 1.2.15. *The curvature* $\mathbf{R}$ *of a d–connection* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in space* $\mathbf{V}^{n+m}$ *is defined as an operator (d–tensor field) adapted to the N–connection structure*

$$\mathbf{R}(X,Y)Z = \mathbf{D}_X\mathbf{D}_Y Z - \mathbf{D}_Y\mathbf{D}_X Z - \mathbf{D}_{[X,Y]}Z. \tag{1.46}$$

[7] see also the original proof for vector bundles in [14]

This Definition is similar to the Definition 1.2.4 being a generalization for the spaces provided with N–connection. One holds certain properties for the h- and v–decompositions of curvature:

$$v\mathbf{R}(X,Y)\,hZ=0,\ h\mathbf{R}(X,Y)\,vZ=0,\ \mathbf{R}(X,Y)\,Z=h\mathbf{R}(X,Y)\,hZ+v\mathbf{R}(X,Y)\,vZ.$$

From (1.46) and the equation $\mathbf{R}(X,Y)=-\mathbf{R}(Y,X)$, we get that the curvature of a d-connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$ is completely determined by the following six d–tensor fields (d–curvatures):

$$\begin{aligned} R^{i}{}_{hjk} &= d^{i}\rfloor\mathbf{R}(\delta_k,\delta_j)\,\delta_h,\ R^{a}{}_{bjk}=\delta^{a}\rfloor\mathbf{R}(\delta_k,\delta_j)\,\partial_b, \\ P^{i}{}_{jkc} &= d^{i}\rfloor\mathbf{R}(\partial_c,\partial_k)\,\delta_j,\ P^{a}{}_{bkc}=\delta^{a}\rfloor\mathbf{R}(\partial_c,\partial_k)\,\partial_b, \\ S^{i}{}_{jbc} &= d^{i}\rfloor\mathbf{R}(\partial_c,\partial_b)\,\delta_j,\ S^{a}{}_{bcd}=\delta^{a}\rfloor\mathbf{R}(\partial_d,\partial_c)\,\partial_b. \end{aligned} \quad (1.47)$$

By a direct computation, using (1.21), (1.22), (1.27), (1.28) and (1.47), we prove

Theorem 1.2.3. *The curvature* $\mathbf{R}^{\alpha}_{.\beta\gamma\tau}=(R^{i}{}_{hjk},R^{a}{}_{bjk},P^{i}{}_{jka},P^{c}{}_{bka},S^{i}{}_{jbc},S^{a}{}_{bcd})$ *of a d–connection* $\mathbf{\Gamma}^{\gamma}_{\alpha\beta}=\left(L^{i}_{jk},L^{a}_{bk},C^{i}_{jc},C^{a}_{bc}\right)$ *(1.26) has the h- v–components (d–curvatures)*

$$\begin{aligned} R^{i}{}_{hjk} &= \frac{\delta L^{i}_{.hj}}{\delta x^{k}}-\frac{\delta L^{i}_{.hk}}{\delta x^{j}}+L^{m}_{.hj}L^{i}_{mk}-L^{m}_{.hk}L^{i}_{mj}-C^{i}_{.ha}\Omega^{a}_{.jk}, \\ R^{a}{}_{bjk} &= \frac{\delta L^{a}_{.bj}}{\delta x^{k}}-\frac{\delta L^{a}_{.bk}}{\delta x^{j}}+L^{c}_{.bj}L^{a}_{.ck}-L^{c}_{.bk}L^{a}_{.cj}-C^{a}_{.bc}\,\Omega^{c}_{.jk}, \\ P^{i}{}_{jka} &= \frac{\partial L^{i}_{.jk}}{\partial y^{k}}-\left(\frac{\partial C^{i}_{.ja}}{\partial x^{k}}+L^{i}_{.lk}C^{l}_{.ja}-L^{l}_{.jk}C^{i}_{.la}-L^{c}_{.ak}C^{i}_{.jc}\right)+C^{i}_{.jb}P^{b}_{.ka}, \\ P^{c}{}_{bka} &= \frac{\partial L^{c}_{.bk}}{\partial y^{a}}-\left(\frac{\partial C^{c}_{.ba}}{\partial x^{k}}+L^{c}_{.dk}C^{d}_{.ba}-L^{d}_{.bk}C^{c}_{.da}-L^{d}_{.ak}C^{c}_{.bd}\right)+C^{c}_{.bd}P^{d}_{.ka}, \\ S^{i}{}_{jbc} &= \frac{\partial C^{i}_{.jb}}{\partial y^{c}}-\frac{\partial C^{i}_{.jc}}{\partial y^{b}}+C^{h}_{.jb}C^{i}_{.hc}-C^{h}_{.jc}C^{i}_{hb}, \\ S^{a}{}_{bcd} &= \frac{\partial C^{a}_{.bc}}{\partial y^{d}}-\frac{\partial C^{a}_{.bd}}{\partial y^{c}}+C^{e}_{.bc}C^{a}_{.ed}-C^{e}_{.bd}C^{a}_{.ec}. \end{aligned} \quad (1.48)$$

The components of the Ricci d-tensor

$$\mathbf{R}_{\alpha\beta}=\mathbf{R}^{\tau}{}_{\alpha\beta\tau}$$

with respect to a locally adapted frame (1.21) has four irreducible h- v–components, $\mathbf{R}_{\alpha\beta}=\{R_{ij},R_{ia},R_{ai},S_{ab}\}$, where

$$\begin{aligned} R_{ij} &= R^{k}{}_{ijk},\quad R_{ia}=-\,{}^{2}P_{ia}=-P^{k}{}_{ika}, \\ R_{ai} &= {}^{1}P_{ai}=P^{b}{}_{aib},\quad S_{ab}=S^{c}{}_{abc}. \end{aligned} \quad (1.49)$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d–tensor is non symmetric.

Having defined a d–metric of type (1.33) in $\mathbf{V}^{n+m}$, we can introduce the scalar curvature of a d–connection $\mathbf{D}$,

$$\overleftarrow{\mathbf{R}} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = R + S, \tag{1.50}$$

where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$ and define the distinguished form of the Einstein tensor (the Einstein d–tensor), see Definition 1.2.7,

$$\mathbf{G}_{\alpha\beta} \doteqdot \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}. \tag{1.51}$$

The Ricci and Bianchi identities (1.40) of d–connections are formulated in h- v-irreducible forms on vector bundle [14]. The same formulas hold for arbitrary metric compatible d–connections on $\mathbf{V}^{n+m}$ (for simplicity, we omit such details in this work).

1.3 Some Classes of Linear and Nonlinear Connections

The geometry of d–connections in a space $\mathbf{V}^{n+m}$ provided with N–connection structure is very reach (works [14] and [23] contain results on generalized Finsler spaces and superspaces). If a triple of fundamental geometric objects $\left(N_i^a(u), \mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}(u), \mathbf{g}_{\alpha\beta}(u)\right)$ is fixed on $\mathbf{V}^{n+m}$, in general, with respect to N–adapted frames, a multi–connection structure is defined (with different rules of covariant derivation). In this Section, we analyze a set of linear connections and associated covariant derivations being very important for investigating spacetimes provided with anholonomic frame structure and generic off–diagonal metrics.

1.3.1 The Levi–Civita connection and N–connections

The Levi–Civita connection $\bigtriangledown = \{\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}\}$ with coefficients

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = g\left(\mathbf{e}_{\alpha}, \bigtriangledown_{\gamma}\mathbf{e}_{\beta}\right) = \mathbf{g}_{\alpha\tau}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}, \tag{1.52}$$

is torsionless,

$$\mathbf{T}^{\alpha}_{\bigtriangledown} \doteqdot \bigtriangledown\vartheta^{\alpha} = d\vartheta^{\alpha} + \mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} \wedge \vartheta^{\beta} = 0,$$

and metric compatible, $\bigtriangledown\mathbf{g} = 0$, see see Definition 1.2.1. The formula (1.52) states that the operator $\bigtriangledown$ can be defined on spaces provided with N–connection structure (we use 'boldfaced' symbols) but this connection is not adapted to the N–connection splitting (1.14). It is defined as a linear connection but not as a d–connection, see Definition 1.2.11. The Levi–Civita connection is usually considered on (pseudo) Riemannian spaces but it can be also introduced, for instance, in (co) vector/tangent bundles both with respect to coordinate and anholonomic frames [14, 34, 35]. One holds a Theorem similar to the Theorem 1.2.1,

Theorem 1.3.4. *If a space $\mathbf{V}^{n+m}$ is provided with both N–connection $\mathbf{N}$ and d–metric $\mathbf{g}$ structures, there is a unique linear symmetric and torsionless connection $\bigtriangledown$, being metric compatible such that $\bigtriangledown_\gamma \mathbf{g}_{\alpha\beta} = 0$ for $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, see (1.33), with the coefficients*

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \mathbf{g}\left(\delta_\alpha, \bigtriangledown_\gamma \delta_\beta\right) = \mathbf{g}_{\alpha\tau}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma},$$

computed as

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_\beta \mathbf{g}_{\alpha\gamma} + \delta_\gamma \mathbf{g}_{\beta\alpha} - \delta_\alpha \mathbf{g}_{\gamma\beta} + \mathbf{g}_{\alpha\tau}\mathbf{w}^{\tau}_{\gamma\beta} + \mathbf{g}_{\beta\tau}\mathbf{w}^{\tau}_{\alpha\gamma} - \mathbf{g}_{\gamma\tau}\mathbf{w}^{\tau}_{\beta\alpha}\right] \tag{1.53}$$

with respect to N–frames $\mathbf{e}_\beta \doteqdot \delta_\beta$ (1.21) and N–coframes $\vartheta^\alpha \doteqdot \delta^\alpha$ (1.22).

The proof is that from Theorem 1.2.1, see also Refs. [45, 46], with $e_\beta \rightarrow \mathbf{e}_\beta$ and $\vartheta^\beta \rightarrow \vartheta^\beta$ substituted directly in formula (1.10).

With respect to coordinate frames ∂_β (1.18) and du^α (1.19), the metric (1.33) transforms equivalently into (1.29) with coefficients (1.34) and the coefficients of (1.53) transform into the usual Christoffel symbols (1.11). We emphasize that we shall use the coefficients just in the form (1.53) in order to compare the properties of different classes of connections given with respect to N–adapted frames. The coordinate form (1.11) is not "N–adapted", being less convenient for geometric constructions on spaces with anholonomic frames and associated N–connection structure.

We can introduce the 1-form formalism and express

$$\mathbf{\Gamma}^{\bigtriangledown}_{\gamma\alpha} = \mathbf{\Gamma}^{\bigtriangledown}_{\gamma\alpha\beta}\vartheta^\beta$$

where

$$\mathbf{\Gamma}^{\bigtriangledown}_{\gamma\alpha} = \frac{1}{2}\left[\mathbf{e}_\gamma \rfloor\ \delta\vartheta_\alpha - \mathbf{e}_\alpha \rfloor\ \delta\vartheta_\gamma - (\mathbf{e}_\gamma \rfloor\ \mathbf{e}_\alpha \rfloor\ \delta\vartheta_\beta) \wedge \vartheta^\beta\right], \tag{1.54}$$

contains h- v-components, $\mathbf{\Gamma}^{\gamma}_{\bigtriangledown\alpha\beta} = \left(L^i_{\bigtriangledown jk}, L^a_{\bigtriangledown bk}, C^i_{\bigtriangledown jc}, C^a_{\bigtriangledown bc}\right)$, defined similarly to (1.27) and (1.28) but using the operator $\bigtriangledown$,

$$L^i_{\bigtriangledown jk} = (\bigtriangledown_k \delta_j)\rfloor d^i, \quad L^a_{\bigtriangledown bk} = (\bigtriangledown_k \partial_b)\rfloor \delta^a, \ C^i_{\bigtriangledown jc} = (\bigtriangledown_c \delta_j)\rfloor d^i, \quad C^a_{\bigtriangledown bc} = (\bigtriangledown_c \partial_b)\rfloor \delta^a.$$

In explicit form, the components $L^{i}_{\bigtriangledown jk}, L^{a}_{\bigtriangledown bk}, C^{i}_{\bigtriangledown jc}$ and $C^{a}_{\bigtriangledown bc}$ are defined by formula (1.54) if we consider N–frame $\mathbf{e}_\gamma = (\delta_i = \partial_i - N^a_i\partial_a, \partial_a)$ and N–coframe $\vartheta^\beta = (dx^i, \delta y^a = dy^a + N^a_i dx^i)$ and a d–metric $\mathbf{g} = (g_{ij,} h_{ab})$. In these formulas, we write $\delta\vartheta_\alpha$ instead of absolute differentials $d\vartheta_\alpha$ from Refs. [4, 38] because the N–connection is considered. The coefficients (1.54) transforms into the usual Levi–Civita (or Christoffel) ones for arbitrary anholonomic frames e_γ and ϑ^β and for a metric

$$g = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$$

if $\mathbf{e}_\gamma \rightarrow e_\gamma$, $\vartheta^\beta \rightarrow \vartheta^\beta$ and $\delta\vartheta_\beta \rightarrow d\vartheta_\beta$.

Finally, we note that if the N–connection structure is not trivial, we can define arbitrary vielbein transforms starting from $\mathbf{e}_\gamma$ and ϑ^β, i. e. $e^{[N]}_\alpha = A_\alpha{}^{\alpha'}(u)\mathbf{e}_{\alpha'}$ and $\vartheta^\beta_{[N]} = A^\beta{}_{\beta'}(u)\vartheta^{\beta'}$ (we put the label $[N]$ in order to emphasize that such object were defined by vielbein transforms starting from certain N–adapted frames). This way we develop a general anholonomic frame formalism adapted to the prescribed N–connection structure. If we consider geometric objects with respect to coordinate frames $\mathbf{e}_{\alpha'} \rightarrow \partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ and coframes $\vartheta^{\beta'} \rightarrow du^{\underline{\beta}}$, the N–connection structure is 'hidden' in the off–diagonal metric coefficients (1.34) and performed geometric constructions, in general, are not N–adapted.

1.3.2 The canonical d–connection and the Levi–Civita connection

The Levi–Civita connection $\bigtriangledown$ is constructed only from the metric coefficients, being torsionless and satisfying the metricity conditions $\bigtriangledown_\alpha g_{\beta\gamma} = 0$. Because the Levi–Civita connection is not adapted to the N–connection structure, we can not state its coefficients in an irreducible form for the h– and v–subspaces. We need a type of d–connection which would be similar to the Levi–Civita connection but satisfy certain metricity conditions adapted to the N–connection.

Proposition 1.3.1. *There are metric d–connections* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in a space* $\mathbf{V}^{n+m}$, *see (1.25), satisfying the metricity conditions if and only if*

$$D^{[h]}_k g_{ij} = 0,\ D^{[v]}_a g_{ij} = 0,\ D^{[h]}_k h_{ab} = 0,\ D^{[h]}_a h_{ab} = 0. \quad (1.55)$$

The general proof of existence of such metric d–connections on vector (super) bundles is given in Ref. [14]. Here we note that the equations (1.55) on $\mathbf{V}^{n+m}$ are just the conditions (1.37). In our case the existence may be proved by constructing an explicit example:

Definition 1.3.16. *The canonical d–connection* $\widehat{\mathbf{D}} = \left(\widehat{D}^{[h]}, \widehat{D}^{[v]}\right)$, *equivalently* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, *is defined by the h– v–irreducible components* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$,

$$\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2}g^{ir}\left(\frac{\delta g_{jk}}{\delta x^{k}} + \frac{\delta g_{kr}}{\delta x^{j}} - \frac{\delta g_{jk}}{\delta x^{r}}\right), \qquad (1.56)\\
\widehat{L}^{a}_{bk} &= \frac{\partial N^{a}_{k}}{\partial y^{b}} + \frac{1}{2}h^{ac}\left(\frac{\delta h_{bc}}{\delta x^{k}} - \frac{\partial N^{d}_{k}}{\partial y^{b}}h_{dc} - \frac{\partial N^{d}_{k}}{\partial y^{c}}h_{db}\right),\\
\widehat{C}^{i}_{jc} &= \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^{c}},\\
\widehat{C}^{a}_{bc} &= \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^{c}} + \frac{\partial h_{cd}}{\partial y^{b}} - \frac{\partial h_{bc}}{\partial y^{d}}\right).
\end{aligned}$$

satisfying the torsionless conditions for the h–subspace and v–subspace, respectively, $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0$.

By straightforward calculations with (1.56) we can verify that the conditions (1.55) are satisfied and that the d–torsions are subjected to the conditions $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0$ (see section 1.2.4)). We emphasize that the canonical d–torsion posses nonvanishing torsion components,

$$\widehat{T}^{a}_{.ji} = -\widehat{T}^{a}_{.ij} = \frac{\delta N^{a}_{i}}{\delta x^{j}} - \frac{\delta N^{a}_{j}}{\delta x^{i}} = \Omega^{a}_{.ji},\ \widehat{T}^{i}_{ja} = -\widehat{T}^{i}_{aj} = \widehat{C}^{i}_{.ja},\ \widehat{T}^{a}_{.bi} = -\widehat{T}^{a}_{.ib} = \widehat{P}^{a}_{.bi} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - \widehat{L}^{a}_{.bj}$$

induced by $\widehat{L}^{a}_{bk}$, $\widehat{C}^{i}_{jc}$ and N–connection coefficients N^{a}_{i} and their partial derivatives $\partial N^{a}_{i}/\partial y^{b}$ (as is to be computed by introducing (1.56) in formulas (1.45)). This is an anholonmic frame effect.

Proposition 1.3.2. *The components of the Levi–Civita connection* $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}$ *and the irreducible components of the canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ *are related by formulas*

$$\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk} - \frac{\partial N^{a}_{k}}{\partial y^{b}}, \widehat{C}^{i}_{jc} + \frac{1}{2}g^{ik}\Omega^{a}_{jk}h_{ca}, \widehat{C}^{a}_{bc}\right), \qquad (1.57)$$

where Ω^{a}_{jk} *is the N–connection curvature (1.20).*

The proof follows from an explicit decomposition of N–adapted frame (1.21) and N–adapted coframe (1.22) in (1.53) (equivalently, in (1.54)) and re–grouping the components as to distinguish the h- and v– irreducible values (1.56) for $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$.

We conclude from (1.57) that, in a trivial case, the Levi–Civita and the canonical d–connection are given by the same h– v– components $\left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$ if $\Omega^{a}_{jk} = 0,$ and $\partial N^{a}_{k}/\partial y^{b} = 0$. This results in zero anholonomy coefficients (1.24) when the anholonomic N–basis is reduced to a holonomic one. It should be also noted that even in this case some components of the anholonomically induced by d–connection torsion $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$ could be nonzero (see formulas (1.95) for $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$). For instance, one holds the

Corollary 1.3.1. *The d–tensor components*

$$\widehat{T}^{a}_{.bi} = -\widehat{T}^{a}_{.ib} = \widehat{P}^{a}_{.bi} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - \widehat{L}^{a}_{.bj} \tag{1.58}$$

for a canonical d–connection (1.56) can be nonzero even $\partial N^{a}_{k}/\partial y^{b} = 0$ *and* $\Omega^{a}_{jk} = 0$ *and a trivial equality of the components of the canonical d–connection and of the Levi–Civita connection,* $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ *holds with respect to coordinate frames.*

This quite surprising fact follows from the anholonomic character of the N–connection structure. If a N–connection is defined, there are imposed specific types of constraints on the frame structure. This is important for definition of d–connections (being adapted to the N–connection structure) but not for the Levi–Civita connection which is not a d–connection. Even such linear connections have the same components with respect to a N–adapted (co) frame, they are very different geometrical objects because they are subjected to different rules of transformation with respect to frame and coordinate transforms. The d–connections' transforms are adapted to those for the N–connection (1.15) but the Levi–Civita connection is subjected to general rules of linear connection transforms (1.2).[8]

Proposition 1.3.3. *A canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ *defined by a N–connection* N^{a}_{i} *and d–metric* $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *has zero d–torsions (1.95) if an only if there are satisfied the*

[8]The Corollary 1.3.1 is important for constructing various classes of exact solutions with generic off–diagonal metrics in Einstein gravity, its higher dimension and/or different gauge, Einstein--Cartan and metric–affine generalizations. Certain type of ansatz were proven to result in completely integrable gravitational field equations for the canonical d–connection (but not for the Levi–Civita one), see details in Refs. [34, 35, 25, 37]. The induced d–torsion (1.58) is contained in the Ricci d–tensor $R_{ai} = {}^{1}P_{ai} = P^{.b}_{a.ib}$, see (1.49), i. e. in the Einstein d–tensor constructed for the canonical d–connection. If a class of solutions were obtained for a d–connection, we can select those subclasses which satisfy the condition $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ with respect to a frame of reference. In this case the nontrivial d–torsion $\widehat{T}^{a}_{.bi}$ (1.58) can be treated as an object constructed from some "pieces" of a generic off–diagonal metric and related to certain components of the N–adapted anholonomic frames.

conditions $\Omega^a_{jk} = 0, \widehat{C}^i_{jc} = 0$ *and* $\widehat{L}^a_{.bj} = \partial N^a_i / \partial y^b$, *i. e.*
$\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk} = \partial N^a_i / \partial y^b, 0, \widehat{C}^a_{bc} \right)$ *which is equivalent to*

$$g^{ik} \frac{\partial g_{jk}}{\partial y^c} = 0, \quad (1.59)$$

$$\frac{\delta h_{bc}}{\delta x^k} - \frac{\partial N^d_k}{\partial y^b} h_{dc} - \frac{\partial N^d_k}{\partial y^c} h_{db} = 0, \quad (1.60)$$

$$\frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b} = 0. \quad (1.61)$$

The Levi–Civita connection defined by the same N–connection and d–metric structure with respect to N–adapted (co) frames has the components
${}^{[0]}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma} = \left(\widehat{L}^i_{jk}, 0, 0, \widehat{C}^a_{bc} \right).$

Proof: The relations (1.59)–(1.61) follows from the condition of vanishing of d–torsion coefficients (1.95) when the coefficients of the canonical d–connection and the Levi–Civita connection are computed respectively following formulas (1.56) and (1.57)

We note a specific separation of variables in the equations (1.59)–(1.61). For instance, the equation (1.59) is satisfied by any $g_{ij} = g_{ij}\left(x^k\right)$. We can search a subclass of N–connections with $N^a_j = \delta_j N^a$, i. e. of 1–forms on the h–subspace, $\widetilde{N}^a = \delta_j N^a dx^i$ which are closed on this subspace,

$$\delta \widetilde{N}^a = \frac{1}{2} \left(\frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b} \right) dx^i \wedge dx^j = 0,$$

satisfying the (1.61). Having defined such N^a_i and computing the values $\partial_c N^a_i$, we may try to solve (1.60) rewritten as a system of first order partial differential equations

$$\frac{\partial h_{bc}}{\partial x^k} = N^e_k \frac{\partial h_{bc}}{\partial y^e} + \partial_b N^d_k \ h_{dc} + \partial_c N^d_k h_{db}$$

with known coefficients.■

We can also associate the nontrivial values of $\widehat{\mathbf{T}}^{\tau}_{\beta\gamma}$ (in particular cases, of $\widehat{T}^a_{.bi}$) to be related to any algebraic equations in the Einstein–Cartan theory or dynamical equations for torsion like in string or supergravity models. But in this case we shall prescribe a specific class of anholonomically constrained dynamics for the N–adapted frames.

Finally, we note that if a (pseudo) Riemannian space is provided with a generic off–diagonal metric structure (see Remark 1.2.1) we can consider alternatively to the Levi–Civita connection an infinite number of metric d–connections, details in the section 1.3.5. Such d–connections have nontrivial d–torsions $\mathbf{T}^{\tau}_{\beta\gamma}$ induced by anholonomic frames and constructed from off–diagonal metric terms and h- and v–components of d–metrics.

1.3.3 The set of metric d–connections

Let us define the set of all possible metric d–connections, satisfying the conditions (1.55) and being constructed only form g_{ij}, h_{ab} and N_i^a and their partial derivatives. Such d–connections satisfy certain conditions for d–torsions that $T^i{}_{jk} = T^a{}_{bc} = 0$ and can be generated by two procedures of deformation of the connection

$$\begin{array}{rl} \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} & \rightarrow \quad {}^{[K]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} + \ {}^{[K]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} \text{ (Kawaguchi's metrization [39])}, \\ \text{or} & \rightarrow \quad {}^{[M]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + \ {}^{[M]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} \text{ (Miron's connections [14]).} \end{array}$$

Theorem 1.3.5. *Every deformation d–tensor (equivalently, distorsion, or deflection)*

$$\begin{array}{rcl} {}^{[K]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} & = & \{ \ {}^{[K]}Z^i{}_{jk} = \frac{1}{2} g^{im} D_j^{[h]} g_{mk}, \ {}^{[K]}Z^a{}_{bk} = \frac{1}{2} h^{ac} D_k^{[h]} h_{cb}, \\ & & {}^{[K]}Z^i{}_{ja} = \frac{1}{2} g^{im} D_a^{[v]} g_{mj}, \ {}^{[K]}Z^a{}_{bc} = \frac{1}{2} h^{ad} D_c^{[v]} h_{db} \} \end{array}$$

transforms a d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc} \right)$ *(1.26) into a metric d–connection*

$${}^{[K]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^i_{jk} + \ {}^{[K]}Z^i{}_{jk}, L^a_{bk} + \ {}^{[K]}Z^a{}_{bk}, C^i_{jc} + \ {}^{[K]}Z^i{}_{ja}, C^a_{bc} + \ {}^{[K]}Z^a{}_{bc} \right).$$

The proof consists from a straightforward verification which demonstrate that the conditions (1.55) are satisfied on $\mathbf{V}^{n+m}$ for ${}^{[K]}\mathbf{D} = \{ {}^{[\mathbf{K}]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta} \}$ and $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$. We note that the Kawaguchi's metrization procedure contains additional covariant derivations of the d–metric coefficients, defined by arbitrary d–connection, not only N–adapted derivatives of the d–metric and N–connection coefficients as in the case of the canonical d–connection.

Theorem 1.3.6. *For a fixed d–metric structure (1.33),* $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, *on a space* $\mathbf{V}^{n+m}$, *the set of metric d–connections* ${}^{[M]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + \ {}^{[M]}\mathbf{Z}^{\gamma}{}_{\alpha\beta}$ *is defined by the deformation d–tensor*

$$\begin{array}{rcl} {}^{[M]}\mathbf{Z}^{\gamma}_{\alpha\beta} & = & \{ \ {}^{[M]}Z^i{}_{jk} = \ {}^{[-]}O^{li}_{km} Y^m_{lj}, \ {}^{[M]}Z^a{}_{bk} = \ {}^{[-]}O^{ea}_{bd} Y^m_{ej}, \\ & & {}^{[M]}Z^i{}_{ja} = \ {}^{[+]}O^{mi}_{jk} Y^k_{mc}, \ {}^{[M]}Z^a{}_{bc} = \ {}^{[+]}O^{ea}_{bd} Y^d_{ec} \} \end{array}$$

where the so–called Obata operators are defined

$${}^{[\pm]}O^{li}_{km} = \frac{1}{2} \left(\delta^l_k \delta^i_m \pm g_{km} g^{li} \right) \ \text{ and } \ {}^{[\pm]}O^{ea}_{bd} = \frac{1}{2} \left(\delta^e_b \delta^a_d \pm h_{bd} h^{ea} \right)$$

and $Y^m_{lj}, Y^m_{ej}, Y^k_{mc}, Y^d_{ec}$ *are arbitrary d–tensor fields.*

The proof consists from a direct verification of the fact that the conditions (1.55) are satisfied on $\mathbf{V}^{n+m}$ for $^{[M]}\mathbf{D} = \{^{[\mathbf{M}]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$. We note that the relation (1.57) between the Levi–Civita and the canonical d–connection is a particular case of $^{[M]}\mathbf{Z}^{\gamma}_{\alpha\beta}$, when Y^{m}_{lj}, Y^{m}_{ej} and Y^{d}_{ec} are zero, but Y^{k}_{mc} is taken to have $^{[+]}O^{mi}_{jk}Y^{k}_{mc} = \frac{1}{2}g^{ik}\Omega^{a}_{jk}h_{ca}$.

There is a very important consequence of the Theorems 1.3.5 and 1.3.6: For a generic off–diagonal metric structure (1.34) we can derive a N–connection structure N^a_i with a d–metric $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ (1.33). So, we may consider an infinite number of d–connections $\{\mathbf{D}\}$, all constructed from the coefficients of the off–diagonal metrics, satisfying the metricity conditions $\mathbf{D}_{\gamma}\mathbf{g}_{\alpha\beta} = 0$ and having partial vanishing torsions, $T^i_{jk} = T^a_{bc} = 0$. The covariant calculi associated to the set $\{\mathbf{D}\}$ are adapted to the N–connection splitting and alternative to the covariant calculus defined by the Levi–Civita connection $\bigtriangledown$, which is not adapted to the N–connection.

1.3.4 Nonmetricity in Finsler Geometry

Usually, the N–connection, d–connection and d–metric in generalized Finsler spaces satisfy certain metric compatibility conditions [14, 15, 16, 17]. Nevertheless, there were considered some classes of d–connections (for instance, related to the Berwald d–connection) with nontrivial components of the nonmetricity d–tensor. Let us consider some such examples modelled on metric–affine spaces.

The Berwald d–connection

A d–connection of Berwald type (see, for instance, Ref. [14] on such configurations in Finsler and Lagrange geometry), $^{[B]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[B]}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, is defined by h- and v–irreducible components

$$^{[B]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(\widehat{L}^{i}{}_{jk}, \frac{\partial N^a_k}{\partial y^b}, 0, \widehat{C}^{a}{}_{bc}\right), \tag{1.62}$$

with $\widehat{L}^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc}$ taken as in (1.56), satisfying only partial metricity compatibility conditions for a d–metric (1.33), $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ on space $\mathbf{V}^{n+m}$

$$^{[B]}D^{[h]}_{k}g_{ij} = 0 \text{ and } {}^{[B]}D^{[v]}_{c}h_{ab} = 0.$$

This is an example of d–connections which may possess nontrivial nonmetricity components, $^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = \left({}^{[B]}Q_{cij},\ {}^{[B]}Q_{iab}\right)$ with

$$^{[B]}Q_{cij} = {}^{[B]}D^{[v]}_{c}g_{ij} \text{ and } {}^{[B]}Q_{iab} = {}^{[B]}D^{[h]}_{i}h_{ab}. \tag{1.63}$$

So, the Berwald d–connection defines a metric–affine space $\mathbf{V}^{n+m}$ with N–connection structure.

If $\widehat{L}^{i}{}_{jk}=0$ and $\widehat{C}^{a}{}_{bc}=0$, we obtain a Berwald type connection

$$^{[N]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta}=\left(0,\frac{\partial N^{a}_{k}}{\partial y^{b}},0,0\right)$$

induced by the N–connection structures. It defines a vertical covariant derivation $^{[N]}D^{[v]}_{c}$ acting in the v–subspace of $\mathbf{V}^{n+m}$, with the coefficients being partial derivatives on v–coordinates y^{a} of the N–connection coefficients N^{a}_{i} [41].

We can generalize the Berwald connection (1.62) to contain any fixed values of d–torsions $T^{i}_{.jk}$ and $T^{a}_{.bc}$ from the h- v–decomposition (1.95). We can check by a straightforward calculations that the d–connection

$$^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}=\left(\widehat{L}^{i}{}_{jk}+\tau^{i}{}_{jk},\frac{\partial N^{a}_{k}}{\partial y^{b}},0,\widehat{C}^{a}{}_{bc}+\tau^{a}{}_{bc}\right) \tag{1.64}$$

with

$$\begin{aligned}\tau^{i}{}_{jk} &= \frac{1}{2}g^{il}\left(g_{kh}T^{h}_{.lj}+g_{jh}T^{h}_{.lk}-g_{lh}T^{h}{}_{jk}\right) \\ \tau^{a}{}_{bc} &= \frac{1}{2}h^{ad}\left(h_{bf}T^{f}{}_{dc}+h_{cf}T^{f}{}_{db}-h_{df}T^{f}{}_{bc}\right)\end{aligned} \tag{1.65}$$

results in $^{[B\tau]}\mathbf{T}^{i}_{jk}=T^{i}_{.jk}$ and $^{[B\tau]}\mathbf{T}^{a}_{bc}=T^{a}_{.bc}$. The d–connection (1.64) has certain nonvanishing irreducible nonmetricity components $^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma}=\left(^{[B\tau]}Q_{cij},\ ^{[B\tau]}Q_{iab}\right)$.

In general, by using the Kawaguchi metrization procedure (see Theorem 1.3.5) we can also construct metric d–connections with prescribed values of d–torsions $T^{i}_{.jk}$ and $T^{a}_{.bc}$, or to express, for instance, the Levi–Civita connection via coefficients of an arbitrary metric d–connection (see details, for vector bundles, in [14]).

Similarly to formulas (1.75), (1.76) and (1.77), we can express a general affine Berwald d–connection $^{[B\tau]}\mathbf{D}$, i. e. $^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha}=\ ^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, via its deformations from the Levi–Civita connection $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$,

$$^{[B\tau]}\mathbf{\Gamma}^{\alpha}{}_{\beta}=\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}+\ ^{[B\tau]}\mathbf{Z}^{\alpha}{}_{\beta}, \tag{1.66}$$

$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ being expressed as (1.54) (equivalently, defined by (1.53)) and

$$\begin{aligned}^{[B\tau]}\mathbf{Z}_{\alpha\beta} &= \mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{T}_{\alpha}-\mathbf{e}_{\alpha}\rfloor\ ^{[B\tau]}\mathbf{T}_{\beta}+\frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\ &+\left(\mathbf{e}_{\alpha}\rfloor\ ^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma}-\left(\mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma}+\frac{1}{2}\ ^{[B\tau]}\mathbf{Q}_{\alpha\beta}.\end{aligned} \tag{1.67}$$

defined with prescribed d–torsions ${}^{[B\tau]}\mathbf{T}^{i}_{jk} = T^{i}_{.jk}$ and ${}^{[B\tau]}\mathbf{T}^{a}_{bc} = T^{a}_{.bc}$. This Berwald d–connection can define a particular subclass of metric–affine connections being adapted to the N–connection structure and with prescribed values of d–torsions.

The canonical/ Berwald metric–affine d–connections

If the deformations of d–metrics in formulas (1.76) and (1.66) are considered not with respect to the Levi–Civita connection $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ but with respect to the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ with h- v–irreducible coefficients (1.56), we can construct a set of canonical metric–affine d–connections. Such metric–affine d–connections $\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ are defined via deformations

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} = \widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + \widehat{\mathbf{Z}}^{\alpha}{}_{\beta}, \tag{1.68}$$

$\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta}$ being the canonical d–connection (1.26) and

$$\begin{aligned} \widehat{\mathbf{Z}}_{\alpha\beta} &= \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\ &+ \left(\mathbf{e}_{\alpha}\rfloor\ {}^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ {}^{[B\tau]}\mathbf{Q}_{\alpha\beta} \end{aligned} \tag{1.69}$$

where $\mathbf{T}_{\alpha}$ and $\mathbf{Q}_{\alpha\beta}$ are arbitrary torsion and nonmetricity structures.

A metric–affine d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha}$ can be also considered as a deformation from the Berwald connection ${}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} + {}^{[B\tau]}\ \widehat{\mathbf{Z}}^{\alpha}{}_{\beta}, \tag{1.70}$$

${}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ being the Berwald d–connection (1.64) and

$$\begin{aligned} {}^{[B\tau]}\ \widehat{\mathbf{Z}}^{\alpha}{}_{\beta} &= \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\ &+ \left(\mathbf{e}_{\alpha}\rfloor\ {}^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ {}^{[B\tau]}\mathbf{Q}_{\alpha\beta} \end{aligned} \tag{1.71}$$

The h- and v–splitting of formulas can be computed by introducing N–frames $\mathbf{e}_{\gamma} = (\delta_i = \partial_i - N^a_i\partial_a, \partial_a)$ and N–coframes $\vartheta^{\beta} = (dx^i, \delta y^a = dy^a + N^a_i dx^i)$ and d–metric $\mathbf{g} = (g_{ij}, h_{ab})$ into (1.54), (1.66) and (1.67) for the general Berwald d–connections. In a similar form we can compute splitting by introducing the N–frames and d–metric into (1.26), (1.68) and (1.69) for the metric affine canonic d–connections and, respectively, into (1.64), (1.70) and (1.71) for the metric–affine Berwald d–connections. For the corresponding classes of d–connections, we can compute the torsion and curvature tensors by introducing respective connections (1.54), (1.76), (1.56), (1.62), (1.64), (1.66), (1.68) and (1.70) into the general formulas for torsion (1.41) and curvature (1.42) on spaces provided with N–connection structure.

1.3.5 N–connections in metric–affine spaces

In order to elaborate a unified MAG and generalized Finsler spaces scheme, it is necessary to explain how the N–connection emerge in a metric–affine space and/or in more particular cases of Riemann–Cartan and (pseudo) Riemann geometry.

Riemann geometry as a Riemann–Cartan geometry with N–connection

It is well known the interpretation of the Riemann–Cartan geometry as a generalization of the Riemannian geometry by distorsions (of the Levi–Civita connection) generated by the torsion tensors [38]. Usually, the Riemann–Cartan geometry is described by certain geometric relations between the torsion tensor, curvature tensor, metric and the Levi–Civita connection on effective Riemann spaces. We can establish new relations between the Riemann and Riemann–Cartan geometry if generic off–diagonal metrics and anholonomic frames of reference are introduced into consideration. Roughly speaking, a generic off–diagonal metric induces alternatively to the well known Riemann spaces a certain class of Riemann–Cartan geometries, with torsions completely defined by off–diagonal metric terms and related anholonomic frame structures.

Theorem 1.3.7. *Any (pseudo) Riemannian spacetime provided with a generic off–diagonal metric, defining the torsionless and metric Levi–Civita connection, can be equivalently modelled as a Riemann–Cartan spacetime provided with a canonical d–connection adapted to N–connection structure.*

Proof:

Let us consider how the data for a (pseudo) Riemannian generic off–diagonal metric $g_{\alpha\beta}$ parametrized in the form (1.34) can generate a Riemann–Cartan geometry. It is supposed that with respect to any convenient anholonomic coframes (1.22) the metric is transformed into a diagonalized form of type (1.33), which gives the possibility to define N_i^a and $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ and to compute the aholonomy coefficients $\mathbf{w}^{\gamma}{}_{\alpha\beta}$ (1.24) and the components of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\right)$ (1.56). This connection has nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{.\beta\gamma}$, see the Theorem 1.2.2 and Corollary 1.3.1. In general, such d–torsions are not zero being induced by the values N_i^a and their partial derivatives, contained in the former off–diagonal components of the metric (1.34). So, the former Riemannian geometry, with respect to anholonomic frames with associated N–connection structure, is equivalently rewritten in terms of a Riemann–Cartan geometry with nontrivial torsion structure.

We can provide an inverse construction when a diagonal d–metric (1.33) is given with respect to an anholonomic coframe (1.22) defined from nontrivial values of N–connection coefficients, N_i^a. The related Riemann–Cartan geometry is defined by the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ possessing nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{.\beta\gamma}$. The data for this geometry with N–connection and torsion can be directly transformed [even with respect to the same N–adapted (co) frames] into the data of related (pseudo) Riemannian geometry by using the relation (1.57) between the components of $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ and of the Levi–Civita connection $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}$.■

Remark 1.3.2.
a) Any generic off–diagonal (pseudo) Riemannian metric $g_{\alpha\beta}[N_i^a] \rightarrow \mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *induces an infinite number of associated Riemann–Cartan geometries defined by sets of d–connections* $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ *which can be constructed according the Kawaguchi's and, respectively, Miron's Theorems 1.3.5 and 1.3.6.*
b) For any metric d–connection $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ *induced by a generic off–diagonal metric (1.34), we can define alternatively to the standard (induced by the Levi–Civita connection) the Ricci d–tensor (1.49),* $\mathbf{R}_{\alpha\beta}$*, and the Einstein d–tensor (1.51),* $\mathbf{G}_{\alpha\beta}$*.*

We emphasize that all Riemann–Cartan geometries induced by metric d–connections $\mathbf{D}$ are characterized not only by nontrivial induced torsions $\mathbf{T}^{\alpha}_{.\beta\gamma}$ but also by corresponding nonsymmetric Ricci d–tensor, $\mathbf{R}_{\alpha\beta}$, and Einstein d–tensor, $\mathbf{G}_{\alpha\beta}$, for which $\mathbf{D}_{\gamma}\mathbf{G}_{\alpha\beta} \neq 0$. This is not a surprising fact, because we transferred the geometrical and physical objects on anholonomic spaces, when the conservation laws should be redefined as to include the anholonomically imposed constraints.

Finally, we conclude that for any generic off–diagonal (pseudo) Riemannian metric we have two alternatives: 1) to choose the approach defined by the Levi–Civita connection $\bigtriangledown$, with vanishing torsion and usually defined conservation laws $\bigtriangledown_{\gamma}\mathbf{G}^{[\bigtriangledown]}_{\alpha\beta} = 0$, or 2) to diagonalize the metric effectively, by respective anholonomic transforms, and transfer the geometric and physical objects into effective Riemann–Cartan geometries defined by corresponding N–connection and d–connection structures. All types of such geometric constructions are equivalent. Nevertheless, one could be defined certain priorities for some physical models like "simplicity" of field equations and definition of conservation laws and/or the possibility to construct exact solutions. We note also that a variant with induced torsions is more appropriate for including in the scheme various type of generalized Finsler structures and/or models of (super) string gravity containing nontrivial torsion fields.

Metric–affine geometry and N–connections

A general affine (linear) connection $D = \bigtriangledown + Z = \{\Gamma^{\gamma}_{\beta\alpha} = \Gamma^{\gamma}_{\bigtriangledown\beta\alpha} + Z^{\gamma}_{\beta\alpha}\}$

$$\Gamma^{\gamma}{}_{\alpha} = \Gamma^{\gamma}_{\alpha\beta}\vartheta^{\beta}, \qquad (1.72)$$

can always be decomposed into the Riemannian $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$ and post–Riemannian Z^{α}_{β} parts (see Refs. [4] and, for irreducible decompositions to the effective Einstein theory, see Ref. [26]),

$$\Gamma^{\alpha}_{\ \beta} = \Gamma^{\alpha}_{\bigtriangledown\ \beta} + Z^{\alpha}_{\ \beta} \qquad (1.73)$$

where the distorsion 1-form Z^{α}_{β} is expressed in terms of torsion and nonmetricity,

$$Z_{\alpha\beta} = e_{\beta}\rfloor T_{\alpha} - e_{\alpha}\rfloor T_{\beta} + \frac{1}{2}\left(e_{\alpha}\rfloor e_{\beta}\rfloor T_{\gamma}\right)\vartheta^{\gamma} + \left(e_{\alpha}\rfloor Q_{\beta\gamma}\right)\vartheta^{\gamma} - \left(e_{\beta}\rfloor Q_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}Q_{\alpha\beta}, \quad (1.74)$$

T_{α} is defined as (1.38) and $Q_{\alpha\beta} \doteqdot -Dg_{\alpha\beta}$. [9] For $Q_{\beta\gamma} = 0$, we obtain from (1.74) the distorsion for the Riemannian–Cartan geometry [38].

By substituting arbitrary (co) frames, metrics and linear connections into N–adapted ones (i. e. performing changes

$$e_{\alpha} \to \mathbf{e}_{\alpha}, \vartheta^{\beta} \to \vartheta^{\beta}, g_{\alpha\beta} \to \mathbf{g}_{\alpha\beta} = \left(g_{ij}, h_{ab}\right), \Gamma^{\gamma}{}_{\alpha} \to \mathbf{\Gamma}^{\gamma}{}_{\alpha}$$

with $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ and $\mathbf{T}^{\alpha}$ as in (1.41)) into respective formulas (1.72), (1.73) and (1.74), we can define an affine connection $\mathbf{D} = \bigtriangledown + \mathbf{Z} = \{\mathbf{\Gamma}^{\gamma}{}_{\beta\alpha}\}$ with respect to N–adapted (co) frames,

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}, \qquad (1.75)$$

with

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta} + \mathbf{Z}^{\alpha}{}_{\beta}, \qquad (1.76)$$

$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ being expressed as (1.54) (equivalently, defined by (1.53)) and $\mathbf{Z}^{\alpha}{}_{\beta}$ expressed as

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor \mathbf{e}_{\beta}\rfloor \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\mathbf{Q}_{\alpha\beta}. \quad (1.77)$$

The h– and v–components of $\mathbf{\Gamma}^{\alpha}_{\ \beta}$ from (1.76) consists from the components of $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ (considered for (1.54)) and of $\mathbf{Z}_{\alpha\beta}$ with $\mathbf{Z}^{\alpha}{}_{\gamma\beta} = \left(Z^{i}_{jk}, Z^{a}_{bk}, Z^{i}_{jc}, Z^{a}_{bc}\right)$ .The values

$$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\gamma\beta} + \mathbf{Z}^{\alpha}{}_{\gamma\beta} = \left(L^{i}_{\bigtriangledown jk} + Z^{i}_{jk}, L^{a}_{\bigtriangledown bk} + Z^{a}_{bk}, C^{i}_{\bigtriangledown jc} + Z^{i}_{jc}, C^{a}_{\bigtriangledown bc} + Z^{a}_{bc}\right)$$

[9] We note that our $\Gamma^{\gamma}{}_{\alpha}$ and Z^{α}_{β} are respectively the $\Gamma^{\gamma}{}_{\alpha}$ and $N_{\alpha\beta}$ from Ref. [26]; in our works we use the symbol N for N–connections.

are defined correspondingly

$$\begin{aligned} L^{i}_{\bigtriangledown jk} + Z^{i}{}_{jk} &= [(\bigtriangledown_{k} + Z_{k})\delta_{j}]\rfloor d^{i}, \quad L^{a}_{\bigtriangledown bk} + Z^{a}{}_{bk} = [(\bigtriangledown_{k} + Z_{k})\partial_{b}]\rfloor \delta^{a}, \\ C^{i}_{\bigtriangledown jc} + Z^{i}{}_{jc} &= [(\bigtriangledown_{c} + Z_{c})\,\delta_{j}]\rfloor d^{i}, \quad C^{a}_{\bigtriangledown bc} + Z^{a}{}_{bc} = [(\bigtriangledown_{c} + Z_{c})\,\partial_{b}]\rfloor \delta^{a}. \end{aligned}$$

and related to (1.77) via h- and v–splitting of N–frames $\mathbf{e}_{\gamma} = (\delta_{i} = \partial_{i} - N^{a}_{i}\partial_{a}, \partial_{a})$ and N–coframes $\vartheta^{\beta} = (dx^{i}, \delta y^{a} = dy^{a} + N^{a}_{i}dx^{i})$ and d–metric $\mathbf{g} = (g_{ij}, h_{ab})$.

We note that for $\mathbf{Q}_{\alpha\beta} = 0$, the distorsion 1–form $\mathbf{Z}_{\alpha\beta}$ defines a Riemann–Cartan geometry adapted to the N–connection structure.

Let us briefly outline the procedure of definition of N–connections in a metric–affine space V^{n+m} with arbitrary metric and connection structures $\left(g^{[od]} = \{g_{\alpha\beta}\}, \underline{\Gamma}^{\gamma}{}_{\beta\alpha}\right)$ and show how the geometric objects may be adapted to the N–connection structure.

Proposition 1.3.4. *Every metric–affine space provided with a generic off–diagonal metric structure admits nontrivial N–connections.*

Proof: We give an explicit example how to introduce the N–connection structure. We write the metric with respect to a local coordinate basis,

$$g^{[od]} = g_{\underline{\alpha}\underline{\beta}} du^{\underline{\alpha}} \otimes du^{\underline{\beta}},$$

where the matrix $g_{\underline{\alpha}\underline{\beta}}$ contains a non–degenerated $(m \times m)$ submatrix h_{ab}, for instance like in ansatz (1.34). Having fixed the block h_{ab}, labelled by running of indices $a, b, ... = n+1, n+2, ..., n+m$, we can define the $(n \times n)$ bloc $g_{\underline{i}\underline{j}}$ with indices $\underline{i}, \underline{j}, ... = 1, 2, ...n$. The next step is to find any nontrivial N^{a}_{i} (the set of coefficients has being defined, we may omit underlying) and find $N^{\underline{e}}_{\underline{j}}$ from the $(n \times m)$ block relations $g_{\underline{j}\underline{a}} = N^{\underline{e}}_{\underline{j}} h_{\underline{a}\underline{e}}$. This is always possible if $g_{\underline{\alpha}\underline{\beta}}$ is generic off–diagonal. The next step is to compute $g_{ij} = g_{\underline{i}\underline{j}} - N^{\underline{a}}_{\underline{i}} N^{\underline{e}}_{\underline{j}} h_{\underline{a}\underline{e}}$ which gives the possibility to transform equivalently

$$g^{[od]} \to \mathbf{g} = g_{ij}\vartheta^{i} \otimes \vartheta^{j} + h_{ab}\vartheta^{a} \otimes \vartheta^{b}$$

where

$$\vartheta^{i} \doteqdot dx^{i},\ \vartheta^{a} \doteqdot \delta y^{a} = dy^{a} + N^{a}_{i}(u)\,dx^{i}$$

are just the N–elongated differentials (1.22) if the local coordinates associated to the block h_{ab} are denoted by y^{a} and the rest ones by x^{i}. We impose a global splitting of the metric–affine spacetime by stating that all geometric objects are subjected to anholonomic frame transforms with vielbein coefficients of type (1.16) and (1.17) defined by $\mathbf{N} = \{N^{a}_{i}\}$. This way, we define on the metric–affine space a vector/covector bundle structure if the coordinates y^{a} are treated as certain local vector/ covector components.■

We note, that having defined the values $\vartheta^{\alpha} = \left(\vartheta^{i}, \vartheta^{b}\right)$ and their duals $\mathbf{e}_{\alpha} = (\mathbf{e}_{i}, \mathbf{e}_{a})$, we can compute the linear connection coefficients with respect to N–adapted (co) frames, $\Gamma^{\gamma}{}_{\beta\alpha} \rightarrow \widetilde{\mathbf{\Gamma}}^{\gamma}{}_{\beta\alpha}$. However, $\widetilde{\mathbf{\Gamma}}^{\gamma}{}_{\beta\alpha}$, in general, is not a d–connection, i. e. it is not adapted to the global splitting $T\mathbf{V}^{n+m} = h\mathbf{V}^{n+m} \oplus v\mathbf{V}^{n+m}$ defined by N–connection, see Definition 1.2.11. If the metric and linear connection are not subjected to any field equations, we are free to consider distorsion tensors in order to be able to apply the Theorems 1.3.5 and/or 1.3.6 with the aim to transform $\widetilde{\mathbf{\Gamma}}^{\gamma}{}_{\beta\alpha}$ into a metric d–connection, or even into a Riemann–Cartan d–connection. Here, we also note that a metric–affine space, in general, admits different classes of N–connections with various nontrivial global splitting $n' + m' = n + m$, where $n' \neq n$.

We can state from the very beginning that a metric–affine space $\mathbf{V}^{n+m}$ is provided with d–metric (1.33) and d–connection structure (1.26) adapted to a class of prescribed vielbein transforms (1.16) and (1.17) and N–elongated frames (1.21) and (1.22). All constructions can be redefined with respect to coordinate frames (1.18) and (1.19) with off–diagonal metric parametrization (1.34) and then subjected to another frame and coordinate transforms hiding the existing N–connection structure and distinguished character of geometric objects. Such 'distinguished' metric–affine spaces are characterized by corresponding N–connection geometries and admit geometric constructions with distinguished objects. They form a particular subclass of metric–affine spaces admitting transformations of the general linear connection $\Gamma^{\gamma}{}_{\beta\alpha}$ into certain classes of d–connections $\mathbf{\Gamma}^{\gamma}{}_{\beta\alpha}$.

Definition 1.3.17. *A distinguished metric–affine space* $\mathbf{V}^{n+m}$ *is a usual metric–affine space additionally enabled with a N–connection structure* $\mathbf{N} = \{N_i^a\}$ *inducing splitting into respective irreducible horizontal and vertical subspaces of dimensions n and m. This space is provided with independent d–metric (1.33) and affine d–connection (1.26) structures adapted to the N–connection.*

The metric–affine spacetimes with stated N–connection structure are also characterized by nontrivial anholonomy relations of type (1.23) with anholonomy coefficients (1.24). This is a very specific type of noncommutative symmetry generated by N–adapted (co) frames defining different anholonomic noncommutative differential calculi (for details with respect to the Einstein and gauge gravity see Ref. [47]).

We construct and analyze explicit examples of metric–affine spacetimes with associated N–connection (noncommutative) symmetry in Refs. [33]. A surprizing fact is that various types of d–metric ansatz (1.33) with associated N–elongated frame (1.21) and coframe (1.22) (or equivalently, respective off–diagonal ansatz (1.34)) can be defined as exact solutions in Einstein gravity of different dimensions and in metric–affine, or Einstein–Cartan gravity and gauge model realizations. Such solutions model also generalized Finsler structures.

1.4 Generalized Finsler–Affine Spaces

The aim of this section is to demonstrate that any well known type of locally anisotropic or locally isotropic spaces can be defined as certain particular cases of distinguished metric–affine spaces. We use the general term of "generalized Finsler–affine spaces" for all type of geometries modelled in MAG as generalizations of the Riemann–Cartan–Finsler geometry, in general, containing nonmetricity fields. A complete classification of such spaces is given by Tables 1–11 in the Appendix.

1.4.1 Spaces with vanishing N–connection curvature

Three examples of such spaces are given by the well known (pseudo) Riemann, Riemann–Cartan or Kaluza–Klein manifolds of dimension $(n+m)$ provided with a generic off–diagonal metric structure $\underline{g}_{\alpha\beta}$ of type (1.34), of corresponding signature, which can be reduced equivalently to the block $(n \times n) \oplus (m \times m)$ form (1.33) via vielbein transforms (1.16). Their N–connection structures may be restricted by the condition $\Omega^a_{ij} = 0$, see (1.20).

Anholonomic (pseudo) Riemannian spaces

The (pseudo) Riemannian manifolds, $\mathbf{V}_R^{n+m}$, provided with a generic off–diagonal metric and anholonomic frame structure effectively diagonalizing such a metric is an anholonomic (pseudo) Riemannian space. The space admits associated N–connection structures with coefficients induced by generic off–diagonal terms in the metric (1.34). If the N–connection curvature vanishes, the Levi–Civita connection is closely defined by the same coefficients as the canonical d–connection (linear connections computed with respect to the N–adapted (co) frames), see Proposition 1.3.2 and related discussions in section 1.3. Following the Theorem 1.3.7, any (pseudo) Riemannian space enabled with generic off–diagonal connection structure can be equivalently modelled as an effective Riemann–Cartan geometry with induced N–connection and d–torsions.

There were constructed a number of exact 'off–diagonal' solutions of the Einstein equations [34, 35, 25], for instance, in five dimensional gravity (with various type restrictions to lower dimensions) with nontrivial N–connection structure with ansatz for metric of type

$$\begin{aligned} \mathbf{g} &= \omega\left(x^i,y^4\right)[g_1(dx^1)^2+g_2\left(x^2,x^3\right)(dx^2)^2+g_3\left(x^2,x^3\right)(dx^3)^2 \\ &+h_4\left(x^2,x^3,y^4\right)\left(\delta y^4\right)^2+h_5\left(x^2,x^3,y^4\right)\left(\delta y^5\right)^2], \end{aligned} \tag{1.78}$$

for $g_1 = const$, where

$$\delta y^a = dy^a + N_k^a\left(x^i,y^4\right)dy^a$$

with indices $i,j,k... = 1,2,3$ and $a = 4,5$. The coefficients $N_i^a\left(x^i,y^4\right)$ were searched as a metric ansatz of type (1.34) transforming equivalently into a certain diagonalized block (1.33) would parametrize generic off–diagonal exact solutions. Such effective N–connections are contained into a corresponding anholonomic moving or static configuration of tetrads/ pentads (vierbeins/funfbeins) defining a conventional splitting of coordinates into n holonomic and m anholonomic ones, where $n+m = 4,5$. The ansatz (1.78) results in exact solutions of vacuum and nonvacuum Einstein equations which can be integrated in general form. Perhaps, all known at present time exact solutions in 3-5 dimensional gravity can be included as particular cases in (1.78) and generalized to anholonomic configurations with running constants and gravitational and matter polarizations (in general, anisotropic on variable y^4) of the metric and frame coefficients.

The vector/ tangent bundle configurations and/or torsion structures can be effectively modelled on such (pseudo) Riemannian spaces by prescribing a corresponding class of anholonomic frames. Such configurations are very different from those, for instance, defined by Killing symmetries and the induced torsion vanishes after frame transforms to coordinate bases. For a corresponding parametrizations of $N_i^a(u)$ and $g_{\alpha\beta}$, we can model Finsler like structures even in (pseudo) Riemannian spacetimes or in gauge gravity [25, 36, 37].

The anholonomic Riemannian spaces $\mathbf{V}_R^{n+m}$ can be considered as a subclass of distinguished metric–affine spaces $\mathbf{V}^{n+m}$ provided with N–connection structure, characterized by the condition that nonmetricity d–filed $\mathbf{Q}_{\alpha\beta\gamma} = 0$ and that a certain type of induced torsions $\mathbf{T}_{\beta\gamma}^{\alpha}$ vanish for the Levi–Civita connection. We can take a generic off–diagonal metric (1.34), transform it into a d–metric (1.33) and compute the h- and v-components of the canonical d–connection (1.26) and put them into the formulas for d–torsions (1.95) and d–curvatures (1.48). The vacuum solutions are defined by d–metrics and N–connections satisfying the condition $\mathbf{R}_{\alpha\beta} = 0$, see the h–, v–components (1.49).

In order to transform certain geometric constructions defined by the canonical d–connection into similar ones for the Levi–Civita connection, we have to constrain the N–connection structure as to have vanishing N–curvature, $\Omega_{ij}^a = 0$, or to see the conditions when the deformation of Levi–Civita connection to any d–connection result in non–deformations of the Einstein equation. We obtain a (pseudo) Riemannian vacuum spacetime with anholonomially induced d–torsion components

$$\widehat{T}^i_{ja} = -\widehat{T}^i_{aj} = \widehat{C}^i_{.ja} \text{ and } \widehat{T}^a_{.bi} = -\widehat{T}^a_{.ib} = \partial N^a_i / \partial y^b - \widehat{L}^a_{.bj}.$$

This torsion can be related algebraically to a spin source like in the usual Riemann–Cartan gravity if we want to give an algebraic motivation to the N–connection splitting. We emphasize that the N–connection and d–metric coefficients can be chosen in order to model on $\mathbf{V}^{n+m}_R$ a special subclass of Finsler/ Lagrange structures (see discussion in section 1.4.2).

Kaluza–Klein spacetimes

Such higher dimension generalizations of the Einstein gravity are characterized by a metric ansatz

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij}(x^\kappa) + A^a_i(x^\kappa) A^b_j(x^\kappa) h_{ab}(x^\kappa, y^a) & A^e_j(x^\kappa) h_{ae}(x^\kappa, y^a) \\ A^e_i(x^\kappa) h_{be}(x^\kappa, y^a) & h_{ab}(x^\kappa, y^a) \end{bmatrix} \tag{1.79}$$

(a particular case of the metric (1.34)) with certain compactifications on extra dimension coordinates y^a. The values $A^a_i(x^\kappa)$ are considered to define gauge fields after compactifications (the electromatgnetic potential in the original extension to five dimensions by Kaluza and Klein, or some non–Abelian gauge fields for higher dimension generalizations). Perhaps, the ansatz (1.79) was originally introduced in Refs. [48] (see [49] as a review of non–supersymmetry models and [50] for supersymmetric theories).

The coefficients $A^a_i(x^\kappa)$ from (1.79) are certain particular parametrizations of the N–connection coefficients $N^a_i(x^\kappa, y^a)$ in (1.34). This suggests a physical interpretation for the N–connection as a specific nonlinear gauge field depending both on spacetime and extra dimension coordinates (in general, noncompactified). In the usual Kaluza–Klein (super) theories, there were not considered anholonomic transforms to block d–metrics (1.33) containing dependencies on variables y^a.

In some more general approaches, with additional anholonomic structures on lower dimensional spacetime, there were constructed a set of exact vacuum five dimensional solutions by reducing ansatz (1.79) and their generalizations of form (1.34) to d–metric ansatz of type (1.78), see Refs. [34, 35, 36, 37, 25, 47]. Such vacuum and nonvacuum solutions describe anisotropically polarized Taub–NUT spaces, wormhole/ flux tube confiugurations, moving four dimensional black holes in bulk five dimensional spacetimes, anisotropically deformed cosmological spacetimes and various type of locally anisotropic spinor–soliton–dialton interactions in generalized Kaluza–Klein and string/ brane gravity.

Teleparallel spaces

Teleparallel theories are usually defined by two geometrical constraints [9] (here, we introduce them for d–connections and nonvanishing N–connection structure),

$$\mathbf{R}^{\alpha}{}_{\beta} = \delta \mathbf{\Gamma}^{\alpha}{}_{\beta} + \mathbf{\Gamma}^{\alpha}{}_{\gamma} \wedge \mathbf{\Gamma}^{\gamma}{}_{\beta} = 0 \tag{1.80}$$

and

$$\mathbf{Q}_{\alpha\beta} = -\mathbf{D}\mathbf{g}_{\alpha\beta} = -\delta \mathbf{g}_{\alpha\beta} + \mathbf{\Gamma}^{\gamma}{}_{\beta}\mathbf{g}_{\alpha\gamma} + \mathbf{\Gamma}^{\gamma}{}_{\alpha}\mathbf{g}_{\beta\gamma} = 0. \tag{1.81}$$

The conditions (1.80) and (1.81) establish a distant paralellism in such spaces because the result of a parallel transport of a vector does not depend on the path (the angles and lengths being also preserved under parallel transports). It is always possible to find such anholonomic transforms $e_{\alpha} = A_{\alpha}^{\ \underline{\beta}} e_{\underline{\beta}}$ and $e_{\underline{\alpha}} = A_{\underline{\alpha}}^{\ \beta} e_{\beta}$, where $A_{\underline{\alpha}}^{\ \beta}$ is inverse to $A_{\alpha}^{\ \underline{\beta}}$ when

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} \rightarrow \mathbf{\Gamma}^{\underline{\alpha}}{}_{\underline{\beta}} = A_{\underline{\beta}}^{\ \beta} \mathbf{\Gamma}^{\alpha}{}_{\beta} A^{\underline{\alpha}}{}_{\alpha} + A^{\underline{\alpha}}{}_{\gamma} \delta A_{\underline{\beta}}^{\ \gamma} = 0$$

and the transformed local metrics becomes the standard Minkowski,

$$g_{\underline{\alpha\beta}} = diag\,(-1, +1, ..., +1)$$

(it can be fixed any signature). If the (co) frame is considered as the only dynamical variable, it is called that the space (and choice of gauge) are of Weitzenbock type. A coframe of type (1.22)

$$\vartheta^{\beta} \doteqdot \left(\delta x^{i} = dx^{i}, \delta y^{a} = dy^{a} + N_{i}^{a}\,(u)\, dx^{i}\right)$$

is defined by N–connection coefficients. If we impose the condition of vanishing the N–connection curvature, $\Omega^{a}_{ij} = 0$, see (1.20), the N–connection defines a specific anholonomic dynamics because of nontrivial anholonomic relations (1.23) with nonzero components (1.24).

By embedding teleparallel configurations into metric–affine spaces provided with N–connection structure we state a distinguished class of (co) frame fields adapted to this structure and open possibilities to include such spaces into Finsler–affine ones, see section 1.4.2. For vielbein fields $\mathbf{e}_{\alpha}^{\ \underline{\alpha}}$ and their inverses $\mathbf{e}^{\alpha}{}_{\underline{\alpha}}$ related to the d–metric (1.33),

$$\mathbf{g}_{\alpha\beta} = \mathbf{e}_{\alpha}^{\ \underline{\alpha}} \mathbf{e}_{\beta}^{\ \underline{\beta}} g_{\underline{\alpha\beta}}$$

we define the Weitzenbock d–connection

$$^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \mathbf{e}^{\alpha}{}_{\underline{\alpha}} \delta_{\gamma} \mathbf{e}_{\beta}^{\ \underline{\alpha}}, \tag{1.82}$$

where δ_γ is the N–elongated partial derivative (1.21). It transforms in the usual Weitzenbock connection for trivial N–connections. The torsion of ${}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ is defined

$$ {}^{[W]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = {}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} - {}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\gamma\beta}. \tag{1.83} $$

It posses h– and v–irreducible components constructed from the components of a d–metric and N–adapted frames. We can express

$$ {}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma} $$

where $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$ is the Levi–Civita connection (1.53) and the contorsion tensor is

$$ \mathbf{Z}_{\alpha\beta} = \mathbf{e}_\beta \rfloor\ {}^{[W]}\mathbf{T}_\alpha - \mathbf{e}_\alpha \rfloor\ {}^{[W]}\mathbf{T}_\beta + \frac{1}{2}\left(\mathbf{e}_\alpha \rfloor \mathbf{e}_\beta \rfloor\ {}^{[W]}\mathbf{T}_\gamma\right)\vartheta^\gamma + \left(\mathbf{e}_\alpha \rfloor \mathbf{Q}_{\beta\gamma}\right)\vartheta^\gamma - \left(\mathbf{e}_\beta \rfloor \mathbf{Q}_{\alpha\gamma}\right)\vartheta^\gamma + \frac{1}{2}\mathbf{Q}_{\alpha\beta}. $$

In formulation of teleparallel alternatives to the general relativity it is considered that $\mathbf{Q}_{\alpha\beta} = 0$.

1.4.2 Finsler and Finsler–Riemann–Cartan spaces

The first approaches to Finsler spaces [15, 16] were developed by generalizing the usual Riemannian metric interval

$$ ds = \sqrt{g_{ij}(x)\, dx^i dx^j} $$

on a manifold M of dimension n into a nonlinear one

$$ ds = F\left(x^i, dx^j\right) \tag{1.84} $$

defined by the Finsler metric F (fundamental function) on $\widetilde{TM} = TM \backslash \{0\}$ (it should be noted an ambiguity in terminology used in monographs on Finsler geometry and on gravity theories with respect to such terms as Minkowski space, metric function and so on). It is also considered a quadratic form on $\mathbb{R}^2$ with coefficients

$$ g^{[F]}_{ij} \to h_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{1.85} $$

defining a positive definite matrix. The local coordinates are denoted $u^\alpha = (x^i, y^a \to y^i)$. There are satisfied the conditions: 1) The Finsler metric on a real manifold M is a function $F : TM \to \mathbb{R}$ which on $\widetilde{TM} = TM \backslash \{0\}$ is of class C^∞ and F is only continuous on the image of the null cross–sections in the tangent bundle to M. 2) $F(x,\chi y)=\chi F(x,y)$ for every $\mathbb{R}_{+}^{*}$. 3) The restriction of F to $\widetilde{TM}$ is a positive function. 4) $rank\left[g_{ij}^{[F]}(x,y)\right]=n.$

There were elaborated a number of models of locally anisotropic spacetime geometry with broken local Lorentz invariance (see, for instance, those based on Finsler geometries [17, 19]). In result, in the Ref. [51], it was ambiguously concluded that Finsler gravity models are very restricted by experimental data. Recently, the subject concerning Lorentz symmetry violations was revived for instance in brane gravity [52] (see a detailed analysis and references on such theoretical and experimental researches in [53]). In this case, the Finsler like geometries broking the local four dimensional Lorentz invariance can be considered as a possible alternative direction for investigating physical models both with local anisotropy and violation of local spacetime symmetries. But it should be noted here that violations of postulates of general relativity is not a generic property of the so–called "Finsler gravity". A subclass of Finsler geometries and their generalizations could be induced by anholonomic frames even in general relativity theory and Riemannian–Cartan or gauge gravity [25, 36, 37, 24]. The idea is that instead of geometric constructions based on straightforward applications of derivatives of (1.85), following from a nonlinear interval (1.84), we should consider d–metrics (1.33) with the coefficients from Finsler geometry (1.85) or their extended variants. In this case, certain type Finsler configurations can be defined even as exact 'off–diagonal' solutions in vacuum Einstein gravity or in string gravity.

Finsler geometry and its almost Kahlerian model

We outline a modern approach to Finsler geometry [14] based on the geometry of nonlinear connections in tangent bundles.

A real (commutative) Finsler space $\mathbf{F}^n=(M,F(x,y))$ can be modelled on a tangent bundle TM enabled with a Finsler metric $F(x^i,y^j)$ and a quadratic form $g_{ij}^{[F]}$ (1.85) satisfying the mentioned conditions and defining the Christoffel symbols (not those from the usual Riemannian geometry)

$$c_{jk}^{\iota}(x,y)=\frac{1}{2}g^{ih}\left(\partial_j g_{hk}^{[F]}+\partial_k g_{jh}^{[F]}-\partial_h g_{jk}^{[F]}\right),$$

where $\partial_j=\partial/\partial x^j$, and the Cartan nonlinear connection

$$^{[F]}\mathbf{N}_j^i(x,y)=\frac{1}{4}\frac{\partial}{\partial y^j}\left[c_{lk}^{\iota}(x,y)y^l y^k\right], \qquad (1.86)$$

where we do not distinguish the v- and h- indices taking on TM the same values.

In Finsler geometry, there were investigated different classes of remarkable Finsler linear connections introduced by Cartan, Berwald, Matsumoto and other geometers (see details in Refs. [15, 17, 16]). Here we note that we can introduce $g_{ij}^{[F]} = g_{ab}$ and ${}^{[F]}\mathbf{N}_j^i(x,y)$ in (1.34) and transfer our considerations to a $(n \times n) \oplus (n \times n)$ blocks of type (1.33) for a metric–affine space V^{n+n}.

A usual Finsler space $\mathbf{F}^n = (M, F(x,y))$ is completely defined by its fundamental tensor $g_{ij}^{[F]}(x,y)$ and the Cartan nonlinear connection ${}^{[F]}\mathbf{N}_j^i(x,y)$ and any chosen d–connection structure (1.26) (see details on different type of d–connections in section 1.3). Additionally, the N–connection allows us to define an almost complex structure I on TM as follows

$$I(\delta_i) = -\partial/\partial y^i \text{ and } I\left(\partial/\partial y^i\right) = \delta_i$$

for which $I^2 = -1$.

The pair $\left(g^{[F]}, I\right)$ consisting from a Riemannian metric on a tangent bundle TM,

$$\mathbf{g}^{[F]} = g_{ij}^{[F]}(x,y)dx^i \otimes dx^j + g_{ij}^{[F]}(x,y)\delta y^i \otimes \delta y^j \tag{1.87}$$

and the almost complex structure I defines an almost Hermitian structure on $\widetilde{TM}$ associated to a 2–form

$$\theta = g_{ij}^{[F]}(x,y)\delta y^i \wedge dx^j.$$

This model of Finsler geometry is called almost Hermitian and denoted H^{2n} and it is proven [14] that is almost Kahlerian, i. e. the form θ is closed. The almost Kahlerian space $\mathbf{K}^{2n} = \left(\widetilde{TM}, \mathbf{g}^{[F]}, I\right)$ is also called the almost Kahlerian model of the Finsler space F^n.

On Finsler spaces (and their almost Kahlerian models), one distinguishes the almost Kahler linear connection of Finsler type, $\mathbf{D}^{[I]}$ on $\widetilde{TM}$ with the property that this covariant derivation preserves by parallelism the vertical distribution and is compatible with the almost Kahler structure $\left(\mathbf{g}^{[F]}, I\right)$, i.e.

$$\mathbf{D}_X^{[I]}\mathbf{g}^{[F]} = 0 \text{ and } \mathbf{D}_X^{[I]}\mathbf{I} = 0$$

for every d–vector field on $\widetilde{TM}$. This d–connection is defined by the data

$${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left({}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk}\right) \tag{1.88}$$

with ${}^{[F]}\widehat{L}^i_{jk}$ and ${}^{[F]}\widehat{C}^i_{jk}$ computed by similar formulas in (1.56) by using $g_{ij}^{[F]}$ as in (1.85) and ${}^{[F]}N_j^i$ from (1.86).

We emphasize that a Finsler space $\mathbf{F}^n$ with a d–metric (1.87) and Cartan's N–connection structure (1.86), or the corresponding almost Hermitian (Kahler) model $\mathbf{H}^{2n}$, can be equivalently modelled on a space of dimension $2n$, $\mathbf{V}^{n+n}$, provided with an off–diagonal metric (1.34) and anholonomic frame structure with associated Cartan's non-linear connection. Such anholonomic frame constructions are similar to modelling of the Einstein–Cartan geometry on (pseudo) Riemannian spaces where the torsion is considered as an effective tensor field. From this viewpoint a Finsler geometry is a Riemannian–Cartan geometry defined on a tangent bundle provided with a respective off–diagonal metric (and a related anholonomic frame structure with associated N–connection) and with additional prescriptions with respect to the type of linear connection chosen to be compatible with the metric and N–connection structures.

Finsler–Kaluza–Klein spaces

In Ref. [37] we defined a 'locally anisotropic' toroidal compactification of the 10 dimensional heterotic string sction [54]. We consider here the corresponding anholonomic frame transforms and off–diagonal metric ansatz. Let (n', m') be the (holonomic, anholonomic) dimensions of the compactified spacetime (as a particular case we can state $n' + m' = 4$, or any integers $n' + m' < 10$, for instance, for brane configurations). There are used such parametrizations of indices and of vierbeinds: Greek indices $\alpha, \beta, ...\mu...$ run values for a 10 dimensional spacetime and split as $\alpha = (\alpha', \widehat{\alpha})$, $\beta = \left(\beta', \widehat{\beta}\right)$, ... when primed indices $\alpha', \beta', ...\mu'...$ run values for compactified spacetime and split into h- and v–components like $\alpha' = (i', a')$, $\beta' = (j', b')$, ...; the frame coefficients are split as

$$e_{\mu}^{\underline{\mu}}(u) = \left(\begin{array}{cc} e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'}) & A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) e_{\widehat{\alpha}}^{\underline{\widehat{\alpha}}}(u^{\beta'}) \\ 0 & e_{\widehat{\alpha}}^{\underline{\widehat{\alpha}}}(u^{\beta'}) \end{array} \right) \quad (1.89)$$

where $e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'})$, in their turn, are taken in the form (1.16),

$$e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'}) = \left(\begin{array}{cc} e_{i'}^{\underline{i}'}(x^{j'}, y^{a'}) & N_{i'}^{a'}(x^{j'}, y^{a'}) e_{a'}^{\underline{a}'}(x^{j'}, y^{a'}) \\ 0 & e_{a'}^{\underline{a}'}(x^{j'}, y^{a'}) \end{array} \right). \quad (1.90)$$

For the metric

$$\mathbf{g} = \underline{g}_{\alpha\beta} du^{\alpha} \otimes du^{\beta} \quad (1.91)$$

we have the recurrent ansatz

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{\alpha'\beta'}(u^{\beta'}) + A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) A_{\beta'}^{\widehat{\beta}}(u^{\beta'}) h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) \\ h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) A_{\beta'}^{\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) \end{array} \right], \quad (1.92)$$

where

$$g_{\alpha'\beta'} = \left[\begin{array}{cc} g_{i'j'}(u^{\beta'}) + N_{i'}^{a'}(u^{\beta'})N_{j'}^{b'}(u^{\beta'})h_{a'b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'})N_{i'}^{a'}(u^{\beta'}) \\ h_{a'b'}(u^{\beta'})N_{j'}^{b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'}) \end{array} \right]. \quad (1.93)$$

After a toroidal compactification on $u^{\widehat{\alpha}}$ with gauge fields $A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'})$, generated by the frame transform (1.89), we obtain a metric (1.91) like in the usual Kaluza–Klein theory (1.79) but containing the values $g_{\alpha'\beta'}(u^{\beta'})$, defined as in (1.93) (in a generic off–diagonal form similar to (1.34), labelled by primed indices), which can be induced as in Finsler geometry. This is possible if $g_{i'j'}(u^{\beta'}), h_{a'b'}(u^{\beta'}) \rightarrow g_{i'j'}^{[F]}(x',y')$ (see (1.85)) and $N_{i'}^{a'}(u^{\beta'}) \rightarrow N_{j'}^{[F]i'}(x',y')$ (see (1.86)) inducing a Finsler space with "primed" labels for objects. Such locally anisotropic spacetimes (in this case we emphasized the Finsler structures) can be generated anisotropic toroidal compactifications from different models of higher dimension of gravity (string, brane, or usual Kaluza–Klein theories). They define a mixed variant of Finsler and Kaluza–Klein spaces.

By using the recurrent ansatz (1.92) and (1.93), we can generate both nontrivial nonmetricity and prescribed torsion structures adapted to a corresponding N–connection $N_{i'}^{a'}$. For instance, (after topological compactification on higher dimension) we can prescribe in the lower dimensional spacetime certain torsion fields $T^{k'}_{\ i'j'}$ and $T^{a'}_{\ b'c'}$ (they could have a particular relation to the so called B–fields in string theory, or connected to other models). The next steps are to compute $\tau^{k'}_{\ i'j'}$ and $\tau^{a'}_{\ b'c'}$ by using formulas (1.65) and define

$$^{[B\tau]}\mathbf{\Gamma}^{\gamma'}_{\ \alpha'\beta'} = \left(L^{i'}_{j'k'} = \widehat{L}^{i'}_{\ j'k'} + \tau^{i'}_{\ j'k'},\ L^{a'}_{.b'k'} = \frac{\partial N^{a'}_{k'}}{\partial y^{b'}}, C^{i'}_{.j'a'} = 0, C^{a'}_{b'c'} = \widehat{C}^{a'}_{\ b'c'} + \tau^{a'}_{\ b'c'} \right) \quad (1.94)$$

as in (1.64) (all formulas being with primed indices and $\widehat{L}^{i'}_{\ j'k'}$ and $\widehat{C}^{a'}_{\ b'c'}$ defined as in (1.56)). This way we can generate from Kaluza–Klein/ string theory a Berwald spacetime with nontrivial N–adapted nonmetricity

$$^{[B\tau]}\mathbf{Q}_{\alpha'\beta'\gamma'} = \ ^{[B\tau]}\mathbf{D}\mathbf{g}_{\beta'\gamma'} = \left(^{[B\tau]}Q_{c'i'j'},\ ^{[B\tau]}Q_{i'a'b'}\right)$$

and torsions $^{[B\tau]}\mathbf{T}^{\alpha'}_{\ \beta'\gamma'}$ with h- and v– irreducible components

$$\begin{aligned} T^{i'}_{.j'k'} &= -T^{i'}_{k'j'} = L^{i'}_{j'k'} - L^{i'}_{k'j'}, \quad T^{i'}_{j'a'} = -T^{i'}_{a'j'} = C^{i'}_{.j'a'},\ T^{a'}_{.i'j'} = \frac{\delta N^{a'}_{i'}}{\delta x^{j'}} - \frac{\delta N^{a'}_{j'}}{\delta x^{i'}}, \\ T^{a'}_{.b'i'} &= -T^{a'}_{.i'b'} = \frac{\partial N^{a'}_{i'}}{\partial y^{b'}} - L^{a'}_{.b'j'},\ T^{a'}_{.b'c'} = -T^{a'}_{.c'b'} = C^{a'}_{b'c'} - C^{a'}_{c'b'}. \end{aligned} \quad (1.95)$$

defined by the h- and v–coefficients of (1.94).

We conclude that if toroidal compactifications are locally anisotropic, defined by a chain of ansatz containing N–connection, the lower dimensional spacetime can be not only with torsion structure (like in low energy limit of string theory) but also with nonmetricity. The anholonomy induced by N–connection gives the possibility to define a more wide class of linear connections adapted to the h- and v–splitting.

Finsler–Riemann–Cartan spaces

Such spacetimes are modelled as Riemann–Cartan geometries on a tangent bundle TM when the metric and anholonomic frame structures distinguished to be of Finsler type (1.87). Both Finsler and Riemann–Cartan spaces possess nontrivial torsion structures (see section 1.2.4 for details on definition and computation torsions of locally anisotropic spaces and Refs. [38] for a review of the Einstein–Cartan gravity). The fundamental geometric objects defining Finsler–Riemann–Cartan spaces consists in the triple $\left(\mathbf{g}^{[F]}, \vartheta^{\alpha}_{[F]}, \mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}\right)$ where $\mathbf{g}^{[F]}$ is a d–metric (1.87),

$$\vartheta^{\alpha}_{[F]} = \left(dx^i, \delta y^j = dy^j + N^j_{[F]k}\left(x^l, y^s\right) dx^k\right)$$

with $N^j_{[F]k}\left(x^l, y^s\right)$ of type (1.86) and $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$ is an arbitrary d–connection (1.26) on TM (we put the label [F] emphasizing that the N–connection is a Finsler type one). The torsion $\mathbf{T}^{\alpha}_{[F]}$ and curvature $\mathbf{R}^{\alpha}_{[F]\beta}$ d–forms are computed following respectively the formulas (1.41) and (1.42) but for $\vartheta^{\alpha}_{[F]}$ and $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$.

We can consider an inverse modelling of geometries when (roughly speaking) the Finsler configurations are 'hidden' in Riemann–Cartan spaces. They can be distinguished for arbitrary Riemann–Cartan manifolds V^{n+n} with coventional split into "horizontal" and "vertical" subspaces and provided with a metric ansatz of type (1.87) and with prescribed procedure of adapting the geometric objects to the Cartan N–connection $N^j_{[F]k}$. Of course, the torsion can not be an arbitrary one but admitting irreducible decompositions with respect to N–frames $\mathbf{e}^{[F]}_{\alpha}$ and N–coframes $\vartheta^{\alpha}_{[F]}$ (see, respectively, the formulas (1.21) and (1.22) when $N^a_i \rightarrow N^j_{[F]i}$). There were constructed and investigated different classes of exact solutions of the Einstein equations with anholonomic variables characterized by anholonomically induced torsions and modelling Finsler like geometries in (pseudo) Riemannian and Riemann–Cartan spaces (see Refs. [34, 35, 25]). All constructions from Finsler–Riemann–Cartan geometry reduce to Finsler–Riemann configurations (in general, we can see metrics of arbitrary signatures) if $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$ is changed into the Levi–Civita metric connection defined with respect to anholonomic frames $\mathbf{e}_{\alpha}$ and coframes ϑ^{α} when the N–connection curvature Ω^i_{jk} and the anholonomically induced torsion vanish.

Teleparallel generalized Finsler geometry

In Refs. [55] the teleparallel Finsler connections, the Cartan–Einstein unification in the teleparallel approach and related moving frames with Finsler structures were investigated. In our analysis of teleparallel geometry we heavily use the results on N–connection geometry in order to illustrate how the teleparallel and metric affine gravity [9] can defined as to include generalized Finsler structures. For a general metric–affine spaces admitting N–connection structure N_i^a, the curvature $\mathbf{R}^{\alpha}_{.\beta\gamma\tau}$ of an arbitrary d–connection $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ splits into h– and v–irreversible components, $\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = (R^i{}_{hjk}, R^a{}_{bjk}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{bcd})$, see (1.48). In order to include Finsler like metrics, we state that the N–connection curvature can be nontrivial $\Omega^a_{jk} \neq 0$, which is quite different from the condition imposed in section 1.4.1. The condition of vanishing of curvature for teleparallel spaces, see (1.80), is to be stated separately for every h-v–irreversible component,

$$R^i{}_{hjk} = 0, R^a{}_{bjk} = 0, P^i{}_{jka} = 0, P^c{}_{bka} = 0, S^i{}_{jbc} = 0, S^a{}_{bcd} = 0.$$

We can define certain types of teleparallel Berwald connections (see sections 1.3.4 and 1.3.4) with certain nontrivial components of nonmetricity d–field (1.63) if we modify the metric compatibility conditions (1.81) into a less strong one when

$$Q_{kij} = -D_k g_{ij} = 0 \text{ and } Q_{abc} = -D_a h_{bc} = 0$$

but with nontrivial components

$$\mathbf{Q}_{\alpha\beta\gamma} = \left(Q_{cij} = -D_c g_{ij},\ Q_{iab} = -D_i h_{ab}\right).$$

The class of teleparallel Finsler spaces is distinguished by Finsler N–connection and d–connection ${}^{[F]}\mathbf{N}^i_j(x,y)$ and ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} = \left({}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk}\right)$, see, respectively, (1.86) and (1.88) with vanishing d–curvature components,

$${}^{[F]}R^i{}_{hjk} = 0,\ {}^{[F]}P^i{}_{jka} = 0,\ {}^{[F]}S^i{}_{jbc} = 0.$$

We can generate teleparallel Finsler affine structures if it is not imposed the condition of vanishing of nonmetricity d–field. In this case, there are considered arbitrary d–connections $\mathbf{D}_\alpha$ that for the induced Finsler quadratic form (1.87) $\mathbf{g}^{[F]}$

$${}^{[F]}\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_\alpha \mathbf{g}^{[F]} \neq 0$$

but $\mathbf{R}^{\alpha}_{.\beta\gamma\tau}(\mathbf{D}) = 0$.

The teleparallel–Finsler configurations are contained as particular cases of Finsler–affine spaces, see section 1.4.2. For vielbein fields $\mathbf{e}_{\alpha}^{\underline{\alpha}}$ and their inverses $\mathbf{e}^{\alpha}_{\ \underline{\alpha}}$ related to the d–metric (1.87),

$$\mathbf{g}^{[F]}_{\alpha\beta} = \tilde{\mathbf{e}}_{\alpha}^{\ \underline{\alpha}}\tilde{\mathbf{e}}_{\beta}^{\ \underline{\beta}} g_{\underline{\alpha\beta}}$$

we define the Weitzenbock–Finsler d–connection

$$^{[WF]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} = \tilde{\mathbf{e}}^{\alpha}_{\ \underline{\alpha}}\delta_{\gamma}\tilde{\mathbf{e}}_{\beta}^{\ \underline{\alpha}} \tag{1.96}$$

where δ_{γ} are the elongated by $^{[F]}\mathbf{N}^{i}_{j}(x,y)$ partial derivatives (1.21). The torsion of $^{[WF]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ is defined

$$^{[WF]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = {}^{[WF]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} - {}^{[WF]}\mathbf{\Gamma}^{\alpha}_{\ \gamma\beta} \tag{1.97}$$

containing h– and v–irreducible components being constructed from the components of a d–metric and N–adapted frames. We can express

$$^{[WF]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} + \mathbf{Z}^{\alpha}_{\ \beta\gamma}$$

where $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$ is the Levi–Civita connection (1.53), $\hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} =^{[F]} \widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma} - \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$, and the contorsion tensor is

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\ {}^{[W]}\mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ {}^{[W]}\mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ {}^{[W]}\mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\mathbf{Q}_{\alpha\beta}.$$

In the non-Berwald standard approaches to the Finsler–teleparallel gravity it is considered that $\mathbf{Q}_{\alpha\beta} = 0$.

Cartan geometry

The theory of Cartan spaces (see, for instance, [16, 56]) can be reformulated as a dual to Finsler geometry [58] (see details and references in [20]). The Cartan space is constructed on a cotangent bundle $T^{*}M$ similarly to the Finsler space on the tangent bundle TM.

Consider a real smooth manifold M, the cotangent bundle $(T^{*}M, \pi^{*}, M)$ and the manifold $\widetilde{T^{*}M} = T^{*}M\backslash\{0\}$.

Definition 1.4.18. *A Cartan space is a pair $C^{n} = (M, K(x,p))$ such that $K : T^{*}M \to \mathbb{R}$ is a scalar function satisfying the following conditions:*

1. *K is a differentiable function on the manifold $\widetilde{T^{*}M} = T^{*}M\backslash\{0\}$ and continuous on the null section of the projection $\pi^{*} : T^{*}M \to M$;*

2. *K is a positive function, homogeneous on the fibers of the T^*M, i. e. $K(x, \lambda p) = \lambda F(x,p), \lambda \in \mathbb{R}$;*

3. *The Hessian of K^2 with elements*

$$\check{g}^{ij}_{[K]}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j} \tag{1.98}$$

is positively defined on $\widetilde{T^*M}$.

The function $K(x,y)$ and $\check{g}^{ij}(x,p)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Cartan space C^n. We use symbols like "$\check{g}$" as to emphasize that the geometrical objects are defined on a dual space.

One considers "anisotropic" (depending on directions, momenta, p_i) Christoffel symbols. For simplicity, we write the inverse to (1.98) as $g^{(K)}_{ij} = \check{g}_{ij}$ and introduce the coefficients

$$\check{\gamma}^i{}_{jk}(x,p) = \frac{1}{2}\check{g}^{ir}\left(\frac{\partial \check{g}_{rk}}{\partial x^j} + \frac{\partial \check{g}_{jr}}{\partial x^k} - \frac{\partial \check{g}_{jk}}{\partial x^r}\right),$$

defining the canonical N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}\}$,

$$\check{N}^{[K]}_{ij} = \check{\gamma}^k{}_{ij}p_k - \frac{1}{2}\gamma^k{}_{nl}p_k p^l \check{\partial}^n \check{g}_{ij} \tag{1.99}$$

where $\check{\partial}^n = \partial/\partial p_n$. The N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}\}$ can be used for definition of an almost complex structure like in (1.87) and introducing on T^*M a d–metric

$$\check{\mathbf{G}}_{[k]} = \check{g}_{ij}(x,p)dx^i \otimes dx^j + \check{g}^{ij}(x,p)\delta p_i \otimes \delta p_j, \tag{1.100}$$

with $\check{g}^{ij}(x,p)$ taken as (1.98).

Using the canonical N–connection (1.99) and Finsler metric tensor (1.98) (or, equivalently, the d–metric (1.100)), we can define the canonical d–connection $\check{\mathbf{D}} = \{\check{\mathbf{\Gamma}}\left(\check{N}_{[k]}\right)\}$

$$\check{\mathbf{\Gamma}}\left(\check{N}_{[k]}\right) = \check{\Gamma}^{\alpha}_{[k]\beta\gamma} = \left(\check{H}^i_{[k]\ jk}, \check{H}^i_{[k]\ jk}, \check{C}_{[k]\ i}^{\ jk}, \check{C}_{[k]\ i}^{\ jk}\right)$$

with the coefficients computed

$$\check{H}^i_{[k]\ jk} = \frac{1}{2}\check{g}^{ir}\left(\check{\delta}_j \check{g}_{rk} + \check{\delta}_k \check{g}_{jr} - \check{\delta}_r \check{g}_{jk}\right),\ \check{C}_{[k]\ i}^{\ jk} = \check{g}_{is}\check{\partial}^s \check{g}^{jk}.$$

The d–connection $\check{\mathbf{\Gamma}}\left(\check{N}_{(k)}\right)$ satisfies the metricity conditions both for the horizontal and vertical components, i. e. $\check{\mathbf{D}}_\alpha \check{\mathbf{g}}_{\beta\gamma} = 0$.

The d–torsions (1.95) and d–curvatures (1.48) are computed like in Finsler geometry but starting from the coefficients in (1.99) and (1.100), when the indices $a, b, c...$ run the same values as indices $i, j, k, ...$ and the geometrical objects are modelled as on the dual tangent bundle. It should be emphasized that in this case all values $\check{g}_{ij}$, $\check{\Gamma}^{\alpha}_{[k]\beta\gamma}$ and $\check{R}^{\cdot\alpha}_{[k]\beta\gamma\delta}$ are defined by a fundamental function $K(x,p)$.

In general, we can consider that a Cartan space is provided with a metric $\check{g}^{ij} = \partial^2 K^2 / 2\partial p_i \partial p_j$, but the N–connection and d–connection could be defined in a different manner, even not be determined by K. If a Cartan space is modelled in a metric–affine space V^{n+n}, with local coordinates $\left(x^i, y^k\right)$, we have to define a procedure of dualization of vertical coordinates, $y^k \rightarrow p_k$.

1.4.3 Generalized Lagrange and Hamilton geometries

The notion of Finsler spaces was extended by J. Kern [57] and R. Miron [60]. It is was elaborated in vector bundle spaces in Refs. [14] and generalized to superspaces [23]. We illustrate how such geometries can be modelled on a space $\mathbf{V}^{n+n}$ provided with N–connection structure.

Lagrange geometry and generalizations

The idea of generalization of the Finsler geometry was to consider instead of the homogeneous fundamental function $F(x,y)$ in a Finsler space a more general one, a Lagrangian $L(x,y)$, defined as a differentiable mapping $L : (x,y) \in TV^{n+n} \rightarrow L(x,y) \in \mathbb{R}$, of class C^{∞} on manifold $\widetilde{TV}^{n+n}$ and continuous on the null section $0 : V^n \rightarrow \widetilde{TV}^{n+n}$ of the projection $\pi : \widetilde{TV}^{n+n} \rightarrow V^n$. A Lagrangian is regular if it is differentiable and the Hessian

$$g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{1.101}$$

is of rank n on V^n.

Definition 1.4.19. *A Lagrange space is a pair $\mathbf{L}^n = (V^n, L(x,y))$ where V^n is a smooth real n–dimensional manifold provided with regular Lagrangian $L(x,y)$ structure $L : TV^n \rightarrow \mathbb{R}$ for which $g_{ij}(x,y)$ from (1.101) has a constant signature over the manifold $\widetilde{TV}^{n+n}$.*

The fundamental Lagrange function $L(x,y)$ defines a canonical N–connection

$${}^{[cL]}N^i_{\ j} = \frac{1}{2}\frac{\partial}{\partial y^j}\left[g^{ik}\left(\frac{\partial^2 L^2}{\partial y^k \partial y^h} y^h - \frac{\partial L}{\partial x^k}\right)\right] \tag{1.102}$$

as well a d-metric

$$\mathbf{g}_{[L]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{1.103}$$

with $g_{ij}(x,y)$ taken as (1.101). As well we can introduce an almost Kählerian structure and an almost Hermitian model of $\mathbf{L}^n$, denoted as $\mathbf{H}^{2n}$ as in the case of Finsler spaces but with a proper fundamental Lagrange function and metric tensor g_{ij}. The canonical metric d–connection $\widehat{\mathbf{D}}_{[L]}$ is defined by the coefficients

$$^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left({}^{[L]}\widehat{L}^{i}{}_{jk},\ {}^{[L]}\widehat{L}^{i}{}_{jk},\ {}^{[L]}\widehat{C}^{i}{}_{jk},\ {}^{[L]}\widehat{C}^{i}{}_{jk}\right) \tag{1.104}$$

computed for $N^i_{[cL]\ j}$ and by respective formulas (1.56) with $h_{ab} \to g^{[L]}_{ij}$ and $\widehat{C}^a_{bc} \to \widehat{C}^i_{ij}$. The d–torsions (1.95) and d–curvatures (1.48) are determined, in this case, by ${}^{[L]}\widehat{L}^{i}{}_{jk}$ and ${}^{[L]}\widehat{C}^{i}{}_{jk}$. We also note that instead of ${}^{[cL]}N^i{}_j$ and ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ we can consider on a L^n–space different N–connections $N^i{}_j$, d–connections $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ which are not defined only by $L(x,y)$ and $g^{[L]}_{ij}$ but can be metric, or non–metric with respect to the Lagrange metric.

The next step of generalization [60] is to consider an arbitrary metric $g_{ij}(x,y)$ on $\mathbf{TV}^{n+n}$ (we use boldface symbols in order to emphasize that the space is enabled with N–connection structure) instead of (1.101) which is the second derivative of "anisotropic" coordinates y^i of a Lagrangian.

Definition 1.4.20. *A generalized Lagrange space is a pair* $\mathbf{GL}^n = (V^n, g_{ij}(x,y))$ *where* $g_{ij}(x,y)$ *is a covariant, symmetric and N–adapted d–tensor field of rank n and of constant signature on* $\widetilde{TV}^{n+n}$.

One can consider different classes of N– and d–connections on TV^{n+n}, which are compatible (metric) or non compatible with (1.103) for arbitrary $g_{ij}(x,y)$ and arbitrary d–metric

$$\mathbf{g}_{[gL]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{1.105}$$

We can apply all formulas for d–connections, N–curvatures, d–torsions and d–curvatures as in sections 1.2.3 and 1.2.4 but reconsidering them on $\mathbf{TV}^{n+n}$, by changing

$$h_{ab} \to g_{ij}(x,y), \widehat{C}^a_{bc} \to \widehat{C}^i_{ij} \text{ and } N^a_i \to N^k{}_i.$$

Prescribed torsions $T^i{}_{jk}$ and $S^i{}_{jk}$ can be introduced on $\mathbf{GL}^n$ by using the d–connection

$$\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left(\widehat{L}^i_{[gL]\ jk} + \tau^i{}_{jk}, \widehat{L}^i_{[gL]\ jk} + \tau^i{}_{jk}, \widehat{C}^i_{[gL]\ jk} + \sigma^i{}_{jk}, \widehat{C}^i_{[gL]\ jk} + \sigma^i{}_{jk}\right) \tag{1.106}$$

with

$$\tau^i{}_{jk} = \frac{1}{2} g^{il} \left(g_{kh} T^h_{.lj} + g_{jh} T^h_{.lk} - g_{lh} T^h{}_{jk} \right) \text{ and } \sigma^i{}_{jk} = \frac{1}{2} g^{il} \left(g_{kh} S^h_{.lj} + g_{jh} S^h_{.lk} - g_{lh} S^h{}_{jk} \right)$$

like we have performed for the Berwald connections (1.64) with (1.65) and (1.94) but in our case

$$^{[aL]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left(\widehat{L}^i_{[gL]\ jk}, \widehat{L}^i_{[gL]\ jk}, \widehat{C}^i_{[gL]\ jk}, \widehat{C}^i_{[gL]\ jk} \right) \tag{1.107}$$

is metric compatible being modelled like on a tangent bundle and with the coefficients computed as in (1.56) with $h_{ab} \rightarrow g^{[L]}_{ij}$ and $\widehat{C}^a_{bc} \rightarrow \widehat{C}^i_{ij}$, by using the d–metric $\mathbf{G}_{[gL]}$ (1.105). The connection (1.106) is a Riemann–Cartan one modelled on effective tangent bundle provided with N–connection structure.

Hamilton geometry and generalizations

The geometry of Hamilton spaces was defined and investigated by R. Miron in the papers [59] (see details and additional references in [20]). It was developed on the cotangent bundle as a dual geometry to the geometry of Lagrange spaces. Here we consider their modelling on couples of spaces $(V^n, {}^*V^n)$, or cotangent bundle T^*M, where ${}^*V^n$ is considered as a 'dual' manifold defined by local coordinates satisfying a duality condition with respect to coordinates on V^n. We start with the definition of generalized Hamilton spaces and then consider the particular cases.

Definition 1.4.21. *A generalized Hamilton space is a pair $\mathbf{GH}^n = (V^n, \check{g}^{ij}(x,p))$ where V^n is a real n–dimensional manifold and $\check{g}^{ij}(x,p)$ is a contravariant, symmetric, nondegenerate of rank n and of constant signature on $\widetilde{T^*V}^{n+n}$.*

The value $\check{g}^{ij}(x,p)$ is called the fundamental (or metric) tensor of the space $\mathbf{GH}^n$. One can define such values for every paracompact manifold V^n. In general, a N–connection on $\mathbf{GH}^n$ is not determined by $\check{g}^{ij}$. Therefore we can consider an arbitrary N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}(x,p)\}$ and define on T^*V^{n+n} a d–metric similarly to (1.33) and/or (1.103)

$$\mathbf{G}_{[gH]}^{\smile} = \check{g}_{\alpha\beta}(\check{u})\, \check{\delta}^\alpha \otimes \check{\delta}^\beta = \check{g}_{ij}(\check{u})\, d^i \otimes d^j + \check{g}^{ij}(\check{u})\, \check{\delta}_i \otimes \check{\delta}_j, \tag{1.108}$$

The N–coefficients $\check{N}_{ij}(x,p)$ and the d–metric structure (1.108) define an almost Kähler model of generalized Hamilton spaces provided with canonical d–connections, d–torsions and d-curvatures (see respectively the formulas d–torsions (1.95) and d–curvatures (1.48) with the fiber coefficients redefined for the cotangent bundle $T^*V^{\ n+n}$).

A generalized Hamilton space $\mathbf{GH}^n$ is called reducible to a Hamilton one if there exists a Hamilton function $H(x,p)$ on $T^*V^{\ n+n}$ such that

$$\check{g}^{ij}_{[H]}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}. \qquad (1.109)$$

Definition 1.4.22. *A Hamilton space is a pair $\mathbf{H}^n = (V^n, H(x,p))$ such that $H : T^*V^n \to \mathbb{R}$ is a scalar function which satisfy the following conditions:*

1. *H is a differentiable function on the manifold $\widetilde{T^*V}^{n+n} = T^*V^{n+n}\backslash\{0\}$ and continuous on the null section of the projection $\pi^* : T^*V^{n+n} \to V^n$;*

2. *The Hessian of H with elements (1.109) is positively defined on $\widetilde{T^*V}^{n+n}$ and $\check{g}^{ij}(x,p)$ is nondegenerate matrix of rank n and of constant signature.*

For Hamilton spaces, the canonical N–connection (defined by H and its Hessian) is introduced as

$$^{[H]}\check{N}_{ij} = \frac{1}{4}\{\check{g}_{ij}, H\} - \frac{1}{2}\left(\check{g}_{ik}\frac{\partial^2 H}{\partial p_k \partial x^j} + \check{g}_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right), \qquad (1.110)$$

where the Poisson brackets, for arbitrary functions f and g on T^*V^{n+n}, act as

$$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial p}{\partial x^i}.$$

The canonical metric d–connection $^{[H]}\widehat{\mathbf{D}}$ is defined by the coefficients

$$^{[H]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma} = \left(\ ^{[c]}\check{H}^i_{\ jk},\ ^{[c]}\check{H}^i_{\ jk},\ ^{[c]}\check{C}^i_{\ jk},\ ^{[c]}\check{C}^i_{\ jk}\right)$$

computed for $^{[H]}\check{N}_{ij}$ and by respective formulas (1.56) with $g_{ij} \to \check{g}_{ij}(\check{u})$, $h_{ab} \to \check{g}^{ij}$ and $\widehat{L}^i_{\ jk} \to\ ^{[c]}\widehat{H}^i_{\ jk},\ \widehat{C}^a_{bc} \to\ ^{[c]}\check{C}_{\ i}^{\ jk}$ when

$$^{[c]}\check{H}^i_{\ jk} = \frac{1}{2}\check{g}^{is}\left(\check{\delta}_j\check{g}_{sk} + \check{\delta}_k\check{g}_{js} - \check{\delta}_s\check{g}_{jk}\right) \text{ and }\ ^{[c]}\check{C}_{\ i}^{\ jk} = -\frac{1}{2}\check{g}_{is}\check{\partial}^j\check{g}^{sk}.$$

In result, we can compute the d–torsions and d–curvatures like on Lagrange or on Cartan spaces. On Hamilton spaces all such objects are defined by the Hamilton function $H(x,p)$ and indices have to be reconsidered for co–fibers of the cotangent bundle.

We note that there were elaborated various type of higher order generalizations (on the higher order tangent and contangent bundles) of the Finsler–Cartan and Lagrange–Hamilton geometry [21] and on higher order supersymmetric (co) vector bundles in

Ref. [23]. We can generalize the d–connection ${}^{[H]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ to any d–connection in $\mathbf{H}^n$ with prescribed torsions, like we have done in previous section for Lagrange spaces, see (1.106). This type of Riemann–Cartan geometry is modelled like on a dual tangent bundle by a Hamilton metric structure (1.109), N–connection ${}^{[H]}\check{N}_{ij}$, and d–connection coefficients ${}^{[c]}\check{H}^{i}{}_{jk}$ and ${}^{[c]}\check{C}_{\ i}^{\ \ jk}$.

1.4.4 Nonmetricity and generalized Finsler–affine spaces

The generalized Lagrange and Finsler geometry may be defined on tangent bundles by using d–connections and d–metrics satisfying metric compatibility conditions [14]. Nonmetricity components can be induced if Berwald type d–connections are introduced into consideration on different type of manifolds provided with N–connection structure, see formulas (1.62), (1.64), (1.70) and (1.94).

We define such spaces as generalized Finsler spaces with nonmetricity.

Definition 1.4.23. *A generalized Lagrange–affine space* $\mathbf{GLa}^n = \left(V^n, g_{ij}(x,y),\ {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ *is defined on manifold* $\mathbf{TV}^{n+n}$*, provided with an arbitrary nontrivial N–connection structure* $\mathbf{N} = \{N^i_j\}$*, as a general Lagrange space* $\mathbf{GL}^n = (V^n, g_{ij}(x,y))$ *(see Definition 1.4.20) enabled with a d–connection structure* ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$$ {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[aL]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[a]}\mathbf{Z}^{\alpha}{}_{\beta}, \tag{1.111} $$

where ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta}$ *is the canonical generalized Lagrange d–connection (1.107) and*

$$ {}^{[a]}\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta}. $$

The d–metric structure on $\mathbf{GLa}^n$ is stated by an arbitrary N–adapted form (1.33) but on $\mathbf{TV}^{n+n}$,

$$ \mathbf{g}_{[a]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j. \tag{1.112} $$

The torsions and curvatures on $\mathbf{GLa}^n$ are computed by using formulas (1.41) and (1.42) with $\mathbf{\Gamma}^{\gamma}{}_{\beta} \rightarrow {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}$,

$$ {}^{[a]}\mathbf{T}^{\alpha} \doteqdot {}^{[a]}\mathbf{D}\vartheta^{\alpha} = \delta\vartheta^{\alpha} + {}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \vartheta^{\beta} \tag{1.113} $$

and

$$ {}^{[a]}\mathbf{R}^{\alpha}{}_{\beta} \doteqdot {}^{[a]}\mathbf{D}({}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}) = \delta({}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}) - {}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\gamma}. \tag{1.114} $$

Modelling in V^{n+n}, with local coordinates $u^\alpha = \left(x^i, y^k\right)$, a tangent bundle structure, we redefine the operators (1.22) and (1.21) respectively as

$$\mathbf{e}_\alpha \doteqdot \delta_\alpha = \left(\delta_i, \tilde{\partial}_k\right) \equiv \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a\left(u\right)\partial_a, \frac{\partial}{\partial y^k}\right) \tag{1.115}$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^\beta \doteqdot \delta^{\ \beta} = \left(d^i, \tilde{\delta}^k\right) \equiv \delta u^\alpha = \left(\delta x^i = dx^i, \delta y^k = dy^k + N_i^k\left(u\right)dx^i\right) \tag{1.116}$$

where Greek indices run the same values, $i, j, ... = 1, 2, ...n$ (we shall use the symbol "$\sim$" if one would be necessary to distinguish operators and coordinates defined on h– and v–subspaces).

Let us define the h– and v–irreducible components of the d–connection ${}^{[a]}\mathbf{\Gamma}^\alpha{}_\beta$ like in (1.27) and (1.28),

$${}^{[a]}\widehat{\mathbf{\Gamma}}^\alpha{}_{\beta\gamma} = \left({}^{[L]}\widehat{L}^i{}_{jk} + \ z^i{}_{jk}, \ {}^{[L]}\widehat{L}^i{}_{jk} \ + \ z^i{}_{jk}, \ {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}, \ {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}\right)$$

with the distorsion d–tensor

$${}^{[a]}\ \mathbf{Z}^\alpha{}_\beta = \left(z^i{}_{jk}, \ z^i{}_{jk}, c^i{}_{jk}, c^i{}_{jk}\right)$$

defined as on a tangent bundle

$$\begin{aligned}
{}^{[a]}L^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\delta_k}\delta_j\right)\rfloor\delta^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\delta_k}\delta_j + \ {}^{[a]}\ \mathbf{Z}_{\delta_k}\delta_j\right)\rfloor\delta^i = \ {}^{[L]}\widehat{L}^i{}_{jk} + \ z^i{}_{jk}, \\
{}^{[a]}\tilde{L}^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\delta_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \left({}^{[L]}\widehat{\mathbf{D}}_k\tilde{\partial}_j + \ {}^{[a]}\ \mathbf{Z}_k\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \ {}^{[L]}\widehat{L}^i{}_{jk} + \ z^i{}_{jk}, \\
{}^{[a]}C^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\tilde{\partial}_k}\delta_j\right)\rfloor\delta^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\tilde{\partial}_k}\delta_j + \ {}^{[a]}\ \mathbf{Z}_{\tilde{\partial}_k}\delta_j\right)\rfloor\delta^i = \ {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}, \\
{}^{[a]}\tilde{C}^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\tilde{\partial}_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\tilde{\partial}_k}\tilde{\partial}_j + \ {}^{[a]}\ \mathbf{Z}_{\tilde{\partial}_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \ {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk},
\end{aligned}$$

where for 'lifts' from the h–subspace to the v–subspace we consider that ${}^{[a]}L^i_{jk} = {}^{[a]}\tilde{L}^i_{jk}$ and ${}^{[a]}C^i_{jk} = {}^{[a]}\tilde{C}^i_{jk}$. As a consequence, on spaces with modelled tangent space structure, the d–connections are distinguished as $\mathbf{\Gamma}^\alpha_{\beta\gamma} = \left(L^i_{jk}, C^i_{jk}\right)$.

Theorem 1.4.8. *The torsion* ${}^{[a]}\mathbf{T}^\alpha$ *(1.113) of a d–connection* ${}^{[a]}\mathbf{\Gamma}^\alpha{}_\beta = \left({}^{[a]}L^i_{jk}, \ {}^{[a]}C^i_{jc}\right)$ *(1.111) has as irreducible h- v–components,* ${}^{[a]}\mathbf{T}^\alpha = \left(T^i_{jk}, \tilde{T}^i_{jk}\right)$*, the d–torsions*

$$\begin{aligned}
T^i_{.jk} &= -T^i_{kj} = \ {}^{[L]}\widehat{L}^i{}_{jk} + \ z^i{}_{jk} - \ {}^{[L]}\widehat{L}^i{}_{kj} - \ z^i{}_{kj}, \\
\tilde{T}^i_{jk} &= -\ \tilde{T}^i_{kj} = \ {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk} - \ {}^{[L]}\widehat{C}^i{}_{kj} - c^i{}_{kj}.
\end{aligned} \tag{1.117}$$

The proof of this Theorem consists from a standard calculus for metric–affine spaces of ${}^{[a]}\mathbf{T}^{\alpha}$ [4] but with N–adapted frames. We note that in $z^{i}{}_{jk}$ and $c^{i}{}_{kj}$ it is possible to include any prescribed values of the d–torsions.

Theorem 1.4.9. *The curvature* ${}^{[a]}\mathbf{R}^{\alpha}{}_{\beta}$ *(1.114) of a d–connection* ${}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ *(1.111) has the h- v–components (d–curvatures),* ${}^{[a]}\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = \{ {}^{[a]}R^{i}{}_{hjk},\ {}^{[a]}P^{i}{}_{jka},\ {}^{[a]}S^{i}{}_{jbc}\}$,

$$\begin{aligned}
{}^{[a]}R^{i}{}_{hjk} &= \frac{\delta\ {}^{[a]}L^{i}_{.hj}}{\delta x^{h}} - \frac{\delta\ {}^{[a]}L^{i}_{.hk}}{\delta x^{j}} + {}^{[a]}L^{m}_{.hj}\ {}^{[a]}L^{i}_{mk} - {}^{[a]}L^{m}_{.hk}\ {}^{[a]}L^{i}_{mj} - {}^{[a]}C^{i}_{.ho}\Omega^{o}_{.jk}, \\
{}^{[a]}P^{i}{}_{jks} &= \frac{\partial\ {}^{[a]}L^{i}_{.jk}}{\partial y^{s}} - \left(\frac{\partial\ {}^{[a]}C^{i}_{.js}}{\partial x^{k}} + {}^{[a]}L^{i}_{.lk}\ {}^{[a]}C^{l}_{.js} - {}^{[a]}L^{l}_{.jk}\ {}^{[a]}C^{i}_{.ls} - {}^{[a]}L^{p}_{.sk}\ {}^{[a]}C^{i}_{.jp}\right) \\
&\quad + {}^{[a]}C^{i}_{.jp}\ {}^{[a]}P^{p}_{.ks}, \\
{}^{[a]}S^{i}{}_{jlm} &= \frac{\partial\ {}^{[a]}C^{i}_{.jl}}{\partial y^{m}} - \frac{\partial\ {}^{[a]}C^{i}_{.jm}}{\partial y^{l}} + {}^{[a]}C^{h}_{.jl}\ {}^{[a]}C^{i}_{.hm} - {}^{[a]}C^{h}_{.jm}\ {}^{[a]}C^{i}_{hl},
\end{aligned}$$

where ${}^{[a]}L^{m}_{.hk} = {}^{[L]}\widehat{L}^{i}{}_{jk} + z^{i}{}_{jk}$, ${}^{[a]}C^{i}_{.jk} = {}^{[L]}\widehat{C}^{i}{}_{jk} + c^{i}{}_{jk}$, $\Omega^{o}_{.jk} = \delta_{j}N^{o}_{i} - \delta_{i}N^{o}_{j}$ *and* ${}^{[a]}P^{p}_{.ks} = \partial N^{p}_{i}/\partial y^{s} - {}^{[a]}L^{p}_{.ks}$.

The proof consists from a straightforward calculus.

Remark 1.4.3. *As a particular case of* $\mathbf{GLa}^{n}$*, we can define a Lagrange–affine space* $\mathbf{La}^{n} = \left(V^{n}, g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$*, provided with a Lagrange quadratic form* $g^{[L]}_{ij}(x,y)$ *(1.101) inducing the canonical N–connection structure* ${}^{[cL]}\mathbf{N} = \{ {}^{[cL]}N^{i}_{j}\}$ *(1.102) as in a Lagrange space* $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ *(see Definition 1.4.19)) but with a d–connection structure* ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$${}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta},$$

where ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta}$ *is the canonical Lagrange d–connection (1.104),*

$${}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta},$$

and the (co) frames $\mathbf{e}_{\beta}$ *and* ϑ^{γ} *are respectively constructed as in (1.21) and (1.22) by using* ${}^{[cL]}N^{i}_{j}$.

Remark 1.4.4. *The Finsler–affine spaces* $\mathbf{Fa}^n = \left(V^n, F(x,y),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ *can be introduced by further restrictions of* $\mathbf{La}^n$ *to a quadratic form* $g_{ij}^{[F]}$ *(1.85) constructed from a Finsler metric* $F(x^i, y^j)$ *inducing the canonical N–connection structure* ${}^{[F]}\mathbf{N} = \{\ {}^{[F]}N^i_j\}$ *(1.86) as in a Finsler space* $\mathbf{F}^n = (V^n, F(x,y))$ *but with a d–connection structure* ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$$^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta},$$

where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta}$ *is the canonical Finsler d–connection (1.88),*

$$^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta},$$

and the (co) frames $\mathbf{e}_{\beta}$ *and* ϑ^{γ} *are respectively constructed as in (1.21) and (1.22) by using* ${}^{[F]}N^i_j$.

Remark 1.4.5. *By similar geometric constructions (see Remarks 1.4.3 and 1.4.4) on spaces modelling cotangent bundles, we can define generalized Hamilton–affine spaces* $\mathbf{GHa}^n = \left(V^n, \check{g}^{ij}(x,p),\ {}^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ *and theirs restrictions to Hamilton–affine* $\mathbf{Ha}^n = (V^n, \check{g}^{ij}_{[H]}(x,p),\ {}^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta})$ *and Cartan–affine spaces* $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p),\ {}^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ *(see sections 1.4.3 and 1.4.2) as to contain distorsions induced by nonmetricity* $\check{\mathbf{Q}}_{\alpha\gamma}$*. The geometric objects have to be adapted to the corresponding N–connection and d–metric/ quadratic form structures (arbitrary* $\check{N}_{ij}(x,p)$ *and d–metric (1.108),* ${}^{[H]}\check{N}_{ij}(x,p)$ *(1.110) and quadratic form* $\check{g}^{ij}_{[H]}$ *(1.109) and* $\check{N}^{[K]}_{ij}$ *(1.99) and* $\check{g}^{ij}_{[K]}$ *(1.98).*

Finally, in this section, we note that Theorems 1.4.8 and 1.4.9 can be reformulated in the forms stating procedures of computing d–torsions and d–curvatures on every type of spaces with nonmetricity and local anisotropy by adapting the abstract symbol and/or coordinate calculations with respect to corresponding N–connection, d–metric and canonical d–connection structures.

1.5 Conclusions

The method of moving anholonomic frames with associated nonlinear connection (N–connection) structure elaborated in this work on metric–affine spaces provides a general framework to deal with any possible model of locally isotropic and/or anisotropic interactions and geometries defined effectively in the presence of generic off–diagonal metric and linear connection configurations, in general, subjected to certain anholonomic constraints. As it has been pointed out, the metric–affine gravity (MAG) contains various types of generalized Finsler–Lagrange–Hamilton–Cartan geometries which can be distinguished by a corresponding N–connection structure and metric and linear connection adapted to the N–connection structure.

As far as the anholonomic frames, nonmetricity and torsion are considered as fundamental quantities, all mentioned geometries can be included into a unique scheme which can be developed on arbitrary manifolds, vector and tangent bundles and their dual bundles (co-bundles) or restricted to Riemann–Cartan and (pseudo) Riemannian spaces. We observe that a generic off–diagonal metric (which can not be diagonalized by any coordinate transform) defining a (pseudo) Riemannian space induces alternatively various type of Riemann–Cartan and Finsler like configuratons modelled by respective frame structures. The constructions are generalized if the linear connection structures are not constrained to metricity conditions. One can regard this as extensions to metric–affine spaces provided with N–connection structure modelling also bundle structures and generalized noncommutative symmetries of metrics and anholonomic frames.

In this paper we have studied the general properties of metric–affine spaces provided with N–connection structure. We formulated and proved the main theorems concerning general metric and nonlinear and linear connections in MAG. There were stated the criteria when the spaces with local isotropy and/or local anisotropy can be modelled in metric–affine spaces and on vector/ tangent bundles. We elaborated the concept of generalized Finsler–affine geometry as a unification of metric–affine (with nontrivial torsion and nonmetricity) and Finsler like spaces (with nontrivial N–connection structure and locally anisotropic metrics and connections).

In a general sense, we note that the generalized Finsler–affine geometries are contained as anhlonomic and noncommutative configurations in extra dimension gravity models (string and brane models and certain limits to the Einstein and gauge gravity defined by off–diagonal metrics and anholonomic constraints). We would like to stress that the N–connection formalism developed for the metric–affine spaces relates the bulk geometry in string and/or MAG to gauge theories in vector/tangent bundles and to various type of non–Riemannian gravity models.

The approach presented here could be advantageous in a triple sense. First, it provides a uniform treatment of all metric and connection geometries, in general, with vector/tangent bundle structures which arise in various type of string and brane gravity models. Second, it defines a complete classification of the generalized Finsler–affine geometries stated in Tables 1-11 from the Appendix. Third, it states a new geometric method of constructing exact solutions with generic off–diagonal metric ansatz, torsions and nonmetricity, depending on 2–5 variables, in string and metric–affine gravity, with limits to the Einstein gravity, see Refs. [33].

1.6 Appendix: Classification of Generalized Finsler–Affine Spaces

We outline and give a brief characterization of the main classes of generalized Finsler–affine spaces (see Tables 1.1–1.11). A unified approach to such very different locally isotropic and anisotropic geometries, defined in the framework of the metric–affine geometry, can be elaborated only by introducing the concept on N–connection (see Definition 1.2.10).

The N–connection curvature is computed following the formula $\Omega^a_{ij} = \delta_{[i}N^a_{j]}$, see (1.20), for any N–connection N^a_i. A d–connection $\mathbf{D} = [\mathbf{\Gamma}^\alpha_{\beta\gamma}] = [L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}]$ (see Definition 1.2.11) defines nontrivial d–torsions $\mathbf{T}^\alpha{}_{\beta\gamma} = [L^i_{[\ jk]}, C^i{}_{ja}, \Omega^a_{ij}, T^a{}_{bj}, C^a{}_{[bc]}]$ and d–curvatures $\mathbf{R}^\alpha{}_{\beta\gamma\tau} = [R^i{}_{jkl}, R^a{}_{bkl}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{dbc}]$ adapted to the N–connection structure (see, respectively, the formulas (1.45) and (1.48)). It is considered that a generic off–diagonal metric $g_{\alpha\beta}$ (see Remark 1.2.1) is associated to a N–connection structure and reprezented as a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ (see formula (1.33)). The components of a N–connection and a d–metric define the canonical d–connection $\mathbf{D} = [\widehat{\mathbf{\Gamma}}^\alpha_{\beta\gamma}] = [\widehat{L}^i{}_{jk}, \widehat{L}^a{}_{bk}, \widehat{C}^i{}_{jc}, \widehat{C}^a{}_{bc}]$ (see (1.56)) with the corresponding values of d–torsions $\widehat{\mathbf{T}}^\alpha{}_{\beta\gamma}$ and d–curvatures $\widehat{\mathbf{R}}^\alpha{}_{\beta\gamma\tau}$. The nonmetricity d–fields are computed by using formula $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_\alpha \mathbf{g}_{\beta\gamma} = [Q_{ijk}, Q_{iab}, Q_{ajk}, Q_{abc}]$, see (1.35).

1.6.1 Generalized Lagrange–affine spaces

The Table 1.1 outlines seven classes of geometries modelled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure. There are emphasized the configurations:

1. Metric–affine spaces (in brief, MA) are those stated by Definition 1.2.9 as certain manifolds V^{n+m} of necessary smoothly class provided with arbitrary metric, $g_{\alpha\beta}$, and linear connection, $\Gamma^\alpha_{\beta\gamma}$, structures. For generic off–diagonal metrics, a MA space always admits nontrivial N–connection structures (see Proposition 1.3.4). Nevertheless, in general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^\alpha_{\beta\gamma}$ can be not adapted to the N–connection structure. As a consequence, the general strength fields $\left(T^\alpha{}_{\beta\gamma}, R^\alpha{}_{\beta\gamma\tau}, Q_{\alpha\beta\gamma}\right)$ can be also not N–adapted. By using the Kawaguchi's metrization process and Miron's

procedure stated by Theorems 1.3.5 and 1.3.6 we can consider alternative geometries with d–connections $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (see Definition 1.2.11) derived from the components of N–connection and d–metric. Such geometries are adapted to the N–connection structure. They are characterized by d–torsion $\mathbf{T}^{\alpha}{}_{\beta\gamma}$, d–curvature $\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$, and nonmetricity d–field $\mathbf{Q}_{\alpha\beta\gamma}$.

2. Distinguished metric–affine spaces (DMA) are defined (see Definition 1.3.17) as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N_i^a, d–metric field (1.33) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$. In this case, all strengths $\left(\mathbf{T}^{\alpha}{}_{\beta\gamma}, \mathbf{R}^{\alpha}{}_{\beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma}\right)$ are N–adapted.

3. Berwald–affine spaces (BA, see section 1.3.4) are metric–affine spaces provided with generic off–diagonal metrics with associated N–connection structure and with a Berwald d–connection ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}{}_{jk}, \partial_b N^a_k, 0, \widehat{C}^{a}{}_{bc}]$, see (1.62), for with the d–torsions ${}^{[B]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [\ {}^{[B]}L^{i}{}_{[jk]}, 0, \Omega^a_{ij}, T^a{}_{bj}, C^a{}_{[bc]}]$ and d–curvatures
$$
{}^{[B]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} ={}^{[B]} [R^{i}{}_{jkl}, R^{a}{}_{bkl}, P^{i}{}_{jka}, P^{c}{}_{bka}, S^{i}{}_{jbc}, S^{a}{}_{dbc}]
$$
are computed by introducing the components of ${}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$, respectively, in formulas (1.45) and (1.48). By definition, this space satisfies the metricity conditions on the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (1.63)).

4. Berwald–affine spaces with prescribed torsion (BAT, see sections 1.3.4 and 1.3.4) are described by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b N^a_k, 0, C^{a}{}_{bc}]$, with more general h– and v–components, $\widehat{L}^{i}{}_{jk} \rightarrow L^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc} \rightarrow C^{a}{}_{bc}$, inducing prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion
$$
{}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^a_{ij}, T^a{}_{bj}, C^a{}_{[bc]} + \tau^{a}{}_{bc}],
$$
see (1.65). The components of curvature ${}^{[BT]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ have to be computed by introducing ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ into (1.48). There are nontrivial components of nonmetricity d–fields, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left({}^{[B\tau]}Q_{cij},\ {}^{[B\tau]}Q_{iab}\right)$.

5. Generalized Lagrange–affine spaces (GLA, see Definition 1.4.23), $\mathbf{GLa}^n = (V^n,\, g_{ij}(x,y),\ {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are modelled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (1.112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(\ {}^{[a]}L^i_{jk},\ {}^{[a]}C^i_{jc}\right)$ (1.111) are similar to those for

the usual Lagrange spaces (see Definition 1.4.20) but with distorsions ${}^{[a]}\ \mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$. The components of d–torsions ${}^{[a]}\mathbf{T}^{\alpha} = \left(T^{i}_{jk}, \tilde{T}^{i}_{jk}\right)$and d–curvatures ${}^{[a]}\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = \{\ {}^{[a]}R^{i}{}_{hjk},\ {}^{[a]}P^{i}{}_{jka},\ {}^{[a]}S^{i}{}_{jbc}\}$ are computed following Theorems 1.4.8 and 1.4.9.

6. Lagrange–affine spaces (LA, see Remark 1.4.3), $\mathbf{La}^{n} = (V^{n}, g^{[L]}_{ij}(x,y),\ \ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^{2}L^{2}}{\partial y^{i}\partial y^{j}}$ (1.101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^{i}_{j}\}$ (1.102) for a Lagrange space $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ (see Definition 1.4.19)) but with a d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta}$. This is a particular case of GLA spaces with prescribed types of N–connection ${}^{[cL]}N^{i}_{j}$ and d–metric to be like in Lagrange geometry.

7. Finsler–affine spaces (FA, see Remark 1.4.4), $\mathbf{Fa}^{n} = \left(V^{n}, F\left(x,y\right),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$, in their turn are introduced by further restrictions of $\mathbf{La}^{n}$ to a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^{2}F^{2}}{\partial y^{i}\partial y^{j}}$ (1.85) constructed from a Finsler metric $F\left(x^{i}, y^{j}\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \{\ {}^{[F]}N^{i}_{j}\}$ (1.86) as in the Finsler space $\mathbf{F}^{n} = (V^{n}, F\left(x,y\right))$ but with a d–connection structure ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta}$,where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection (1.88).

1.6.2 Generalized Hamilton–affine spaces

The Table 1.2 outlines geometries modelled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure splitting the space into any conventional a horizontal subspace and vertical subspace being isomorphic to a dual vector space provided with respective dual coordinates. We can use respectively the classification from Table 1.1 when the v–subspace is transformed into dual one as we noted in Remark 1.4.5 For simplicity, we label such spaces with symbols like $\check{N}_{ai}$ instead N^{a}_{i} where "inverse hat" points that the geometric object is defined for a space containing a dual subspaces. The local h–coordinates are labelled in the usual form, x^{i}, with $i = 1,2,...,n$ but the v–coordinates are certain dual vectors $\check{y}^{a} = p_{a}$ with $a = n{+}1, n{+}2, ..., n{+}m$. The local coordinates are denoted $\check{u}^{\alpha} = (x^{i}, \check{y}^{a}) = (x^{i}, p_{a})$. The curvature of a N–connection $\check{N}_{ai}$ is computed as $\check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$. The h– v–irreducible components of a general d–connection are parametrized $\check{\mathbf{D}} = [\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}] = [L^{i}{}_{jk}, L_{a}{}^{b}{}_{k}, \check{C}^{i}{}_{j}{}^{c}, \check{C}_{a}{}^{bc}]$, the d–torsions are

$\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma} = [L^i_{\ [jk]}, L_a^{\ b}{}_{\ k}, \check{C}^i{}_j^{\ c}, \check{C}_a^{\ [bc]}]$ and the d–curvatures

$$^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = [R^i_{\ jkl}, \check{R}_a^{\ b}{}_{kl}, \check{P}^i{}_{jk}^{\ \ a}, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a^{\ dbc}].$$

The nonmetricity d–fields are stated $\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk}, \check{Q}_i^{\ ab}, \check{Q}^a_{\ jk}, \check{Q}^{abc}]$. There are also considered additional labels for the Berwald, Cartan and another type d–connections.

1. Metric–dual–affine spaces (in brief, MDA) are usual metric–affine spaces with a prescribed structure of "dual" local coordinates.

2. Distinguished metric–dual-affine spaces (DMDA) are provided with d–metric and d–connection structures adapted to a N–connection $\check{N}_{ai}$ defining a global splitting into a usual h–subspace and a v–dual–subspace being dual to a usual v–subspace.

3. Berwald–dual–affine spaces (BDA) are Berwald–affine spaces with a dual v–subspace. Their Berwald d–connection is stated in the form

$$^{[B]}\check{\mathbf{D}} = [^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}] = [\widehat{L}^i_{\ jk}, \partial_b \check{N}_{ai}, 0, \check{C}_a^{\ [bc]}]$$

   with induced d–torsions $^{[B]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma} = [L^i_{[jk]}, 0, \check{\Omega}_{iaj}, \check{T}_a^{\ b}{}_j, \check{C}_a^{\ [bc]}]$ and d–curvatures

$$^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = [R^i_{\ jkl}, \check{R}_a^{\ b}{}_{kl}, , \check{P}^i{}_{jk}^{\ \ a}, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a^{\ dbc}]$$

   computed by introducing the components of $^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$, respectively, in formulas (1.45) and (1.48) re–defined for dual v–subspaces. By definition, this d–connection satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $\check{Q}^{abc} = 0$ but with nontrivial components of
   $^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ ^{[B]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk} = 0, \check{Q}_i^{\ ab}, \check{Q}^a_{\ jk}, \check{Q}^{abc} = 0]$.

4. Berwald–dual–affine spaces with prescribed torsion (BDAT) are described by a more general class of d–connections $^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^i_{\ jk}, \partial_b \check{N}_{ai}, 0, \check{C}_a^{\ bc}]$, inducing prescribed values $\tau^i_{\ jk}$ and $\check{\tau}_a^{\ bc}$ for d–torsions

$$^{[BT]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma} = [L^i_{\ [jk]} + \tau^i_{\ jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_a^{\ b}{}_j, \check{C}_a^{\ [bc]} + \check{\tau}_a^{\ bc}].$$

   The components of d–curvatures

$$^{[BT]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = \ [R^i_{\ jkl}, \check{R}_a^{\ b}{}_{kl}, \check{P}^i{}_{jk}^{\ \ a}, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a^{\ dbc}]$$

   have to be computed by introducing $^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ into dual form of formulas (1.48). There are nontrivial components of nonmetricity d–field,

$$^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = -\ ^{[BT]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = (Q_{ijk} = 0, \check{Q}_i^{\ ab}, \check{Q}^a_{\ jk}, \check{Q}^{abc} = 0).$$

5. Generalized Hamilton–affine spaces (GHA), $\mathbf{GHa}^n = \left(V^n, \check{g}^{ij}(x,p),\ {}^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, are modelled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a cotangent bundle structure. The d–metric $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ab}]$ and the d–connection ${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\ {}^{[a]}L^{i}_{jk},\ {}^{[a]}\check{C}_{i}{}^{jc})$ are similar to those for usual Hamilton spaces (see section 1.4.3) but with distorsions ${}^{[a]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The components of d–torsion and d–curvature, respectively, ${}^{[a]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_{a}{}^{[bc]}]$ and ${}^{[a]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = [R^{i}{}_{jkl}, \check{P}^{i}{}_{jk}^{\ a}, \check{S}_{a}{}^{dbc}]$, are computed following Theorems 1.4.8 and 1.4.9 reformulated for cotangent bundle structures.

6. Hamilton–affine spaces (HA, see Remark 1.4.5), $\mathbf{Ha}^n = (V^n,\ \check{g}^{ij}_{[H]}(x,p),\ {}^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta})$, are provided with Hamilton N–connection ${}^{[H]}\check{N}_{ij}\,(x,p)$ (1.110) and quadratic form $\check{g}^{ij}_{[H]}$ (1.109) for a Hamilton space $\mathbf{H}^n = (V^n, H(x,p))$ (see section 1.4.3)) but with a d–connection structure ${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[H]}[L^{i}{}_{jk}, \check{C}_{a}{}^{bc}]$ distorted by arbitrary torsion, $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$, and nonmetricity d–fields, $\check{\mathbf{Q}}_{\beta\gamma\alpha}$, when $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta} = {}^{[H]}\widehat{\check{\mathbf{\Gamma}}}^{\alpha}_{\beta} + {}^{[H]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$. This is a particular case of GHA spaces with prescribed types of N–connection ${}^{[H]}\check{N}_{ij}$ and d–metric $\check{\mathbf{g}}^{[H]}_{\alpha\beta} = [g^{ij}_{[H]} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_i}]$ to be like in the Hamilton geometry.

7. Cartan–affine spaces (CA, see Remark 1.4.5), $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p),\ {}^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, are dual to the Finsler spaces $\mathbf{Fa}^n = \left(V^n, F\,(x,y)\,,\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$. The CA spaces are introduced by further restrictions of $\mathbf{Ha}^n$ to a quadratic form $\check{g}^{ij}_{[C]}$ (1.98) and canonical N–connection $\check{N}^{[C]}_{ij}$ (1.99). They are like usual Cartan spaces, see section 1.4.2) but contain distorsions induced by nonmetricity $\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The d–metric is parametrized $\check{\mathbf{g}}^{[C]}_{\alpha\beta} = [g^{ij}_{[C]} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_i}]$ and the curvature ${}^{[C]}\check{\Omega}_{iaj}$ of N–connection ${}^{[C]}\check{N}_{ia}$ is computed ${}^{[C]}\check{\Omega}_{iaj} = \delta_{[i}\ {}^{[C]}\check{N}_{j]a}$. The Cartan's d–connection

$$
{}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[C]}[L^{i}{}_{jk}, L^{i}{}_{jk}, \check{C}_{a}{}^{bc}, \check{C}_{a}{}^{bc}]
$$

possess nontrivial d–torsions ${}^{[C]}\check{\mathbf{T}}_{\alpha}{}^{\beta\gamma} = [L^{i}{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_{a}{}^{[bc]}]$ and d–curvatures ${}^{[C]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = [R^{i}{}_{jkl},\ \check{P}^{i}{}_{jk}^{\ a},\ \check{S}_{a}{}^{dbc}]$ computed following Theorems 1.4.8 and 1.4.9 reformulated on cotangent bundles with explicit type of N–connection $\check{N}^{[C]}_{ij}$ d–metric $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$ and d–connection ${}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. The nonmetricity d–fields are not trivial for such spaces, ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[C]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk},\ \check{Q}_{i}{}^{ab},\ \check{Q}^{a}{}_{jk}, \check{Q}^{abc}]$.

1.6.3 Teleparallel Lagrange–affine spaces

We considered the main properties of teleparallel Finsler–affine spaces in section 1.4.2 (see also section 1.4.1 on locally isotropic teleparallel spaces). Every type of teleparallel spaces is distinguished by the condition that the curvature tensor vanishes but the torsion plays a cornerstone role. Modelling generalized Finsler structures on metric–affine spaces, we do not impose the condition on vanishing nonmetricity (which is stated for usual teleparallel spaces). For $\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$, the classification of spaces from Table 1.1 trasforms in that from Table 1.3.

1. Teleparallel metric–affine spaces (in brief, TMA) are usual metric–affine ones but with vanishing curvature, modelled on manifolds V^{n+m} of necessary smoothly class provided, for instance, with the Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ (1.82). For generic off–diagonal metrics, a TMA space always admits nontrivial N–connection structures (see Proposition 1.3.4). We can model teleparallel geometries with local anisotropy by distorting the Levi–Civita or the canonical d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (see Definition 1.2.11) both constructed from the components of N–connection and d–metric. In general, such geometries are characterized by d–torsion $\mathbf{T}^{\alpha}{}_{\beta\gamma}$ and nonmetricity d–field $\mathbf{Q}_{\alpha\beta\gamma}$ both constrained to the condition to result in zero d–curvatures.

2. Distinguished teleparallel metric–affine spaces (DTMA) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (1.33) and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with vanishing d–curvatures defined by Weitzenbock–affine d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions by nonmetricity d–fields preserving the condition of zero values for d–curvatures.

3. Teleparallel Berwald–affine spaces (TBA) are defined by distorsions of the Weitzenbock connection to any Berwald like structure, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. All constructions with generic off–diagonal metrics can be adapted to the N–connection and considered for d–objects. By definition, such spaces satisfy the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (1.63)).

4. Teleparallel Berwald–affine spaces with prescribed torsion (TBAT) are defined by a more general class of distorsions resulting in the Weitzenbock type d–connections, ${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$, with more general h– and v–components, $\widehat{L}^{i}{}_{jk} \rightarrow$

$L^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc} \rightarrow C^{a}{}_{bc}$, having prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion

$$^{[WB]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}]$$

and characterized by the condition $^{[WB\tau]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$ with notrivial components of nonmetricity $^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij},\ Q_{iab})$.

5. Teleparallel generalized Lagrange–affine spaces (TGLA) are distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure with vanishing d–curvature. The Weitzenblock–Lagrange d–connection $^{[Wa]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(^{[Wa]}L^{i}_{jk},\ ^{[Wa]}C^{i}_{jc}\right)$, where

$$^{[WaL]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$$

is defined by a d–metric $\mathbf{g}_{[a]}$ (1.112) $\mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields $^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$ and $^{[Wa]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$.

6. Teleparallel Lagrange–affine spaces (TLA 1.4.3) consist a subclass of spaces $\mathbf{La}^{n} = \left(V^{n}, g^{[L]}_{ij}(x,y),\ ^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^{2}L^{2}}{\partial y^{i}\partial y^{j}}$ (1.101) inducing the canonical N–connection structure $^{[cL]}\mathbf{N} = \{\ ^{[cL]}N^{i}_{j}\}$ (1.102) for a Lagrange space $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure $^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, $^{[W]}\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when $^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$. This is a generalization of teleparallel Finsler affine spaces (see section (1.4.2)) when $g^{[L]}_{ij}(x,y)$ is considered instead of $g^{[F]}_{ij}(x,y)$.

7. Teleparallel Finsler–affine spaces (TFA) are particular cases of spaces of type $\mathbf{Fa}^{n} = (V^{n}, F(x,y),\ ^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, defined by a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^{2}F^{2}}{\partial y^{i}\partial y^{j}}$ (1.85) constructed from a Finsler metric $F(x^{i}, y^{j})$. They are provided with a canonical N–connection structure $^{[F]}\mathbf{N} = \{\ ^{[F]}N^{i}_{j}\}$ (1.86) as in the Finsler space $\mathbf{F}^{n} = (V^{n}, F(x,y))$ but with a Finsler–Weitzenbock d–connection structure $^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$, respective d–torsion, $^{[WF]}\mathbf{T}_{\beta}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields,

$$^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma},$$

where $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ contains distorsions from the canonical Finsler d–connection (1.88). Such distorsions are constrained to satisfy the condition of vanishing curvature d–tensors (see section (1.4.2)).

1.6.4 Teleparallel Hamilton–affine spaces

This class of metric–affine spaces is similar to that outlined in previous subsection, see Table 1.3 but derived on spaces with dual vector bundle structure and induced generalized Hamilton–Cartan geometry (section 1.4.3 and Remark 1.4.5). We outline the main denotations for such spaces and note that they are characterized by the condition $\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau} = 0.$

1. Teleparallel metric dual affine spaces (in brief, TMDA) define teleparalles structures on metric–affine spaces provided with generic off–diagonal metrics and associated N–connections modelling splitting with effective dual vector bundle structures.

2. Distinguished teleparallel metric dual affine spaces (DTMDA) are spaces provided with independent d–metric, d–connection structures adapted to a N–connection in an effective dual vector bundle and resulting in zero d–curvatures.

3. Teleparallel Berwald dual affine spaces (TBDA) .

4. Teleparallel dual Berwald–affine spaces with prescribed torsion (TDBAT).

5. Teleparallel dual generalized Hamilton–affine spaces (TDGHA).

6. Teleparallel dual Hamilton–affine spaces (TDHA, see section 1.4.3).

7. Teleparallel dual Cartan–affine spaces (TDCA).

1.6.5 Generalized Finsler–Lagrange spaces

This class of geometries is modelled on vector/tangent bundles [14] (see subsections 1.4.2 and 1.4.3) or on metric–affine spaces provided with N–connection structure. There are also alternative variants when metric–affine structures are defined for vector/tangent bundles with independent generic off–diagonal metrics and linear connection structures. The standard approaches to generalized Finsler geometries emphasize the connections satisfying the metricity conditions. Nevertheless, the Berwald type connections admit certain nonmetricity d–fields. The classification stated in Table 1.5 is similar to that from Table 1.1 with that difference that the spaces are defined from the very beginning to be any vector or tangent bundles. The local coordinates x^i are considered for base subspaces and y^a are for fiber type subspaces. We list the short denotations and main properties of such spaces:

1. Metric affine vector bundles (in brief, MAVB) are provided with arbitrary metric $g_{\alpha\beta}$ and linear connection $\Gamma^{\alpha}_{\beta\gamma}$ structure. For generic off–diagonal metrics, we can introduce associated nontrivial N–connection structures. In general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^{\alpha}_{\beta\gamma}$ may be not adapted to the N–connection structure. As a consequence, the general strength fields $\left(T^{\alpha}_{\ \beta\gamma}, R^{\alpha}_{\ \beta\gamma\tau}, Q_{\alpha\beta\gamma} = 0\right)$, defined in the total space of the vector bundle are also not N–adapted. We can consider a metric–affine (MA) structure on the total space if $Q_{\alpha\beta\gamma} \neq 0$.

2. Distinguished metric–affine vector bundles (DMAVB) are provided with N–connection structure N^a_i, d–metric field and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$. In this case, all strengths $(\mathbf{T}^{\alpha}_{\ \beta\gamma}, \mathbf{R}^{\alpha}_{\ \beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma} = 0)$ are N–adapted. A distinguished metric–affine (DMA) structure on the total space is considered if $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$.

3. Berwald metric–affine tangent bundles (BMATB) are provided with Berwald d–connection structure ${}^{[B]}\mathbf{\Gamma}$. By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} do not vanish (see formulas (1.63)).

4. Berwald metric–affine bundles with prescribed torsion (BMATBT) are described by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^{i}_{\ jk}, \partial_b N^a_k, 0, C^{a}_{\ bc}]$ inducing prescribed values $\tau^{i}_{\ jk}$ and $\tau^{a}_{\ bc}$ in d–torsion
$$^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = [L^{i}_{\ [jk]}, +\tau^{i}_{\ jk}, 0, \Omega^a_{ij}, T^{a}_{\ bj}, C^{a}_{\ [bc]} + \tau^{a}_{\ bc}],$$
see (1.65). There are nontrivial nonmetricity d–fields, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij}, Q_{iab})$.

5. Generalized Lagrange metric–affine bundles (GLMAB) are modelled as $\mathbf{GLa}^n = (V^n, g_{ij}(x,y), {}^{[a]}\mathbf{\Gamma}^{\alpha}_{\ \beta})$ spaces on tangent bundles provided with generic off–diagonal metrics with associated N–connection. If the d–connection is a canonical one, $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$, the nonmetricity vanish. But we can consider arbitrary d–connections $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with nontrivial nonmetricity d–fields.

6. Lagrange metric–affine bundles (LMAB) are defined on tangent bundles as spaces $\mathbf{La}^n = \left(V^n, g^{[L]}_{ij}(x,y), {}^{[b]}\mathbf{\Gamma}^{\alpha}_{\ \beta}\right)$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{ {}^{[cL]}N^i_j\}$ for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ (see Definition 1.4.19)) but with a d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity

d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta}$. This is a particular case of GLA spaces with prescribed types of N–connection ${}^{[cL]}N^{i}_{j}$ and d–metric to be like in Lagrange geometry.

7. Finsler metric–affine bundles (FMAB), are modelled on tangent bundles as spaces $\mathbf{Fa}^{n} = \left(V^{n}, F\left(x,y\right), {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ with quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^{2}F^{2}}{\partial y^{i}\partial y^{j}}$ (1.85) constructed from a Finsler metric $F\left(x^{i}, y^{j}\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N^{i}_{j}\}$ as in the Finsler space $\mathbf{F}^{n} = \left(V^{n}, F\left(x,y\right)\right)$ but with a d–connection structure ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta}$,where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection (1.88).

1.6.6 Generalized Hamilton–Cartan spaces

Such spaces are modelled on vector/tangent dual bundles (see sections subsections 1.4.3 and 1.4.2) as metric–affine spaces provided with N–connection structure. The classification stated in Table 1.6 is similar to that from Table 1.2 with that difference that the geometry is modelled from the very beginning as vector or tangent dual bundles. The local coordinates x^{i} are considered for base subspaces and $y^{a} = p_{a}$ are for cofiber type subspaces. So, the spaces from Table 1.6 are dual to those from Table 1.7, when the respective Lagrange–Finsler structures are changed into Hamilton–Cartan structures. We list the short denotations and main properties of such spaces:

1. The metric–affine dual vector bundles (in brief, MADVB) are defined by metric–affine independent metric and linear connection structures stated on dual vector bundles. For generic off–diagonal metrics, there are nontrivial N–connection structures. The linear connection may be not adapted to the N–connection structure.

2. Distinguished metric-affine dual vector bundles (DMADVB) are provided with d–metric and d–connection structures adapted to a N–connection $\check{N}_{ai}$.

3. Berwald metric–affine dual bundles (BMADB) are provided with a Berwald d–connection
$$ {}^{[B]}\check{\mathbf{D}} = [\ {}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}{}_{jk}, \partial_{b}\check{N}_{ai}, 0, \check{C}_{a}{}^{[bc]}]. $$
By definition, on such spaces, there are satisfied the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $\check{Q}^{abc} = 0$ but with nontrivial components of ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[B]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk} = 0, \check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}, \check{Q}^{abc} = 0]$.

4. Berwald metrical–affine dual bundles with prescribed torsion (BMADBT) are described by a more general class of d–connections ${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^i{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_a{}^{bc}]$ inducing prescribed values $\tau^i{}_{jk}$ and $\check{\tau}_a{}^{bc}$ for d–torsions
$$^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^i{}_{[jk]} + \tau^i{}_{jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_a{}^b{}_j, \check{C}_a{}^{[bc]} + \check{\tau}_a{}^{bc}].$$
There are nontrivial components of nonmetricity d–field, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = {}^{[BT]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = \left(Q_{ijk} = 0, \check{Q}_i{}^{ab}, \check{Q}^a{}_{jk}, \check{Q}^{abc} = 0\right)$.

5. Generalized metric–affine Hamilton bundles (GMAHB) are modelled on dual vector bundles as spaces $\mathbf{GHa}^n = \left(V^n, \check{g}^{ij}(x,p), {}^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, provided with generic off–diagonal metrics with associated N–connection inducing a cotangent bundle structure. The d–metric $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ab}]$ and the d–connection ${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left({}^{[a]}L^i_{jk}, {}^{[a]}\check{C}_i{}^{jc}\right)$ are similar to those for usual Hamilton spaces, with distorsions ${}^{[a]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$. For canonical configurations, ${}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$, we obtain ${}^{[GH]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$.

6. Metric–affine Hamilton bundles (MAHB) are defied on dual bundles as spaces $\mathbf{Ha}^n = \left(V^n, \check{g}^{ij}_{[H]}(x,p), {}^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, provided with Hamilton N–connection ${}^{[H]}\check{N}_{ij}(x,p)$ and quadratic form $\check{g}^{ij}_{[H]}$ for a Hamilton space $\mathbf{H}^n = (V^n, H(x,p))$ (see section 1.4.3) with a d–connection structure ${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[H]}[L^i{}_{jk}, \check{C}_a{}^{bc}]$ distorted by arbitrary torsion, $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$, and nonmetricity d–fields, $\check{\mathbf{Q}}_{\beta\gamma\alpha}$, when $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta} = {}^{[H]}\widehat{\check{\mathbf{\Gamma}}}^{\alpha}_{\ \beta} + {}^{[H]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$. This is a particular case of GMAHB spaces with prescribed types of N–connection ${}^{[H]}\check{N}_{ij}$ and d–metric $\check{\mathbf{g}}^{[H]}_{\alpha\beta} = [g^{ij}_{[H]} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_i}]$ to be like in the Hamilton geometry but with nontrivial nonmetricity.

7. Metric–affine Cartan bundles (MACB) are modelled on dual tangent bundles as spaces $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p), {}^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ being dual to the Finsler spaces. They are like usual Cartan spaces, see section 1.4.2) but may contain distorsions induced by nonmetricity $\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The d–metric is parametrized $\check{\mathbf{g}}^{[C]}_{\alpha\beta} = [g^{ij}_{[C]} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_i}]$ and the curvature ${}^{[C]}\check{\Omega}_{iaj}$ of N–connection ${}^{[C]}\check{N}_{ia}$ is computed ${}^{[C]}\check{\Omega}_{iaj} = \delta_{[i}\ {}^{[C]}\check{N}_{j]a}$. The Cartan's d–connection ${}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[C]}[L^i{}_{jk}, L^i{}_{jk}, \check{C}_a{}^{bc}, \check{C}_a{}^{bc}]$ possess nontrivial d–torsions ${}^{[C]}\check{\mathbf{T}}_{\alpha}^{\ \beta\gamma} = [L^i{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_a{}^{[bc]}]$ and d–curvatures ${}^{[C]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = {}^{[C]}[R^i{}_{jkl}, \check{P}^i{}_{jk}{}^a, \check{S}_a{}^{dbc}]$ computed following Theorems 1.4.8 and 1.4.9 reformulated on cotangent bundles with explicit type of N–connection $\check{N}^{[C]}_{ij}$ d–metric $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$ and d–connection ${}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. Distorsions result in d–connection $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} = {}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} + {}^{[C]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$. The nonmetricity d–fields are not trivial for such spaces.

1.6.7 Teleparallel Finsler–Lagrange spaces

The teleparallel configurations can be modelled on vector and tangent bundles (the teleparallel Finsler–affine spaces are defined in section 1.4.2, see also section 1.4.1 on locally isotropic teleparallel spaces) were constructed as subclasses of metric–affine spaces on manifolds of necessary smoothly class. The classification from Table 1.7 is a similar to that from Table 1.3 but for direct vector/ tangent bundle configurations with vanishing nonmetricity. Nevertheless, certain nonzero nonmetricity d–fields can be present if the Berwald d–connection is considered or if we consider a metric–affine geometry in bundle spaces.

1. Teleparallel vector bundles (in brief, TVB) are provided with independent metric and linear connection structures like in metric–affine spaces satisfying the condition of vanishing curvature. The N–connection is associated to generic off–diagonal metrics. The TVB spaces can be provided with a Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$(1.82) which can be transformed in a d–connection one with respect to N–adapted frames. We can model teleparallel geometries with local anisotropy by distorting the Levi–Civita or the canonical d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (see Definition 1.2.11) both constructed from the components of N–connection and d–metric. In general, such vector (in particular cases, tangent) bundle geometries are characterized by d–torsions $\mathbf{T}^{\alpha}{}_{\beta\gamma}$ and nonmetricity d–fields $\mathbf{Q}_{\alpha\beta\gamma}$ both constrained to the condition to result in zero d–curvatures.

2. Distinguished teleparallel vector bundles (DTVB, or vect. b.) are provided with N–connection structure N^a_i, d–metric field (1.33) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with vanishing d–curvatures. The geometric constructions are stated by the Weitzenbock–affine d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions by nonmetricity d–fields preserving the condition of zero values for d–curvatures. The standard constructions from Finsler geometry and generalizations are with vanishing nonmetricity.

3. Teleparallel Berwald vector bundles (TBVB) are defined by Weitzenbock connections of Berwald type structure, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. By definition, such spaces satisfy the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} do not vanish (see formulas (1.63)).

4. Teleparallel Berwald vector bundles with prescribed torsion (TBVBT) are defined by a more general class of distorsions resulting in the Weitzenbock d–connection,

$^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion,

$$^{[WB]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}],$$

characterized by the condition $^{[WB\tau]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$ and nontrivial components of nonmetricity d–field, $^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij}, Q_{iab})$.

5. Teleparallel generalized Lagrange spaces (TGL) are modelled on tangent bundles (tang. b.) provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure being enabled with zero d–curvature. The Weitzenblock–Lagrange d–connections

$$^{[Wa]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left({}^{[Wa]}L^{i}_{jk},\ {}^{[Wa]}C^{i}_{jc} \right),\quad {}^{[WaL]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$$

are defined by a d–metric $\mathbf{g}_{[a]}$ (1.112) $\mathbf{Z}^{\alpha}{}_{\beta}$ inducing $^{[Wa]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$. For simplicity, we consider the configurations when nonmetricity d–fields $^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$.

6. Teleparallel Lagrange spaces (TL) are modelled on tangent bundles provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (1.101) inducing the canonical N–connection structure $^{[cL]}\mathbf{N} = \{\ ^{[cL]}N^i_j\}$ (1.102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure $^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, $^{[W]}\mathbf{T}_{\beta}$ when $^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$. For simplicity, we can consider configurations with zero nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$.

7. Teleparallel Finsler spaces (TF) are modelled on tangent bundles provided with a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (1.85) constructed from a Finsler metric $F(x^i, y^j)$. They are also enabled with a canonical N–connection structure $^{[F]}\mathbf{N} = \{\ ^{[F]}N^i_j\}$ (1.86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ but with a Finsler–Weitzenbock d–connection structure $^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$, respective d–torsion, $^{[WF]}\mathbf{T}_{\beta}$. We can write

$$^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma},$$

where $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ contains distorsions from the canonical Finsler d–connection (1.88). Such distorsions are constrained to satisfy the condition of vanishing curvature d–tensors (see section (1.4.2)) and, for simplicity, of vanishing nonmetricity, $\mathbf{Q}_{\beta\gamma\tau} = 0$.

1.6.8 Teleparallel Hamilton–Cartan spaces

This subclass of Hamilton–Cartan spaces is modelled on dual vector/ tangent bundles being similar to that outlined in Table 1.4 (on generalized Hamilton–Cartan geometry, see section 1.4.3 and Remark 1.4.5) and dual to the subclass outlined in Table 1.7. We outline the main denotations and properties of such spaces and note that they are characterized by the condition $\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau} = 0$ and $\check{\mathbf{Q}}^{\alpha}{}_{\beta\gamma} = 0$ with that exception that there are nontrivial nonmetricity d–fields for Berwald configuratons.

1. Teleparallel dual vector bundles (TDVB, or d. vect. b.) are provided with generic off–diagonal metrics and associated N–connections. In general, $\check{Q}^{\alpha}{}_{\beta\gamma} \neq 0$.

2. Distinguished teleparallel dual vector bundles spaces (DTDVB) are provided with independent d–metric, d–connection structures adapted to a N–connection in an effective dual vector bundle and resulting in zero d–curvatures. In general, $\check{\mathbf{Q}}^{\alpha}{}_{\beta\gamma} \neq 0$.

3. Teleparallel Berwald dual vector bundles (TBDVB) are provided with Berwald–Weitzenbock d–connection structure resulting in vanishing d–curvature.

4. Teleparallel Berwald dual vector bundles with prescribed d–torsion (TBDVB) are with d–connections ${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_{a}{}^{bc}]$ inducing prescribed values $\tau^{i}{}_{jk}$ and $\check{\tau}_{a}{}^{bc}$ for d–torsions ${}^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]} + \tau^{i}{}_{jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_{a}{}^{b}{}_{j}, \check{C}_{a}{}^{[bc]} + \check{\tau}_{a}{}^{bc}]$. They are described by certain distorsions to a Weitzenbock d–connection.

5. Teleparallel generalized Hamilton spaces (TGH) consist a subclass of generalized Hamilton spaces with vanishing d–curvature structure, defined on dual tangent bundles (d. tan. b.). They are described by distorsions to a Weitzenbock d–connection ${}^{[Wa]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. In the simplest case, we consider ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$.

6. Teleparallel Hamilton spaces (TH, see section 1.4.3), as a particular subclass of TGH, are provided with d–connection and N–connection structures corresponding to Hamilton configurations.

7. Teleparallel Cartan spaces (TC) are particular Cartan configurations with absolut teleparallelism.

1.6.9 Distinguished Riemann–Cartan spaces

A wide class of generalized Finsler geometries can be modelled on Riemann–Cartan spaces by using generic off–diagonal metrics and associated N–connection structures.

The locally anisotropic metric–affine configurations from Table 1.1 transform into a Riemann–Cartan ones if we impose the condition of metricity. For the Berwald type connections one could be certain nontrivial nonmetricity d–fields on intersection of h- and v–subspaces. The local coordinates x^i are considered as certain holonomic ones and y^a are anholonomic. We list the short denotations and main properties of such spaces:

1. Riemann–Cartan spaces (in brief, RC, see related details in section 1.3.5) are certain manifolds V^{n+m} of necessary smoothly class provided with metric structure $g_{\alpha\beta}$ and linear connection structure $\Gamma^{\alpha}_{\beta\gamma}$ (constructed as a distorsion by torsion of the Levi–Civita connection) both satisfying the conditions of metric compatibility, $Q_{\alpha\beta\gamma} = 0$. For generic off–diagonal metrics, a RC space always admits nontrivial N–connection structures (see Proposition 1.3.4 reformulated for the case of vanishing nonmetricity). In general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^{\alpha}_{\beta\gamma}$ may be not adapted to the N–connection structure.

2. Distinguished Riemann–Cartan spaces (DRC) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (1.33) and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (a distorsion of the Levi–Civita connection, or of the canonical d–connection) satisfying the condition $\mathbf{Q}_{\alpha\beta\gamma} = 0$. In this case, the strengths $\left(\mathbf{T}^{\alpha}{}_{\beta\gamma}, \mathbf{R}^{\alpha}{}_{\beta\gamma\tau}\right)$ are N–adapted.

3. Berwald Riemann–Cartan (BRC) are modelled if a N–connection structure is defined in a Riemann–Cartan space and distorting the connection to a Berwald d–connection ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}{}_{jk}, \partial_b N^a_k, 0, \widehat{C}^{a}{}_{bc}]$, see (1.62). By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (1.63)). Nonmetricities vanish with respect to holonomic frames.

4. Berwald Riemann–Cartan spaces with prescribed torsion (BRCT) are defined by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b N^a_k, 0, C^{a}{}_{bc}]$ inducing prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion ${}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^a_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}]$, see (1.65). The nontrivial components of nonmetricity d–fields are ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij}, Q_{iab})$. Such components vanish with respect to holonomic frames.

5. Generalized Lagrange Riemann–Cartan spaces (GLRC) are modelled as distinguished Riemann–Cartan spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (1.112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(\ {}^{[a]}L^i_{jk},\ {}^{[a]}C^i_{jc}\right)$ (1.111) are those for the usual Lagrange spaces (see Definition 1.4.20).

6. Lagrange Riemann–Cartan spaces (LRC, see Remark 1.4.3) are provided with a Lagrange quadratic form $g_{ij}^{[L]}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (1.101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{ {}^{[cL]}N_j^i \}$ (1.102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ (see Definition 1.4.19)) and, for instance, with a canonical d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ satisfying metricity conditions for the d–metric defined by $g_{ij}^{[L]}(x,y)$.

7. Finsler Riemann–Cartan spaces (FRC, see Remark 1.4.4) are defined by a quadratic form $g_{ij}^{[F]} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (1.85) constructed from a Finsler metric $F(x^i, y^j)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N_j^i \}$ (1.86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ with ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ being the canonical Finsler d–connection (1.88).

1.6.10 Distinguished (pseudo) Riemannian spaces

Sections 1.3.5 and 1.4.1 are devoted to modelling of locally anisotropic geometric configurations in (pseudo) Riemannian spaces enabled with generic off–diagonal metrics and associated N–connection structure. Different classes of generalized Finsler metrics can be embedded in (pseudo) Riemannian spaces as certain anholonomic frame configurations. Every such space is characterized by a corresponding off–diagonal metric ansatz and Levi–Civita connection stated with respect to coordinate frames or, alternatively (see Theorem 1.3.7), by certain N–connection and induced d–metric and d–connection structures related to the Levi–Cevita connection with coefficients defined with respect to N–adapted anholonomic (co) frames. We characterize every such type of (pseudo) Riemannian spaces both by Levi–Civita and induced canonical/or Berwald d–connections which contain also induced (by former off–diagonal metric terms) nontrivial d–torsion and/or nonmetricity d–fields.

1. (Pseudo) Riemann spaces (in brief, pR) are certain manifolds V^{n+m} of necessary smoothly class provided with generic off–diagonal metric structure $g_{\alpha\beta}$ of arbitrary signature inducing the unique torsionless and metric Levi–Civita connection $\Gamma^{\alpha}_{\bigtriangledown\beta\gamma}$. We can effectively diagonalize such metrics by anholonomic frame transforms with associated N–connection structure. We can also consider alternatively the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}]$ (1.56)defined by the coefficients of d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ and N–connection N_i^a. We have nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$, but $T^{\alpha}_{\bigtriangledown\beta\gamma} = 0, Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ and $\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$. The simplest anholonomic configurations are characterized by associated N–connections with vanishing N–connection curvature, $\Omega^a_{ij} = \delta_{[i}N^a_{j]} = 0$. The d–torsions $\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [\widehat{L}^i_{[\ jk]}, \widehat{C}^i{}_{ja}, \Omega^a_{ij}, \widehat{T}^a{}_{bj}, \widehat{C}^a{}_{[bc]}]$ and

d–curvatures $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = [\widehat{R}^{i}_{\ jkl}, \widehat{R}^{a}_{\ bkl}, \widehat{P}^{i}_{\ jka}, \widehat{P}^{c}_{\ bka}, \widehat{S}^{i}_{\ jbc}, \widehat{S}^{a}_{\ dbc}]$ are computed by introducing the components of $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$, respectively, in formulas (1.45) and (1.48).

2. Distinguished (pseudo) Riemannian spaces (DpR) are defined as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N_i^a, d–metric field and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (a distorsion of the Levi–Civita connection, or of the canonical d–connection) satisfying the condition $\mathbf{Q}_{\alpha\beta\gamma} = 0$.

3. Berwald (pseudo) Riemann spaces (pRB) are modelled if a N–connection structure is defined by a generic off–diagonal metric. The Levi–Civita connection is distorted to a Berwald d–connection ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}_{\ jk}, \partial_b N_k^a, 0, \widehat{C}^{a}_{\ bc}]$, see (1.62). By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (1.63)). Such nonmetricities vanish with respect to holonomic frames. The torsion is zero for the Levi–Civita connection but ${}^{[B]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = [L^{i}_{\ [jk]}, 0, \Omega^a_{ij}, T^{a}_{\ bj}, C^{a}_{\ [bc]}]$ is not trivial.

4. Berwald (pseudo) Riemann spaces with prescribed d–torsion (pRBT) are defined by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^{i}_{\ jk}, \partial_b N_k^a, 0, C^{a}_{\ bc}]$ inducing prescribed values $\tau^{i}_{\ jk}$ and $\tau^{a}_{\ bc}$ in d–torsion ${}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = [L^{i}_{\ [jk]}, +\tau^{i}_{\ jk}, 0, \Omega^a_{ij}, T^{a}_{\ bj}, C^{a}_{\ [bc]} + \tau^{a}_{\ bc}]$, see (1.65). The nontrivial components of nonmetricity d–fields are ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left({}^{[B\tau]}Q_{cij},\ {}^{[B\tau]}Q_{iab}\right)$. Such components vanish with respect to holonomic frames.

5. Generalized Lagrange (pseudo) Riemannian spaces (pRGL) are modelled as distinguished Riemann spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (1.112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = \left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ (1.111) are those for the usual Lagrange spaces (see Definition 1.4.20) but on a (pseudo) Riemann manifold with prescribed N–connection structure.

6. Lagrange (pseudo) Riemann spaces (pRL) are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (1.101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^i_j\}$ (1.102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ and, for instance, provided with a canonical d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}$ satisfying metricity conditions for the d–metric defined by $g^{[L]}_{ij}(x,y)$. There is an alternative construction with Levi–Civita connection.

7. Finsler (pseudo) Riemann (FpR) are defined by a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (1.85) constructed from a Finsler metric $F\left(x^i, y^j\right)$. It is induced the canonical

N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N^i_j \}$ (1.86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ with ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ being the canonical Finsler d–connection (1.88).

1.6.11 Teleparallel spaces

Teleparallel spaces were considered in sections 1.4.1 and 1.4.2. Here we classify what type of locally isotropic and anisotropic structures can be modelled in by anholonomic transforms of (pseudo) Riemannian spaces to teleparallel ones. The anholonomic frame structures are with associated N–connection with the components defined by the off–diagonal metric coefficients.

1. Teleparallel spaces (in brief, T) are usual ones with vanishing curvature, modelled on manifolds V^{n+m} of necessary smoothly class provided, for instance, with the Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$(1.82) which can be transformed in a d–connection one with respect to N–adapted frames. In general, such geometries are characterized by torsion ${}^{[W]}T^{\alpha}_{\beta\gamma}$ constrained to the condition to result in zero d–curvatures. The simplest theories are with vanishing nonmetricity.

2. Distinguished teleparallel spaces (DT) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (1.33) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with vanishing d–curvatures. The geometric constructions are stated by the Weitzenbock d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions without nonmetricity d–fields preserving the condition of zero values for d–curvatures.

3. Teleparallel Berwald spaces (TB) are defined by distorsions of the Weitzenbock connection on a manifold V^{n+m} to any Berwald like structure, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. All constructions with effective off–diagonal metrics can be adapted to the N–connection and considered for d–objects. Such spaces satisfy the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields, Q_{iab} and Q_{ajk}.

4. Teleparallel Berwald spaces with prescribed torsion (TBT) are defined by a more general class of distorsions resulting in the Weitzenbock d–connection,
$${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma},$$
having prescribed values $\tau^i{}_{jk}$ and $\tau^a{}_{bc}$ in d–torsion
$${}^{[WB]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^i{}_{[jk]}, +\tau^i{}_{jk}, 0, \Omega^a_{ij}, T^a{}_{bj}, C^a{}_{[bc]} + \tau^a{}_{bc}]$$

and characterized by the condition ${}^{[WB\tau]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$ with certain nontrivial nonmetricity d–fields, ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left({}^{[WB\tau]}Q_{cij},\ {}^{[WB\tau]}Q_{iab}\right)$.

5. Teleparallel generalized Lagrange spaces (TGL) are modelled as Riemann–Cartan spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure with zero d–curvature. The Weitzenblock–Lagrange d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (\ {}^{[Wa]}L^{i}_{jk},$ ${}^{[Wa]}C^{i}_{jc})$, where ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$, are defined by a d–metric $\mathbf{g}_{[a]}$ (1.112) with $\mathbf{Z}^{\alpha}{}_{\beta}$ inducing zero nonmetricity d–fields, ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ and zero d–curvature, ${}^{[Wa]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$.

6. Teleparallel Lagrange spaces (TL, see section 1.4.3) are Riemann–Cartan spaces $\mathbf{V}^{n+n}$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (1.101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^{i}_{j}\}$ (1.102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, ${}^{[W]}\mathbf{T}_{\beta}$, but zero nonmetricity d–fields, ${}^{[WL]}\mathbf{Q}_{\beta\gamma\alpha} = 0$, when ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$.

7. Teleparallel Finsler spaces (TF) are Riemann–Cartan manifolds $\mathbf{V}^{n+n}$ defined by a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (1.85) and a Finsler metric $F\left(x^i, y^j\right)$. They are provided with a canonical N–connection structure ${}^{[F]}\mathbf{N} = \{\ {}^{[F]}N^{i}_{j}\}$ (1.86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ but with a Finsler–Weitzenbock d–connection structure ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$, respective d–torsion, ${}^{[WF]}\mathbf{T}_{\beta}$, and vanishing nonmetricity, ${}^{[WF]}\mathbf{Q}_{\beta\gamma\tau} = 0$, d–fields, ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$,where $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ contains a distorsion from the canonical Finsler d–connection (1.88).

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^\alpha_{\beta\gamma}$ $T^\alpha_{\ \beta\gamma}$	$R^\alpha_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma}$
2. DMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ $\mathbf{T}^\alpha_{\ \beta\gamma}$	$\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma}$
3. BA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[B]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[B]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BAT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[BT]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[BT]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLA	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[a]}\mathbf{\Gamma}^\gamma_{\ \alpha\beta}$ ${}^{[a]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[a]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$
6. LA	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[b]}\mathbf{\Gamma}^\gamma_{\ \alpha\beta}$ ${}^{[b]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[b]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[b]}\mathbf{D}_\alpha \mathbf{g}_{\beta\gamma}^{[L]}$
7. FA	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[f]}\mathbf{\Gamma}^\gamma_{\ \alpha\beta}$ ${}^{[f]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[f]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[f]}\mathbf{D}_\alpha \mathbf{g}_{\beta\gamma}^{[F]}$

Table 1.1: Generalized Lagrange–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\beta\gamma}$ $\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [0, \check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}, 0]$
4. BDAT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}^{[BT]}_{\alpha\beta\gamma} = [0, \check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}, 0]$
5. GHA	$\dim i = \dim a$. $\check{N}_{ia}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ij}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
6. HA	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia}$, ${}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[H]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[L]}_{\beta\gamma}$
7. CA	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia}$; ${}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[C]}\mathbf{D}_{\alpha}\mathbf{g}^{[C]}_{\beta\gamma}$

Table 1.2: Generalized Hamilton–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TMA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ ${}^{[W]}T^{\alpha}_{\ \beta\gamma}$	${}^{[W]}R^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[W]}Q_{\alpha\beta\gamma}$
2. DTMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma}$
3. TBA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[WB]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. TBAT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[WB\tau]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. TGLA	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[Wa]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma}$
6. TLA	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WL]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WL]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[L]}$
7. TFA	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WF]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WF]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[F]}$

Table 1.3: Teleparallel Lagrange–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TMDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[W]}\check{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[W]}\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\check{Q}_{\alpha\beta\gamma}$
2. DTMDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. TBDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
4. TDBAT	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB\tau]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB\tau]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
5. TDGHA	$\dim i = \dim a$. $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ij}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
6. TDHA	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia}$, ${}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[WH]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[H]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[L]}_{\beta\gamma}$
7. TDCA	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia}$; ${}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[CW]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[CW]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[CW]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[CW]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[CW]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[C]}_{\beta\gamma}$

Table 1.4: Teleparallel Hamilton–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MAVB	N_i^a, Ω_{ij}^a,off.d.m vect.bundle $g_{\alpha\beta}$, total space $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^\alpha_{\beta\gamma}$, total space $T^\alpha_{\ \beta\gamma}$	$R^\alpha_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma} = 0$; $Q_{\alpha\beta\gamma} \neq 0$ for MA str.
2. DMAVB	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^\alpha_{\beta\gamma}$ $\mathbf{T}^\alpha_{\ \beta\gamma}$	$\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma} = 0$; $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$ for DMA str.
3. BMATB	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^\alpha_{\beta\gamma}$ ${}^{[B]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[B]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BMATBT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^\alpha_{\beta\gamma}$ ${}^{[BT]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[BT]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLMAB	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[gL]} = [g_{ij}, h_{kl}]$	$\widehat{\mathbf{\Gamma}}^\alpha_{\beta\gamma}, \mathbf{\Gamma}^\alpha_{\beta\gamma}$ $\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}, \mathbf{T}^\alpha_{\ \beta\gamma}$	$\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\tau}, \mathbf{R}^\alpha_{\ \beta\gamma\tau}$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
6. LMAB	$\dim i = \dim a$ ${}^{[L]}N_i^a$, ${}^{[L]}\Omega_{ij}^a$ d–metr.$\mathbf{g}^{[L]}_{\alpha\beta} =$ $[g^{[L]}_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}]$	${}^{[L]}\mathbf{\Gamma}^\alpha_{\beta\gamma}$ ${}^{[b]}\mathbf{\Gamma}^\alpha_{\beta\gamma}$ ${}^{[L]}\mathbf{T}^\alpha_{\ \beta\gamma}$, ${}^{[b]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[L]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[b]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[L]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
7. FMAB	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}^{[F]}_{\alpha\beta} =$ $[g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$	${}^{[F]}\widehat{\mathbf{\Gamma}}^\alpha_{\beta\gamma}$ ${}^{[f]}\mathbf{\Gamma}^\alpha_{\beta\gamma}$ ${}^{[F]}\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}$ ${}^{[f]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[F]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[f]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[F]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$

Table 1.5: Generalized Finsler–Lagrange spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ total space off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\ \beta\gamma}$ $\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BMADB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
4. BMADBT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
5. GMAHB	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, {}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, {}^{[GH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, {}^{[GH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. MAHB	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia}, \ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, \check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$
7. MACB	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia}; \quad {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$

Table 1.6: Generalized Hamilton–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ total space off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\ \beta\gamma}$ $\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BMADB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
4. BMADBT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
5. GMAHB	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, {}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, {}^{[GH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, {}^{[GH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. MAHB	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia},\ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, \check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$
7. MACB	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia};\ \ {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$

Table 1.7: Teleparallel Finsler–Lagrange spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{g}_{\alpha\beta}$, d. vect. b. $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$,	${}^{[W]}\check{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[W]}\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\check{Q}_{\alpha\beta\gamma}$
2. DTDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. TBDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{iab}, \check{Q}_{ajk}]$
4. TBDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB\tau]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB\tau]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{iab}, \check{Q}_{ajk}]$
5. TGH	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$, d. tan. b. $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ij}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. TH	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia}$, ${}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[WH]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WH]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
7. TC	$\dim i = \dim a.$ ${}^{[C]}\check{N}_{ia}$; ${}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[CW]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[CW]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[CW]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[CW]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$

Table 1.8: Teleparallel Hamilton–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. RC	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^{\alpha}_{\beta\gamma}$ $T^{\alpha}{}_{\beta\gamma}$	$R^{\alpha}{}_{\beta\gamma\tau}$ $Q_{\alpha\beta\gamma} = 0$
2. DRC	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ $\mathbf{T}^{\alpha}{}_{\beta\gamma}$	$\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma} = 0$
3. BRC	N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[B]}\mathbf{T}^{\alpha}{}_{\beta\gamma}$	${}^{[B]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BRCT	N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma}$	${}^{[BT]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLRC	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[a]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[a]}\mathbf{T}^{\alpha}{}_{\beta\gamma}$	${}^{[a]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
6. LRC	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}^{[L]}_{\alpha\beta}$	$\hat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ $\hat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	$\hat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ $\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
7. FRC	$\dim i = \dim a.$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}^{[F]}_{\alpha\beta}$	${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ ${}^{[F]}\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[F]}\hat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[F]}\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$

Table 1.9: Distinguished Riemann–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. pR	N_i^a, off-d.metr. $\Omega_{ij}^a = 0, \neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} =$ $[g_{ij}, h_{ab}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\widehat{\mathbf{D}} = [\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ $\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma} \neq 0$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
2. DpR	N_i^a, off-d.metr. $\Omega_{ij}^a = 0, \neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} =$ $[g_{ij}, h_{ab}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\mathbf{D} = [\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ $\mathbf{T}^{\alpha}_{\ \beta\gamma} \neq 0$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} = 0$
3. pRB	N_i^a, off-d.metr. $\Omega_{ij}^a = 0, \neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} =$ $[g_{ij}, h_{ab}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ ${}^{[B]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[B]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
4. pRBT	N_i^a, off-d.metr. $\Omega_{ij}^a = 0, \neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} =$ $[g_{ij}, h_{ab}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[BT]}\mathbf{D} = [{}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ ${}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[BT]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
5. pRGL	$N_i^a; \dim i = \dim a$ $\Omega_{ij}^a = 0, \neq 0$ $\mathbf{g}_{\alpha\beta} =$ $[g_{ij}, h_{ab}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\widehat{\mathbf{D}} = [\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ $\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. pRL	${}^{[L]}N_i^a; \dim i = \dim a$ ${}^{[L]}\Omega_{ij}^a = 0, \neq 0$ $\mathbf{g}^{[L]}_{\alpha\beta} = [g^{[L]}_{ij}, g^{[L]}_{ij}]$ $[g^{[L]}_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[L]}\mathbf{D} = [\ {}^{[L]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ ${}^{[L]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[L]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ ${}^{[L]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
7. pRF	${}^{[F]}N_i^a; \dim i = \dim a$ ${}^{[F]}\Omega_{ij}^a = 0, \neq 0$ $\mathbf{g}^{[F]}_{\alpha\beta} = [g^{[F]}_{ij}]$ $[g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$	$\bigtriangledown = [\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[F]}\widehat{\mathbf{D}} = [\ {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma} = 0$ ${}^{[F]}\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[F]}\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ ${}^{[F]}\widehat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$

Table 1.10: Distinguished (pseudo) Riemannian spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. T	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ ${}^{[W]}T^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \beta\gamma\tau} = 0$ $Q_{\alpha\beta\gamma} = 0$
2. DT	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
3. TB	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. TBT	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB\tau]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. TGL	$\dim i = \dim a$ N_i^a, Ω_{ij}^a $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[Wa]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
6. TL	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WL]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WL]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WL]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
7. TF	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WF]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WF]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WF]}\mathbf{Q}_{\alpha\beta\gamma} = 0$

Table 1.11: Teleparallel spaces

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Chapter 2

A Method of Constructing Off–Diagonal Solutions in Metric–Affine and String Gravity

Abstract [1]

The anholonomic frame method is generalized for non–Riemannian gravity models defined by string corrections to the general relativity and metric–affine gravity (MAG) theories. Such spacetime configurations are modelled as metric–affine spaces provided with generic off–diagonal metrics (which can not be diagonalized by coordinate transforms) and anholonomic frames with associated nonlinear connection (N–connection) structure. We investigate the field equations of MAG and string gravity with mixed holonomic and anholonomic variables. There are proved the main theorems on irreducible reduction to effective Einstein–Proca equations with respect to anholonomic frames adapted to N–connections. String corrections induced by the antisymmetric H–fields are considered. There are also proved the theorems and criteria stating a new method of constructing exact solutions with generic off–diagonal metric ansatz depending on 3-5 variables and describing various type of locally anisotropic gravitational configurations with torsion, nonmetricity and/or generalized Finsler–affine effective geometry. We analyze solutions, generated in string gravity, when generalized Finsler–affine metrics, torsion and nonmetricity interact with three dimensional solitons.

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2.1 Introduction

Nowadays, there exists an interest to non–Riemannian descriptions of gravity interactions derived in the low energy string theory [1] and/or certain noncommutative [2] and quantum group generalizations [3] of gravity and field theory. Such effective models can be expressed in terms of geometries with torsion and nonmetricity in the framework of metric–affine gravity (MAG) [4] and a subclass of such theories can be expressed as an effective Einstein–Proca gravity derived via irreducible decompositions [5].

In a recent work [6] we developed a unified scheme to the geometry of anholonomic frames with associated nonlinear connection (N–connection) structure for a large number of gauge and gravity models with locally isotropic and anisotropic interactions and nontrivial torsion and nonmetricity contributions and effective generalized Finsler–Weyl–Riemann–Cartan geometries derived from MAG. The synthesis of metric–affine and Finsler like theories was inspired by a number of exact solutions parametrized by generic off–diagonal metrics and anholonomic frames in Einstein, Einstein–Cartan, gauge and string gravity [7, 8]. The resulting formalism admits inclusion of locally anisotropic spinor interactions and extensions to noncommutative geometry and string/brane gravity [9, 10]. We concluded that the geometry of metric–affine spaces enabled with an additional N–connection structure is sufficient not only to model the bulk of physically important non–Riemannian geometries on (pseudo) Riemannian spaces but also states the conditions when such effective spaces with generic anisotropy can be defined as certain generalized Finsler–affine geometric configurations constructed as exact solutions of field equations. It was elaborated a detailed classification of such spacetimes provided with N–connection structure.

If in the Ref. [6] we paid attention to the geometrical (pre–dynamical) aspects of the generalized Finsler–affine configurations derived in MAG, the aim of this paper (the second partner) is to formulate a variatonal formalism of deriving field equations on metric–affine spaces provided with N–connection structure and to state the main theorems for constructing exact off–diagonal solutions in such generalized non–Riemannian gravity theories. We emphasize that generalized Finsler metrics can be generated in string gravity connected to anholonomic metric–affine configurations. In particular, we investigate how the so–called Obukhov's equivalence theorem [5] should be modified as to include various type of Finsler–Lagrange–Hamilton–Cartan metrics, see Refs. [11, 12, 13]. The results of this paper consist a theoretical background for constructing exact solutions in MAG and string gravity in the third partner paper [14] derived as exact solutions of gravitational and matter field equations parametrized by generic off–diagonal metrics (which can not be diagonalized by local coordinate transforms) and anholonomic frames with associated N–connection structure. Such solutions depending on 3-5 variables (generalizing to MAG the results from [7, 8, 9, 10, 15]) differ substantially from those elaborated in Refs. [16]; they define certain extensions to nontrivial torsion and nonmetricity fields of certain generic off–diagonal metrics in general relativity theory.

The plan of the paper is as follows: In Sec. 2 we outline the necessary results on Finsler–affine geometry. Next, in Sec. 3, we formulate the field equations on metric–affine spaces provided with N–connection structure. We consider Lagrangians and derive geometrically the field equations of Finsler–affine gravity. We prove the main theorems for the Einstein–Proca systems distinguished by N–connection structure and analyze possible string gravity corrections by H–fields from the bosonic string theory. There are defined the restrictions on N–connection structures resulting in Einstein–Cartan and Einstein gravity. Section 4 is devoted to extension of the anholonomic frame method in MAG and string gravity. We formulate and prove the main theorems stating the possibility of constructing exact solutions parametrized by generic off–diagonal metrics, nontrivial torsion and nonmetricity structures and possible sources of matter fields. In Sec. 5 we construct three classes of exact solutions. The first class of solutions is stated for five subclasses of two dimensional generalized Finsler geometries modelled in MAG with possible string corrections. The second class of solutions is for MAG with effective variable and inhomogeneous cosmological constant. The third class of solutions are for the string Finsler–affine gravity (i. e. string gravity containing in certain limits Finsler like metrics) with possible nonlinear three dimensional solitonic interactions, Proca fields with almost vanishing masses, nontrivial torsions and nonmetricity. In Sec. 6 we present the final remarks. In Appendices A, B and C we give respectively the details on the proof of the Theorem 4.1 (stating the components of the Ricci tensor for generalized Finsler–affine spaces), analyze the reduction of nonlinear solutions from five to four dimensions and present a short characterization of five classes of generalized Finsler–affine spaces.

Our basic notations and conventions are those from Ref. [6] and contain an interference of approaches elaborated in MAG and generalized Finsler geometry. The spacetime is considered to be a manifold V^{n+m} of necessary smoothly class of dimension $n+m$. The Greek indices $\alpha, \beta, ...$ split into subclasses like $\alpha = (i,a)$, $\beta = (j,b)$ ... where the Latin indices $i, j, k, ...$ run values $1, 2, ...n$ and $a, b, c, ...$ run values $n+1, n+2, ..., n+m$. We follow the Penrose convention on abstract indices [17] and use underlined indices like $\underline{\alpha} = (\underline{i}, \underline{a})$, for decompositions with respect to coordinate frames. The symbol " $\doteqdot$" will be used is some formulas will be introduced by definition and the end of proofs will be stated by symbol $\blacksquare$. The notations for connections $\Gamma^{\alpha}{}_{\beta\gamma}$, metrics $g_{\alpha\beta}$ and frames e_α and coframes ϑ^β, or another geometrical and physical objects, are the standard ones from MAG if a nonlinear connection (N–connection) structure is not emphasized on the spacetime. If a N–connection and corresponding anholonomic frame structure are prescribed, we use "boldfaced" symbols with possible splitting of the objects and indices like

$$\mathbf{V}^{n+m},\ \mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right),\ \mathbf{g}_{\alpha\beta} = \left(g_{ij}, h_{ab}\right),\ \mathbf{e}_{\alpha} = \left(e_{i}, e_{a}\right), ...$$

being distinguished by N–connection (in brief, there are used the terms d–objects, d–tensor, d–connection).

2.2 Metric–Affine and Generalized Finsler Gravity

In this section we recall some basic facts on metric–affine spaces provided with nonlinear connection (N–connection) structure and generalized Finsler–affine geometry [6].

The spacetime is modelled as a manifold V^{n+m} of dimension $n+m$, with $n \geq 2$ and $m \geq 1$, admitting (co) vector/ tangent structures. It is denoted by $\pi^{T} : TV^{n+m} \rightarrow TV^{n}$ the differential of the map $\pi : V^{n+m} \rightarrow V^{n}$ defined as a fiber–preserving morphism of the tangent bundle $(TV^{n+m}, \tau_{E}, V^{n})$ to V^{n+m} and of tangent bundle (TV^{n}, τ, V^{n}). We consider also the kernel of the morphism π^{T} as a vector subbundle of the vector bundle $(TV^{n+m}, \tau_{E}, V^{n+m})$. The kernel defines the vertical subbundle over V^{n+m}, s denoted as $(vV^{n+m}, \tau_{V}, V^{n+m})$. We parametrize the local coordinates of a point $u \in V^{n+m}$ as $u^{\alpha} = (x^{i}, y^{a})$, where the values of indices are $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = n+1, n+2, ..., n+m$. The inclusion mapping is written as $i : vV^{n+m} \rightarrow TV^{n+m}$.

A nonlinear connection (N–connection) $\mathbf{N}$ in a space (V^{n+m}, π, V^{n}) is a morphism of manifolds $N : TV^{n+m} \rightarrow vV^{n+m}$ defined by the splitting on the left of the exact sequence

$$0 \rightarrow vV^{n+m} \rightarrow TV^{n+m}/vV^{n+m} \rightarrow 0. \tag{2.1}$$

The kernel of the morphism $\mathbf{N}$ is a subbundle of $(TV^{n+m}, \tau_{E}, V^{n+m})$, called the horizontal subspace and denoted by $(hV^{n+m}, \tau_{H}, V^{n+m})$. Every tangent bundle $(TV^{n+m}$, τ_{E}, $V^{n+m})$ provided with a N–connection structure is a Whitney sum of the vertical and horizontal subspaces (in brief, h- and v– subspaces), i. e.

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \tag{2.2}$$

We note that the exact sequence (2.1) defines the N–connection in a global coordinate free form resulting in invariant splitting (2.2) (see details in Refs. [18, 12] stated for vector and tangent bundles and generalizations on covector bundles, superspaces and noncommutative spaces [13] and [9]).

A N–connection structure prescribes a class of vielbein transforms

$$A_{\alpha}^{\ \underline{\alpha}}(u) = \mathbf{e}_{\alpha}^{\ \underline{\alpha}} = \begin{bmatrix} e_i^{\ \underline{i}}(u) & N_i^b(u)e_b^{\ \underline{a}}(u) \\ 0 & e_a^{\ \underline{a}}(u) \end{bmatrix}, \tag{2.3}$$

$$A^{\beta}_{\ \underline{\beta}}(u) = \mathbf{e}^{\beta}_{\ \underline{\beta}} = \begin{bmatrix} e^i_{\ \underline{i}}(u) & -N_k^b(u)e^k_{\ \underline{i}}(u) \\ 0 & e^a_{\ \underline{a}}(u) \end{bmatrix}, \tag{2.4}$$

in particular case $e_i^{\ \underline{i}} = \delta_i^{\underline{i}}$ and $e_a^{\ \underline{a}} = \delta_a^{\underline{a}}$ with $\delta_i^{\underline{i}}$ and $\delta_a^{\underline{a}}$ being the Kronecker symbols, defining a global splitting of $\mathbf{V}^{n+m}$ into "horizontal" and "vertical" subspaces with the N–vielbein structure

$$\mathbf{e}_\alpha = \mathbf{e}_{\alpha}^{\ \underline{\alpha}}\partial_{\underline{\alpha}} \text{ and } \vartheta^\beta = \mathbf{e}^{\beta}_{\ \underline{\beta}}du^{\underline{\beta}}.$$

We adopt the convention that for the spaces provided with N–connection structure the geometrical objects can be denoted by "boldfaced" symbols if it would be necessary to distinguish such objects from similar ones for spaces without N–connection.

A N–connection $\mathbf{N}$ in a space $\mathbf{V}^{n+m}$ is parametrized by its components $N_i^a(u) = N_i^a(x,y)$,

$$\mathbf{N} = N_i^a(u)d^i \otimes \partial_a$$

and characterized by the N–connection curvature

$$\mathbf{\Omega} = \frac{1}{2}\Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

with N–connection curvature coefficients

$$\Omega_{ij}^a = \delta_{[j}N_{i]}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \tag{2.5}$$

On spaces provided with N–connection structure, we have to use 'N–elongated' operators like δ_j in (2.5) instead of usual partial derivatives. They are defined by the vielbein configuration induced by the N–connection, the N–elongated partial derivatives (in brief, N–derivatives)

$$\mathbf{e}_\alpha \doteqdot \delta_\alpha = (\delta_i, \partial_a) \equiv \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(u)\,\partial_a, \frac{\partial}{\partial y^a} \right) \tag{2.6}$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^\beta \doteqdot \delta^{\ \beta} = \left(d^i, \delta^a\right) \equiv \delta u^\alpha = \left(\delta x^i = dx^i, \delta y^a = dy^a + N_i^a(u)\,dx^i\right) \tag{2.7}$$

called also, respectively, the N–frame and N–coframe. There are used both type of denotations $\mathbf{e}_\alpha \doteqdot \delta_\alpha$ and $\vartheta^\beta \doteqdot \delta^{\ \alpha}$ in order to preserve a connection to denotations from Refs. [12, 7, 8, 9]. The 'boldfaced' symbols $\mathbf{e}_\alpha$ and ϑ^β will be considered in order to emphasize that they define N–adapted vielbeins but the symbols δ_α and $\delta^{\ \beta}$ will be used for the N–elongated partial derivatives and, respectively, differentials.

The N–coframe (2.7) satisfies the anholonomy relations

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\,\delta_\gamma \tag{2.8}$$

with nontrivial anholonomy coefficients $\mathbf{w}^\alpha_{\beta\gamma}(u)$ computed as

$$\mathbf{w}^a{}_{ji} = -\mathbf{w}^a{}_{ij} = \Omega^a_{ij},\ \mathbf{w}^b{}_{ia} = -\mathbf{w}^b{}_{ai} = \partial_a N^b_i. \tag{2.9}$$

The distinguished objects (by a N–connection on a spaces $\mathbf{V}^{n+m}$) are introduced in a coordinate free form as geometric objects adapted to the splitting (2.2). In brief, they are called d–objects, d–tensor, d–connections, d–metrics....

There is an important class of linear connections adapted to the N–connection structure:

A d–connection $\mathbf{D}$ on a space $\mathbf{V}^{n+m}$ is defined as a linear connection D conserving under a parallelism the global decomposition (2.2).

The N–adapted components $\mathbf{\Gamma}^\alpha_{\beta\gamma}$ of a d-connection $\mathbf{D}_\alpha = (\delta_\alpha \rfloor \mathbf{D})$ are defined by the equations

$$\mathbf{D}_\alpha\delta_\beta = \mathbf{\Gamma}^\gamma{}_{\alpha\beta}\delta_\gamma,$$

from which one immediately follows

$$\mathbf{\Gamma}^\gamma{}_{\alpha\beta}(u) = (\mathbf{D}_\alpha\delta_\beta)\rfloor\delta^\gamma. \tag{2.10}$$

The operations of h- and v-covariant derivations, $D^{[h]}_k = \{L^i_{jk}, L^a_{bk}\}$ and $D^{[v]}_c = \{C^i_{jk}, C^a_{bc}\}$ are introduced as corresponding h- and v–parametrizations of (2.10),

$$L^i_{jk} = (\mathbf{D}_k\delta_j)\rfloor d^i, \quad L^a_{bk} = (\mathbf{D}_k\partial_b)\rfloor\delta^a,\ C^i_{jc} = (\mathbf{D}_c\delta_j)\rfloor d^i, \quad C^a_{bc} = (\mathbf{D}_c\partial_b)\rfloor\delta^a.$$

The components $\mathbf{\Gamma}^\gamma{}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ completely define a d–connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$.

A metric structure $\mathbf{g}$ on a space $\mathbf{V}^{n+m}$ is defined as a symmetric covariant tensor field of type $(0,2)$, $g_{\alpha\beta}$, being nondegenerate and of constant signature on $\mathbf{V}^{n+m}$. A N–connection $\mathbf{N} = \{N^{\underline{b}}_{\underline{i}}(u)\}$ and a metric structure $\mathbf{g} = g_{\underline{\alpha}\underline{\beta}}du^{\underline{\alpha}}\otimes du^{\underline{\beta}}$ on $\mathbf{V}^{n+m}$ are mutually compatible if there are satisfied the conditions

$$\mathbf{g}\left(\delta_{\underline{i}}, \partial_{\underline{a}}\right) = 0, \text{ or equivalently, } g_{\underline{ia}}(u) - N^{\underline{b}}_{\underline{i}}(u)\,h_{\underline{ab}}(u) = 0,$$

where $h_{\underline{ab}} \doteqdot \mathbf{g}\left(\partial_{\underline{a}}, \partial_{\underline{b}}\right)$ and $g_{\underline{ia}} \doteqdot \mathbf{g}\left(\partial_{\underline{i}}, \partial_{\underline{a}}\right)$ resulting in

$$N_i^b\left(u\right) = h^{ab}\left(u\right) g_{ia}\left(u\right)$$

(the matrix h^{ab} is inverse to h_{ab}; for simplicity, we do not underline the indices in the last formula). In consequence, we define an invariant h–v–decomposition of metric (in brief, a d–metric)

$$\mathbf{g}(X,Y) = h\mathbf{g}(X,Y) + v\mathbf{g}(X,Y).$$

With respect to a N–coframe (2.7), the d–metric is written

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}\left(u\right)\delta^{\alpha} \otimes \delta^{\beta} = g_{ij}\left(u\right) d^i \otimes d^j + h_{ab}\left(u\right)\delta^a \otimes \delta^b, \tag{2.11}$$

where $g_{ij} \doteqdot \mathbf{g}\left(\delta_i, \delta_j\right)$. The d–metric (2.11) can be equivalently written in "off–diagonal" with respect to a coordinate basis defined by usual local differentials $du^{\alpha} = \left(dx^i, dy^a\right)$,

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array}\right]. \tag{2.12}$$

A metric, for instance, parametrized in the form (2.12) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. The anholonomy coefficients (2.9) do not vanish for the off–diagonal form (2.12) and the equivalent d–metric (2.11).

The nonmetricity d–field

$$\mathcal{Q} = \mathbf{Q}_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta} = \mathbf{Q}_{\alpha\beta}\delta^{\ \alpha} \otimes \delta^{\beta}$$

on a space $\mathbf{V}^{n+m}$ provided with N–connection structure is defined by a d–tensor field with the coefficients

$$\mathbf{Q}_{\alpha\beta} \doteqdot -\mathbf{D}\mathbf{g}_{\alpha\beta} \tag{2.13}$$

where the covariant derivative $\mathbf{D}$ is for a d–connection (2.10) $\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ with $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$.

A linear connection D_X is compatible with a d–metric $\mathbf{g}$ if

$$D_X\mathbf{g} = 0, \tag{2.14}$$

i. e. if $Q_{\alpha\beta} \equiv 0$. In a space provided with N–connection structure, the metricity condition (2.14) may split into a set of compatibility conditions on h- and v– subspaces,

$$D^{[h]}(h\mathbf{g}) = 0, D^{[v]}(h\mathbf{g}) = 0, D^{[h]}(v\mathbf{g}) = 0, D^{[v]}(v\mathbf{g}) = 0. \tag{2.15}$$

For instance, if $D^{[v]}(h\mathbf{g}) = 0$ and $D^{[h]}(v\mathbf{g}) = 0$, but, in general, $D^{[h]}(h\mathbf{g}) \neq 0$ and $D^{[v]}(v\mathbf{g}) \neq 0$ we have a nontrivial nonmetricity d–field $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ with irreducible h–v–components $\mathbf{Q}_{\gamma\alpha\beta} = (Q_{ijk}, Q_{abc})$.

In a metric–affine space, by acting on forms with a covariant derivative D, we can also define another very important geometric objects (the 'gravitational field potentials', the torsion and, respectively, curvature; see [4]):

$$\mathbf{T}^{\alpha} \doteqdot \mathbf{D}\vartheta^{\alpha} = \delta\vartheta^{\alpha} + \mathbf{\Gamma}^{\gamma}_{\ \beta} \wedge \vartheta^{\beta} \tag{2.16}$$

and

$$\mathbf{R}^{\alpha}_{\ \beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}_{\ \beta} = \delta\mathbf{\Gamma}^{\alpha}_{\ \beta} - \mathbf{\Gamma}^{\gamma}_{\ \beta} \wedge \mathbf{\Gamma}^{\alpha}_{\ \ \gamma} \tag{2.17}$$

For spaces provided with N–connection structures, we consider the same formulas but for "boldfaced" symbols and change the usual differential d into N-adapted operator δ.

A general affine (linear) connection $D = \bigtriangledown + Z = \{\Gamma^{\gamma}_{\beta\alpha} = \Gamma^{\gamma}_{\bigtriangledown\beta\alpha} + Z^{\gamma}_{\beta\alpha}\}$

$$\Gamma^{\gamma}_{\ \alpha} = \Gamma^{\gamma}_{\alpha\beta}\vartheta^{\beta}, \tag{2.18}$$

can always be decomposed into the Riemannian $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$ and post–Riemannian $Z^{\alpha}_{\ \beta}$ parts [4, 5],

$$\Gamma^{\alpha}_{\ \beta} = \Gamma^{\alpha}_{\bigtriangledown\ \beta} + Z^{\alpha}_{\ \beta}. \tag{2.19}$$

The distorsion 1-form $Z^{\alpha}_{\ \beta}$ from (2.19) is expressed in terms of torsion and nonmetricity,

$$Z_{\alpha\beta} = e_{\beta}\rfloor T_{\alpha} - e_{\alpha}\rfloor T_{\beta} + \frac{1}{2}\left(e_{\alpha}\rfloor e_{\beta}\rfloor T_{\gamma}\right)\vartheta^{\gamma} + \left(e_{\alpha}\rfloor Q_{\beta\gamma}\right)\vartheta^{\gamma} - \left(e_{\beta}\rfloor Q_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}Q_{\alpha\beta} \tag{2.20}$$

where T_{α} is defined as (2.16) and $Q_{\alpha\beta} \doteqdot -Dg_{\alpha\beta}$. (We note that $Z^{\alpha}_{\ \beta}$ are $N_{\alpha\beta}$ from Ref. [5], but in our works we use the symbol N for N–connections .) For $Q_{\beta\gamma} = 0$, we obtain from (2.20) the distorsion for the Riemannian–Cartan geometry [19].

By substituting arbitrary (co) frames, metrics and linear connections into N–adapted ones,

$$e_{\alpha} \to \mathbf{e}_{\alpha}, \vartheta^{\beta} \to \vartheta^{\beta}, g_{\alpha\beta} \to \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}), \Gamma^{\gamma}_{\ \alpha} \to \mathbf{\Gamma}^{\gamma}_{\ \alpha},$$

with $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ and $\mathbf{T}^{\alpha}$ as in (2.16), into respective formulas (2.18), (2.19) and (2.20), we can define an affine connection $\mathbf{D} = \bigtriangledown + \mathbf{Z} = [\mathbf{\Gamma}^{\gamma}_{\ \beta\alpha}]$ with respect to N–adapted (co) frames,

$$\mathbf{\Gamma}^{\gamma}_{\ \ \alpha} = \mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}, \tag{2.21}$$

with

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta} + \mathbf{Z}^{\alpha}_{\ \ \beta}, \tag{2.22}$$

where

$$\Gamma^{\bigtriangledown}_{\ \gamma\alpha} = \frac{1}{2}\left[\mathbf{e}_{\gamma}\rfloor\ \delta\vartheta_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \delta\vartheta_{\gamma} - (\mathbf{e}_{\gamma}\rfloor\ \mathbf{e}_{\alpha}\rfloor\ \delta\vartheta_{\beta}) \wedge \vartheta^{\beta}\right], \tag{2.23}$$

and

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor \mathbf{e}_{\beta}\rfloor \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\mathbf{Q}_{\alpha\beta}. \tag{2.24}$$

The h– and v–components of $\Gamma^{\alpha}_{\ \beta}$ from (2.22) consists from the components of $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$ (considered for (2.23)) and of $\mathbf{Z}_{\alpha\beta}$ with $\mathbf{Z}^{\alpha}_{\ \ \gamma\beta} = \left(Z^{i}_{jk}, Z^{a}_{bk}, Z^{i}_{jc}, Z^{a}_{bc}\right)$. We note that for $\mathbf{Q}_{\alpha\beta} = 0$, the distorsion 1–form $\mathbf{Z}_{\alpha\beta}$ defines a Riemann–Cartan geometry adapted to the N–connection structure.

A distinguished metric–affine space $\mathbf{V}^{n+m}$ is defined as a usual metric–affine space additionally enabled with a N–connection structure $\mathbf{N} = \{N^{a}_{i}\}$ inducing splitting into respective irreducible horizontal and vertical subspaces of dimensions n and m. This space is provided with independent d–metric (2.11) and affine d–connection (2.10) structures adapted to the N–connection.

If a space $\mathbf{V}^{n+m}$ is provided with both N–connection $\mathbf{N}$ and d–metric $\mathbf{g}$ structures, there is a unique linear symmetric and torsionless connection $\bigtriangledown$, called the Levi–Civita connection, being metric compatible such that $\bigtriangledown_{\gamma}\mathbf{g}_{\alpha\beta} = 0$ for $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, see (2.11), with the coefficients

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \mathbf{g}\left(\delta_{\alpha}, \bigtriangledown_{\gamma}\delta_{\beta}\right) = \mathbf{g}_{\alpha\tau}\Gamma^{\tau}_{\bigtriangledown\beta\gamma},$$

computed as

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_{\beta}\mathbf{g}_{\alpha\gamma} + \delta_{\gamma}\mathbf{g}_{\beta\alpha} - \delta_{\alpha}\mathbf{g}_{\gamma\beta} + \mathbf{g}_{\alpha\tau}\mathbf{w}^{\tau}_{\gamma\beta} + \mathbf{g}_{\beta\tau}\mathbf{w}^{\tau}_{\alpha\gamma} - \mathbf{g}_{\gamma\tau}\mathbf{w}^{\tau}_{\beta\alpha}\right] \tag{2.25}$$

with respect to N–frames $\mathbf{e}_{\beta} \doteqdot \delta_{\beta}$ (2.6) and N–coframes $\vartheta^{\alpha} \doteqdot \delta^{\alpha}$ (2.7).

We note that the Levi–Civita connection is not adapted to the N–connection structure. Se, we can not state its coefficients in an irreducible form for the h– and v–subspaces. There is a type of d–connections which are similar to the Levi–Civita connection but satisfying certain metricity conditions adapted to the N–connection. They are introduced as metric d–connections $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ in a space $\mathbf{V}^{n+m}$ satisfying the metricity conditions if and only if

$$D^{[h]}_{k} g_{ij} = 0,\ D^{[v]}_{a} g_{ij} = 0,\ D^{[h]}_{k} h_{ab} = 0,\ D^{[h]}_{a} h_{ab} = 0. \tag{2.26}$$

Let us consider an important example: The canonical d–connection $\widehat{\mathbf{D}} = \left(\widehat{D}^{[h]}, \widehat{D}^{[v]}\right)$, equivalently $\widehat{\Gamma}^{\gamma}_{\ \alpha} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}$, is defined by the h– v–irreducible components $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta} =$

$(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$, where

$$\begin{aligned} \widehat{L}^i_{jk} &= \frac{1}{2}g^{ir}\left(\frac{\delta g_{jk}}{\delta x^k}+\frac{\delta g_{kr}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^r}\right), \qquad (2.27)\\ \widehat{L}^a_{bk} &= \frac{\partial N^a_k}{\partial y^b}+\frac{1}{2}h^{ac}\left(\frac{\delta h_{bc}}{\delta x^k}-\frac{\partial N^d_k}{\partial y^b}h_{dc}-\frac{\partial N^d_k}{\partial y^c}h_{db}\right),\\ \widehat{C}^i_{jc} &= \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^c},\\ \widehat{C}^a_{bc} &= \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^c}+\frac{\partial h_{cd}}{\partial y^b}-\frac{\partial h_{bc}}{\partial y^d}\right). \end{aligned}$$

satisfying the torsionless conditions for the h–subspace and v–subspace, respectively, $\widehat{T}^i_{jk} = \widehat{T}^a_{bc} = 0$.

The components of the Levi–Civita connection $\mathbf{\Gamma}^\tau_{\bigtriangledown\beta\gamma}$ and the irreducible components of the canonical d–connection $\widehat{\mathbf{\Gamma}}^\tau{}_{\beta\gamma}$ are related by formulas

$$\mathbf{\Gamma}^\tau_{\bigtriangledown\beta\gamma} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk} - \frac{\partial N^a_k}{\partial y^b}, \widehat{C}^i_{jc} + \frac{1}{2}g^{ik}\Omega^a_{jk}h_{ca}, \widehat{C}^a_{bc}\right), \qquad (2.28)$$

where Ω^a_{jk} is the N–connection curvature (2.5).

We can define and calculate the irreducible components of torsion and curvature in a space $\mathbf{V}^{n+m}$ provided with additional N–connection structure (these could be any metric–affine spaces [4], or their particular, like Riemann–Cartan [19], cases with vanishing nonmetricity and/or torsion, or any (co) vector / tangent bundles like in Finsler geometry and generalizations).

The torsion

$$\mathbf{T}^\alpha_{.\beta\gamma} = (T^i_{.jk}, T^i_{ja}, T^a_{.ij}, T^a_{.bi}, T^a_{.bc})$$

of a d–connection $\mathbf{\Gamma}^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$ (2.10) has irreducible h- v–components (d–torsions)

$$\begin{aligned} T^i_{.jk} &= -T^i_{kj} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{.ja},\ T^a_{.ji} = -T^a_{.ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i} = \Omega^a_{.ji},\\ T^a_{.bi} &= -T^a_{.ib} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj},\ T^a_{.bc} = -T^a_{.cb} = S^a_{.bc} = C^a_{bc} - C^a_{cb}. \qquad (2.29) \end{aligned}$$

We note that on (pseudo) Riemanian spacetimes the d–torsions can be induced by the N–connection coefficients and reflect an anholonomic frame structure. Such objects vanish when we transfer our considerations with respect to holonomic bases for a trivial N–connection and zero "vertical" dimension.

The curvature

$$\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = (R^i{}_{hjk}, R^a{}_{bjk}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{bcd})$$

of a d–connection $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$ (2.10) has irreducible h- v–components (d–curvatures)

$$\begin{aligned}
R^i{}_{hjk} &= \frac{\delta L^i_{.hj}}{\delta x^k} - \frac{\delta L^i_{.hk}}{\delta x^j} + L^m_{.hj} L^i_{mk} - L^m_{.hk} L^i_{mj} - C^i_{.ha} \Omega^a_{.jk}, \qquad (2.30)\\
R^a{}_{bjk} &= \frac{\delta L^a_{.bj}}{\delta x^k} - \frac{\delta L^a_{.bk}}{\delta x^j} + L^c_{.bj} L^a_{.ck} - L^c_{.bk} L^a_{.cj} - C^a_{.bc}\ \Omega^c_{.jk},\\
P^i{}_{jka} &= \frac{\partial L^i_{.jk}}{\partial y^k} - \left(\frac{\partial C^i_{.ja}}{\partial x^k} + L^i_{.lk} C^l_{.ja} - L^l_{.jk} C^i_{.la} - L^c_{.ak} C^i_{.jc} \right) + C^i_{.jb} P^b_{.ka},\\
P^c{}_{bka} &= \frac{\partial L^c_{.bk}}{\partial y^a} - \left(\frac{\partial C^c_{.ba}}{\partial x^k} + L^c_{.dk} C^d_{.ba} - L^d_{.bk} C^c_{.da} - L^d_{.ak} C^c_{.bd} \right) + C^c_{.bd} P^d_{.ka},\\
S^i{}_{jbc} &= \frac{\partial C^i_{.jb}}{\partial y^c} - \frac{\partial C^i_{.jc}}{\partial y^b} + C^h_{.jb} C^i_{.hc} - C^h_{.jc} C^i_{hb},\\
S^a{}_{bcd} &= \frac{\partial C^a_{.bc}}{\partial y^d} - \frac{\partial C^a_{.bd}}{\partial y^c} + C^e_{.bc} C^a_{.ed} - C^e_{.bd} C^a_{.ec}.
\end{aligned}$$

The components of the Ricci tensor

$$\mathbf{R}_{\alpha\beta} = \mathbf{R}^{\tau}{}_{\alpha\beta\tau}$$

with respect to a locally adapted frame (2.6) has four irreducible h- v–components $\mathbf{R}_{\alpha\beta} = (R_{ij}, R_{ia}, R_{ai}, S_{ab})$, where

$$\begin{aligned}
R_{ij} &= R^k{}_{ijk}, \quad R_{ia} = -\ {}^2P_{ia} = -P^k{}_{ika}, \qquad (2.31)\\
R_{ai} &= {}^1P_{ai} = P^b{}_{aib}, \quad S_{ab} = S^c{}_{abc}.
\end{aligned}$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d–tensor is non symmetric.

Having defined a d–metric of type (2.11) in $\mathbf{V}^{n+m}$, we can introduce the scalar curvature of a d–connection $\mathbf{D}$,

$$\overleftarrow{\mathbf{R}} = \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = R + S, \qquad (2.32)$$

where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$ and define the distinguished form of the Einstein tensor (the Einstein d–tensor),

$$\mathbf{G}_{\alpha\beta} \doteqdot \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}. \quad (2.33)$$

The introduced geometrical objects are extremely useful in definition of field equations of MAG and string gravity with nontrivial N–connection structure.

2.3 N–Connections and Field Equations

The field equations of metric–affine gravity (in brief, MAG) [4, 5] can be reformulated with respect to frames and coframes consisting from mixed holonomic and anholonomic components defined by the N–connection structure. In this case, various type of (pseudo) Riemannian, Riemann–Cartan and generalized Finsler metrics and additional torsion and nonmetricity sructures with very general local anisotropy can be embedded into MAG. It is known that in a metric–affine spacetime the curvature, torsion and nonmetricity have correspondingly eleven, three and four irreducible pieces. If the N–connection is defined in a metric–affine spacetime, every irreducible component of curvature splits additionally into six h- and v– components (2.30), every irreducible component of torsion splits additionally into five h- and v– components (2.29) and every irreducible component of nonmetricity splits additionally into two h- and v– components (defined by splitting of metrics into block ansatz (2.11)).

2.3.1 Lagrangians and field equations for Finsler–affine theories

For an arbitrary d–connection $\mathbf{\Gamma}^{\alpha}{}_{\beta}$ in a metric–affine space $\mathbf{V}^{n+m}$ provided with N–connection structure (for simplicity, we can take $n + m = 4$) one holds the respective decompositions for d–torsion and nonmetricity d–field,

$$\begin{aligned} {}^{(2)}\mathbf{T}^{\alpha} &\doteqdot \frac{1}{3}\vartheta^{\alpha} \wedge \mathbf{T}, \text{ for } \mathbf{T} \doteqdot \mathbf{e}_{\alpha}\rfloor\mathbf{T}^{\alpha}, \\ {}^{(3)}\mathbf{T}^{\alpha} &\doteqdot \frac{1}{3} * (\vartheta^{\alpha} \wedge \mathbf{P}), \text{ for } \mathbf{P} \doteqdot *(\mathbf{T}^{\alpha} \wedge \vartheta_{\alpha}), \\ {}^{(1)}\mathbf{T}^{\alpha} &\doteqdot \mathbf{T}^{\alpha} -{}^{(2)}\mathbf{T}^{\alpha} -{}^{(3)}\mathbf{T}^{\alpha} \end{aligned} \quad (2.34)$$

and

$$
\begin{aligned}
{}^{(2)}\mathbf{Q}_{\alpha\beta} &\doteqdot \frac{1}{3} * (\vartheta_\alpha \wedge \mathbf{S}_\beta + \vartheta_\beta \wedge \mathbf{S}_\alpha), \ {}^{(4)}\mathbf{Q}_{\alpha\beta} \doteqdot \mathbf{g}_{\alpha\beta}\mathbf{Q}, \\
{}^{(3)}\mathbf{Q}_{\alpha\beta} &\doteqdot \frac{2}{9}\left[(\vartheta_\alpha \mathbf{e}_\beta + \vartheta_\beta \mathbf{e}_\alpha)\rfloor\mathbf{\Lambda} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{\Lambda}\right], \qquad (2.35)\\
{}^{(1)}\mathbf{Q}_{\alpha\beta} &\doteqdot \mathbf{Q}_{\alpha\beta} - {}^{(2)}\mathbf{Q}_{\alpha\beta} - {}^{(3)}\mathbf{Q}_{\alpha\beta} - {}^{(4)}\mathbf{Q}_{\alpha\beta},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{Q} &\doteqdot \frac{1}{4}\mathbf{g}^{\alpha\beta}\mathbf{Q}_{\alpha\beta}, \ \mathbf{\Lambda} \doteqdot \vartheta^\alpha \mathbf{e}^\beta \rfloor \left(\mathbf{Q}_{\alpha\beta} - \mathbf{Q}\mathbf{g}_{\alpha\beta}\right), \\
\mathbf{\Theta}_\alpha &\doteqdot * \left[\left(\mathbf{Q}_{\alpha\beta} - \mathbf{Q}\mathbf{g}_{\alpha\beta}\right) \wedge \vartheta^\beta\right], \\
\mathbf{S}_\alpha &\doteqdot \mathbf{\Theta}_\alpha - \frac{1}{3}\mathbf{e}_\alpha\rfloor\left(\vartheta^\beta \wedge \mathbf{\Theta}_\beta\right)
\end{aligned}
$$

and the Hodge dual "$*$" is such that $\eta \doteqdot *1$ is the volume 4–form and

$$
\eta_\alpha \doteqdot \mathbf{e}_\alpha\rfloor\eta = *\vartheta_\alpha, \ \eta_{\alpha\beta} \doteqdot \mathbf{e}_\alpha\rfloor\eta_\beta = *(\vartheta_\alpha \wedge \vartheta_\beta), \ \eta_{\alpha\beta\gamma} \doteqdot \mathbf{e}_\gamma\rfloor\eta_{\alpha\beta}, \ \eta_{\alpha\beta\gamma\tau} \doteqdot \mathbf{e}_\tau\rfloor\eta_{\alpha\beta\gamma}
$$

with $\eta_{\alpha\beta\gamma\tau}$ being totally antisymmetric. In higher dimensions, we have to consider $\eta \doteqdot *1$ as the volume $(n+m)$–form. For N–adapted h- and v–constructions, we have to consider couples of 'volume' forms $\eta \doteqdot \left(\eta^{[g]} = *^{[g]}1, \eta^{[h]} = *^{[h]}1\right)$ defined correspondingly by $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$.

With respect to N–adapted (co) frames $\mathbf{e}_\beta = (\delta_i, \partial_a)$ (2.6) and $\vartheta^\alpha = (d^i, \delta^a)$ (2.7), the irreducible decompositions (2.34) split into h- and v–components ${}^{(A)}\mathbf{T}^\alpha = \left({}^{(A)}\mathbf{T}^i, \ {}^{(A)}\mathbf{T}^a\right)$ for every $A = 1, 2, 3, 4$. Because, by definition, $\mathbf{Q}_{\alpha\beta} \doteqdot \mathbf{D}\mathbf{g}_{\alpha\beta}$ and $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ is a d–metric field, we conclude that in a similar form can be decomposed the nonmetricity, $\mathbf{Q}_{\alpha\beta} = (Q_{ij}, Q_{ab})$. The symmetrizations in formulas (2.35) hide splitting for ${}^{(1)}\mathbf{Q}_{\alpha\beta}, {}^{(2)}\mathbf{Q}_{\alpha\beta}$ and ${}^{(3)}\mathbf{Q}_{\alpha\beta}$. Nevertheless, the h– and v– decompositions can be derived separately on h– and v–subspaces by distinguishing the interior product $\rfloor = \left(\rfloor^{[h]}, \rfloor^{[v]}\right)$ as to have $\eta_\alpha = (\eta_i = \delta_i\rfloor\eta, \ \eta_a = \partial_a\rfloor\eta)$...and all formulas after decompositions with respect to N–adapted frames (co resulting into a separate relations in h– and v–subspaces, when ${}^{(A)}\mathbf{Q}_{\alpha\beta} = \left({}^{(A)}\mathbf{Q}_{ij}, \ {}^{(A)}\mathbf{Q}_{ab}\right)$ for every $A = 1, 2, 3, 4$.

A generalized Finsler–affine theory is described by a Lagrangian

$$
\mathcal{L} = \mathcal{L}_{GFA} + \mathcal{L}_{mat},
$$

where $\mathcal{L}_{mat}$ represents the Lagrangian of matter fields and

$$\begin{aligned}
\mathcal{L}_{GFA} &= \frac{1}{2\kappa}[-a_{0[Rh]}\mathbf{R}^{ij}\wedge\eta_{ij} - a_{0[Rv]}\mathbf{R}^{ab}\wedge\eta_{ab} - a_{0[Ph]}\mathbf{P}^{ij}\wedge\eta_{ij} - a_{0[Pv]}\mathbf{P}^{ab}\wedge\eta_{ab} \\
&-a_{0[Sh]}\mathbf{S}^{ij}\wedge\eta_{ij} - a_{0[Sv]}\mathbf{S}^{ab}\wedge\eta_{ab} - 2\lambda_{[h]}\eta_{[h]} - 2\lambda_{[v]}\eta_{[v]} \\
&+\mathbf{T}^{i}\wedge *^{[h]}\left(\sum_{[A]=1}^{3} a_{[hA]}\ {}^{[A]}\mathbf{T}_{i}\right) + \mathbf{T}^{a}\wedge *^{[v]}\left(\sum_{[A]=1}^{3} a_{[vA]}\ {}^{[A]}\mathbf{T}_{a}\right) \\
&+2\left(\sum_{[I]=2}^{4} c_{[hI]}\ {}^{[I]}\mathbf{Q}_{ij}\right)\wedge\vartheta^{i}\wedge *^{[h]}\ \mathbf{T}^{j} + 2\left(\sum_{[I]=2}^{4} c_{[vI]}\ {}^{[I]}\mathbf{Q}_{ab}\right)\wedge\vartheta^{a}\wedge *^{[v]}\mathbf{T}^{b} \\
&+\mathbf{Q}_{ij}\wedge\left(\sum_{[I]=1}^{4} b_{[hI]}\ {}^{[I]}\mathbf{Q}^{ij}\right) + \mathbf{Q}_{ab}\wedge\left(\sum_{[I]=1}^{4} b_{[vI]}\ {}^{[I]}\mathbf{Q}^{ab}\right) \\
&+b_{[h5]}\left({}^{[3]}\mathbf{Q}_{ij}\wedge\vartheta^{i}\right)\wedge *^{[h]}\left({}^{[4]}\mathbf{Q}^{kj}\wedge\vartheta_{k}\right) + b_{[v5]}\left({}^{[3]}\mathbf{Q}_{ij}\wedge\vartheta^{i}\right)\wedge *^{[v]}\left({}^{[4]}\mathbf{Q}^{kj}\wedge\vartheta_{k}\right)]
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
&-\frac{1}{2\rho_{[Rh]}}\mathbf{R}^{ij}\wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[RhI]}\ ({}^{[I]}\mathbf{R}_{ij} - {}^{[I]}\mathbf{R}_{ji}) + w_{[Rh7]}\vartheta_{i}\wedge[\mathbf{e}_{k}\rfloor^{[h]}\ ({}^{[5]}\mathbf{R}^{k}{}_{j} - {}^{[5]}\mathbf{R}_{j}{}^{k})] \\
&+\sum_{[I]=1}^{5} z_{[RhI]}\ ({}^{[I]}\mathbf{R}_{ij} + {}^{[I]}\mathbf{R}_{ji}) + z_{[Rh6]}\vartheta_{k}\wedge[\mathbf{e}_{i}\rfloor^{[h]}\ ({}^{[2]}\mathbf{R}^{k}{}_{j} - {}^{[2]}\mathbf{R}_{j}{}^{k})] \\
&+\sum_{[I]=7}^{9} z_{[RhI]}\vartheta_{i}\wedge[\mathbf{e}_{k}\rfloor^{[h]}\ ({}^{[I-4]}\mathbf{R}^{k}{}_{j} - {}^{[I-4]}\mathbf{R}_{j}{}^{k})]\}
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{2\rho_{[Rv]}}\mathbf{R}^{ab}\wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[RvI]}\ ({}^{[I]}\mathbf{R}_{ab} - {}^{[I]}\mathbf{R}_{ba}) + w_{[Rv7]}\vartheta_{a}\wedge[\mathbf{e}_{c}\rfloor^{[v]}\ ({}^{[5]}\mathbf{R}^{a}{}_{b} - {}^{[5]}\mathbf{R}_{b}{}^{a})] \\
&+\sum_{[I]=1}^{5} z_{[RvI]}\ ({}^{[I]}\mathbf{R}_{ab} + {}^{[I]}\mathbf{R}_{ba}) + z_{[Rv6]}\vartheta_{c}\wedge[\mathbf{e}_{a}\rfloor^{[v]}\ ({}^{[2]}\mathbf{R}^{c}{}_{b} - {}^{[2]}\mathbf{R}_{b}{}^{c})] \\
&+\sum_{[I]=7}^{9} z_{[RvI]}\vartheta_{a}\wedge[\mathbf{e}_{c}\rfloor^{[v]}\ ({}^{[I-4]}\mathbf{R}^{c}{}_{b} - {}^{[I-4]}\mathbf{R}_{b}{}^{c})]\}
\end{aligned}$$

$$-\frac{1}{2\rho_{[Ph]}}\mathbf{P}^{ij}\wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[PhI]}\ ({}^{[I]}\mathbf{P}_{ij}-\ {}^{[I]}\mathbf{P}_{ji})+w_{[Ph7]}\vartheta_i\wedge\lfloor\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[5]}\mathbf{P}^{k}{}_{\ j}-\ {}^{[5]}\mathbf{P}_{j}{}^{k})]$$

$$+\sum_{[I]=1}^{5} z_{[PhI]}\ ({}^{[I]}\mathbf{P}_{ij}+\ {}^{[I]}\mathbf{P}_{ji})+z_{[Ph6]}\vartheta_k\wedge\lfloor\mathbf{e}_i\rfloor^{[h]}\ (\ {}^{[2]}\mathbf{P}^{k}{}_{\ j}-\ {}^{[2]}\mathbf{P}_{j}{}^{k})]$$

$$+\sum_{[I]=7}^{9} z_{[PhI]}\vartheta_i\wedge\lfloor\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[I-4]}\mathbf{P}^{k}{}_{\ j}-\ {}^{[I-4]}\mathbf{P}_{j}{}^{k})]\}-$$

$$\frac{1}{2\rho_{[Pv]}}\mathbf{P}^{ab}\wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[PvI]}\ ({}^{[I]}\mathbf{P}_{ab}-\ {}^{[I]}\mathbf{P}_{ba})+w_{[Pv7]}\vartheta_a\wedge\lfloor\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[5]}\mathbf{P}^{a}{}_{\ b}-\ {}^{[5]}\mathbf{P}_{b}{}^{a})]$$

$$+\sum_{[I]=1}^{5} z_{[PvI]}\ ({}^{[I]}\mathbf{P}_{ab}+\ {}^{[I]}\mathbf{P}_{ba})+z_{[Pv6]}\vartheta_c\wedge\lfloor\mathbf{e}_a\rfloor^{[v]}\ (\ {}^{[2]}\mathbf{P}^{c}{}_{\ b}-\ {}^{[2]}\mathbf{P}_{b}{}^{c})]$$

$$+\sum_{[I]=7}^{9} z_{[PvI]}\vartheta_a\wedge\lfloor\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[I-4]}\mathbf{P}^{c}{}_{\ b}-\ {}^{[I-4]}\mathbf{P}_{b}{}^{c})]\}$$

$$-\frac{1}{2\rho_{[Sh]}}\mathbf{S}^{ij}\wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[ShI]}\ ({}^{[I]}\mathbf{S}_{ij}-\ {}^{[I]}\mathbf{S}_{ji})+w_{[Sh7]}\vartheta_i\wedge\lfloor\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[5]}\mathbf{S}^{k}{}_{\ j}-\ {}^{[5]}\mathbf{S}_{j}{}^{k})]$$

$$+\sum_{[I]=1}^{5} z_{[ShI]}\ ({}^{[I]}\mathbf{S}_{ij}+\ {}^{[I]}\mathbf{S}_{ji})+z_{[Sh6]}\vartheta_k\wedge\lfloor\mathbf{e}_i\rfloor^{[h]}\ (\ {}^{[2]}\mathbf{S}^{k}{}_{\ j}-\ {}^{[2]}\mathbf{S}_{j}{}^{k})]$$

$$+\sum_{[I]=7}^{9} z_{[ShI]}\vartheta_i\wedge\lfloor\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[I-4]}\mathbf{S}^{k}{}_{\ j}-\ {}^{[I-4]}\mathbf{S}_{j}{}^{k})]\}-$$

$$\frac{1}{2\rho_{[Sv]}}\mathbf{S}^{ab}\wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[SvI]}\ ({}^{[I]}\mathbf{S}_{ab}-\ {}^{[I]}\mathbf{S}_{ba})+w_{[Sv7]}\vartheta_a\wedge\lfloor\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[5]}\mathbf{S}^{a}{}_{\ b}-\ {}^{[5]}\mathbf{S}_{b}{}^{a})]$$

$$+\sum_{[I]=1}^{5} z_{[SvI]}\ ({}^{[I]}\mathbf{S}_{ab}+\ {}^{[I]}\mathbf{S}_{ba})+z_{[Sv6]}\vartheta_c\wedge\lfloor\mathbf{e}_a\rfloor^{[v]}\ (\ {}^{[2]}\mathbf{S}^{c}{}_{\ b}-\ {}^{[2]}\mathbf{S}_{b}{}^{c})]$$

$$+\sum_{[I]=7}^{9} z_{[SvI]}\vartheta_a\wedge\lfloor\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[I-4]}\mathbf{S}^{c}{}_{\ b}-\ {}^{[I-4]}\mathbf{S}_{b}{}^{c})]\}.$$

Let us explain the denotations used in (2.36): The signature is adapted in the form $(-+++)$ and there are considered two Hodge duals, $*^{[h]}$for h–subspace and $*^{[v]}$for v–subspace, and respectively two cosmological constants, $\lambda_{[h]}$ and $\lambda_{[v]}$. The strong gravity coupling constants $\rho_{[Rh]}, \rho_{[Rv]}, \rho_{[Ph]}, ...,$ the constants $a_{0[Rh]}, a_{0[Rv]}, a_{0[Ph]}, ..., a_{[hA]}, a_{[vA]}, ...$ $c_{[hI]}, c_{[vI]}, ...$ are dimensionless and provided with labels $[R], [P], [h], [v]$, emphasizing that the constants are related, for instance, to respective invariants of curvature, torsion, nonmetricity and their h- and v–decompositions.

The action (2.36) describes all possible models of Einstein, Einstein–Cartan and all type of Finsler–Lagrange–Cartan–Hamilton gravities which can be modelled on metric affine spaces provided with N–connection structure (i. e. with generic off–diagonal metrics) and derived from quadratic MAG–type Lagrangians.

We can reduce the number of constants in $\mathcal{L}_{GFA} \to \mathcal{L}'_{GFA}$ if we select the limit resulting in the usual quadratic MAG–Lagrangian [4] for trivial N–connection structure. In this case, all constants for h– and v– decompositions coincide with those from MAG without N–connection structure, for instance,

$$a_0 = a_{0[Rh]} = a_{0[Rv]} = a_{0[Ph]} = ..., \; a_{[A]} = a_{[hA]} = a_{[vA]} = ..., ..., c_{[I]} = c_{[hI]} = c_{[vI]}, ...$$

The Lagrangian (2.36) can be reduced to a more simple one written in terms of boldfaced symbols (emphasizing a nontrivial N–connection structure) provided with Greek indices,

$$\begin{aligned}
\mathcal{L}'_{GFA} &= \frac{1}{2\kappa}[-a_0 \mathbf{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\eta + \mathbf{T}^i \wedge * \left(\sum_{[A]=1}^{3} a_{[A]} \ ^{[A]}\mathbf{T}_i \right) \\
&+2 \left(\sum_{[I]=2}^{4} c_{[I]} \ ^{[I]}\mathbf{Q}_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge * \ \mathbf{T}^\beta + \mathbf{Q}_{\alpha\beta} \wedge \left(\sum_{[I]=1}^{4} b_{[I]} \ ^{[I]}\mathbf{Q}^{\alpha\beta} \right) \\
&+\mathbf{Q}_{\alpha\beta} \wedge \left(\sum_{[I]=1}^{4} b_{[I]} \ ^{[I]}\mathbf{Q}^{\alpha\beta} \right) + b_{[5]} \left(^{[3]}\mathbf{Q}_{\alpha\beta} \wedge \vartheta^\alpha \right) \wedge * \left(^{[4]}\mathbf{Q}^{\gamma\beta} \wedge \vartheta_\gamma \right)] \\
&-\frac{1}{2\rho}\mathbf{R}^{\alpha\beta} \wedge *[\sum_{[I]=1}^{6} w_{[I]} \ ^{[I]}\mathbf{W}_{\alpha\beta} + w_{[7]}\vartheta_\alpha \wedge \left(\mathbf{e}_\gamma \rfloor \ ^{[5]}\mathbf{W}^\gamma{}_\beta \right) \\
&+\sum_{[I]=1}^{5} z_{[I]} \ ^{[I]}\mathbf{Y}_{\alpha\beta} + z_{[6]}\vartheta_\gamma \wedge \left(\mathbf{e}_\alpha \rfloor \ ^{[2]}\mathbf{Y}^\gamma{}_\beta \right) + \sum_{[I]=7}^{9} z_{[I]}\vartheta_\alpha \wedge \left(\mathbf{e}_\gamma \rfloor \ ^{[I-4]}\mathbf{Y}^\gamma{}_\beta \right)].
\end{aligned} \qquad (2.37)$$

where $^{[I]}\mathbf{W}_{\alpha\beta} = \ ^{[I]}\mathbf{R}_{\alpha\beta} - \ ^{[I]}\mathbf{R}_{\beta\alpha}$ and $^{[I]}\mathbf{Y}_{\alpha\beta} = \ ^{[I]}\mathbf{R}_{\alpha\beta} + \ ^{[I]}\mathbf{R}_{\beta\alpha}$. This action is just for the MAG quadratic theory but with $\mathbf{e}_\alpha$ and ϑ^β being adapted to the N–connection structure as in (2.6) and (2.7) with a corresponding splitting of geometrical objects.

The field equations of a metric–affine space provided with N–connection structure,

$$\mathbf{V}^{n+m} = \left[N_i^a, \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}), \mathbf{\Gamma}^\gamma_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)\right],$$

can be obtained by the Noether procedure in its turn being N–adapted to (co) frames $\mathbf{e}_\alpha$ and ϑ^β. At the first step, we parametrize the generalized Finsler–affine Lagrangian and matter Lagrangian respectively as

$$\mathcal{L}'_{GFA} = \mathcal{L}_{[fa]}\left(N_i^a, \mathbf{g}_{\alpha\beta}, \vartheta^\gamma, \mathbf{Q}_{\alpha\beta}, \mathbf{T}^\alpha,\ \mathbf{R}^\alpha{}_\beta\right)$$

and

$$\mathcal{L}_{mat} = \mathcal{L}_{[m]}\left(N_i^a, \mathbf{g}_{\alpha\beta}, \vartheta^\gamma, \mathbf{\Psi}, \mathbf{D\Psi}\right),$$

where $\mathbf{T}^\alpha$ and $\mathbf{R}^\alpha{}_\beta$ are the curvature of arbitrary d–connection $\mathbf{D}$ and $\mathbf{\Psi}$ represents the matter fields as a p–form. The action $\mathcal{S}$ on $\mathbf{V}^{n+m}$ is written

$$\mathcal{S} = \int \delta^{n+m}u\sqrt{|\mathbf{g}_{\alpha\beta}|}\left[\mathcal{L}_{[fa]} + \mathcal{L}_{[m]}\right] \tag{2.38}$$

which results in the matter and gravitational (generalized Finsler–affine type) field equations.

Theorem 2.3.1. *The Yang–Mills type field equations of the generalized Finsler–affine gravity with matter derived by a variational procedure adapted to the N–connection structure are defined by the system*

$$\begin{aligned}
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[m]}}{\partial(\mathbf{D\Psi})}\right) - (-1)^p\frac{\partial\mathcal{L}_{[m]}}{\partial\mathbf{\Psi}} &= 0, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{Q}_{\alpha\beta}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{g}_{\alpha\beta}} &= -\sigma^{\alpha\beta}, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{T}^\alpha}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\vartheta^\alpha} &= -\mathbf{\Sigma}_\alpha, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{R}^\alpha{}_\beta}\right) + \vartheta^\beta\wedge\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{T}^\alpha} &= -\mathbf{\Delta}_\alpha{}^\beta,
\end{aligned} \tag{2.39}$$

where the material currents are defined

$$\sigma^{\alpha\beta} \doteqdot 2\frac{\delta\mathcal{L}_{[m]}}{\delta\mathbf{g}_{\alpha\beta}},\ \mathbf{\Sigma}_\alpha \doteqdot \frac{\delta\mathcal{L}_{[m]}}{\delta\vartheta^\alpha},\ \mathbf{\Delta}_\alpha{}^\beta = \frac{\delta\mathcal{L}_{[m]}}{\delta\mathbf{\Gamma}^\alpha{}_\beta}$$

for variations "boldfaced" $\delta\mathcal{L}_{[m]}/\delta$ computed with respect to N–adapted (co) frames.

The proof of this theorem consists from N–adapted variational calculus. The equations (2.39) transforms correspondingly into "MATTER, ZEROTH, FIRST, SECOND" equations of MAG [4] for trivial N–connection structures.

Corollary 2.3.1. *The system (2.39) has respectively the h– and v–irreducible components*

$$D^{[h]}\left(\frac{\partial \mathcal{L}_{[m]}}{\partial\left(D^{[h]}\mathbf{\Psi}\right)}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[m]}}{\partial\left(D^{[v]}\mathbf{\Psi}\right)}\right)-(-1)^{p}\frac{\partial \mathcal{L}_{[m]}}{\partial \mathbf{\Psi}}=0,$$

$$\begin{aligned} D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial Q_{ij}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial Q_{ij}}\right)+2\frac{\partial \mathcal{L}_{[fa]}}{\partial g_{ij}} &= -\sigma^{ij}, \\ D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial Q_{ab}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial Q_{ab}}\right)+2\frac{\partial \mathcal{L}_{[fa]}}{\partial g_{ab}} &= -\sigma^{ab}, \end{aligned} \quad (2.40)$$

$$\begin{aligned} D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{i}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{i}}\right)+2\frac{\partial \mathcal{L}_{[fa]}}{\partial \vartheta^{i}} &= -\Sigma_{i}, \\ D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{a}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{a}}\right)+2\frac{\partial \mathcal{L}_{[fa]}}{\partial \vartheta^{a}} &= -\Sigma_{a}, \end{aligned}$$

$$\begin{aligned} D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{i}{}_{j}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{i}{}_{j}}\right)+\vartheta^{j}\wedge\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{i}} &= -\mathbf{\Delta}_{i}{}^{j}, \\ D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{a}{}_{b}}\right)+D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{a}{}_{b}}\right)+\vartheta^{b}\wedge\frac{\partial \mathcal{L}_{[fa]}}{\partial T^{a}} &= -\mathbf{\Delta}_{a}{}^{b}, \end{aligned}$$

where

$$\begin{aligned} \sigma^{\alpha\beta} &= \left(\sigma^{ij},\sigma^{ab}\right) \ \mathit{for}\ \ \sigma^{ij}\doteqdot 2\frac{\delta \mathcal{L}_{[m]}}{\delta g_{ij}},\ \sigma^{ab}\doteqdot 2\frac{\delta \mathcal{L}_{[m]}}{\delta h_{ab}}, \\ \mathbf{\Sigma}_{\alpha} &= \left(\Sigma_{i},\Sigma_{a}\right) \ \mathit{for}\ \ \Sigma_{i}\doteqdot\frac{\delta \mathcal{L}_{[m]}}{\delta \vartheta^{i}},\ \Sigma_{a}\doteqdot\frac{\delta \mathcal{L}_{[m]}}{\delta \vartheta^{a}}, \\ \mathbf{\Delta}_{\alpha}{}^{\beta} &= \left(\Delta_{i}{}^{j},\Delta_{a}{}^{b}\right) \ \mathit{for}\ \ \Delta_{i}{}^{j}=\frac{\delta \mathcal{L}_{[m]}}{\delta \Gamma^{i}{}_{j}},\ \Delta_{a}{}^{b}=\frac{\delta \mathcal{L}_{[m]}}{\delta \Gamma^{a}{}_{b}}\ . \end{aligned}$$

It should be noted that the complete h– v–decomposition of the system (2.40) can be obtained if we represent the d–connection and curvature forms as

$$\Gamma^{i}{}_{j}\ = L^{i}{}_{jk}dx^{j}+C^{i}{}_{ja}\delta y^{a} \text{ and } \Gamma^{a}{}_{b}\ = L^{a}{}_{bk}dx^{k}+C^{a}{}_{bc}\,\delta y^{c},$$

see the d–connection components (2.10) and

$$\begin{aligned} 2R^i{}_j &= R^i{}_{jkl}dx^k \wedge dx^l + P^i{}_{jka}dx^k \wedge \delta y^a + S^i{}_{jba}\delta y^b \wedge \delta y^a, \\ 2R^e{}_f &= R^e{}_{fkl}dx^k \wedge dx^l + P^e{}_{fka}dx^k \wedge \delta y^a + S^e{}_{fba}\delta y^a \wedge \delta y^a, \end{aligned}$$

see the d–curvature components (2.30).

Remark 2.3.1. *For instance, a Finsler configuration can be modelled on a metric affine space provided with N–connection structure,* $\mathbf{V}^{n+m} = \left[N_i^a, \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}),\ {}^{[F]}\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}\right]$, *if* $n = m$, *the ansatz for N–connection is of Cartan–Finsler type*

$$N_j^a \rightarrow\ {}^{[F]}N_j^i = \frac{1}{8}\frac{\partial}{\partial y^j}\left[y^l y^k g^{ih}_{[F]}\left(\frac{\partial g^{[F]}_{hk}}{\partial x^l} + \frac{\partial g^{[F]}_{lh}}{\partial x^k} - \frac{\partial g^{[F]}_{lk}}{\partial x^h}\right)\right],$$

the d–metric $\mathbf{g}_{\alpha\beta} = \mathbf{g}^{[F]}_{\alpha\beta}$ *is defined by (2.11) with*

$$g^{[F]}_{ij} = g_{ij} = h_{ij} = \frac{1}{2}\partial^2 F/\partial y^i \partial y^j$$

and ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}$ *is the Finsler canonical d–connection computed as (2.27). The data should define an exact solution of the system of field equation (2.40) (equivalently of (2.39)).*

Similar Remarks hold true for all types of generalized Finsler–affine spaces considered in Tables 1–11 from Ref. [6]. We shall analyze the possibility of modelling various type of locally anisotropic geometries by the Einstein–Proca systems and in string gravity in next subsection.

2.3.2 Effective Einstein–Proca systems and N–connections

Any affine connection can always be decomposed into (pseudo) Riemannian, $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$, and post–Riemannian, $Z^{\alpha}{}_{\beta}$, parts as $\Gamma^{\alpha}_{\ \beta} = \Gamma^{\alpha}_{\bigtriangledown\ \beta} + Z^{\alpha}{}_{\beta}$, see formulas (2.19) and (2.20) (or (2.22) and (2.24) if any N–connection structure is prescribed). This mean that it is possible to split all quantities of a metric–affine theory into (pseudo) Riemannian and post–Riemannian pieces, for instance,

$$R^{\alpha}_{\ \beta} = R^{\alpha}_{\bigtriangledown\ \beta} + \bigtriangledown Z^{\alpha}{}_{\beta} + Z^{\alpha}{}_{\gamma} \wedge Z^{\gamma}{}_{\beta}. \tag{2.41}$$

Under certain assumptions one holds the Obukhov's equivalence theorem according to which the field vacuum metric–affine gravity equations are equivalent to Einstein's equations with an energy–momentum tensor determined by a Proca field [5, 20]. We can generalize the constructions and reformulate the equivalence theorem for generalized Finsler–affine spaces and effective spaces provided with N–connection structure.

Theorem 2.3.2. *The system of effective field equations of MAG on spaces provided with N–connection structure (2.39) (equivalently, (2.40)) for certain ansatz for torsion and nonmetricity fields (see (2.34) and (2.35))*

$$\begin{aligned} {}^{(1)}\mathbf{T}^{\alpha} &= {}^{(2)}\mathbf{T}^{\alpha} = 0, \quad {}^{(1)}\mathbf{Q}_{\alpha\beta} = {}^{(2)}\mathbf{Q}_{\alpha\beta} = 0, \\ \mathbf{Q} &= k_0\phi, \ \mathbf{\Lambda} = k_1\phi, \ \mathbf{T} = k_2\phi, \end{aligned} \tag{2.42}$$

where $k_0, k_1, k_2 = const$ and the Proca 1–form is $\phi = \phi_\alpha \vartheta^\alpha = \phi_i dx^i + \phi_a \delta y^a$, reduces to the Einstein–Proca system of equations for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ (2.27) and massive d–field ϕ_α,

$$\begin{aligned} \frac{a_0}{2}\eta_{\alpha\beta\gamma} \wedge \widehat{\mathbf{R}}^{\beta\gamma} &= k\ \mathbf{\Sigma}_\alpha, \\ \delta\left(*\mathbf{H}\right) + \mu^2\phi &= \mathbf{0}, \end{aligned} \tag{2.43}$$

where $\mathbf{H} \doteqdot \delta\phi$, the mass $\mu = const$ and the energy–momentum is given by

$$\mathbf{\Sigma}_\alpha = \mathbf{\Sigma}_\alpha^{[\phi]} + \mathbf{\Sigma}_\alpha^{[\mathbf{m}]},$$

$$\mathbf{\Sigma}_\alpha^{[\phi]} \doteqdot \frac{z_4 k_0^2}{2\rho}\{(\mathbf{e}_\alpha \rfloor\ \mathbf{H}) \wedge *\mathbf{H} - (\mathbf{e}_\alpha \rfloor * \mathbf{H}) \wedge \mathbf{H} + \mu^2[(\mathbf{e}_\alpha \rfloor\ \phi) \wedge *\phi - (\mathbf{e}_\alpha \rfloor * \phi) \wedge \phi]\}$$

is the energy–momentum current of the Proca d–field and $\mathbf{\Sigma}_\alpha^{[\mu]}$ is the energy–momentum current of the additional matter d–fields satisfying the corresponding Euler–Largange equations.

The proof of the Theorem is just the reformulation with respect to N–adapted (co) frames (2.6) and (2.7) of similar considerations in Refs. [5, 20]. The constants $k_0, k_1 ....$ are taken in terms of the gravitational coupling constants like in [21] as to have connection to the usual MAG and Einstein theory for trivial N–connection structures and for the dimension $m \to 0$. We use the triplet ansatz sector (2.42) of MAG theories [5, 20]. It is a remarkable fact that the equivalence Theorem 2.3.2 holds also in presence of arbitrary N–connections i. e. for all type of anholonomic generalizations of the Einstein, Einstein–Cartan and Finsler–Lagrange and Cartan–Hamilton geometries by introducing canonical d–connections (we can also consider Berwald type d–connections).

Corollary 2.3.2. *In abstract index form, the effective field equations for the generalized Finsler–affine gravity following from (2.43) are written*

$$\begin{aligned} \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} &= \tilde{\kappa}\left(\mathbf{\Sigma}_{\alpha\beta}^{[\phi]} + \mathbf{\Sigma}_{\alpha\beta}^{[\mathbf{m}]}\right), \\ \widehat{\mathbf{D}}_\nu \mathbf{H}^{\nu\mu} &= \mu^2\phi^\mu, \end{aligned} \tag{2.44}$$

with $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_\nu \phi_\mu - \widehat{\mathbf{D}}_\mu \phi_\nu + w^{\gamma}_{\mu\nu}\phi_\gamma$ *being the field strengths of the Abelian Proca field* $\phi^\mu, \tilde{\kappa} = const,$ *and*

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \mathbf{H}_\alpha{}^\mu \mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}_{\mu\nu}\mathbf{H}^{\mu\nu} + \mu^2\phi_\alpha\phi_\beta - \frac{\mu^2}{2}\mathbf{g}_{\alpha\beta}\phi_\mu\phi^\mu. \tag{2.45}$$

The Ricci d–tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ and scalar $\overleftarrow{\hat{\mathbf{R}}}$ from (2.44) can be decomposed in irreversible h– and v–invariant components like (2.31) and (2.32),

$$\widehat{R}_{ij} - \frac{1}{2}g_{ij}\left(\widehat{R} + \widehat{S}\right) = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ij} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ij}\right), \tag{2.46}$$

$$\widehat{S}_{ab} - \frac{1}{2}h_{ab}\left(\widehat{R} + \widehat{S}\right) = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ab} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ab}\right), \tag{2.47}$$

$$^1P_{ai} = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ai} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ai}\right), \tag{2.48}$$

$$-^2P_{ia} = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ia} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ia}\right). \tag{2.49}$$

The constants are those from [5] being related to the constants from (2.37),

$$\mu^2 = \frac{1}{z_k\kappa}\left(-4\beta_4 + \frac{k_1}{2k_0}\beta_5 + \frac{k_2}{k_0}\gamma_4\right),$$

where

$$k_0 = 4\alpha_2\beta_3 - 3(\gamma_3)^2 \neq 0,\ k_1 = 9\left(\frac{1}{2}\alpha_5\beta_5 - \gamma_3\gamma_4\right),\ k_2 = 3\left(4\beta_3\gamma_4 - \frac{3}{2}\beta_5\gamma_3\right),$$

$$\alpha_2 = a_2 - 2a_0,\ \beta_3 = b_3 + \frac{a_0}{8},\ \beta_4 = b_4 - \frac{3a_0}{8},\ \gamma_3 = c_3 + a_0,\ \gamma_4 = c_4 + a_0.$$

If

$$\beta_4 \to \frac{1}{4k_0}\left(\frac{1}{2}\beta_5 k_1 + k_2\gamma_4\right), \tag{2.50}$$

the mass of Proca field $\mu^2 \to 0$. The system becomes like the Einstein–Maxwell one with the source (2.45) defined by the antisymmetric field $\mathbf{H}_{\mu\nu}$ in its turn being determined by a solution of $\widehat{\mathbf{D}}_\nu \widehat{\mathbf{D}}^\nu \phi_\alpha = 0$ (a wave like equation in a curved space provided with N–connection). Even in this case the nonmetricity and torsion can be nontrivial, for instance, oscillating (see (2.42)).

We note that according the Remark 2.3.1, the system (2.44) defines, for instance, a Finsler configuration if the d–metric $\mathbf{g}_{\alpha\beta}$, the d–connection $\widehat{\mathbf{D}}_\nu$ and the N–connection are of Finsler type (or contains as imbedding such objects).

2.3.3 Einstein–Cartan gravity and N–connections

The Einstein–Cartan gravity contains gravitational configurations with nontrivial N–connection structure. The simplest model with local anisotropy is to write on a space $\mathbf{V}^{n+m}$ the Einstein equations for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ (2.27) introduced in the Einstein d–tensor (2.33),

$$\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta},$$

or in terms of differential forms,

$$\eta_{\alpha\beta\gamma} \wedge \widehat{\mathbf{R}}^{\beta\gamma} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha} \qquad (2.51)$$

which is a particular case of equations (2.43). The model contains nontrivial d–torsions, $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$, computed by introducing the components of (2.27) into formulas (2.29). We can consider that specific distributions of "spin dust/fluid" of Weyssenhoff and Raabe type, or any generalizations, adapted to the N–connection structure, can constitute the source of certain algebraic equations for torsion (see details in Refs. [19]) or even to consider generalizations for dynamical equations for torsion like in gauge gravity theories [22]. A more special case is defined by the theories when the d–torsions $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$ are induced by specific frame effects of N–connection structures. Such models contain all possible distorsions to generalized Finsler–Lagrange–Cartan spacetimes of the Einstein gravity and emphasize the conditions when such generalizations to locally anisotropic gravity preserve the local Lorentz invariance or even model Finsler like configurations in the framework of general relativity.

Let us express the 1–form of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha}$ as the deformation of the Levi–Civita connection $\mathbf{\Gamma}^{\gamma}_{\bigtriangledown\ \alpha}$,

$$\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}_{\bigtriangledown\ \alpha} + \widehat{\mathbf{Z}}^{\gamma}{}_{\alpha} \qquad (2.52)$$

where

$$\widehat{\mathbf{Z}}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\widehat{\mathbf{T}}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\widehat{\mathbf{T}}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\widehat{\mathbf{T}}_{\gamma}\right)\vartheta^{\gamma} \qquad (2.53)$$

being a particular case of formulas (2.22) and (2.24) when nonmetricity vanishes, $\mathbf{Q}_{\alpha\beta} = 0$. This induces a distorsion of the curvature tensor like (2.41) but for d–objects, expressing (2.51) in the form

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{R}^{\beta\gamma}_{\bigtriangledown} + \eta_{\alpha\beta\gamma} \wedge \mathbf{Z}^{\beta\gamma}_{\bigtriangledown} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha} \qquad (2.54)$$

where

$$\mathbf{Z}^{\beta}_{\bigtriangledown\ \gamma} = \bigtriangledown\mathbf{Z}^{\beta}{}_{\gamma} + \mathbf{Z}^{\beta}{}_{\alpha} \wedge \mathbf{Z}^{\alpha}{}_{\gamma}.$$

Theorem 2.3.3. *The Einstein equations (2.51) for the canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha}$ *constructed for a d–metric field* $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *(2.11) and N–connection* N_i^a *is equivalent to the gravitational field equations for the Einstein–Cartan theory with torsion* $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha}$ *defined by the N–connection, see formulas (2.29).*

Proof: The proof is trivial and follows from decomposition (2.52).

Remark 2.3.2. *Every type of generalized Finsler–Lagrange geometries is characterized by a corresponding N– and d–connection and d–metric structures, see Tables 1–11 in Ref. [6]. For the canonical d–connection such locally anisotropic geometries can be modelled on Riemann–Cartan manifolds as solutions of (2.51) for a prescribed type of d–torsions (2.29).*

Corollary 2.3.3. *A generalized Finsler geometry can be modelled in a (pseudo) Riemann spacetime by a d–metric* $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *(2.11), equivalently by generic off–diagonal metric (2.12), satisfying the Einstein equations for the Levi–Civita connection,*

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{R}_{\bigtriangledown}^{\beta\gamma} = \kappa \mathbf{\Sigma}_{\alpha}^{[\mathbf{m}]} \qquad (2.55)$$

if and only if

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{Z}_{\bigtriangledown}^{\beta\gamma} = 0. \qquad (2.56)$$

The proof follows from equations (2.54). We emphasize that the conditions (2.56) are imposed for the deformations of the Ricci tensors computed from distorsions of the Levi–Civita connection to the canonical d–connection. In general, a solution $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ of the Einstein equations (2.55) can be characterized alternatively by d–connections and N–connections as follows from relation (2.28). The alternative geometric description contains nontrivial torsion fields. The simplest such anholonomic configurations can be defined by the condition of vanishing of N–connection curvature (2.5), $\Omega^a_{ij} = 0,$ but even in such cases there are nontrivial anholonomy coefficients, see (2.9), $\mathbf{w}^b{}_{ia} = -\mathbf{w}^b{}_{ai} = \partial_a N_i^b$, and nonvanishing d–torsions (2.29),

$$\widehat{T}^i_{ja} = -\widehat{T}^i_{aj} = \widehat{C}^i_{.ja} \text{ and } \widehat{T}^a_{.bi} = -\widehat{T}^a_{.ib} = \widehat{P}^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - \widehat{L}^a_{.bj},$$

being induced by off–diagonal terms in the metric (2.12).

2.3.4 String gravity and N–connections

The subjects concerning generalized Finsler (super) geometry, spinors and (super) strings are analyzed in details in Refs. [9]. Here, we consider the simplest examples when Finsler like geometries can be modelled in string gravity and related to certain metric–affine structures.

For instance, in the sigma model for bosonic string (see, [1]), the background connection is taken to be not the Levi–Civita one, but a certain deformation by the strength (torsion) tensor

$$H_{\mu\nu\rho} \doteqdot \delta_\mu B_{\nu\rho} + \delta_\rho B_{\mu\nu} + \delta_\nu B_{\rho\mu}$$

of an antisymmetric field $B_{\nu\rho}$, defined as

$$\mathcal{D}_\mu = \bigtriangledown_\mu + \frac{1}{2} H_{\mu\nu}{}^{\rho}.$$

We consider the H–field defined by using N–elongated operators (2.6) in order to compute the coefficients with respect to anholonomic frames.

The condition of the Weyl invariance to hold in two dimensions in the lowest nontrivial approximation in string constant α', see [9], turn out to be

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{4} H_\mu{}^{\nu\rho} H_{\nu\lambda\rho} + 2 \bigtriangledown_\mu \bigtriangledown_\nu \Phi, \\
\bigtriangledown_\lambda H^\lambda{}_{\mu\nu} &= 2 \left(\bigtriangledown_\lambda \Phi\right) H^\lambda{}_{\mu\nu}, \\
\left(\bigtriangledown \Phi\right)^2 &= \bigtriangledown_\lambda \bigtriangledown^\lambda \Phi + \frac{1}{4} R + \frac{1}{48} H_{\mu\nu\rho} H^{\mu\nu\rho}.
\end{aligned}$$

where Φ is the dilaton field. For trivial dilaton configurations, $\Phi = 0$, we may write

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{4} H_\mu{}^{\nu\rho} H_{\nu\lambda\rho}, \\
\bigtriangledown_\lambda H^\lambda{}_{\mu\nu} &= 0.
\end{aligned}$$

In Refs. [9] we analyzed string gravity models derived from superstring effective actions, for instance, from the 4D Neveu-Schwarz action. In this paper we consider, for simplicity, a model with zero dilaton field but with nontrivial H–field related to the d–torsions induced by the N–connection and canonical d–connection.

A class of Finsler like metrics can be derived from the bosonic string theory if $\mathbf{H}_{\nu\lambda\rho}$ and $\mathbf{B}_{\nu\rho}$ are related to the d–torsions components, for instance, with $\widehat{\mathbf{T}}^\gamma{}_{\alpha\beta}$. Really, we can take an ansatz

$$\mathbf{B}_{\nu\rho} = [B_{ij}, B_{ia}, B_{ab}]$$

and consider that

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}\ _{\nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho} \tag{2.57}$$

where $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$ is the distorsion of the Levi–Civita connection induced by $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$, see (2.53). In this case the induced by N–connection torsion structure is related to the antisymmetric H–field and correspondingly to the B–field from string theory. The equations

$$\bigtriangledown^{\nu}\mathbf{H}_{\nu\lambda\rho} = \bigtriangledown^{\nu}(\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) = 0 \tag{2.58}$$

impose certain dynamical restrictions to the N–connection coefficients N_i^a and d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ contained in $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$. If on the background space it is prescribed the canonical d–connection $\widehat{\mathbf{D}}$, we can state a model with (2.58) redefined as

$$\widehat{\mathbf{D}}^{\nu}\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{D}}^{\nu}(\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) = 0, \tag{2.59}$$

where $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ are computed for stated values of $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$. For trivial N–connections when $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} \to 0$ and $\widehat{\mathbf{D}}^{\nu} \to \bigtriangledown^{\nu}$, the $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ transforms into usual H–fields.

Proposition 2.3.1. *The dynamics of generalized Finsler–affine string gravity is defined by the system of field equations*

$$\begin{aligned}
\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\right), \\
\widehat{\mathbf{D}}_{\nu}\mathbf{H}^{\nu\mu} &= \mu^2\phi^{\mu}, \\
\widehat{\mathbf{D}}^{\nu}(\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) &= 0
\end{aligned} \tag{2.60}$$

with $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_{\nu}\phi_{\mu} - \widehat{\mathbf{D}}_{\mu}\phi_{\nu} + w^{\gamma}_{\mu\nu}\phi_{\gamma}$ *being the field strengths of the Abelian Proca field* $\phi^{\mu}, \tilde{\kappa} = const$,

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \mathbf{H}_{\alpha}{}^{\mu}\mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}_{\mu\nu}\mathbf{H}^{\mu\nu} + \mu^2\phi_{\alpha}\phi_{\beta} - \frac{\mu^2}{2}\mathbf{g}_{\alpha\beta}\phi_{\mu}\phi^{\mu},$$

and

$$\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta} = \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\left(\widehat{\mathbf{T}}, \Phi\right)$$

contains contributions of $\widehat{\mathbf{T}}$ *and* Φ *fields.*

Proof: It follows as an extension of the Corollary 2.3.2 to sources induced by string corrections. The system (2.60) should be completed by the field equations for the matter fields present in $\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta}$.

Finally, we note that the equations (2.60) reduce to equations of type (2.54) (for Riemann–Cartan configurations with zero nonmetricity),

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{R}^{\beta\gamma}_{\bigtriangledown} + \eta_{\alpha\beta\gamma} \wedge \mathbf{Z}^{\beta\gamma}_{\bigtriangledown} = \kappa\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha},$$

and to equations of type (2.55) and (2.56) (for (pseudo) Riemannian configurations)

$$\begin{aligned} \eta_{\alpha\beta\gamma} \wedge \mathbf{R}_{\triangledown}^{\beta\gamma} &= \kappa \mathbf{\Sigma}_{\alpha}^{[\mathbf{T}]}, \\ \eta_{\alpha\beta\gamma} \wedge \mathbf{Z}_{\triangledown}^{\beta\gamma} &= 0 \end{aligned} \quad (2.61)$$

with sources defined by torsion (related to N–connection) from string theory.

2.4 The Anholonomic Frame Method in MAG and String Gravity

In a series of papers, see Refs. [7, 8, 10, 15], the anholonomic frame method of constructing exact solutions with generic off–diagonal metrics (depending on 2-4 variables) in general relativity, gauge gravity and certain extra dimension generalizations was elaborated. In this section, we develop the method in MAG and string gravity with applications to different models of five dimensional (in brief, 5D) generalized Finsler–affine spaces.

We consider a metric–affine space provided with N–connection structure $\mathbf{N} = [N_i^4(u^\alpha),\ N_i^5(u^\alpha)]$ where the local coordinates are labelled $u^\alpha = (x^i, y^4 = v, y^5)$, for $i = 1,2,3$. We state the general condition when exact solutions of the field equations of the generalized Finsler–affine string gravity depending on holonomic variables x^i and on one anholonomic (equivalently, anisotropic) variable $y^4 = v$ can be constructed in explicit form. Every coordinate from a set u^α can may be time like, 3D space like, or extra dimensional. For simplicity, the partial derivatives are denoted $a^\times = \partial a/\partial x^1, a^\bullet = \partial a/\partial x^2, a' = \partial a/\partial x^3, a^* = \partial a/\partial v.$

The 5D metric

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}\left(x^i, v\right) du^\alpha \otimes du^\beta \quad (2.62)$$

has the metric coefficients $\mathbf{g}_{\alpha\beta}$ parametrized with respect to the coordinate dual basis by an off–diagonal matrix (ansatz)

$$\begin{bmatrix} g_1 + {w_1}^2 h_4 + {n_1}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix}, \quad (2.63)$$

with the coefficients being some necessary smoothly class functions of type

$$\begin{aligned} g_1 &= \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \\ w_i &= w_i(x^i, v), n_i = n_i(x^i, v), \end{aligned}$$

where the N–coefficients from (2.6) and (2.7) are parametrized $N_i^4 = w_i$ and $N_i^5 = n_i$.

Theorem 2.4.4. *The nontrivial components of the 5D Ricci d–tensors (2.31),* $\widehat{\mathbf{R}}_{\alpha\beta} = (\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{S}_{ab})$, *for the d–metric (2.11) and canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$*(2.27) both defined by the ansatz (2.63), computed with respect to anholonomic frames (2.6) and (2.7), consist from h- and v–irreducible components:*

$$R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}], \qquad (2.64)$$

$$S_4^4 = S_5^5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)]^{*}\right], \qquad (2.65)$$

$$R_{4i} = -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5}, \qquad (2.66)$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right], \qquad (2.67)$$

where

$$\alpha_i = \partial_i h_5^{*} - h_5^{*}\partial_i \ln\sqrt{|h_4h_5|}, \beta = h_5^{**} - h_5^{*}[\ln\sqrt{|h_4h_5|}]^{*}, \gamma = 3h_5^{*}/2h_5 - h_4^{*}/h_4 \qquad (2.68)$$

$h_4^{*} \neq 0,\ h_5^{*} \neq 0$ *cases with vanishing* h_4^{*} *and/or* h_5^{*} *should be analyzed additionally.*

The proof of Theorem 2.4.4 is given in Appendix 2.7.

We can generalize the ansatz (2.63) by introducing a conformal factor $\omega(x^i, v)$ and additional deformations of the metric via coefficients $\zeta_{\hat{i}}(x^i, v)$ (here, the indices with 'hat' take values like $\hat{i} = 1, 2, 3, 5$), i. e. for metrics of type

$$\mathbf{g}^{[\omega]} = \omega^2(x^i, v)\hat{\mathbf{g}}_{\alpha\beta}\left(x^i, v\right) du^{\alpha} \otimes du^{\beta}, \qquad (2.69)$$

were the coefficients $\hat{\mathbf{g}}_{\alpha\beta}$ are parametrized by the ansatz

$$\left[\begin{array}{ccccc} g_1 + (w_1{}^2 + \zeta_1{}^2)h_4 + n_1{}^2 h_5 & (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_1 + \zeta_1)h_4 & n_1h_5 \\ (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & g_2 + (w_2{}^2 + \zeta_2{}^2)h_4 + n_2{}^2h_5 & (w_2w_3 + \zeta_2\zeta_3)h_4 + n_2n_3h_5 & (w_2 + \zeta_2)h_4 & n_2h_5 \\ (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_2w_3 + \zeta_2\zeta_3)h_4 + n_2n_3h_5 & g_3 + (w_3{}^2 + \zeta_3{}^2)h_4 + n_3{}^2h_5 & (w_3 + \zeta_3)h_4 & n_3h_5 \\ (w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & h_4 & 0 \\ n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5 + \zeta_5h_4 \end{array}\right]. \qquad (2.70)$$

Such 5D metrics have a second order anisotropy [9, 13] when the N–coefficients are parametrized in the first order anisotropy like $N_i^4 = w_i$ and $N_i^5 = n_i$ (with three anholonomic, x^i, and two anholonomic, y^4 and y^5, coordinates) and in the second order anisotropy (on the second 'shell', with four holonomic, (x^i, y^5), and one anholonomic,y^4, coordinates) with $N_{\hat{i}}^5 = \zeta_{\hat{i}}$, in this work we state, for simplicity, $\zeta_5 = 0$. For trivial values $\omega = 1$ and $\zeta_{\hat{i}} = 0$, the metric (2.69) transforms into (2.62).

The Theorem 2.4.4 can be extended as to include the ansatz (2.69):

Theorem 2.4.5. *The nontrivial components of the 5D Ricci d–tensors (2.31),* $\widehat{\mathbf{R}}_{\alpha\beta} = (\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{S}_{ab})$*, for the metric (2.11) and canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ *(2.27) defined by the ansatz (2.70), computed with respect to the anholonomic frames (2.6) and (2.7), are given by the same formulas (2.64)–(2.67) if there are satisfied the conditions*

$$\hat{\delta}_i h_4 = 0 \text{ and } \hat{\delta}_i \omega = 0 \tag{2.71}$$

for $\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\,\partial_4 + n_i \partial_5$ *when the values* $\zeta_{\tilde{i}} = (\zeta_i, \zeta_5 = 0)$ *are to be defined as any solutions of (2.71).*

The proof of Theorem 2.4.5 consists from a straightforward calculation of the components of the Ricci tensor (2.31) like in Appendix 2.7. The simplest way to do this is to compute the deformations by the conformal factor of the coefficients of the canonical connection (2.27) and then to use the calculus for Theorem 2.4.4. Such deformations induce corresponding deformations of the Ricci tensor (2.31). The condition that we have the same values of the Ricci tensor for the (2.12) and (2.70) results in equations (2.71) which are compatible, for instance, if for instance, if

$$\omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}), \tag{2.72}$$

and ζ_i satisfy the equations

$$\partial_i \omega - (w_i + \zeta_i)\omega^* = 0. \tag{2.73}$$

There are also different possibilities to satisfy the condition (2.71). For instance, if $\omega = \omega_1\ \omega_2$, we can consider that $h_4 = \omega_1^{q_1/q_2}\ \omega_2^{q_3/q_4}$ for some integers q_1, q_2, q_3 and q_4 ■

There are some important consequences of the Theorems 2.4.4 and 2.4.5:

Corollary 2.4.4. *The non–trivial components of the Einstein tensor [see (2.33) for the canonical d–connection]* $\widehat{\mathbf{G}}^{\alpha}{}_{\beta} = \widehat{\mathbf{R}}^{\alpha}{}_{\beta} - \frac{1}{2}\overleftarrow{\hat{\mathbf{R}}}\delta^{\alpha}_{\beta}$ *for the ansatz (2.63) and (2.70) given with respect to the N–adapted (co) frames are*

$$G^1_1 = -\left(R^2_2 + S^4_4\right), G^2_2 = G^3_3 = -S^4_4, G^4_4 = G^5_5 = -R^2_2. \tag{2.74}$$

The relations (2.74) can be derived following the formulas for the Ricci tensor (2.64)–(2.67). They impose the condition that the dynamics of such gravitational fields is defined by two independent components R^2_2 and S^4_4 and result in

Corollary 2.4.5. *The system of effective 5D Einstein–Proca equations on spaces provided with N–connection structure (2.44) (equivalently, the system (2.46)–(2.49)_is compatible for the generic off–diagonal ansatz (2.63) and (2.70) if the energy–momentum* *tensor* $\mathbf{\Upsilon}_{\alpha\beta} = \tilde{\kappa}(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta})$ *of the Proca and matter fields given with respect to N–frames is diagonal and satisfies the conditions*

$$\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v), \ \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^2, x^3), \ \text{and } \Upsilon_1 = \Upsilon_2 + \Upsilon_4. \tag{2.75}$$

Remark 2.4.3. *Instead of the energy–momentum tensor* $\mathbf{\Upsilon}_{\alpha\beta} = \tilde{\kappa}(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta})$ *for the Proca and matter fields we can consider any source, for instance, with string corrections, when* $\mathbf{\Upsilon}^{[str]}_{\alpha\beta} = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\right)$ *like in (2.60) satisfying the conditions (2.75).*

If the conditions of the Corollary 2.4.5, or Remark 2.4.3, are satisfied, the h- and v– irreducible components of the 5D Einstein–Proca equations (2.46) and (2.49), or of the string gravity equations (2.60), for the ansatz (2.63) and (2.70) transform into the system

$$R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = -\Upsilon_4(x^2, x^3), \tag{2.76}$$

$$S_4^4 = S_5^5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)^{*}\right] = -\Upsilon_2(x^2, x^3, v). \tag{2.77}$$

$$R_{4i} = -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \tag{2.78}$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right] = 0. \tag{2.79}$$

A very surprising result is that we are able to construct exact solutions of the 5D Einstein–Proca equations with anholonomic variables and generic off–diagonal metrics:

Theorem 2.4.6. *The system of second order nonlinear partial differential equations (2.76)–(2.79) and (2.73) can be solved in general form if there are given certain values of functions* $g_2(x^2, x^3)$ *(or, inversely,* $g_3(x^2, x^3)$*),* $h_4(x^i, v)$ *(or, inversely,* $h_5(x^i, v)$*),* $\omega(x^i, v)$ *and of sources* $\Upsilon_2(x^2, x^3, v)$ *and* $\Upsilon_4(x^2, x^3)$*.*

We outline the main steps of constructing exact solutions and proving this Theorem.

- The general solution of equation (2.76) can be written in the form

$$\varpi = g_{[0]}\exp[a_2\widetilde{x}^2\left(x^2, x^3\right) + a_3\widetilde{x}^3\left(x^2, x^3\right)], \tag{2.80}$$

  were $g_{[0]}, a_2$ and a_3 are some constants and the functions $\widetilde{x}^{2,3}(x^2, x^3)$ define any coordinate transforms $x^{2,3} \rightarrow \widetilde{x}^{2,3}$ for which the 2D line element becomes conformally flat, i. e.

$$g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \varpi(x^2, x^3)\left[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\right], \tag{2.81}$$

where $\epsilon = \pm 1$ for a corresponding signature. In coordinates $\widetilde{x}^{2,3}$, the equation (2.76) transform into

$$\varpi\left(\ddot{\varpi}+\varpi''\right)-\dot{\varpi}-\varpi' = 2\varpi^2\Upsilon_4(\tilde{x}^2,\tilde{x}^3)$$

or

$$\ddot{\psi}+\psi'' = 2\Upsilon_4(\tilde{x}^2,\tilde{x}^3), \tag{2.82}$$

for $\psi = \ln|\varpi|$. The integrals of (2.82) depends on the source Υ_4. As a particular case we can consider that $\Upsilon_4 = 0$. There are three alternative possibilities to generate solutions of (2.76). For instance, we can prescribe that $g_2 = g_3$ and get the equation (2.82) for $\psi = \ln|g_2| = \ln|g_3|$. If we suppose that $g_2^{'} = 0$, for a given $g_2(x^2)$, we obtain from (2.76)

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} = 2g_2g_3\Upsilon_4(x^2,x^3)$$

which can be integrated explicitly for given values of Υ_4. Similarly, we can generate solutions for a prescribed $g_3(x^3)$ in the equation

$$g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2} = 2g_2g_3\Upsilon_4(x^2,x^3).$$

We note that a transform (2.81) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case with $\Upsilon_4 = 0$ the equation (2.76) is solved by arbitrary two functions $g_2(x^3)$ and $g_3(x^2)$.

- For $\Upsilon_2 = 0$, the equation (2.77) relates two functions $h_4\left(x^i,v\right)$ and $h_5\left(x^i,v\right)$ following two possibilities:

  a) to compute

$$\begin{aligned}\sqrt{|h_5|} &= h_{5[1]}\left(x^i\right)+h_{5[2]}\left(x^i\right)\int\sqrt{|h_4\left(x^i,v\right)|}dv,\ h_4^*\left(x^i,v\right)\neq 0;\\ &= h_{5[1]}\left(x^i\right)+h_{5[2]}\left(x^i\right)v,\ h_4^*\left(x^i,v\right)=0,\end{aligned} \tag{2.83}$$

  for some functions $h_{5[1,2]}\left(x^i\right)$ stated by boundary conditions;

  b) or, inversely, to compute h_4 for a given $h_5\left(x^i,v\right), h_5^*\neq 0$,

$$\sqrt{|h_4|} = h_{[0]}\left(x^i\right)(\sqrt{|h_5\left(x^i,v\right)|})^*, \tag{2.84}$$

with $h_{[0]}(x^i)$ given by boundary conditions. We note that the sourceless equation (2.77) is satisfied by arbitrary pairs of coefficients $h_4(x^i, v)$ and $h_{5[0]}(x^i)$. Solutions with $\Upsilon_2 \neq 0$ can be found by ansatz of type

$$h_5[\Upsilon_2] = h_5, h_4[\Upsilon_2] = \varsigma_4\left(x^i, v\right) h_4, \tag{2.85}$$

where h_4 and h_5 are related by formula (2.83), or (2.84). Substituting (2.85), we obtain

$$\varsigma_4\left(x^i, v\right) = \varsigma_{4[0]}\left(x^i\right) - \int \Upsilon_2(x^2, x^3, v)\frac{h_4 h_5}{4h_5^*}dv, \tag{2.86}$$

where $\varsigma_{4[0]}(x^i)$ are arbitrary functions.

- The exact solutions of (2.78) for $\beta \neq 0$ are defined from an algebraic equation, $w_i\beta + \alpha_i = 0$, where the coefficients β and α_i are computed as in formulas (2.68) by using the solutions for (2.76) and (2.77). The general solution is

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \tag{2.87}$$

  with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients w_k could be arbitrary functions on (x^i, v). For the vacuum Einstein equations this is a degenerated case imposing the the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the h_4 and h_5 are related as in the formula (2.84) but with $h_{[0]}(x^i) = const$.

- Having defined h_4 and h_5 and computed γ from (2.68) we can solve the equation (2.79) by integrating on variable "v" the equation $n_i^{**} + \gamma n_i^* = 0$. The exact solution is

$$\begin{aligned} n_k &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int [h_4/(\sqrt{|h_5|})^3]dv, \ h_5^* \neq 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int h_4 dv, \qquad h_5^* = 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int [1/(\sqrt{|h_5|})^3]dv, \ h_4^* = 0, \end{aligned} \tag{2.88}$$

  for some functions $n_{k[1,2]}(x^i)$ stated by boundary conditions.

The exact solution of (2.73) is given by some arbitrary functions $\zeta_i = \zeta_i(x^i, v)$ if both $\partial_i\omega = 0$ and $\omega^* = 0$, we chose $\zeta_i = 0$ for $\omega = const$, and

$$\begin{aligned} \zeta_i &= -w_i + (\omega^*)^{-1}\partial_i\omega, \quad \omega^* \neq 0, \\ &= (\omega^*)^{-1}\partial_i\omega, \quad \omega^* \neq 0, \ \text{for vacuum solutions.} \end{aligned} \tag{2.89}$$

The Theorem 2.4.6 states a general method of constructing exact solutions in MAG, of the Einstein–Proca equations and various string gravity generalizations with generic off–diagonal metrics. Such solutions are with associated N–connection structure. This method can be also applied in order to generate, for instance, certain Finsler or Lagrange configurations as v-irreducible components. The 5D ansatz can not be used to generate standard Finsler or Lagrange geometries because the dimension of such spaces can not be an odd number. Nevertheless, the anholonomic frame method can be applied in order to generate 4D exact solutions containing Finsler–Lagrange configurations, see Appendix 2.8.

Summarizing the results for the nondegenerated cases when $h_4^* \neq 0$ and $h_5^* \neq 0$ and (for simplicity, for a trivial conformal factor ω), we derive an explicit result for 5D exact solutions with local coordinates $u^\alpha = (x^i, y^a)$ when $x^i = \left(x^1, x^{\widehat{i}}\right), x^{\widehat{i}} = (x^2, x^3), y^a = (y^4 = v, y^a)$ and arbitrary signatures $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ (where $\epsilon_\alpha = \pm 1$) :

Corollary 2.4.6. *Any off–diagonal metric*

$$\begin{aligned} \delta s^2 &= \epsilon_1 (dx^1)^2 + \epsilon_{\widehat{k}} g_{\widehat{k}}\left(x^{\widehat{i}}\right)(dx^{\widehat{k}})^2 + \\ & \epsilon_4 h_0^2(x^i)\left[f^*\left(x^i,v\right)\right]^2 |\varsigma_\Upsilon\left(x^i,v\right)| (\delta v)^2 + \epsilon_5 f^2\left(x^i,v\right)\left(\delta y^5\right)^2, \\ \delta v &= dv + w_k\left(x^i,v\right)dx^k,\ \ \delta y^5 = dy^5 + n_k\left(x^i,v\right)dx^k, \end{aligned} \tag{2.90}$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ *is a solution of the 2D equation (2.76) for a given source* $\Upsilon_4\left(x^{\widehat{i}}\right)$,

$$\varsigma_\Upsilon\left(x^i,v\right) = \varsigma_4\left(x^i,v\right) = \varsigma_{4[0]}\left(x^i\right) - \frac{\epsilon_4}{16} h_0^2(x^i)\int \Upsilon_2(x^{\widehat{k}},v)[f^2\left(x^i,v\right)]^2 dv,$$

and the N–connection coefficients $N_i^4 = w_i(x^k,v)$ *and* $N_i^5 = n_i(x^k,v)$ *are*

$$w_i = -\frac{\partial_i \varsigma_\Upsilon\left(x^k,v\right)}{\varsigma_\Upsilon^*\left(x^k,v\right)} \tag{2.91}$$

and

$$n_k = n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int \frac{\left[f^*\left(x^i,v\right)\right]^2}{\left[f\left(x^i,v\right)\right]^2}\varsigma_\Upsilon\left(x^i,v\right)dv, \tag{2.92}$$

define an exact solution of the system of Einstein equations with holonomic and anholonomic variables (2.76)–(2.79) for arbitrary nontrivial functions $f\left(x^i,v\right)$ *(with* $f^* \neq 0$*),*

$h_0^2(x^i)$, $\varsigma_{4[0]}(x^i), n_{k[1]}(x^i)$ and $n_{k[2]}(x^i)$, and sources $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$ and any integration constants and signatures $\epsilon_\alpha = \pm 1$ to be defined by certain boundary conditions and physical considerations.

Any metric (2.90) with $h_4^* \neq 0$ and $h_5^* \neq 0$ has the property to be generated by a function of four variables $f(x^i, v)$ with emphasized dependence on the anisotropic coordinate v, because $f^* \doteqdot \partial_v f \neq 0$ and by arbitrary sources $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$. The rest of arbitrary functions not depending on v have been obtained in result of integration of partial differential equations. This fix a specific class of metrics generated by using the relation (2.84) and the first formula in (2.88). We can generate also a different class of solutions with $h_4^* = 0$ by considering the second formula in (2.83) and respective formulas in (2.88). The "degenerated" cases with $h_4^* = 0$ but $h_5^* \neq 0$ and inversely, $h_4^* \neq 0$ but $h_5^* = 0$ are more special and request a proper explicit construction of solutions. Nevertheless, such type of solutions are also generic off–diagonal and they could be of substantial interest.

The sourceless case with vanishing Υ_2 and Υ_4 is defined following

Remark 2.4.4. *Any off–diagonal metric (2.90) with $\varsigma_\Upsilon = 1$, $h_0^2(x^i) = h_0^2 = const$, $w_i = 0$ and n_k computed as in (2.92) but for $\varsigma_\Upsilon = 1$, defines a vacuum solution of 5D Einstein equations for the canonical d–connection (2.27) computed for the ansatz (2.90).*

By imposing additional constraints on arbitrary functions from $N_i^5 = n_i$ and $N_i^5 = w_i$, we can select off–diagonal gravitational configurations with distorsions of the Levi–Civita connection resulting in canonical d–connections with the same solutions of the vacuum Einstein equations. For instance, we can model Finsler like geometries in general relativity, see Corollary 2.3.3. Under similar conditions the ansatz (2.63) was used for constructing exact off–diagonal solutions in the 5D Einstein gravity, see Refs. [7, 8, 9].

Let us consider the procedure of selecting solutions with off–diagonal metrics from an ansatz (2.90) with trivial N–connection curvature (such metrics consists a simplest subclass which can be restricted to (pseudo) Riemannian ones). The corresponding nontrivial coefficients the N–connection curvature (2.5) are computed

$$\Omega_{ij}^4 = \partial_i w_j - \partial_j w_i + w_i w_j^* - w_j w_i^* \text{ and } \Omega_{ij}^5 = \partial_i n_j - \partial_j n_i + w_i n_j^* - w_j n_i^*.$$

So, there are imposed six constraints, $\Omega_{ij}^4 = \Omega_{ij}^5 = 0$, for $i, j... = 1, 2, 4$ on six functions w_i and n_i computed respectively as (2.92) and (2.92) which can be satisfied by a corresponding subclass of functions $f(x^i, v)$ (with $f^* \neq 0$), $h_0^2(x^i)$, $\varsigma_{4[0]}(x^i), n_{k[1]}(x^i)$, $n_{k[2]}(x^i)$ and $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$ (in general, we have to solve certain first order partial derivative equations with may be reduced to algebraic relations by corresponding parametrizations). For instance, in the vacuum case when $w_j = 0$, we obtain $\Omega^5_{ij} = \partial_i n_j - \partial_j n_i$. The simplest example when condition $\Omega^5_{\widehat{i}\widehat{j}} = \partial_{\widehat{i}} n_{\widehat{j}} - \partial_{\widehat{j}} n_{\widehat{i}} = 0$, with $\widehat{i},\widehat{j} = 2,3$ (reducing the metric (2.90) to a 4D one trivially embedded into 5D) is satisfied is to take $n_{3[1]} = n_{3[2]} = 0$ in (2.92) and consider that $f = f\left(x^2, v\right)$ with $n_{2[1]} = n_{2[1]}\left(x^2\right)$ and $n_{2[2]} = n_{2[2]}\left(x^2\right)$, i. e. by eliminating the dependence of the coefficients on x^3. This also results in a generic off–diagonal solution, because the anholonomy coefficients (2.9) are not trivial, for instance, $w^5_{24} = n^*_2$ and $w^5_{14} = n^*_1$.

Another interesting remark is that even we have reduced the canonical d–connection to the Levi–Civita one [with respect to N–adapted (co) frames; this imposes the metric to be (pseudo) Riemannian] by selecting the arbitrary functions as to have $\Omega^a_{ij} = 0$, one could be nonvanishing d–torsion components like $T^5_{41} = P^5_{41}$ and $T^5_{41} = P^5_{41}$ in (2.29). Such objects, as well the anholonomy coefficients w^5_{24} and w^5_{14} (which can be also considered as torsion like objects) are constructed by taking certain "scarps" from the coefficients of off–diagonal metrics and anholonomic frames. They are induced by the frame anholonomy (like "torsions" in rotating anholonomic systems of reference for the Newton gravity and mechanics with constraints) and vanish if we transfer the constructions with respect to any holonomic basis.

The above presented results are for generic 5D off–diagonal metrics, anholonomic transforms and nonlinear field equations. Reductions to a lower dimensional theory are not trivial in such cases. We emphasize some specific points of this procedure in the Appendix 2.8 (see details in [15]).

2.5 Exact Solutions

There were found a set of exact solutions in MAG [16, 20, 5] describing various configuration of Einstein–Maxwell of dilaton gravity emerging from low energy string theory, soliton and multipole solutions and generalized Plebanski–Demianski solutions, colliding waves and static black hole metrics. In this section we are going to look for some classes of 4D and 5D solutions of the Einstein–Proca equations in MAG related to string gravity modelling generalized Finsler–affine geometries and extending to such spacetimes some our previous results [7, 8, 9].

2.5.1 Finsler–Lagrange metrics in string and metric–affine gravity

As we discussed in section 2, the generalized Finsler–Lagrange spaces can be modelled in metric–affine spacetimes provided with N–connection structure. In this subsection, we show how such two dimensional Finsler like spaces with d–metrics depending on one anisotropic coordinate $y^4 = v$ (denoted as $\mathbf{F}^2 = [V^2, F(x^2, x^3, y)]$, $\mathbf{L}^2 = [V^2, L(x^2, x^3, y)]$ and $\mathbf{GL}^2 = [V^2, g_{ij}(x^2, x^3, y)]$ according to Ref. [6]) can be modelled by corresponding diad transforms on spacetimes with 5D (or 4D) d–metrics being exact solutions of the field equations for the generalized Finsler–affine string gravity (2.60) (as a particular case we can consider the Einstein–Proca system (2.76)–(2.79) and (2.73)). For every particular case of locally anisotropic spacetime, for instance, outlined in Appendix C, see Table 2.1, the quadratic form $\tilde{g}_{ij}$, d–metric $\tilde{\mathbf{g}}_{\alpha\beta} = [\tilde{g}_{ij}, \tilde{g}_{ij}]$ and N–connection $\tilde{N}^a_j$ one holds

Theorem 2.5.7. *Any 2D locally anisotropic structure given by* $\tilde{\mathbf{g}}_{\alpha\beta}$ *and* $\tilde{N}^a_j$ *can be modelled on the space of exact solutions of the 5D (or 4D) the generalized Finsler–affine string gravity system defined by the ansatz (2.70) (or (2.126)).*

We give the proof via an explicit construction. Let us consider

$$\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}] = \left[\omega g_2\left(x^2, x^3\right), \omega g_3\left(x^2, x^3\right), \omega h_4\left(x^2, x^3, v\right), \omega h_5\left(x^2, x^3, v\right)\right]$$

for $\omega = \omega(x^2, x^3, v)$ and

$$N^a_i = \left[N^4_i = w_i\left(x^2, x^3, v\right), N^5_i = n_i\left(x^2, x^3, v\right)\right],$$

where indices are running the values $a = 4, 5$ and $i = 2, 3$ define an exact 4D solution of the equations (2.60) (or, in the particular case, of the system (2.76)–(2.79), for simplicity, we put $\omega(x^2, x^3, v) = 1$). We can relate the data $(\mathbf{g}_{\alpha\beta}, N^a_i)$ to any data $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}^a_j\right)$ via nondegenerate diadic transforms $e^{i'}_i = e^{i'}_i(x^2, x^3, v)$, $l^{i'}_a = l^{i'}_a(x^2, x^3, v)$ and $q^{i'}_a = q^{i'}_a(x^2, x^3, v)$ (and theirs inverse matrices)

$$g_{ij} = e^{i'}_i e^{j'}_j \tilde{g}_{i'j'},\ h_{ab} = l^{i'}_a l^{i'}_b \tilde{g}_{i'j'}, N^a_{i'} = q^a_{a'} \tilde{N}^{a'}_{j'}\ . \tag{2.93}$$

Such transforms may be associated to certain tetradic transforms of the N–elongated (co) frames ((2.7)) (2.6). If for the given data $(\mathbf{g}_{\alpha\beta}, N^a_i)$ and $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}^a_j\right)$ in (2.93), we can solve the corresponding systems of quadratic algebraic equations and define nondegenerate matrices $\left(e^{i'}_i\right)$, $\left(l^{i'}_a\right)$ and $\left(q^a_{i'}\right)$, we argue that the 2D locally anisotropic spacetime $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}^a_j\right)$ (really, it is a 4D spacetime with generic off–diagonal metric and associated N–connection structure) can be modelled on by a class of exact solutions of effective Einstein–Proca equations for MAG.■

The d–metric with respect to transformed N–adapted diads is written in the form

$$\mathbf{g} = \tilde{g}_{i'j'}\mathbf{e}^{i'} \otimes \mathbf{e}^{j'} + \tilde{g}_{i'j'}\tilde{\mathbf{e}}^{i'} \otimes \tilde{\mathbf{e}}^{j'} \tag{2.94}$$

where

$$\mathbf{e}^{i'} = e^{i'}_{i} dx^i,\ \tilde{\mathbf{e}}^{i'} = l^{i'}_{a}\tilde{\mathbf{e}}^a,\ \tilde{\mathbf{e}}^a = dy^a + \tilde{N}^a_{j'}\tilde{\mathbf{e}}^{j'}_{[N]},\ \tilde{\mathbf{e}}^{j'}_{[N]} = q^{j'}_{i} dx^i.$$

The d–metric (2.94) has the coefficients corresponding to generalized Finsler–Lagrange spaces and emphasizes that any quadratic form $\tilde{g}_{i'j'}$ from Table 2.1 can be related via an exact solution $(g_{ij}, h_{ab}, N^a_{i'})$.

We note that we can define particular cases of imbedding with $h_{ab} = l^{i'}_{a} l^{i'}_{b} \tilde{g}_{i'j'}$ and $N^a_{j'} = q^a_{i'}\tilde{N}^{i'}_{j'}$ for a prescribed value of $g_{ij} = \tilde{g}_{i'j'}$ and try to model only the quadratic form $\tilde{h}_{i'j'}$ in MAG. Similar considerations were presented for particular cases of modelling Finsler structures and generalizations in Einstein and Einstein–Cartan spaces [7, 9], see the conditions (2.61).

2.5.2 Solutions in MAG with effective variable cosmological constant

A class of 4D solutions in MAG with local anisotropy can be derived from (2.44) for $\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} = 0$ and almost vanishing mass $\mu \rightarrow 0$ of the Proca field in the source $\mathbf{\Sigma}^{[\phi]}_{\alpha\beta}$. This holds in absence of matter fields and when the constant in the action for the Finsler–affine gravity are subjected to the condition (2.50). We consider that $\phi_\mu = \left(\phi_{\widehat{i}}\left(x^{\widehat{k}}\right), \phi_a = 0\right)$, where $\widehat{i}, \widehat{k}, ... = 2,3$ and $a, b, ... = 4,5$, with respect to a N–adapted coframe (2.7) and choose a metric ansatz of type (2.124) with $g_2 = 1$ and $g_3 = -1$ which select a flat h–subspace imbedded into a general anholonomic 4D background with nontrivial h_{ab} and N–connection structure N^a_i. The h–covariant derivatives are $\widehat{D}^{[h]}\ \phi_{\widehat{i}} = (\partial_2\phi_{\widehat{i}}, \partial_3\phi_{\widehat{i}})$ because the coefficients $\widehat{L}^i{}_{jk}$ and $\widehat{C}^i{}_{ja}$ are zero in (2.27) and any contraction with $\phi_a = 0$ results in zero values. In this case the Proca equations, $\widehat{\mathbf{D}}_\nu \mathbf{H}^{\nu\mu} = \mu^2\phi^\mu$, transform in a Maxwell like equation,

$$\partial_2(\partial_2\phi_{\widehat{i}}) - \partial_3(\partial_3\phi_{\widehat{i}}) = 0, \tag{2.95}$$

for the potential $\phi_{\widehat{i}}$, with the dynamics in the h–subspace distinguished by a N–connection structure to be defined latter. We note that ϕ_i is not an electromagnetic field, but a component of the metric–affine gravity related to nonmetricity and torsion. The relation $\mathbf{Q} = k_0\phi,\ \mathbf{\Lambda} = k_1\phi,\ \mathbf{T} = k_2\phi$ from (2.42) transforms into $Q_{\widehat{i}} = k_0\phi_{\widehat{i}}, \Lambda_{\widehat{i}} = k_1\phi_{\widehat{i}}, T_i = k_2\phi_{\widehat{i}}$, and vanishing Q_a, Λ_a and T_a, defined, for instance, by a wave solution of (2.95),

$$\phi_{\widehat{i}} = \phi_{[0]\widehat{i}} \cos\left(\varrho_i x^i + \varphi_{[0]}\right) \tag{2.96}$$

for any constants $\phi_{[0]2,3}$, $\varphi_{[0]}$ and $(\varrho_2)^2 - (\varrho_3)^2 = 0$. In this simplified model we have related plane waves of nonmetricity and torsion propagating on an anholonomic background provided with N–connection. Such nonmetricity and torsion do not vanish even $\mu \rightarrow 0$ and the Proca field is approximated by a massless vector field defined in the h–subspace.

The energy–momentum tensor $\mathbf{\Sigma}^{[\phi]}_{\alpha\beta}$ for the massless field (2.96) is defined by a nontrivial value

$$H_{23} = \partial_2\phi_3 - \partial_3\phi_2 = \varepsilon_{23}\lambda_{[h]} \sin\left(\varrho_i x^i + \varphi_{[0]}\right)$$

with antisymmetric $\varepsilon_{23}, \varepsilon_{23} = 1$, and constant $\lambda_{[h]}$ taken for a normalization $\varepsilon_{23}\lambda_{[h]} = \varrho_2\phi_{[0]3} - \varrho_3\phi_{[0]2}$. This tensor is diagonal with respect to N–adapted (co) frames, $\mathbf{\Sigma}^{[\phi]\beta}_{\alpha} = \{\Upsilon_2, \Upsilon_2, 0, 0\}$ with

$$\Upsilon_2\left(x^2, x^3\right) = -\lambda^2_{[h]} \sin^2\left(\varrho_i x^i + \varphi_{[0]}\right). \qquad (2.97)$$

So, we have the case from (2.135) and (2.136) with $\Upsilon_2\left(x^2, x^2, v\right) \rightarrow \Upsilon_2\left(x^2, x^2\right)$ and Υ_4, i. e.

$$G^2_2 = G^3_3 = -S^4_4 = \Upsilon_2\left(x^2, x^2\right) \text{ and } G^4_4 = G^4_4 = -R^2_2 = 0. \qquad (2.98)$$

There are satisfied the compatibility conditions from Corollary 2.4.5. For the above stated ansatz for the d–metric and ϕ–field, the system (2.44) reduces to a particular case of (2.76)–(2.79), when the first equation is trivially satisfied by $g_2 = 1$ and $g_3 = -1$ but the second one is

$$S^4_4 = S^5_5 = -\frac{1}{2h_4 h_5}\left[h^{**}_5 - h^*_5\left(\ln\sqrt{|h_4 h_5|}\right)^*\right]\Big] = \lambda^2_{[h]} \sin^2\left(\varrho_i x^i + \varphi_{[0]}\right). \qquad (2.99)$$

The right part of this equation is like a "cosmological constant", being nontrivial in the h–subspace and polarized by a nonmetricity and torsion wave (we can state $x^2 = t$ and choose the signature $(- + - - -)$).

The exact solution of (2.99) exists according the Theorem 2.4.6 (see formulas (2.83)–(2.86)). Taking any $h_4 = h_4[\lambda_{[h]} = 0]$ and $h_5 = h_5[\lambda_{[h]} = 0]$ solving the equation with $\lambda_{[h]} = 0$, for instance, like in (2.84), we can express the general solution with nontrivial source like

$$h_5[\lambda_{[h]}] = h_5,\ h_4[\lambda_{[h]}] = \varsigma_{[\lambda]}\left(x^i, v\right) h_4,$$

where (for an explicit source (2.97) in (2.86))

$$\varsigma_{[\lambda]}\left(t, x^3, v\right) = \varsigma_{4[0]}\left(t, x^3\right) - \frac{\lambda^2_{[h]}}{4} \sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right) \int \frac{h_4 h_5}{h^*_5} dv,$$

where $\varsigma_{4[0]}\left(t, x^3\right) = 1$ if we want to have $\varsigma_{[\lambda]}$ for $\lambda^2_{[h]} \rightarrow 0$. A particular class of 4D off–diagonal exact solutions with $h^*_{4,5} \neq 0$ (see the Corollary 2.4.6 with $x^2 = t$ stated to be the time like coordinate and x^1 considered as the extra 5th dimensional one to be eliminated for reductions 5D→4D) is parametrized by the generic off–diagonal metric

$$\begin{aligned} \delta s^2 &= (dt)^2 - \left(dx^3\right) - h_0^2(t,x^3)\left[f^*\left(t,x^3,v\right)\right]^2 |\varsigma_{[\lambda]}\left(t,x^3,v\right)|\,(\delta v)^2 - f^2\left(t,x^3,v\right)\left(\delta y^5\right)^2 \\ \delta v &= dv + w_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}},\ \delta y^5 = dy^5 + n_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}}, \end{aligned} \tag{2.100}$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ is a solution of the 2D equation (2.76) for a given source $\Upsilon_4\left(x^{\widehat{i}}\right)$,

$$\varsigma_{[\lambda]}\left(t,x^3,v\right) = 1 + \frac{\lambda_{[h]}^2}{16}h_0^2(t,x^3)\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right)f^2\left(t,x^3,v\right),$$

and the N–connection coefficients $N_{\widehat{i}}^4 = w_{\widehat{i}}(t,x^3,v)$ and $N_{\widehat{i}}^5 = n_{\widehat{i}}(t,x^3,v)$ are

$$w_{2,3} = -\frac{\partial_{2,3}\varsigma_{[\lambda]}\left(t,x^3,v\right)}{\varsigma_{[\lambda]}^*\left(t,x^3,v\right)}$$

and

$$n_{2,3}\left(t,x^3,v\right) = n_{2,3[1]}\left(t,x^3\right) + n_{2,3[2]}\left(t,x^3\right)\int\frac{\left[f^*\left(t,x^3,v\right)\right]^2}{\left[f\left(t,x^3,v\right)\right]^2}\varsigma_{[\lambda]}\left(t,x^3,v\right)dv,$$

define an exact 4D solution of the system of Einstein–Proca equations (2.46)–(2.49) for vanishing mass $\mu \to 0$, with holonomic and anholonomic variables and 1-form field

$$\phi_\mu = \left[\phi_{\widehat{i}} = \phi_{[0]\widehat{i}}\cos\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right), \phi_4 = 0, \phi_0 = 0\right]$$

for arbitrary nontrivial functions $f\left(t,x^3,v\right)$ (with $f^* \neq 0$), $h_0^2(t,x^3)$, $n_{k[1,2]}\left(t,x^3\right)$ and sources $\Upsilon_2\left(t,x^3\right) = -\lambda_{[h]}^2\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right)$ and $\Upsilon_4 = 0$ and any integration constants to be defined by certain boundary conditions and additional physical arguments. For instance, we can consider ellipsoidal symmetries for the set of space coordinates $(x^3, y^4 = v, y^5)$ considered on possibility to be ellipsoidal ones, or even with topologically nontrivial configurations like torus, with toroidal coordinates. Such exact solutions emphasize anisotropic dependencies on coordinate v and do not depend on y^5.

2.5.3 3D solitons in string Finsler–affine gravity

The d–metric (2.100) can be extended as to define a class of exact solutions of generalized Finsler affine string gravity (2.60), for certain particular cases describing 3D solitonic configurations.

We start with the the well known ansatz in string theory (see, for instance, [24]) for the H–field (2.57) when

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}\ _{\nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} \tag{2.101}$$

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_{[H]} = const$, which satisfies the field equations for $\mathbf{H}_{\nu\lambda\rho}$, see (2.59). The ansatz (2.101) is chosen for a locally anisotropic background with $\widehat{\mathbf{Z}}\ _{\nu\lambda\rho}$ defined by the d–torsions for the canonical d–connection. So, the values $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ are constrained to solve the equations (2.101) for a fixed value of the cosmological constant $\lambda_{[H]}$ effectively modelling some corrections from string gravity. In this case, the source (2.97) is modified to

$$\mathbf{\Sigma}_{\alpha}^{[\phi]\beta} + \mathbf{\Sigma}_{\alpha}^{[\mathbf{H}]\beta} = \{\Upsilon_2 + \frac{\lambda_{[H]}^2}{4}, \Upsilon_2 + \frac{\lambda_{[H]}^2}{4}, \frac{\lambda_{[H]}^2}{4}, \frac{\lambda_{[H]}^2}{4}\}$$

and the equations (2.98) became more general,

$$G_2^2 = G_3^3 = -S_4^4 = \Upsilon_2\left(x^2, x^2\right) + \frac{\lambda_{[H]}^2}{4} \text{ and } G_4^4 = G_4^4 = -R_2^2 = \frac{\lambda_{[H]}^2}{4}, \tag{2.102}$$

or, in component form

$$R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = -\frac{\lambda_{[H]}^2}{4}, \tag{2.103}$$

$$S_4^4 = S_5^5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)^{*}\right] = -\frac{\lambda_{[H]}^2}{4} + \lambda_{[h]}^2\sin^2\left(\varrho_i x^i + \varphi_{[0]}\right) \tag{2.104}$$

The solution of (2.103) can be found as in the case for (2.82), when $\psi = \ln|g_2| = \ln|g_3|$ is a solution of

$$\ddot{\psi} + \psi'' = -\frac{\lambda_{[H]}^2}{2}, \tag{2.105}$$

where, for simplicity we choose the h–variables $x^2 = \tilde{x}^2$ and $x^3 = \tilde{x}^3$.

The solution of (2.104) can be constructed similarly to the equation (2.99) but for a modified source (see Theorem 2.4.6 and formulas (2.83)–(2.86)). Taking any $h_4 = h_4[\lambda_{[h]} = 0, \lambda_{[H]} = 0]$ and $h_5 = h_5[\lambda_{[h]} = 0, \lambda_{[H]} = 0]$ solving the equation with $\lambda_{[h]} = 0$ and $\lambda_{[H]} = 0$ like in (2.84), we can express the general solution with nontrivial source like

$$h_5[\lambda_{[h]}, \lambda_{[H]}] = h_5,\ h_4[\lambda_{[h]}, \lambda_{[H]}] = \varsigma_{[\lambda,H]}\left(x^i, v\right)h_4,$$

where (for an explicit source from (2.104) in (2.86))

$$\varsigma_{[\lambda,H]}\left(t,x^{3},v\right)=\varsigma_{4[0]}\left(t,x^{3}\right)-\frac{1}{4}\left[\lambda_{[h]}^{2}\sin^{2}\left(\varrho_{2}t+\varrho_{3}x^{3}+\varphi_{[0]}\right)-\frac{\lambda_{[H]}^{2}}{4}\right]\int\frac{h_{4}h_{5}}{h_{5}^{*}}dv,$$

where $\varsigma_{4[0]}\left(t,x^{3}\right)=1$ if we want to have $\varsigma_{[\lambda]}$ for $\lambda_{[h]}^{2},\lambda_{[H]}^{2}\rightarrow 0$.

We define a class of 4D off–diagonal exact solutions of the system (2.60) with $h_{4,5}^{*}\neq 0$ (see the Corollary 2.4.6 with $x^{2}=t$ stated to be the time like coordinate and x^{1} considered as the extra 5th dimensional one to be eliminated for reductions 5D→4D) is parametrized by the generic off–diagonal metric

$$\begin{aligned}\delta s^{2} &= e^{\psi(t,x^{3})}(dt)^{2}-e^{\psi(t,x^{3})}\left(dx^{3}\right)-f^{2}\left(t,x^{3},v\right)\left(\delta y^{5}\right)^{2}\\ &\quad -h_{0}^{2}(t,x^{3})\left[f^{*}\left(t,x^{3},v\right)\right]^{2}|\varsigma_{[\lambda,H]}\left(t,x^{3},v\right)|\,(\delta v)^{2},\\ \delta v &= dv+w_{\widehat{k}}\left(t,x^{3},v\right)dx^{\widehat{k}},\ \delta y^{5}=dy^{5}+n_{\widehat{k}}\left(t,x^{3},v\right)dx^{\widehat{k}},\end{aligned}\qquad(2.106)$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ is a solution of the 2D equation (2.76) for a given source $\Upsilon_{4}\left(x^{\widehat{i}}\right)$,

$$\varsigma_{[\lambda,H]}\left(t,x^{3},v\right)=1+\frac{h_{0}^{2}(t,x^{3})}{16}\left[\lambda_{[h]}^{2}\sin^{2}\left(\varrho_{2}t+\varrho_{3}x^{3}+\varphi_{[0]}\right)-\frac{\lambda_{[H]}^{2}}{4}\right]f^{2}\left(t,x^{3},v\right),$$

and the N–connection coefficients $N_{\widehat{i}}^{4}=w_{\widehat{i}}(t,x^{3},v)$ and $N_{\widehat{i}}^{5}=n_{\widehat{i}}(t,x^{3},v)$ are

$$w_{2,3}=-\frac{\partial_{2,3}\varsigma_{[\lambda,H]}\left(t,x^{3},v\right)}{\varsigma_{[\lambda,H]}^{*}\left(t,x^{3},v\right)}$$

and

$$n_{2,3}\left(t,x^{3},v\right)=n_{2,3[1]}\left(t,x^{3}\right)+n_{2,3[2]}\left(t,x^{3}\right)\int\frac{\left[f^{*}\left(t,x^{3},v\right)\right]^{2}}{\left[f\left(t,x^{3},v\right)\right]^{2}}\varsigma_{[\lambda,H]}\left(t,x^{3},v\right)dv,$$

define an exact 4D solution of the system of generalized Finsler–affine gravity equations (2.60) for vanishing Proca mass $\mu\rightarrow 0$, with holonomic and anholonomic variables, 1-form field

$$\phi_{\mu}=\left[\phi_{\widehat{i}}=\phi_{[0]\widehat{i}}(t,x^{3})\cos\left(\varrho_{2}t+\varrho_{3}x^{3}+\varphi_{[0]}\right),\phi_{4}=0,\phi_{0}=0\right]\qquad(2.107)$$

and nontrivial effective H–field $\mathbf{H}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho}$ for arbitrary nontrivial functions $f\left(t,x^3,v\right)$ (with $f^* \neq 0$), $h_0^2(t,x^3)$, $n_{k[1,2]}\left(t,x^3\right)$ and sources

$$\Upsilon_2\left(t,x^3\right) = \lambda_{[H]}^2/4 - \lambda_{[h]}^2(t,x^3)\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right) \text{ and } \Upsilon_4 = \lambda_{[H]}^2/4$$

and any integration constants to be defined by certain boundary conditions and additional physical arguments. The function $\phi_{[0]\widehat{i}}(t,x^3)$ in (2.107) is taken to solve the equation

$$\partial_2[e^{-\psi\left(t,x^3\right)}\partial_2\phi_k] - \partial_3[e^{-\psi\left(t,x^3\right)}\partial_3\phi_k] = L_{ki}^j\partial^i\phi_j - L_{ij}^i\partial^j\phi_k \tag{2.108}$$

where L_{ki}^j are computed for the d–metric (2.106) following the formulas (2.27). For $\psi = 0$, we obtain just the plane wave equation (2.95) when $\phi_{[0]\widehat{i}}$ and $\lambda_{[h]}^2(t,x^3)$ reduce to constant values. We do not fix here any value of $\psi\left(t,x^3\right)$ solving (2.105) in order to define explicitly a particular solution of (2.108). We note that for any value of $\psi\left(t,x^3\right)$ we can solve the inhomogeneous wave equation (2.108) by using solutions of the homogeneous case.

For simplicity, we do not present here the explicit value of $\sqrt{|\mathbf{g}_{\alpha\beta}|}$ computed for the d–metric (2.106) as well the values for distorsions $\widehat{\mathbf{Z}}\ _{\nu\lambda\rho}$, defined by d–torsions of the canonical d–connection, see formulas (2.53) and (2.29) (the formulas are very cumbersome and do not reflect additional physical properties). Having defined $\widehat{\mathbf{Z}}\ _{\nu\lambda\rho}$, we can compute

$$\widehat{\mathbf{H}}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} - \widehat{\mathbf{Z}}\ _{\nu\lambda\rho}.$$

We note that the torsion $\widehat{\mathbf{T}}^{\nu}{}_{\lambda\rho}$ contained in $\widehat{\mathbf{Z}}\ _{\nu\lambda\rho}$, related to string corrections by the H–field, is different from the torsion $\mathbf{T} = k_2\phi$ and nontrivial nonmetricity $\mathbf{Q} = k_0\phi,\ \mathbf{\Lambda} = k_1\phi$, from the metric–affine part of the theory, see (2.42).

We can choose the function $f\left(t,x^3,v\right)$ from (2.106), or (2.100), as it would be a solution of the Kadomtsev–Petviashvili (KdP) equation [25], i. e. to satisfy

$$f^{\bullet\bullet} + \epsilon\left(f' + 6ff^* + f^{***}\right)^* = 0,\ \epsilon = \pm 1,$$

or, for another locally anisotropic background, to satisfy the $(2+1)$–dimensional sine–Gordon (SG) equation,

$$-f^{\bullet\bullet} + f'' + f^{**} = \sin f,$$

see Refs. [26] on gravitational solitons and theory of solitons. In this case, we define a nonlinear model of gravitational plane wave and 3D solitons in the framework of the MAG with string corrections by H–field. Such solutions generalized those considered in Refs. [7] for 4D and 5D gravity.

We can also consider that $F/L = f^2(t, x^3, v)$ is just the generation function for a 2D model of Finsler/Lagrange geometry (being of any solitonic or another type nature). In this case, the geometric background is characterized by this type locally anisotropic configurations (for Finsler metrics we shall impose corresponding homogeneity conditions on coordinates).

2.6 Final Remarks

In this paper we have investigated the dynamical aspects of metric–affine gravity (MAG) with certain additional string corrections defined by the antisymmetric H–field when the metric structure is generic off–diagonal and the spacetime is provided with an anholonomic frame structure with associated nonlinear connection (N–connection). We analyzed the corresponding class of Lagrangians and derived the field equations of MAG and string gravity with mixed holonomic and anholonomic variables. The main motivation for this work is to determine the place and significance of such models of gravity which contain as exact solutions certain classes of metrics and connections modelling Finsler like geometries even in the limits to the general relativity theory.

The work supports the results of Refs. [7, 8] where various classes of exact solutions in Einstein, Einstein–Cartan, gauge and string gravity modelling Finsler–Lagrange configurations were constructed. We provide an irreducible decomposition techniques (in our case with additional N–connection splitting) and study the dynamics of MAG fields generating the locally anisotropic geometries and interactions classified in Ref. [6]. There are proved the main theorems on irreducible reduction to effective Einstein–Proca equations with string corrections and formulated a new method of constructing exact solutions.

As explicit examples of the new type of locally anisotropic configurations in MAG and string gravity, we have elaborated three new classes of exact solutions depending on 3-4 variables possessing nontrivial torsion and nonmetricity fields, describing plane wave and three dimensional soliton interactions and induced generalized Finsler–affine effective configurations.

Finally, it seems worthwhile to note that such Finsler like configurations do not violates the postulates of the general relativity theory in the corresponding limits to the four dimensional Einstein theory because such metrics transform into exact solutions of this theory. The anisotropies are modelled by certain anholonomic frame constraints on a (pseudo) Riemannian spacetime. In this case the restrictions imposed on physical applications of the Finsler geometry, derived from experimental data on possible limits for brocken local Lorentz invariance (see, for instance, Ref. [27]), do not hold.

2.7 Appendix A: Proof of Theorem 2.4.4

We give some details on straightforward calculations outlined in Ref. [15] for (pseudo) Riemannian and Riemann–Cartan spaces. In brief, the proof of Theorem 2.4.4 is to be performed in this way: Introducing $N_i^4 = w_i$ and $N_i^5 = n_i$ in (2.6) and (2.7) and re–writing (2.63) into a diagonal (in our case) block form (2.11), we compute the h- and v–irreducible components of the canonical d–connection (2.27). The next step is to compute d–curvatures (2.30) and by contracting of indices to define the components of the Ricci d–tensor (2.31) which results in (2.64)–(2.67). We emphasize that such computations can not be performed directly by applying any Tensor, Maple of Mathematica macros because, in our case, we consider canonical d–connections instead of the Levi–Civita connection [23]. We give the details of such calculus related to N–adapted anholonomic frames.

The five dimensional (5D) local coordinates are x^i and $y^a = (v, y)$, i. e. $y^4 = v$, $y^5 = y$, were indices $i, j, k... = 1, 2, 3$ and $a, b, c, ... = 4, 5$. Our reductions to 4D will be considered by excluding dependencies on the variable x^1 and for trivial embedding of 4D off–diagonal ansatz into 5D ones. The signatures of metrics could be arbitrary ones. In general, the spacetime could be with torsion, but we shall always be interested to define the limits to (pseudo) Riemannian spaces.

The d–metric (2.11) for an ansatz (2.63) with $g_1 = const$, is written

$$\begin{aligned}
\delta s^2 &= g_1 (dx^1)^2 + g_2\left((x^2, x^3\right)(dx^2)^2 + g_3\left(x^k\right)(dx^3)^2 \\
&\quad + h_4\left(x^k, v\right)(\delta v)^2 + h_5\left(x^k, v\right)(\delta y)^2, \\
\delta v &= dv + w_i\left(x^k, v\right) dx^i,\ \delta y = dy + n_i\left(x^k, v\right) dx^i
\end{aligned} \tag{2.109}$$

when the generic off–diagonal metric (2.62) is associated to a N–connection structure N_i^a with $N_i^4 = w_i\left(x^k, v\right)$ and $N_i^5 = n_i\left(x^k, v\right)$. We note that the metric (2.109) does not depend on variable $y^5 = y$, but emphasize the dependence on "anisotropic" variable $y^4 = v$.

If we regroup (2.109) with respect to true differentials $du^\alpha = (dx^i, dy^a)$ we obtain just the ansatz (2.63). It is a cumbersome task to perform tensor calculations (for instance, of curvature and Ricci tensors) with such generic off–diagonal ansatz but the formulas simplify substantially with respect to N–adapted frames of type(2.6) and (2.7) and for effectively diagonalized metrics like (2.109).

So, the metric (2.62) transform in a diagonal one with respect to the pentads (frames, funfbeins)

$$e^i = dx^i, e^4 = \delta v = dv + w_i\left(x^k, v\right) dx^i, e^5 = \delta y = dy + n_i\left(x^k, v\right) dx^i \tag{2.110}$$

or

$$\delta u^{\alpha} = \left(dx^i, \delta y^a = dy^a + N_i^a dx^i\right)$$

being dual to the N–elongated partial derivative operators,

$$\begin{aligned} e_1 &= \delta_1 = \frac{\partial}{\partial x^1} - N_1^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^1} - w_1 \frac{\partial}{\partial v} - n_1 \frac{\partial}{\partial y}, \\ e_2 &= \delta_2 = \frac{\partial}{\partial x^2} - N_2^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^2} - w_2 \frac{\partial}{\partial v} - n_2 \frac{\partial}{\partial y}, \\ e_3 &= \delta_3 = \frac{\partial}{\partial x^3} - N_3^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^3} - w_3 \frac{\partial}{\partial v} - n_3 \frac{\partial}{\partial y}, \\ e_4 &= \frac{\partial}{\partial y^4} = \frac{\partial}{\partial v},\ e_5 = \frac{\partial}{\partial y^5} = \frac{\partial}{\partial y} \end{aligned} \tag{2.111}$$

when $\delta_\alpha = \frac{\delta}{\partial u^\alpha} = \left(\frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$.

The N–elongated partial derivatives of a function $f(u^\alpha) = f(x^i, y^a) = f(x, r, v, y)$ are computed in the form when the N–elongated derivatives are

$$\delta_2 f = \frac{\delta f}{\partial u^2} = \frac{\delta f}{\partial x^2} = \frac{\delta f}{\partial x} = \frac{\partial f}{\partial x} - N_2^a \frac{\partial f}{\partial y^a} = \frac{\partial f}{\partial x} - w_2 \frac{\partial f}{\partial v} - n_2 \frac{\partial f}{\partial y} = f^\bullet - w_2\ f' - n_2\ f^*$$

where

$$f^\bullet = \frac{\partial f}{\partial x^2} = \frac{\partial f}{\partial x},\ f' = \frac{\partial f}{\partial x^3} = \frac{\partial f}{\partial r},\ f^* = \frac{\partial f}{\partial y^4} = \frac{\partial f}{\partial v}.$$

The N–elongated differential is

$$\delta f = \frac{\delta f}{\partial u^\alpha} \delta u^\alpha.$$

The N–elongated differential calculus should be applied if we work with respect to N–adapted frames.

2.7.1 Calculation of N–connection curvature

We compute the coefficients (2.5) for the d–metric (2.109) (equivalently, the ansatz (2.63)) defining the curvature of N–connection N_i^a, by substituting $N_i^4 = w_i\left(x^k, v\right)$ and $N_i^5 = n_i\left(x^k, v\right)$, where $i = 2, 3$ and $a = 4, 5$. The result for nontrivial values is

$$\begin{aligned} \Omega_{23}^4 &= -\Omega_{23}^4 = w_2' - w_3^{\cdot} - w_3 w_2^* - w_2 w_3^*, \\ \Omega_{23}^5 &= -\Omega_{23}^5 = n_2' - n_3^{\cdot} - w_3 n_2^* - w_2 n_3^*. \end{aligned} \tag{2.112}$$

The canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$ (2.27) defines the covariant derivative $\widehat{\mathbf{D}}$, satisfying the metricity conditions $\widehat{\mathbf{D}}_{\alpha}\mathbf{g}_{\gamma\delta} = 0$ for $\mathbf{g}_{\gamma\delta}$ being the metric (2.109) with the coefficients written with respect to N–adapted frames. $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ has nontrivial d–torsions.

We compute the Einstein tensors for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ defined by the ansatz (2.109) with respect to N–adapted frames (2.110) and (2.111). This results in exactly integrable vacuum Einstein equations and certain type of sources. Such solutions could be with nontrivial torsion for different classes of linear connections from Riemann–Cartan and generalized Finsler geometries. So, the anholonomic frame method offers certain possibilities to be extended to in string gravity where the torsion could be not zero. But we can always select the limit to Levi–Civita connections, i. e. to (pseudo) Riemannian spaces by considering additional constraints, see Corollary 2.3.3 and/or conditions (2.61).

2.7.2 Calculation of the canonical d–connection

We compute the coefficients (2.27) for the d–metric (2.109) (equivalently, the ansatz (2.63)) when $g_{jk} = \{g_j\}$ and $h_{bc} = \{h_b\}$ are diagonal and g_{ik} depend only on x^2 and x^3 but not on y^a.

We have

$$\begin{aligned} \delta_k g_{ij} &= \partial_k g_{ij} - w_k g^{*}_{ij} = \partial_k g_{ij},\ \delta_k h_b = \partial_k h_b - w_k h^{*}_b \\ \delta_k w_i &= \partial_k w_i - w_k w^{*}_i,\ \delta_k n_i = \partial_k n_i - w_k n^{*}_i \end{aligned} \tag{2.113}$$

resulting in formulas

$$\widehat{L}^{i}_{jk} = \frac{1}{2} g^{ir}\left(\frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right) = \frac{1}{2} g^{ir}\left(\frac{\partial g_{jk}}{\delta x^k} + \frac{\partial g_{kr}}{\delta x^j} - \frac{\partial g_{jk}}{\delta x^r}\right)$$

The nontrivial values of $\widehat{L}^{i}_{jk}$ are

$$\begin{aligned} \widehat{L}^{2}_{22} &= \frac{g^{\bullet}_{2}}{2g_2} = \alpha^{\bullet}_{2},\ \widehat{L}^{2}_{23} = \frac{g^{\prime}_{2}}{2g_2} = \alpha^{\prime}_{2},\ \widehat{L}^{2}_{33} = -\frac{g^{\bullet}_{3}}{2g_2} \\ \widehat{L}^{3}_{22} &= -\frac{g^{\prime}_{2}}{2g_3},\ \widehat{L}^{3}_{23} = \frac{g^{\bullet}_{3}}{2g_3} = \alpha^{\bullet}_{3},\ \ \widehat{L}^{3}_{33} = \frac{g^{\prime}_{3}}{2g_3} = \alpha^{\prime}_{3}. \end{aligned} \tag{2.114}$$

In a similar form we compute the components

$$\widehat{L}^{a}_{bk} = \partial_b N^{a}_{k} + \frac{1}{2} h^{ac}\left(\partial_k h_{bc} - N^{d}_{k}\frac{\partial h_{bc}}{\partial y^d} - h_{dc}\partial_b N^{d}_{k} - h_{db}\partial_c N^{d}_{k}\right)$$

having nontrivial values

$$\widehat{L}^4_{42} = \frac{1}{2h_4}\left(h_4^\bullet - w_2 h_4^*\right) = \delta_2 \ln\sqrt{|h_4|} \doteqdot \delta_2\beta_4, \quad (2.115)$$
$$\widehat{L}^4_{43} = \frac{1}{2h_4}\left(h_4^{'} - w_3 h_4^*\right) = \delta_3 \ln\sqrt{|h_4|} \doteqdot \delta_3\beta_4$$

$$\widehat{L}^4_{5k} = -\frac{h_5}{2h_4} n_k^*,\ \widehat{L}^5_{bk} = \partial_b n_k + \frac{1}{2h_5}\left(\partial_k h_{b5} - w_k h_{b5}^* - h_5 \partial_b n_k\right), \quad (2.116)$$

$$\widehat{L}^5_{4k} = n_k^* + \frac{1}{2h_5}\left(-h_5 n_k^*\right) = \frac{1}{2} n_k^*, \quad (2.117)$$
$$\widehat{L}^5_{5k} = \frac{1}{2h_5}\left(\partial_k h_5 - w_k h_5^*\right) = \delta_k \ln\sqrt{|h_4|} = \delta_k\beta_4.$$

We note that

$$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik}\frac{\partial g_{jk}}{\partial y^c} \doteqdot 0 \quad (2.118)$$

because $g_{jk} = g_{jk}(x^i)$ for the considered ansatz.

The values

$$\widehat{C}^a_{bc} = \frac{1}{2} h^{ad}\left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right)$$

for $h_{bd} = [h_4, h_5]$ from the ansatz (2.63) have nontrivial components

$$\widehat{C}^4_{44} = \frac{h_4^*}{2h_4} \doteqdot \beta_4^*, \widehat{C}^4_{55} = -\frac{h_5^*}{2h_4}, \widehat{C}^5_{45} = \frac{h_5^*}{2h_5} \doteqdot \beta_5^*. \quad (2.119)$$

The set of formulas (2.114)–(2.119) define the nontrivial coefficients of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\right)$ (2.27) for the 5D ansatz (2.109).

2.7.3 Calculation of torsion coefficients

We should put the nontrivial values (2.114)– (2.119) into the formulas for d–torsion (2.29).

One holds $T^i_{.jk} = 0$ and $T^a_{.bc} = 0$, because of symmetry of coefficients L^i_{jk} and C^a_{bc}.

We have computed the nontrivial values of $\Omega^a_{.ji}$, see (2.112) resulting in

$$T^4_{23} = \Omega^4_{23} = -\Omega^4_{23} = w_2' - w_3^\bullet - w_3 w_2^* - w_2 w_3^*, \quad (2.120)$$
$$T^5_{23} = \Omega^5_{23} = -\Omega^5_{23} = n_2' - n_3^\bullet - w_3 n_2^* - w_2 n_3^*.$$

One follows

$$T^i_{ja} = -T^i_{aj} = C^i_{.ja} = \widehat{C}^i_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} \doteqdot 0,$$

see (2.118).

For the components

$$T^a_{.bi} = -T^a_{.ib} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj},$$

i. e. for

$$\widehat{P}^4_{.bi} = \frac{\partial N^4_i}{\partial y^b} - \widehat{L}^4_{.bj} = \partial_b w_i - \widehat{L}^4_{.bj} \text{ and } \widehat{P}^5_{.bi} = \frac{\partial N^5_i}{\partial y^b} - \widehat{L}^5_{.bj} = \partial_b n_i - \widehat{L}^5_{.bj},$$

we have the nontrivial values

$$\begin{aligned} \widehat{P}^4_{.4i} &= w^*_i - \frac{1}{2h_4}\left(\partial_i h_4 - w_i h^*_4\right) = w^*_i - \delta_i \beta_4,\ \widehat{P}^4_{.5i} = \frac{h_5}{2h_4} n^*_i, \\ \widehat{P}^5_{.4i} &= \frac{1}{2} n^*_i,\ \widehat{P}^5_{.5i} = -\frac{1}{2h_5}\left(\partial_i h_5 - w_i h^*_5\right) = -\delta_i \beta_5. \end{aligned} \tag{2.121}$$

The formulas (2.120) and (2.121) state the nontrivial coefficients of the canonical d–connection for the chosen ansatz (2.109).

2.7.4 Calculation of the Ricci tensor

Let us compute the value $R_{ij} = R^k{}_{ijk}$ as in (2.31) for

$$R^i{}_{hjk} = \frac{\delta L^i_{.hj}}{\delta x^k} - \frac{\delta L^i_{.hk}}{\delta x^j} + L^m_{.hj} L^i_{mk} - L^m_{.hk} L^i_{mj} - C^i_{.ha} \Omega^a_{.jk},$$

from (2.30). It should be noted that $C^i_{.ha} = 0$ for the ansatz under consideration, see (2.118). We compute

$$\frac{\delta L^i_{.hj}}{\delta x^k} = \partial_k L^i_{.hj} + N^a_k \partial_a L^i_{.hj} = \partial_k L^i_{.hj} + w_k \left(L^i_{.hj}\right)^* = \partial_k L^i_{.hj}$$

because $L^i_{.hj}$ do not depend on variable $y^4 = v$.

Derivating (2.114), we obtain

$$\begin{aligned}
\partial_2 L^2{}_{22} &= \frac{g_2^{\bullet\bullet}}{2g_2} - \frac{(g_2^{\bullet})^2}{2\,(g_2)^2},\ \partial_2 L^2{}_{23} = \frac{g_2^{\bullet\prime}}{2g_2} - \frac{g_2^{\bullet} g_2^{\prime}}{2\,(g_2)^2},\ \partial_2 L^2{}_{33} = -\frac{g_3^{\bullet\bullet}}{2g_2} + \frac{g_2^{\bullet} g_3^{\bullet}}{2\,(g_2)^2}, \\
\partial_2 L^3{}_{22} &= -\frac{g_2^{\bullet\prime}}{2g_3} + \frac{g_2^{\bullet} g_3^{\prime}}{2\,(g_3)^2}, \partial_2 L^3{}_{23} = \frac{g_3^{\bullet\bullet}}{2g_3} - \frac{(g_3^{\bullet})^2}{2\,(g_3)^2},\ \partial_2 L^3{}_{33} = \frac{g_3^{\bullet\prime}}{2g_3} - \frac{g_3^{\bullet} g_3^{\prime}}{2\,(g_3)^2}, \\
\partial_3 L^2{}_{22} &= \frac{g_2^{\bullet\prime}}{2g_2} - \frac{g_2^{\bullet} g_2^{\prime}}{2\,(g_2)^2}, \partial_3 L^2{}_{23} = \frac{g_2^{ll}}{2g_2} - \frac{\left(g_2^{l}\right)^2}{2\,(g_2)^2}, \partial_3 L^2{}_{33} = -\frac{g_3^{\bullet\prime}}{2g_2} + \frac{g_3^{\bullet} g_2^{\prime}}{2\,(g_2)^2}, \\
\partial_3 L^3{}_{22} &= -\frac{g_2^{\prime\prime}}{2g_3} + \frac{g_2^{\bullet} g_2^{\prime}}{2\,(g_3)^2}, \partial_3 L^3{}_{23} = \frac{g_3^{\bullet\prime}}{2g_3} - \frac{g_3^{\bullet} g_3^{\prime}}{2\,(g_3)^2}, \partial_3 L^3{}_{33} = \frac{g_3^{ll}}{2g_3} - \frac{\left(g_3^{l}\right)^2}{2\,(g_3)^2}.
\end{aligned}$$

For these values and (2.114), there are only 2 nontrivial components,

$$\begin{aligned}
R^2{}_{323} &= \frac{g_3^{\bullet\bullet}}{2g_2} - \frac{g_2^{\bullet} g_3^{\bullet}}{4\,(g_2)^2} - \frac{(g_3^{\bullet})^2}{4g_2 g_3} + \frac{g_2^{ll}}{2g_2} - \frac{g_2^{l} g_3^{l}}{4g_2 g_3} - \frac{\left(g_2^{l}\right)^2}{4\,(g_2)^2} \\
R^3{}_{223} &= -\frac{g_3^{\bullet\bullet}}{2g_3} + \frac{g_2^{\bullet} g_3^{\bullet}}{4g_2 g_3} + \frac{(g_3^{\bullet})^2}{4(g_3)^2} - \frac{g_2^{ll}}{2g_3} + \frac{g_2^{l} g_3^{l}}{4(g_3)^2} + \frac{\left(g_2^{l}\right)^2}{4g_2 g_3}
\end{aligned}$$

with

$$R_{22} = -R^3{}_{223} \text{ and } R_{33} = R^2{}_{323},$$

or

$$R_2^2 = R_3^3 = -\frac{1}{2g_2 g_3}\left[g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{\prime\prime} - \frac{g_2^{l} g_3^{l}}{2g_3} - \frac{\left(g_2^{l}\right)^2}{2g_2}\right]$$

which is (2.64).

Now, we consider

$$\begin{aligned}
P^c{}_{bka} &= \frac{\partial L^c_{.bk}}{\partial y^a} - \left(\frac{\partial C^c_{.ba}}{\partial x^k} + L^c_{.dk} C^d_{.ba} - L^d_{.bk} C^c_{.da} - L^d_{.ak} C^c_{.bd}\right) + C^c_{.bd} P^d_{.ka} \\
&= \frac{\partial L^c_{.bk}}{\partial y^a} - C^c_{.ba|k} + C^c_{.bd} P^d_{.ka}
\end{aligned}$$

from (2.30). Contracting indices, we have

$$R_{bk} = P^a{}_{bka} = \frac{\partial L^a_{.bk}}{\partial y^a} - C^a_{.ba|k} + C^a_{.bd} P^d_{.ka}$$

Let us denote $C_b = C^c_{.ba}$ and write

$$C_{.b|k} = \delta_k C_b - L^d{}_{bk} C_d = \partial_k C_b - N^e_k \partial_e C_b - L^d{}_{bk} C_d = \partial_k C_b - w_k C^*_b - L^d{}_{bk} C_d.$$

We express

$$R_{bk} = {}_{[1]}R_{bk} + {}_{[2]}R_{bk} + {}_{[3]}R_{bk}$$

where

$$\begin{aligned} {}_{[1]}R_{bk} &= \left(L^4_{bk}\right)^*, \\ {}_{[2]}R_{bk} &= -\partial_k C_b + w_k C^*_b + L^d{}_{bk} C_d, \\ {}_{[3]}R_{bk} &= C^a_{.bd} P^d_{.ka} = C^4_{.b4} P^4_{.k4} + C^4_{.b5} P^5_{.k4} + C^5_{.b4} P^4_{.k5} + C^5_{.b5} P^5_{.k5} \end{aligned}$$

and

$$\begin{aligned} C_4 &= C^4_{44} + C^5_{45} = \frac{h^*_4}{2h_4} + \frac{h^*_5}{2h_5} = \beta^*_4 + \beta^*_5, \qquad (2.122) \\ C_5 &= C^4_{54} + C^5_{55} = 0 \end{aligned}$$

see(2.119) .

We compute

$$R_{4k} = {}_{[1]}R_{4k} + {}_{[2]}R_{4k} + {}_{[3]}R_{4k}$$

with

$$\begin{aligned} {}_{[1]}R_{4k} &= \left(L^4_{4k}\right)^* = (\delta_k \beta_4)^* \\ {}_{[2]}R_{4k} &= -\partial_k C_4 + w_k C^*_4 + L^4{}_{4k} C_4, L^4{}_{4k} = \delta_k \beta_4 \text{ see (2.115} \\ &= -\partial_k \left(\beta^*_4 + \beta^*_5\right) + w_k \left(\beta^*_4 + \beta^*_5\right)^* + L^4{}_{4k} \left(\beta^*_4 + \beta^*_5\right) \\ {}_{[3]}R_{4k} &= C^4_{.44} P^4_{.k4} + C^4_{.45} P^5_{.k4} + C^5_{.44} P^4_{.k5} + C^5_{.45} P^5_{.k5} \\ &= \beta^*_4 \left(w^*_k - \delta_k \beta_4\right) - \beta^*_5 \delta_k \beta_5 \end{aligned}$$

Summarizing, we get

$$R_{4k} = w_k \left[\beta^{**}_5 + \left(\beta^*_5\right)^2 - \beta^*_4 \beta^*_5\right] + \beta^*_5 \partial_k \left(\beta_4 + \beta_5\right) - \partial_k \beta^*_5$$

or, for

$$\beta^*_4 = \frac{h^*_4}{2h_4}, \partial_k \beta_4 = \frac{\partial_k h_4}{2h_4}, \beta^*_5 = \frac{h^*_5}{2h_5}, \beta^{**}_5 = \frac{h^{**}_5 h_5 - \left(h^*_5\right)^2}{2\left(h_5\right)^5},$$

we can write

$$2h_5 R_{4k} = w_k \left[h^{**}_5 - \frac{\left(h^*_5\right)^2}{2h_5} - \frac{h^*_4 h^*_5}{2h_4}\right] + \frac{h^*_5}{2} \left(\frac{\partial_k h_4}{h_4} + \frac{\partial_k h_5}{h_5}\right) - \partial_k h^*_5$$

which is equivalent to (2.66)

In a similar way, we compute

$$R_{5k} = {}_{[1]}R_{5k} + {}_{[2]}R_{5k} + {}_{[3]}R_{5k}$$

with

$$\begin{aligned}
{}_{[1]}R_{5k} &= \left(L^4_{5k}\right)^*, \\
{}_{[2]}R_{5k} &= -\partial_k C_5 + w_k C_5^* + L^4{}_{5k} C_4, \\
{}_{[3]}R_{5k} &= C^4_{.54} P^4_{.k4} + C^4_{.55} P^5_{.k4} + C^5_{.54} P^4_{.k5} + C^5_{.55} P^5_{.k5}.
\end{aligned}$$

We have

$$\begin{aligned}
R_{5k} &= \left(L^4_{5k}\right)^* + L^4{}_{5k} C_4 + C^4_{.55} P^5_{.k4} + C^5_{.54} P^4_{.k5} \\
&= \left(-\frac{h_5}{h_4} n_k^*\right)^* - \frac{h_5}{h_4} n_k^* \left(\frac{h_4^*}{2h_4} + \frac{h_5^*}{2h_5}\right) + \frac{h_5^*}{2h_5}\frac{h_5}{2h_4} n_k^* - \frac{h_5^*}{2h_4}\frac{1}{2} n_k^*
\end{aligned}$$

which can be written

$$2h_4 R_{5k} = h_5 n_k^{**} + \left(\frac{h_5}{h_4} h_4^* - \frac{3}{2} h_5^*\right) n_k^*$$

i. e. (2.67)

For the values

$$P^i{}_{jka} = \frac{\partial L^i_{.jk}}{\partial y^k} - \left(\frac{\partial C^i_{.ja}}{\partial x^k} + L^i_{.lk} C^l_{.ja} - L^l_{.jk} C^i_{.la} - L^c_{.ak} C^i_{.jc}\right) + C^i_{.jb} P^b_{.ka}$$

from (2.30), we obtain zeros because $C^i_{.jb} = 0$ and $L^i_{.jk}$ do not depend on y^k. So,

$$R_{ja} = P^i{}_{jia} = 0.$$

Taking

$$S^a{}_{bcd} = \frac{\partial C^a_{.bc}}{\partial y^d} - \frac{\partial C^a_{.bd}}{\partial y^c} + C^e_{.bc} C^a_{.ed} - C^e_{.bd} C^a_{.ec}.$$

from (2.30) and contracting the indices in order to obtain the Ricci coefficients,

$$R_{bc} = \frac{\partial C^d_{.bc}}{\partial y^d} - \frac{\partial C^d_{.bd}}{\partial y^c} + C^e_{.bc} C^d_{.ed} - C^e_{.bd} C^d_{.ec}$$

with $C^d_{.bd} = C_b$ already computed, see (2.122), we obtain

$$R_{bc} = \left(C^4_{.bc}\right)^* - \partial_c C_b + C^4_{.bc} C_4 - C^4_{.b4} C^4_{.4c} - C^4_{.b5} C^5_{.4c} - C^5_{.b4} C^4_{.5c} - C^5_{.b5} C^5_{.5c}.$$

There are nontrivial values,

$$\begin{aligned} R_{44} &= \left(C^4_{.44}\right)^* - C^*_4 + C^4_{44}(C_4 - C^4_{44}) - \left(C^5_{.45}\right)^2 \\ &= \beta^{**}_4 - (\beta^*_4 + \beta^*_5)^* + \beta^*_4\left(\beta^*_4 + \beta^*_5 - \beta^*_4\right) - (\beta^*_5)^* \\ R_{55} &= \left(C^4_{.55}\right)^* - C^4_{.55}\left(-C_4 + 2C^5_{.45}\right) \\ &= -\left(\frac{h^*_5}{2h_4}\right)^* + \frac{h^*_5}{2h_4}\left(2\beta^*_5 + \beta^*_4 - \beta^*_5\right) \end{aligned}$$

Introducing

$$\beta^*_4 = \frac{h^*_4}{2h_4}, \beta^*_5 = \frac{h^*_5}{2h_5}$$

we get

$$R^4_4 = R^5_5 = \frac{1}{2h_4h_5}\left[-h^{**}_5 + \frac{\left(h^*_5\right)^2}{2h_5} + \frac{h^*_4h^*_5}{2h_4}\right]$$

which is just (2.65).

Theorem 2.4.4 is proven.

2.8 Appendix B: Reductions from 5D to 4D

To construct a $5D \to 4D$ reduction for the ansatz (2.63) and (2.70) is to eliminate from formulas the variable x^1 and to consider a 4D space (parametrized by local coordinates (x^2, x^3, v, y^5)) being trivially embedded into 5D space (parametrized by local coordinates (x^1, x^2, x^3, v, y^5) with $g_{11} = \pm 1, g_{1\widehat{\alpha}} = 0, \widehat{\alpha} = 2,3,4,5)$ with possible 4D conformal and anholonomic transforms depending only on variables (x^2, x^3, v). We suppose that the 4D metric $g_{\widehat{\alpha}\widehat{\beta}}$ could be of arbitrary signature. In order to emphasize that some coordinates are stated just for a such 4D space we put "hats" on the Greek indices, $\widehat{\alpha}, \widehat{\beta}, ...$ and on the Latin indices from the middle of alphabet, $\widehat{i}, \widehat{j}, ... = 2, 3$, where $u^{\widehat{\alpha}} = \left(x^{\widehat{i}}, y^a\right) = (x^2, x^3, y^4, y^5)$.

In result, the Theorems 2.4.4 and 2.4.5, Corollaries 2.4.4 and 2.4.5 and Theorem 2.4.6 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of a such reduction.

- The metric (2.62) with ansatz (2.63) and metric (2.69) with (2.70) are respectively transformed on 4D spaces to the values:

  The first type 4D off–diagonal metric is taken

$$\mathbf{g} = \mathbf{g}_{\widehat{\alpha}\widehat{\beta}}\left(x^{\widehat{i}}, v\right) du^{\widehat{\alpha}} \otimes du^{\widehat{\beta}} \qquad (2.123)$$

with the metric coefficients $g_{\widehat{\alpha}\widehat{\beta}}$ parametrized

$$\left[\begin{array}{cccc} g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 \end{array} \right], \qquad (2.124)$$

where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g_{2,3} &= g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^{\widehat{k}}, v), \\ w_{\widehat{i}} &= w_{\widehat{i}}(x^{\widehat{k}}, v), n_{\widehat{i}} = n_{\widehat{i}}(x^{\widehat{k}}, v);\ \widehat{i}, \widehat{k} = 2, 3. \end{aligned}$$

The anholonomically and conformally transformed 4D off–diagonal metric is

$$\mathbf{g} = \omega^2(x^{\widehat{i}}, v) \hat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}} \left(x^{\widehat{i}}, v\right) du^{\widehat{\alpha}} \otimes du^{\widehat{\beta}}, \qquad (2.125)$$

were the coefficients $\hat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}}$ are parametrized by the ansatz

$$\left[\begin{array}{cccc} g_2 + ({w_2}^2 + {\zeta_2}^2) h_4 + {n_2}^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\ (w_2 w_3 + + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & g_3 + ({w_3}^2 + {\zeta_3}^2) h_4 + {n_3}^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\ (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4 \end{array} \right] \qquad (2.126)$$

where $\zeta_{\widehat{i}} = \zeta_{\widehat{i}} \left(x^{\widehat{k}}, v\right)$ and we shall restrict our considerations for $\zeta_5 = 0$.

- We obtain a quadratic line element

$$\delta s^2 = g_2 (dx^2)^2 + g_3 (dx^3)^2 + h_4 (\delta v)^2 + h_5 (\delta y^5)^2, \qquad (2.127)$$

written with respect to the anholonomic co–frame $\left(dx^{\widehat{i}}, \delta v, \delta y^5\right)$, where

$$\delta v = dv + w_{\widehat{i}} dx^{\widehat{i}} \text{ and } \delta y^5 = dy^5 + n_{\widehat{i}} dx^{\widehat{i}} \qquad (2.128)$$

is the dual of $(\delta_{\widehat{i}}, \partial_4, \partial_5)$, where

$$\delta_{\widehat{i}} = \partial_{\widehat{i}} + w_{\widehat{i}} \partial_4 + n_{\widehat{i}} \partial_5. \qquad (2.129)$$

- If the conditions of the 4D variant of the Theorem 2.4.4 are satisfied, we have the same equations (2.76) –(2.79) were we substitute $h_4 = h_4\left(x^{\widehat{k}}, v\right)$ and $h_5 = h_5\left(x^{\widehat{k}}, v\right)$. As a consequence we have $\alpha_i\left(x^k, v\right) \to \alpha_{\widehat{i}}\left(x^{\widehat{k}}, v\right), \beta = \beta\left(x^{\widehat{k}}, v\right)$ and $\gamma = \gamma\left(x^{\widehat{k}}, v\right)$ resulting in $w_{\widehat{i}} = w_{\widehat{i}}\left(x^{\widehat{k}}, v\right)$ and $n_{\widehat{i}} = n_{\widehat{i}}\left(x^{\widehat{k}}, v\right)$.

- The 4D line element with conformal factor (2.127) subjected to an anhlonomic map with $\zeta_5 = 0$ transforms into

$$\delta s^2 = \omega^2(x^{\widehat{i}}, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\delta y^5)^2], \tag{2.130}$$

given with respect to the anholonomic co–frame $\left(dx^{\widehat{i}}, \hat{\delta} v, \delta y^5\right)$, where

$$\delta v = dv + (w_{\widehat{i}} + \zeta_{\widehat{i}})dx^{\widehat{i}} \text{ and } \delta y^5 = dy^5 + n_{\widehat{i}} dx^{\widehat{i}} \tag{2.131}$$

is dual to the frame $\left(\hat{\delta}_{\widehat{i}}, \partial_4, \hat{\partial}_5\right)$ with

$$\hat{\delta}_{\widehat{i}} = \partial_{\widehat{i}} - (w_{\widehat{i}} + \zeta_{\widehat{i}})\partial_4 + n_{\widehat{i}}\partial_5, \hat{\partial}_5 = \partial_5. \tag{2.132}$$

- The formulas (2.71) and (2.73) from Theorem 2.4.5 must be modified into a 4D form

$$\hat{\delta}_{\widehat{i}} h_4 = 0 \text{ and } \hat{\delta}_{\widehat{i}} \omega = 0 \tag{2.133}$$

and the values $\zeta_{\widehat{i}} = (\widehat{\zeta_i}, \zeta_5 = 0)$ are found as to be a unique solution of (2.71); for instance, if

$$\omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_{\widehat{i}}$ satisfy the equations

$$\partial_{\widehat{i}}\omega - (w_{\widehat{i}} + \zeta_{\widehat{i}})\omega^* = 0. \tag{2.134}$$

- One holds the same formulas (2.83)-(2.88) from the Theorem 2.4.6 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\widehat{i} \to i$, and holonomic coordinates, $x^i \to x^{\widehat{i}}$, i. e. in the 4D solutions there is not contained the variable x^1.

- The formulae (2.74) for the nontrivial coefficients of the Einstein tensor in 4D stated by the Corollary 2.4.4 are written

$$G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2. \tag{2.135}$$

- For symmetries of the Einstein tensor (2.135), we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 2.4.5 by the conditions (2.75), which in 4D are transformed into

$$\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v), \ \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^2, x^3). \tag{2.136}$$

The 4D dimensional off–diagonal ansatz may model certain generalized Lagrange configurations and Lagrange–affine solutions. They can also include certain 3D Finsler or Lagrange metrics but with 2D frame transforms of the corresponding quadratic forms and N–connections.

2.9 Appendix C: Generalized Lagrange–Affine Spaces

We outline and give a brief characterization of five classes of generalized Finsler–affine spaces (contained in the Table 1 from Ref. [6]; see also in that work the details on classification of such geometries). We note that the N–connection curvature is computed following the formula $\Omega_{ij}^a = \delta_{[i}N_{j]}^a$, see (2.5), for any N–connection N_i^a. A d–connection $\mathbf{D} = [\mathbf{\Gamma}_{\beta\gamma}^\alpha] = [L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}]$ defines nontrivial d–torsions $\mathbf{T}^\alpha_{\ \beta\gamma} = [L^i_{[\ jk]}, C^i_{\ ja}, \Omega^a_{ij}, T^a_{\ bj}, C^a_{\ [bc]}]$ and d–curvatures $\mathbf{R}^\alpha_{\ \beta\gamma\tau} = [R^i_{\ jkl}, R^a_{\ bkl}, P^i_{\ jka}, P^c_{\ bka}, S^i_{\ jbc}, S^a_{\ dbc}]$ adapted to the N–connection structure (see, respectively, the formulas (2.29) and (2.30)). Any generic off–diagonal metric $g_{\alpha\beta}$ is associated to a N–connection structure and represented as a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ (see formula (2.11)). The components of a N–connection and a d–metric define the canonical d–connection $\mathbf{D} = [\widehat{\mathbf{\Gamma}}_{\beta\gamma}^\alpha] =$ $[\widehat{L}^i_{\ jk}, \widehat{L}^a_{\ bk}, \widehat{C}^i_{\ jc}, \widehat{C}^a_{\ bc}]$ (see (2.27)) with the corresponding values of d–torsions $\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}$ and d–curvatures $\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\tau}$. The nonmetricity d–fields are computed by using formula $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_\alpha \mathbf{g}_{\beta\gamma} = [Q_{ijk}, Q_{iab}, Q_{ajk}, Q_{abc}]$, see (2.13).

The Table 2.1 outlines five classes of geometries modelled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure (for simplicity, we omitted the Berwald configurations, see Ref. [6]).

1. Metric–affine spaces (in brief, MA) are those stated as certain manifolds V^{n+m} of necessary smoothly class provided with arbitrary metric, $g_{\alpha\beta}$, and linear connection, $\Gamma^\alpha_{\beta\gamma}$, structures. For generic off–diagonal metrics, a MA space always admits nontrivial N–connection structures. Nevertheless, in general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^\alpha_{\beta\gamma}$ can be not

adapted to the N–connection structure. As a consequence, the general strength fields $\left(T^{\alpha}_{\beta\gamma}, R^{\alpha}{}_{\beta\gamma\tau}, Q_{\alpha\beta\gamma}\right)$ can be also not N–adapted.

2. Distinguished metric–affine spaces (DMA) are defined as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N_i^a, d–metric field (2.11) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$. In this case, all strengths $\left(\mathbf{T}^{\alpha}{}_{\beta\gamma}, \mathbf{R}^{\alpha}{}_{\beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma}\right)$ are N–adapted.

3. Generalized Lagrange–affine spaces (GLA), $\mathbf{GLa}^n = (V^n, g_{ij}(x,y), \ ^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are modelled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ and the d–connection $\ ^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(\ ^{[a]}L^i_{jk}, \ ^{[a]}C^i_{jc}\right)$ are similar to those for the usual Lagrange spaces but with distorsions $\ ^{[a]}\ \mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields $^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$.

4. Lagrange–affine spaces (LA), $\mathbf{La}^n = (V^n, g^{[L]}_{ij}(x,y), \ ^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ inducing the canonical N–connection structure $^{[cL]}\mathbf{N} = \{\ ^{[cL]}N^i_j\}$ for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ but with a d–connection structure $\ ^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \ ^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when $^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \ ^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + \ ^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta}$. This is a particular case of GLA spaces with prescribed types of N–connection $^{[cL]}N^i_j$ and d–metric to be like in Lagrange geometry.

5. Finsler–affine spaces (FA), $\mathbf{Fa}^n = \left(V^n, F\left(x,y\right), \ ^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$, in their turn are introduced by further restrictions of $\mathbf{La}^n$ to a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ constructed from a Finsler metric $F\left(x^i, y^j\right)$. It is induced the canonical N–connection structure $\ ^{[F]}\mathbf{N} = \{\ ^{[F]}N^i_j\}$ as in the Finsler space $\mathbf{F}^n = (V^n, F\left(x,y\right))$ but with a d–connection structure $\ ^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, $\ ^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \ ^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + \ ^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta}$,where $\ ^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection.

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta}=[g_{ij},h_{ab}]$	$\Gamma_{\beta\gamma}^{\alpha}$ $T^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma}$
2. DMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta}=[g_{ij},h_{ab}]$	$\mathbf{\Gamma}_{\beta\gamma}^{\alpha}$ $\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma}$
3. GLA	$\dim i=\dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]}=[g_{ij},h_{kl}]$	${}^{[a]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$
4. LA	$\dim i=\dim a$ ${}^{[cL]}N_j^i,\ {}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[b]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[b]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma}=-\,{}^{[b]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[L]}$
5. FA	$\dim i=\dim a$ ${}^{[F]}N_j^i;\ {}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[f]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[f]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[f]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma}=-\,{}^{[f]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[F]}$

Table 2.1: Generalized Finsler/Lagrange–affine spaces

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Chapter 3

Noncommutative Symmetries and Stability of Black Ellipsoids in Metric–Affine and String Gravity

Abstract [1]

We construct new classes of exact solutions in metric–affine gravity (MAG) with string corrections by the antisymmetric H–field. The solutions are parametrized by generic off–diagonal metrics possessing noncommutative symmetry associated to anholonomy frame relations and related nonlinear connection (N–connection) structure. We analyze the horizon and geodesic properties of a class of off–diagonal metrics with deformed spherical symmetries. The maximal analytic extension of ellipsoid type metrics are constructed and the Penrose diagrams are analyzed with respect to adapted frames. We prove that for small deformations (small eccentricities) there are such metrics that the geodesic behaviour is similar to the Schwarzcshild one. We conclude that some static and stationary ellipsoid configurations may describe black ellipsoid objects. The new class of spacetimes do not possess Killing symmetries even in the limits to the general relativity and, in consequence, they are not prohibited by black hole uniqueness theorems. Such static ellipsoid (rotoid) configurations are compatible with the cosmic cenzorship criteria. We study the perturbations of two classes of static black ellipsoid solutions of four dimensional gravitational field equations. The analysis is performed in the approximation of small eccentricity deformations of the Schwarzschild solution. We conclude that such anisotropic black hole objects may be stable with respect to

the perturbations parametrized by the Schrodinger equations in the framework of the one–dimensional inverse scattering theory. We emphasize that the anholonomic frame method of generating exact solutions is a general one for off–diagonal metrics (and linear and nonlinear connections) depending on 2-3 variables, in various types of gravity theories.

3.1 Introduction

In the past much effort has been made to construct and investigate exact solutions of gravitational field equations with spherical/cylindrical symmetries and/or with time dependence, paramertrized by metrics diagonalizable by certain coordinate transforms. Recently, the off–diagonal metrics were considered in a new manner by diagonalizing them with respect to anholonomic frames with associated nonlinear connection structure [1, 2, 3, 4]. There were constructed new classes of exact solutions of Einstein's equations in three (3D), four (4D) and five (5D) dimensions. Such solutions posses a generic geometric local anisotropy (*e.g.* static black hole and/or cosmological solutions with ellipsoidal or toroidal symmetry, various soliton–dilaton 2D and 3D configurations in 4D gravity, and wormholes and flux tubes with anisotropic polarizations and/or running constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy).

A number of ansatz with off–diagonal metrics were investigated in higher dimensional gravity (see, for instance, the Salam, Strathee, Percacci and Randjbar–Daemi works [5]) which showed that off–diagonal components in higher dimensional metrics are equivalent to including $U(1), SU(2)$ and $SU(3)$ gauge fields. There are various generalizations of the Kaluza–Klein gravity when the compactifications of off–diagonal metrics are considered with the aim to reduce the vacuum 5D gravity to effective Einstein gravity and Abelian or non–Abelian gauge theories. There were also constructed 4D exact solutions of Einstein equations with matter fields and cosmological constants like black torus and black strings induced from some 3D black hole configurations by considering 4D off–diagonal metrics whose curvature scalar splits equivalently into a curvature term for a diagonal metric together with a cosmological constant term and/or a Lagrangian for gauge (electromagnetic) field [6].

We can model certain effective (diagonal metric) gravitational and matter fields interactions for some particular off–diagonal metric ansatz and redefinitions of Lagrangians. However, in general, the vacuum gravitational dynamics can not be associated to any matter field contributions. This holds true even if we consider non–Riemanian generalizations from string and/or metric–affine gravity (MAG) [7]. In this work (being the third partner of the papers [8, 9]), we prove that such solutions are not with usual Killing symmetries but admit certain anholonomic noncommutative symmetries and preserve such properties if the constructions are extended to MAG and string gravity (see also [10] for extensions to complex and/or noncommutative gravity).

There are constructed the maximal analytic extension of a class of static metrics with deformed spherical symmetry (containing as particular cases ellipsoid configurations). We analyze the Penrose diagrams and compare the results with those for the Reissner–Nordstrom solution. Then we state the conditions when the geodesic congruence with 'ellipsoid' type symmetry can be reduced to the Schwarzschild configuration. We argue that in this case we may generate some static black ellipsoid solutions which, for corresponding parametrizations of off–diagonal metric coefficients, far away from the horizon, satisfy the asymptotic conditions of the Minkowski spacetime.

For the new classes of "off–diagonal" spacetimes possessing noncommutative symmetries, we extend the methods elaborated to investigate the perturbations and stability of black hole metrics. The theory of perturbations of the Schwarzschild spacetime black holes was initiated in Ref. [11], developed in a series of works, e. g. Refs [12, 13], and related [14] to the theory of inverse scattering and its ramifications (see, for instance, Refs. [15]). The results on the theory of perturbations and stability of the Schwarzschild, Reissner–Nordstrom and Kerr solutions are summarized in a monograph [16]. As alternative treatments of the stability of black holes we cite Ref. [17].

Our first aim is to investigate such off–diagonal gravitational configurations in MAG and string gravity (defined by anholonomic frames with associated nonlinear connection structure) which describe black hole solutions with deformed horizons, for instance, with a static ellipsoid hypersurface. The second aim is to study perturbations of black ellipsoids and to prove that there are such static ellipsoid like configurations which are stable with respect to perturbations of a fixed type of anisotropy (i. e. for certain imposed anholonomic constraints). The main idea of a such proof is to consider small (ellipsoidal, or another type) deformations of the Schwarzschild metric and than to apply the already developed methods of the theory of perturbations of classical black hole solutions, with a re–definition of the formalism for adapted anholonomic frames.

We note that the solutions defining black ellipsoids are very different from those defining ellipsoidal shapes in general relativity (see Refs. [18]) associated to some perfect–fluid bodies, rotating configurations or to some families of confocal ellipsoids in Reimannian spaces. Our black ellipsoid metrics are parametrized by generic off–diagonal ansatz with anholonomically deformed Killing symmetry and not subjected to uniqueness theorems. Such ansatz are more general than the class of vacuum solutions which can not be written in diagonal form [19] (see details in Refs. [20, 10]).

The paper is organized as follows: In Sec. 2 we outline the necessary results on off–

diagonal metrics and anhlonomic frames with associated nonlinear connection structure. We write the system of Einstein–Proca equations from MAG with string corrections of the antisymmetric H–tensor from bosonic string theory. We introduce a general off–diagonal metric ansatz and derive the corresponding system of Einstein equations with anholonomic variables. In Sec. 3 we argue that noncommutative anholonomic geometries can be associated to real off–diagonal metrics and show two simple realizations within the algebra for complex matrices. Section 4 is devoted to the geometry and physics of four dimensional static black ellipsoids. We illustrate how such solutions can be constructed by using anholonomic deformations of the Schwarzshild metric, define analytic extensions of black ellipsoid metrics and analyze the geodesic behaviour of the static ellipsoid backgrounds. We conclude that black ellipsoid metrics posses specific noncommutative symmetries. We outline a perturbation theory of anisotropic black holes and prove the stability of black ellipsoid objects in Sec. 5. Then, in Sec. 6 we discuss how the method of anholonomic frame transforms can be related solutions for ellipsoidal shapes and generic off–diagonal solutions constructed by F. Canfora and H. -J. Schmidt. We outline the work and present conclusions in Sec. 7.

There are used the basic notations and conventions stated in Refs. [8, 9].

3.2 Anholonomic Frames and Off–Diagonal Metrics

We consider a 4D manifold V^{3+1} (for MAG and string gravity with possible torsion and nonmetricity structures [7, 8, 9]) enabled with local coordinates $u^\alpha = (x^i, y^a)$ where the indices of type $i, j, k, ...$ run values 1 and 2 and the indices $a, b, c, ...$ take values 3 and 4; $y^3 = v = \varphi$ and $y^4 = t$ are considered respectively as the "anisotropic" and time like coordinates (subjected to some constraints). It is supposed that such spacetimes can also admit nontrivial torsion structures induced by certain frame transforms.

The quadratic line element

$$ds^2 = g_{\alpha\beta}\left(x^i, v\right) du^\alpha du^\beta, \tag{3.1}$$

is parametrized by a metric ansatz

$$g_{\alpha\beta} = \begin{bmatrix} g_1 + {w_1}^2 h_3 + {n_1}^2 h_4 & w_1 w_2 h_3 + n_1 n_2 h_4 & w_1 h_3 & n_1 h_4 \\ w_1 w_2 h_3 + n_1 n_2 h_4 & g_2 + {w_2}^2 h_3 + {n_2}^2 h_4 & w_2 h_3 & n_2 h_4 \\ w_1 h_3 & w_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix}, \tag{3.2}$$

with $g_i = g_i\left(x^i\right), h_a = h_{ai}\left(x^k, v\right)$ and $n_i = n_i\left(x^k, v\right)$ being some functions of necessary smoothly class or even singular in some points and finite regions. The coefficients g_i

depend only on "holonomic" variables x^i but the rest of coefficients may also depend on one "anisotropic" (anholonomic) variable $y^3 = v$; the ansatz does not depend on the time variable $y^4 = t$; we shall search for static solutions.

The spacetime may be provided with a general anholonomic frame structure of tetrads, or vierbiends,

$$e_\alpha = A_\alpha^\beta \left(u^\gamma\right) \partial/\partial u^\beta, \tag{3.3}$$

satisfying some anholonomy relations

$$e_\alpha e_\beta - e_\beta e_\alpha = w_{\alpha\beta}^\gamma \left(u^\varepsilon\right) e_\gamma, \tag{3.4}$$

where $w_{\alpha\beta}^\gamma \left(u^\varepsilon\right)$ are called the coefficients of anholonomy. A 'holonomic' frame, for instance, a coordinate frame, $e_\alpha = \partial_\alpha = \partial/\partial u^\alpha$, is defined as a set of tetrads satisfying the holonomy conditions

$$\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0.$$

We can 'effectively' diagonalize the line element (3.1),

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3(\delta v)^2 + h_4(\delta y^4)^2, \tag{3.5}$$

with respect to the anholonomic co–frame

$$\delta^\alpha = (d^i = dx^i, \delta^a = dy^a + N_i^a dx^i) = \left(d^i, \delta v = dv + w_i dx^i, \delta y^4 = dy^4 + n_i dx^i\right) \tag{3.6}$$

which is dual to the anholonomic frame

$$\delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_b) = (\delta_i = \partial_i - w_i \partial_3 - n_i \partial_4, \partial_3, \partial_4)\,, \tag{3.7}$$

where $\partial_i = \partial/\partial x^i$ and $\partial_b = \partial/\partial y^b$ are usual partial derivatives. The tetrads δ_α and δ^α are anholonomic because, in general, they satisfy the anholonomy relations (3.4) with some non–trivial coefficients,

$$w_{ij}^a = \delta_i N_j^a - \delta_j N_i^a,\ w_{ia}^b = -\ w_{ai}^b = \partial_a N_i^b. \tag{3.8}$$

The anholonomy is induced by the coefficients $N_i^3 = w_i$ and $N_i^4 = n_i$ which "elongate" partial derivatives and differentials if we are working with respect to anholonomic frames. This results in a more sophisticate differential and integral calculus (as in the tetradic and/or spinor gravity), but simplifies substantially the tensor computations, because we are dealing with diagonalized metrics. In order to construct exact 'off–diagonal' solutions with 4D metrics depending on 3 variables $\left(x^k, v\right)$ it is more convenient to work with respect to anholonomic frames (3.7) and (3.6) for diagonalized metrics (3.5) than to consider directly the ansatz (3.1) [1, 2, 3, 4].

There is a 'preferred' linear connection constructed only from the components $\left(g_i, h_a, N_k^b\right)$, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i. e. being constructed only from the metric components and satisfying the metricity conditions. It is parametrized by the coefficients, $\Gamma^{\alpha}{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$ stated with respect to the anholonomic frames (3.7) and (3.6) as

$$
\begin{aligned}
L^i{}_{jk} &= \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \\
L^a{}_{bk} &= \partial_b N_k^a + \frac{1}{2} h^{ac} \left(\delta_k h_{bc} - h_{dc} \partial_b N_k^d - h_{db} \partial_c N_k^d\right), \\
C^i{}_{jc} &= \frac{1}{2} g^{ik} \partial_c g_{jk},\ C^a{}_{bc} = \frac{1}{2} h^{ad} \left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right),
\end{aligned} \tag{3.9}
$$

computed for the ansatz (3.2). This induces a linear covariant derivative locally adapted to the nonlinear connection structure (N–connection, see details, for instance, in Refs. [21, 1, 20]). By straightforward calculations, we can verify that for D_α defined by $\Gamma^{\alpha}{}_{\beta\gamma}$ with the components (3.9) the condition $D_\alpha g_{\beta\gamma} = 0$ is satisfied.

We note that on (pseudo) Riemannian spaces the N–connection is an object completely defined by anholonomic frames, when the coefficients of tetradic transform (3.3), $A_\alpha^\beta \left(u^\gamma\right)$, are parametrized explicitly via certain values $\left(N_i^a, \delta_i^j, \delta_b^a\right)$, where δ_i^j and δ_b^a are the Kronecker symbols. By straightforward calculations we can compute (see, for instance Ref. [22]) that the coefficients of the Levi–Civita metric connection

$$
\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = g\left(\delta_\alpha, \bigtriangledown_\gamma \delta_\beta\right) = g_{\alpha\tau} \Gamma^{\bigtriangledown\tau}_{\beta\gamma},
$$

associated to a covariant derivative operator $\bigtriangledown$, satisfying the metricity condition $\bigtriangledown_\gamma g_{\alpha\beta} = 0$ for $g_{\alpha\beta} = \left(g_{ij}, h_{ab}\right)$,

$$
\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\beta\alpha} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau} w^\tau_{\gamma\beta} + g_{\beta\tau} w^\tau_{\alpha\gamma} - g_{\gamma\tau} w^\tau_{\beta\alpha}\right], \tag{3.10}
$$

are given with respect to the anholonomic basis (3.6) by the coefficients

$$
\Gamma^{\bigtriangledown\tau}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk} - \frac{\partial N_k^a}{\partial y^b}, C^i{}_{jc} + \frac{1}{2} g^{ik} \Omega^a_{jk} h_{ca}, C^a{}_{bc}\right), \tag{3.11}
$$

where $\Omega^a_{jk} = \delta_k N_j^a - \delta_j N_k^a$. The anholonomic frame structure may induce on (pseudo) Riemannian spacetimes nontrivial torsion structures. For instance, the canonical connection (3.9), in general, has nonvanishing torsion components

$$
T^i_{ja} = -T^i_{aj} = C^i_{ja}, T^a_{jk} = -T^a_{kj} = \Omega^a_{kj}, T^a_{bk} = -T^a_{kb} = \partial_b N_k^a - L^a_{bk}. \tag{3.12}
$$

This is a "pure" anholonomic frame effect. We can conclude that the Einstein theory transforms into an effective Einstein–Cartan theory with anholonomically induced torsion if the general relativity is formulated with respect to general anholonomic frame bases. In this paper we shall also consider distorsions of the Levi–Civita connection induced by nonmetricity.

A very specific property of off–diagonal metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [23] (Details can be found in Ref. [21], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N–connection structure.)

The Levi–Civita connection does not play an exclusive role on non–Riemannian spaces. For instance, the torsion on spaces provided with N–connection is induced by anholonomy relation and both linear connections (3.9) and (3.11) are compatible with the same metric and transform into the usual Levi–Civita coefficients for vanishing N–connection and "pure" holonomic coordinates (see related details in Refs. [8, 9]). This means that to an off–diagonal metric we can associated different covariant differential calculi, all being compatible with the same metric structure (like in noncommutative geometry, which is not a surprising fact because the anolonomic frames satisfy by definition some noncommutative relations (3.4)). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N–coefficients.

The dynamics of generalized Finsler–affine string gravity is defined by the system of field equations (see Proposition 3.1 in Ref. [9])

$$\begin{aligned} \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\right), \qquad (3.13)\\ \widehat{\mathbf{D}}_{\nu}\mathbf{H}^{\nu\mu} &= \mu^2\phi^{\mu},\\ \widehat{\mathbf{D}}^{\nu}\mathbf{H}_{\nu\lambda\rho} &= 0 \end{aligned}$$

for

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}\ _{\nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}$$

being the antisymmetric torsion field

$$\mathbf{H}_{\nu\lambda\rho} = \delta_{\nu}\mathbf{B}_{\lambda\rho} + \delta_{\rho}\mathbf{B}_{\nu\lambda} + \delta_{\lambda}\mathbf{B}_{\nu\rho}$$

of the antisymmetric $\mathbf{B}_{\lambda\rho}$ in bosonic string theory (for simplicity, we restrict our considerations to the sigma model with H–field corrections and zero dilatonic field). The covariant derivative $\widehat{\mathbf{D}}_\nu$ is defined by the coefficients (3.9) (we use in our references the "boldfaced" indices when it is necessary to emphasize that the spacetime is provided with N–connection structure. The distorsion $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$ of the Levi–Civita connection, when

$$\Gamma^\tau_{\beta\gamma} = \Gamma^\tau_{\bigtriangledown\beta\gamma} + \widehat{\mathbf{Z}}^\tau_{\beta\gamma},$$

from (3.13) is defined by the torsion $\widehat{\mathbf{T}}$ with the components computed for $\widehat{\mathbf{D}}$ by applying the formulas (3.12),

$$\widehat{\mathbf{Z}}_{\alpha\beta} = \delta_\beta \rfloor \widehat{\mathbf{T}}_\alpha - \delta_\alpha \rfloor \widehat{\mathbf{T}}_\beta + \frac{1}{2}\left(\delta_\alpha \rfloor \delta_\beta \rfloor \widehat{\mathbf{T}}_\gamma\right)\delta^\gamma,$$

see Refs. [8, 9] on definition of the interior product "$\rfloor$" and differential forms like $\widehat{\mathbf{T}}_\beta$ on spaces provided with N–connection structure. The tensor $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_\nu\phi_\mu - \widehat{\mathbf{D}}_\mu\phi_\nu + w^\gamma_{\mu\nu}\phi_\gamma$ is the field strengths of the Abelian–Proca field ϕ^α, with $\mu, \tilde{\kappa} = const$,

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \mathbf{H}_\alpha^{\ \mu}\mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}_{\mu\nu}\mathbf{H}^{\mu\nu} + \mu^2\phi_\alpha\phi_\beta - \frac{\mu^2}{2}\mathbf{g}_{\alpha\beta}\phi_\mu\phi^\mu,$$

where the source

$$\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta} = \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\left(\widehat{\mathbf{T}}, \mathbf{H}_{\nu\lambda\rho}\right)$$

contains contributions of the torsion fields $\widehat{\mathbf{T}}$ and $\mathbf{H}_{\nu\lambda\rho}$. The field ϕ_α is defined by certain irreducible components of torsion and nonmetricity in MAG, see [7] and Theorem 3.2 in [9].

Our aim is to elaborate a method of constructing exact solutions of equations (3.13) for vanishing matter fields, $\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} = 0$. The ansatz for the field ϕ_μ is taken in the form

$$\phi_\mu = \left[\phi_i\left(x^k\right), \phi_a = 0\right]$$

for $i, j, k... = 1, 2$ and $a, b, ... = 3, 4$. The Proca equations $\widehat{\mathbf{D}}_\nu\mathbf{H}^{\nu\mu} = \mu^2\phi^\mu$ for $\mu \to 0$ (for simplicity) transform into

$$\partial_1\left[(g_1)^{-1}\partial^1\phi_k\right] + \partial_2\left[(g_2)^{-1}\partial^2\phi_k\right] = L^j_{ki}\partial^i\phi_j - L^i_{ij}\partial^j\phi_k. \tag{3.14}$$

Two examples of solutions of this equation are considered in Ref. [9]. In this paper, we do not state any particular configurations and consider that it is possible always to define certain $\phi_i\left(x^k\right)$ satisfying the wave equation (3.14). The energy–momentum tensor $\mathbf{\Sigma}^{[\phi]}_{\alpha\beta}$ is computed for one nontrivial value

$$H_{12} = \partial_1\phi_2 - \partial_2\phi_1.$$

In result, we can represent the source of the fields ϕ_k as

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \left[\Psi_2\left(H_{12}, x^k\right), \Psi_2\left(H_{12}, x^k\right), 0, 0\right].$$

The ansatz for the H–field is taken in the form

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} \tag{3.15}$$

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_{[H]} = const$. This ansatz satisfies the field equations $\widehat{\mathbf{D}}^{\nu}\mathbf{H}_{\nu\lambda\rho} = 0$ because the metric $\mathbf{g}_{\alpha\beta}$ is compatible with $\widehat{\mathbf{D}}$. The values $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ have to be defined in a form to satisfy the condition (3.15) for any $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$ derived from $\mathbf{g}_{\alpha\beta}$ and, as a consequence, from (3.9) and (3.12), for instance, to compute them as

$$\widehat{\mathbf{H}}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} - \widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$$

for defined values of $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}, \lambda_{[H]}$ and $\mathbf{g}_{\alpha\beta}$. In result, we obtain the effective energy–momentum tensor in the form

$$\mathbf{\Sigma}^{[\phi]\beta}_{\alpha} + \mathbf{\Sigma}^{[H]\beta}_{\alpha} = \left[\Upsilon_2\left(x^k\right) + \frac{\lambda^2_{[H]}}{4}, \Upsilon_2\left(x^k\right) + \frac{\lambda^2_{[H]}}{4}, \frac{\lambda^2_{[H]}}{4}, \frac{\lambda^2_{[H]}}{4}\right]. \tag{3.16}$$

For the source (3.16), the system of field equations (3.13) defined for the metric (3.5) and connection (3.9), with respect to anholonomic frames (3.6) and (3.7), transform into a system of partial differential equations with anholonomic variables [1, 2, 20], see also details in the section 5.3 in Ref. [9],

$$R^1_1 = R^2_2 = -\frac{1}{2g_1g_2}\left[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'}g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}\right] = -\frac{\lambda^2_{[H]}}{4}, \tag{3.17}$$

$$R^3_3 = R^4_4 = -\frac{\beta}{2h_3h_4} = -\frac{1}{2h_3h_4}\left[h_4^{**} - h_4^{*}\left(\ln\sqrt{|h_3h_4|}\right)^{*}\right] = -\frac{\lambda^2_{[H]}}{4} - \Upsilon_2\left(x^k\right), \tag{3.18}$$

$$R_{3i} = -w_i\frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0, \tag{3.19}$$

$$R_{4i} = -\frac{h_4}{2h_3}\left[n_i^{**} + \gamma n_i^{*}\right] = 0, \tag{3.20}$$

where

$$\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln \sqrt{|h_3 h_4|},\ \gamma = 3h_4^*/2h_4 - h_3^*/h_3, \tag{3.21}$$

and the partial derivatives are denoted $g_1^\bullet = \partial g_1/\partial x^1, g_1^{'} = \partial g_1/\partial x^2$ and $h_3^* = \partial h_3/\partial v$. We can additionally impose the condition $\delta_i N_j^a = \delta_j N_i^a$ in order to have $\Omega_{jk}^a = 0$ which may be satisfied, for instance, if

$$w_1 = w_1\left(x^1, v\right), n_1 = n_1\left(x^1, v\right), w_2 = n_2 = 0,$$

or, inversely, if

$$w_1 = n_1 = 0, w_2 = w_2\left(x^2, v\right), n_2 = n_2\left(x^2, v\right).$$

In this paper we shall select a class of static solutions parametrized by the conditions

$$w_1 = w_2 = n_2 = 0. \tag{3.22}$$

The system of equations (3.17)–(3.20) can be integrated in general form [9, 20]. Physical solutions are selected following some additional boundary conditions, imposed types of symmetries, nonlinearities and singular behavior and compatibility in the locally anisotropic limits with some well known exact solutions.

Finally, we note that there is a difference between our approach and the so–called "tetradic" gravity (see basic details and references in [22]) when the metric coefficients $g_{\alpha\beta}(u^\gamma)$ are substituted by tetradic fields $e_{\underline{\alpha}}^{\alpha}(u^\gamma)$, mutually related by formula $g_{\alpha\beta} = e_{\underline{\alpha}}^{\alpha} e_{\underline{\beta}}^{\beta} \eta_{\underline{\alpha}\underline{\beta}}$ with $\eta_{\underline{\alpha}\underline{\beta}}$ chosen, for instance, to be the Minkowski metric. In our case we partially preserve some metric dynamics given by diagonal effective metric coefficients (g_i, h_a) but also adapt the calculus to tetrads respectively defied by $\left(N_i^a, \delta_i^j, \delta_b^a\right)$, see (3.6) and (3.7). This substantially simplifies the method of constructing exact solutions and also reflects new type symmetries of such classes of metrics.

3.3 Anholonomic Noncommutative Symmetries

The nontrivial anholonomy coefficients, see (3.4) and (3.8) induced by off–diagonal metric (3.1) (and associated N–connection) coefficients emphasize a kind of Lie algebra noncommutativity relation. In this section, we analyze a simple realizations of noncommutative geometry of anholonomic frames within the algebra of complex $k \times k$ matrices, $M_k(\mathbb{C}, u^\alpha)$ depending on coordinates u^α on spacetime V^{n+m} connected to complex Lie algebras $SL(k, \mathbb{C})$ (see Ref. [10] for similar constructions with the group SU_k).

We consider matrix valued functions of necessary smoothly class derived from the anholonomic frame relations (3.4) (being similar to the Lie algebra relations) with the coefficients (3.8) induced by off–diagonal metric terms in (3.2) and by N–connection coefficients N_i^a. We use algebras of complex matrices in order to have the possibility for some extensions to complex solutions and to relate the constructions to noncommutative/complex gravity). For commutative gravity models, the anholonomy coefficients $w^{\gamma}{}_{\alpha\beta}$ are real functions but there are considered also complex metrics and tetrads related to noncommutative gravity [24].

Let us consider the basic relations for the simplest model of noncommutative geometry realized with the algebra of complex $(k \times k)$ noncommutative matrices [25], $M_k(\mathbb{C})$. Any element $M \in M_k(\mathbb{C})$ can be represented as a linear combination of the unit $(k \times k)$ matrix I and $(k^2 - 1)$ hermitian traceless matrices $q_{\underline{\alpha}}$ with the underlined index $\underline{\alpha}$ running values $1, 2, ..., k^2 - 1$, i. e.

$$M = \alpha\ I + \sum \beta^{\underline{\alpha}} q_{\underline{\alpha}}$$

for some constants α and $\beta^{\underline{\alpha}}$. It is possible to chose the basis matrices $q_{\underline{\alpha}}$ satisfying the relations

$$q_{\underline{\alpha}} q_{\underline{\beta}} = \frac{1}{k} \rho_{\underline{\alpha}\underline{\beta}} I + Q^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} q_{\underline{\gamma}} - \frac{i}{2} f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} q_{\underline{\gamma}}, \tag{3.23}$$

where $i^2 = -1$ and the real coefficients satisfy the properties

$$Q^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} = Q^{\underline{\gamma}}_{\underline{\beta}\underline{\alpha}},\ Q^{\underline{\gamma}}_{\underline{\gamma}\underline{\beta}} = 0,\ f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} = -f^{\underline{\gamma}}_{\underline{\beta}\underline{\alpha}}, f^{\underline{\gamma}}_{\underline{\gamma}\underline{\alpha}} = 0$$

with $f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$ being the structure constants of the Lie group $SL\,(k, \mathbb{C})$ and the Killing–Cartan metric tensor $\rho_{\underline{\alpha}\underline{\beta}} = f^{\underline{\tau}}_{\underline{\alpha}\underline{\gamma}} f^{\underline{\gamma}}_{\underline{\tau}\underline{\beta}}$. This algebra admits a formalism of interior derivatives $\widehat{\partial}_{\underline{\gamma}}$ defied by relations

$$\widehat{\partial}_{\underline{\gamma}} q_{\underline{\beta}} = ad\left(i q_{\underline{\gamma}}\right) q_{\underline{\beta}} = i[q_{\underline{\gamma}}, q_{\underline{\beta}}] = f^{\underline{\alpha}}_{\underline{\gamma}\underline{\beta}} q_{\underline{\alpha}} \tag{3.24}$$

and

$$\widehat{\partial}_{\underline{\alpha}} \widehat{\partial}_{\underline{\beta}} - \widehat{\partial}_{\underline{\beta}} \widehat{\partial}_{\underline{\alpha}} = f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} \widehat{\partial}_{\underline{\gamma}} \tag{3.25}$$

(the last relation follows the Jacoby identity and is quite similar to (3.4) but with constant values $f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$).

Our idea is to associate a noncommutative geometry starting from the anholonomy relations of frames (3.4) by adding to the structure constants $f^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$ the anholonomy coefficients $w^{[N]\tau}_{\alpha\gamma}$ (3.8) (we shall put the label [N] if would be necessary to emphasize that the anholonomic coefficients are induced by a nonlinear connection. Such deformed structure constants consist from N–connection coefficients N_i^a and their first partial derivatives, i. e. they are induced by some off–diagonal terms in the metric (3.2) being a solution of the gravitational field equations.

We emphasize that there is a rough analogy between formulas (3.25) and (3.4) because the anholonomy coefficients do not satisfy, in general, the condition $w_{\tau\alpha}^{[N]\tau} = 0$. There is also another substantial difference because the anholonomy relations are defined for a manifold of dimension $n+m$, with Greek indices $\alpha, \beta, ...$ running values from 1 to $n+m$ but the matrix noncommutativity relations are stated for traceless matrices labelled by underlined indices $\underline{\alpha}, \underline{\beta}$, running values from 1 to k^2-1. It is not possible to satisfy the condition $k^2-1 = n+m$ by using integer numbers for arbitrary $n+m$. We may extend the dimension of spacetime from $n+m$ to any $n' \geq n$ and $m' \geq m$ when the condition $k^2-1 = n'+m'$ can be satisfied by a trivial embedding of the metric (3.2) into higher dimension, for instance, by adding the necessary number of unities on the diagonal and writing

$$\widehat{g}_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} 1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 1 & 0 & 0 \\ 0 & ... & 0 & g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ 0 & ... & 0 & N_i^e h_{be} & h_{ab} \end{bmatrix}$$

and $e_{\underline{\alpha}}^{[N]} = \delta_{\underline{\alpha}} = \left(1, 1, ..., e_{\alpha}^{[N]}\right)$. For simplicity, we preserve the same type of underlined Greek indices, $\underline{\alpha}, \underline{\beta} ... = 1, 2, ..., k^2-1 = n'+m'$.

The anholonomy coefficients $w_{\alpha\beta}^{[N]\gamma}$ can be extended with some trivial zero components and for consistency we rewrite them without labelled indices, $w_{\alpha\beta}^{[N]\gamma} \rightarrow W_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}}$. The set of anholonomy coefficients $w_{\alpha\beta}^{[N]\gamma}$(3.4) may result in degenerated matrices, for instance for certain classes of exact solutions of the Einstein equations. So, it would not be a well defined construction if we shall substitute the structure Lie algebra constants directly by $w_{\alpha\beta}^{[N]\gamma}$. We can consider a simple extension $w_{\alpha\beta}^{[N]\gamma} \rightarrow W_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}}$ when the coefficients $w_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}}(u^{\underline{\tau}})$ for any fixed value $u^{\underline{\tau}} = u_{[0]}^{\underline{\tau}}$ would be some deformations of the structure constants of the Lie algebra $SL(k, \mathbb{C})$, like

$$W_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}} = f_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}} + w_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}}, \tag{3.26}$$

being nondegenerate.

Instead of the matrix algebra $M_k(\mathbb{C})$, constructed from constant complex elements, we have also to introduce dependencies on coordinates $u^{\underline{\alpha}} = (0, ..., u^{\alpha})$, for instance, like a trivial matrix bundle on $V^{n'+m'}$, and denote this space $M_k(\mathbb{C}, u^{\underline{\alpha}})$. Any element $B(u^{\underline{\alpha}}) \in M_k(\mathbb{C}, u^{\underline{\alpha}})$ with a noncommutative structure induced by $W_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}}$ is represented as a linear combination of the unit $(n'+m') \times (n'+m')$ matrix I and the $[(n'+m')^2-1]$ hermitian traceless matrices $q_{\underline{\alpha}}(u^{\underline{\tau}})$ with the underlined index $\underline{\alpha}$ running values $1,2,...,(n'+m')^2-1$,

$$B(u^{\underline{\tau}}) = \alpha(u^{\underline{\tau}})\ I + \sum \beta^{\underline{\alpha}}(u^{\underline{\tau}})\, q_{\underline{\alpha}}(u^{\underline{\tau}})$$

under condition that the following relation holds:

$$q_{\underline{\alpha}}(u^{\underline{\tau}})\, q_{\underline{\beta}}(u^{\underline{\gamma}}) = \frac{1}{n'+m'}\rho_{\underline{\alpha}\underline{\beta}}(u^{\underline{\nu}}) + Q^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} q_{\underline{\gamma}}(u^{\underline{\mu}}) - \frac{i}{2} W^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} q_{\underline{\gamma}}(u^{\underline{\mu}})$$

with the same values of $Q^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$ from the Lie algebra for $SL(k,\mathbb{C})$ but with the Killing–Cartan like metric tensor defined by anholonomy coefficients, i. e. $\rho_{\underline{\alpha}\underline{\beta}}(u^{\underline{\nu}}) = W^{\underline{\tau}}_{\underline{\alpha}\underline{\gamma}}(u^{\underline{\alpha}})$ $W^{\underline{\gamma}}_{\underline{\tau}\underline{\beta}}(u^{\underline{\alpha}})$. For complex spacetimes, we shall consider that the coefficients $N^{\underline{a}}_{\underline{i}}$ and $W^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$ may be some complex valued functions of necessary smooth (in general, with complex variables) class. In result, the Killing–Cartan like metric tensor $\rho_{\underline{\alpha}\underline{\beta}}$ can be also complex.

We rewrite (3.4) as

$$e_{\underline{\alpha}} e_{\underline{\beta}} - e_{\underline{\beta}} e_{\underline{\alpha}} = W^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} e_{\underline{\gamma}} \tag{3.27}$$

being equivalent to (3.25) and defining a noncommutative anholonomic structure (for simplicity, we use the same symbols $e_{\underline{\alpha}}$ as for some 'N–elongated' partial derivatives, but with underlined indices). The analogs of derivation operators (3.24) are stated by using $W^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}$,

$$e_{\underline{\alpha}} q_{\underline{\beta}}(u^{\underline{\gamma}}) = ad\left[iq_{\underline{\alpha}}(u^{\underline{\gamma}})\right] q_{\underline{\beta}}(u^{\underline{\gamma}}) = i\left[q_{\underline{\alpha}}(u^{\underline{\gamma}})\, q_{\underline{\beta}}(u^{\underline{\gamma}})\right] = W^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} q_{\underline{\gamma}} \tag{3.28}$$

The operators (3.28) define a linear space of anholonomic derivations satisfying the conditions (3.27), denoted $AderM_k(\mathbb{C}, u^{\underline{\alpha}})$, elongated by N–connection and distinguished into irreducible h– and v–components, respectively, into $e_{\underline{i}}$ and $e_{\underline{b}}$, like $e_{\underline{\alpha}} = \left(e_{\underline{i}} = \partial_{\underline{i}} - N^{\underline{a}}_{\underline{i}} e_{\underline{a}}, e_{\underline{b}} = \partial_{\underline{b}}\right)$. The space $AderM_k(\mathbb{C}, u^{\underline{\alpha}})$ is not a left module over the algebra $M_k(\mathbb{C}, u^{\underline{\alpha}})$ which means that there is a a substantial difference with respect to the usual commutative differential geometry where a vector field multiplied on the left by a function produces a new vector field.

The duals to operators (3.28), $e^{\underline{\mu}}$, found from $e^{\underline{\mu}}(e_{\underline{\alpha}}) = \delta^{\underline{\mu}}_{\underline{\alpha}} I$, define a canonical basis of 1–forms $e^{\underline{\mu}}$ connected to the N–connection structure. By using these forms, we can span a left module over $M_k(\mathbb{C}, u^{\underline{\alpha}})$ following $q_{\underline{\alpha}} e^{\underline{\mu}}\left(e_{\underline{\beta}}\right) = q_{\underline{\alpha}} \delta^{\underline{\mu}}_{\underline{\beta}} I = q_{\underline{\alpha}} \delta^{\underline{\mu}}_{\underline{\beta}}$. For an arbitrary vector field

$$Y = Y^{\alpha} e_{\alpha} \to Y^{\underline{\alpha}} e_{\underline{\alpha}} = Y^{\underline{i}} e_{\underline{i}} + Y^{\underline{a}} e_{\underline{a}},$$

it is possible to define an exterior differential (in our case being N–elongated), starting with the action on a function φ (equivalent, a 0–form),

$$\delta\ \varphi(Y) = Y\varphi = Y^{\underline{i}} \delta_{\underline{i}} \varphi + Y^{\underline{a}} \partial_{\underline{a}} \varphi$$

when

$$(\delta \ I)\,(e_{\underline{\alpha}}) = e_{\underline{\alpha}} I = ad\,(ie_{\underline{\alpha}})\,I = i\,[e_{\underline{\alpha}}, I] = 0, \text{ i. e. } \delta I = 0,$$

and

$$\delta q_{\underline{\mu}}(e_{\underline{\alpha}}) = e_{\underline{\alpha}}(e_{\underline{\mu}}) = i[e_{\underline{\mu}}, e_{\underline{\alpha}}] = W^{\underline{\gamma}}_{\underline{\alpha}\underline{\mu}} e_{\underline{\gamma}}. \tag{3.29}$$

Considering the nondegenerated case, we can invert (3.29) as to obtain a similar expression with respect to $e^{\underline{\mu}}$,

$$\delta(e_{\underline{\alpha}}) = W^{\underline{\gamma}}_{\underline{\alpha}\underline{\mu}} e_{\underline{\gamma}} e^{\underline{\mu}}, \tag{3.30}$$

from which a very important property follows by using the Jacobi identity, $\delta^2 = 0$, resulting in a possibility to define a usual Grassman algebra of p–forms with the wedge product $\wedge$ stated as

$$e^{\underline{\mu}} \wedge e^{\underline{\nu}} = \frac{1}{2}\left(e^{\underline{\mu}} \otimes e^{\underline{\nu}} - e^{\underline{\nu}} \otimes e^{\underline{\mu}}\right).$$

We can write (3.30) as

$$\delta(e^{\underline{\alpha}}) = -\frac{1}{2} W^{\underline{\alpha}}_{\underline{\beta\mu}} e^{\underline{\beta}} e^{\underline{\mu}}$$

and introduce the canonical 1–form $e = q_{\underline{\alpha}} e^{\underline{\alpha}}$ being coordinate–independent and adapted to the N–connection structure and satisfying the condition $\delta e + e \wedge e = 0$.

In a standard manner, we can introduce the volume element induced by the canonical Cartan–Killing metric and the corresponding star operator $\star$ (Hodge duality). We define the volume element σ by using the complete antisymmetric tensor $\epsilon_{\underline{\alpha}_1 \underline{\alpha}_2 \cdots \underline{\alpha}_{k^2-1}}$ as

$$\sigma = \frac{1}{[(n'+m')^2 - 1]!} \epsilon_{\underline{\alpha}_1 \underline{\alpha}_2 \cdots \underline{\alpha}_{n'+m'}} e^{\underline{\alpha}_1} \wedge e^{\underline{\alpha}_2} \wedge \ldots \wedge e^{\underline{\alpha}_{n'+m'}}$$

to which any (k^2-1)–form is proportional $(k^2 - 1 = n' + m')$. The integral of such a form is defined as the trace of the matrix coefficient in the from σ and the scalar product introduced for any couple of p–forms ϖ and ψ

$$(\varpi, \psi) = \int (\varpi \wedge \star\psi).$$

Let us analyze how a noncommutative differential form calculus (induced by an anholonomic structure) can be developed and related to the Hamiltonian classical and quantum mechanics and Poisson bracket formalism:

For a p–form $\varpi^{[p]}$, the anti–derivation i_Y with respect to a vector field $Y \in Ader M_k(\mathbb{C}, u^{\underline{\alpha}})$ can be defined as in the usual formalism,

$$i_Y \varpi^{[p]}\,(X_1, X_2, ..., X_{p-1}) = \varpi^{[p]}\,(Y, X_1, X_2, ..., X_{p-1})$$

where $X_1, X_2, ..., X_{p-1} \in AderM_k(\mathbb{C}, u^{\underline{\alpha}})$. By a straightforward calculus we can check that for a 2–form $\Xi = \delta\ e$ one holds

$$\delta\Xi = \delta^2 e = 0 \text{ and } L_Y\Xi = 0$$

where the Lie derivative of forms is defined as $L_Y \varpi^{[p]} = (i_Y\ \delta + \delta\ i_Y)\, \varpi^{[p]}$.

The Hamiltonian vector field $H_{[\varphi]}$ of an element of algebra $\varphi \in M_k(\mathbb{C}, u^{\underline{\alpha}})$ is introduced following the equality $\Xi\left(H_{[\varphi]}, Y\right) = Y\varphi$ which holds for any vector field. Then, we can define the Poisson bracket of two functions (in a quantum variant, observables) φ and χ, $\{\varphi, \chi\} = \Xi\left(H_{[\varphi]}, H_{[\chi]}\right)$ when

$$\{e_{\underline{\alpha}}, e_{\underline{\beta}}\} = \Xi\left(e_{\underline{\alpha}}, e_{\underline{\beta}}\right) = i[e_{\underline{\alpha}}, e_{\underline{\beta}}].$$

This way, a simple version of noncommutative classical and quantum mechanics (up to a factor like the Plank constant, $\hbar$) is proposed, being derived by anholonomic relations for a certain class of exact 'off–diagonal' solutions in commutative gravity.

In order to define the algebra of forms $\Omega^*\left[M_k(\mathbb{C}, u^{\underline{\alpha}})\right]$ over $M_k(\mathbb{C}, u^{\underline{\alpha}})$ we put $\Omega^0 = M_k$ and write

$$\delta\varphi\left(e_{\underline{\alpha}}\right) = e_{\underline{\alpha}}(\varphi)$$

for every matrix function $\varphi \in M_k(\mathbb{C}, u^{\underline{\alpha}})$. As a particular case, we have

$$\delta q^{\underline{\alpha}}\left(e_{\underline{\beta}}\right) = -W^{\underline{\alpha}}_{\ \underline{\beta\gamma}} q^{\underline{\gamma}}$$

where indices are raised and lowered with the anholonomically deformed metric $\rho_{\underline{\alpha\beta}}(u^{\underline{\lambda}})$. This way, we can define the set of 1–forms $\Omega^1\left[M_k(\mathbb{C}, u^{\underline{\alpha}})\right]$ to be the set of all elements of the form $\varphi\delta\beta$ with φ and β belonging to $M_k(\mathbb{C}, u^{\underline{\alpha}})$. The set of all differential forms define a differential algebra $\Omega^*\left[M_k(\mathbb{C}, u^{\underline{\alpha}})\right]$ with the couple $\left(\Omega^*\left[M_k(\mathbb{C}, u^{\underline{\alpha}})\right], \delta\right)$ said to be a differential calculus in $M_k(\mathbb{C}, u^{\underline{\alpha}})$ induced by the anholonomy of certain exact solutions (with off–diagonal metrics and associated N–connections) in a gravity theory.

We can also find a set of generators $e^{\underline{\alpha}}$ of $\Omega^1\left[M_k(\mathbb{C}, u^{\underline{\alpha}})\right]$, as a left/ right –module completely characterized by the duality equations $e^{\underline{\mu}}\left(e_{\underline{\alpha}}\right) = \delta^{\underline{\mu}}_{\underline{\alpha}} I$ and related to $\delta q^{\underline{\alpha}}$,

$$\delta q^{\underline{\alpha}} = W^{\underline{\alpha}}_{\ \underline{\beta\gamma}} q^{\underline{\beta}} q^{\underline{\gamma}} \text{ and } e^{\underline{\mu}} = q_{\underline{\gamma}} q^{\underline{\mu}} \delta q^{\underline{\gamma}}.$$

Similarly to the formalism presented in details in Ref. [27], we can elaborate a differential calculus with derivations by introducing a linear torsionless connection

$$\mathcal{D}e^{\underline{\mu}} = -\omega^{\underline{\mu}}_{\ \underline{\gamma}} \otimes e^{\underline{\gamma}}$$

with the coefficients $\omega^{\mu}_{\ \underline{\gamma}} = -\frac{1}{2}W^{\mu}_{\ \underline{\gamma\beta}}e^{\underline{\gamma}}$, resulting in the curvature 2–form

$$\mathcal{R}^{\mu}_{\ \underline{\gamma}} = \frac{1}{8}W^{\mu}_{\ \underline{\gamma\beta}}W^{\beta}_{\ \underline{\alpha\tau}}e^{\underline{\alpha}}e^{\underline{\tau}}.$$

This is a surprising fact that 'commutative' curved spacetimes provided with off–diagonal metrics and associated anhlonomic frames and N–connections may be characterized by a noncommutative 'shadow' space with a Lie algebra like structure induced by the frame anholonomy. We argue that such metrics possess anholonomic noncommutative symmetries and conclude that for the 'holonomic' solutions of the Einstein equations, with vanishing $w^{\gamma}_{\ \underline{\alpha\beta}}$, any associated noncommutative geometry or $SL(k, \mathbb{C})$ decouples from the off–diagonal (anholonomic) gravitational background and transforms into a trivial one defined by the corresponding structure constants of the chosen Lie algebra. The anholonomic noncommutativity and the related differential geometry are induced by the anholonomy coefficients. All such structures reflect a specific type of symmetries of generic off–diagonal metrics and associated frame/ N–connection structures.

Considering exact solutions of the gravitational field equations, we can assert that we constructed a class of vacuum or nonvacuum metrics possessing a specific noncommutative symmetry instead, for instance, of any usual Killing symmetry. In general, we can introduce a new classification of spacetimes following anholonomic noncommutative algebraic properties of metrics and vielbein structures (see Ref. [28, 10]). In this paper, we analyze the simplest examples of such spacetimes.

3.4 4D Static Black Ellipsoids in MAG and String Gravity

We outline the black ellipsoid solutions [29, 30] and discuss their associated anholonomic noncommutative symmetries [10]. We note that such solutions can be extended for the (anti) de Sitter spaces, in gauge gravity and string gravity with effective cosmological constant [31]. In this paper, the solutions are considered for 'real' metric–affine spaces and extended to nontrivial cosmological constant. We emphasize the possibility to construct solutions with locally "anisotropic" cosmological constants (such configurations may be also induced, for instance, from string/ brane gravity).

3.4.1 Anholonomic deformations of the Schwarzschild metric

We consider a particular case of effectively diagonalized (3.5) (and corresponding off–diagonal metric ansatz (3.1)) when

$$\begin{aligned} \delta s^2 &= [-\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right)^{-1} dr^2 - r^2 q(r) d\theta^2 \\ &\quad -\eta_3\left(r,\varphi\right) r^2 \sin^2\theta d\varphi^2 + \eta_4\left(r,\varphi\right)\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right)\delta t^2] \end{aligned} \tag{3.31}$$

where the "polarization" functions $\eta_{3,4}$ are decomposed on a small parameter $\varepsilon, 0<\varepsilon\ll 1$,

$$\begin{aligned} \eta_3\left(r,\varphi\right) &= \eta_{3[0]}\left(r,\varphi\right)+\varepsilon\lambda_3\left(r,\varphi\right)+\varepsilon^2\gamma_3\left(r,\varphi\right)+..., \\ \eta_4\left(r,\varphi\right) &= 1+\varepsilon\lambda_4\left(r,\varphi\right)+\varepsilon^2\gamma_4\left(r,\varphi\right)+..., \end{aligned} \tag{3.32}$$

and

$$\delta t = dt + n_1\left(r,\varphi\right) dr$$

for $n_1 \sim \varepsilon ... + \varepsilon^2$ terms. The functions $q(r), \eta_{3,4}\left(r,\varphi\right)$ and $n_1\left(r,\varphi\right)$ will be found as the metric will define a solution of the gravitational field equations generated by small deformations of the spherical static symmetry on a small positive parameter ε (in the limits $\varepsilon \rightarrow 0$ and $q, \eta_{3,4} \rightarrow 1$ we have just the Schwarzschild solution for a point particle of mass m). The metric (3.31) is a particular case of a class of exact solutions constructed in [1, 2, 20]. Its complexification by complex valued N–coefficients is investigated in Ref. [10].

We can state a particular symmetry of the metric (3.31) by imposing a corresponding condition of vanishing of the metric coefficient before the term δt^2. For instance, the constraints that

$$\begin{aligned}
\eta_4(r,\varphi)\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right) &= 1-\frac{2m}{r}+\varepsilon\frac{\Phi_4}{r^2}+\varepsilon^2\Theta_4=0, \\
\Phi_4 &= \lambda_4\left(r^2-2mr\right)+1 \\
\Theta_4 &= \gamma_4\left(1-\frac{2m}{r}\right)+\lambda_4,
\end{aligned} \tag{3.33}$$

define a rotation ellipsoid configuration if

$$\lambda_4=\left(1-\frac{2m}{r}\right)^{-1}(\cos\varphi-\frac{1}{r^2})$$

and

$$\gamma_4=-\lambda_4\left(1-\frac{2m}{r}\right)^{-1}.$$

In the first order on ε one obtains a zero value for the coefficient before δt^2 if

$$r_+=\frac{2m}{1+\varepsilon\cos\varphi}=2m[1-\varepsilon\cos\varphi], \tag{3.34}$$

which is the equation for a 3D ellipsoid like hypersurface with a small eccentricity ε. In general, we can consider arbitrary pairs of functions $\lambda_4(r,\theta,\varphi)$ and $\gamma_4(r,\theta,\varphi)$ (for φ–anisotropies, $\lambda_4(r,\varphi)$ and $\gamma_4(r,\varphi)$) which may be singular for some values of r, or on some hypersurvaces $r=r(\theta,\varphi)$ $(r=r(\varphi))$.

The simplest way to define the condition of vanishing of the metric coefficient before the value δt^2 is to choose γ_4 and λ_4 as to have $\Theta=0$. In this case we find from (3.33)

$$r_\pm=m\pm m\sqrt{1-\varepsilon\frac{\Phi}{m^2}}=m\left[1\pm\left(1-\varepsilon\frac{\Phi_4}{2m^2}\right)\right] \tag{3.35}$$

where $\Phi_4(r,\varphi)$ is taken for $r=2m$.

For a new radial coordinate

$$\xi=\int dr\sqrt{|1-\frac{2m}{r}+\frac{\varepsilon}{r^2}|} \tag{3.36}$$

and

$$h_3=-\eta_3(\xi,\varphi)r^2(\xi)\sin^2\theta,\ h_4=1-\frac{2m}{r}+\varepsilon\frac{\Phi_4}{r^2}, \tag{3.37}$$

when $r = r(\xi)$ is inverse function after integration in (3.36), we write the metric (3.31) as

$$
\begin{aligned}
ds^2 &= -d\xi^2 - r^2(\xi)\, q(\xi)\, d\theta^2 + h_3(\xi,\theta,\varphi)\,\delta\varphi^2 + h_4(\xi,\theta,\varphi)\,\delta t^2, \\
\delta t &= dt + n_1(\xi,\varphi)\, d\xi,
\end{aligned} \tag{3.38}
$$

where the coefficient n_1 is taken to solve the equation (3.20) and to satisfy the (3.22). The next step is to state the conditions when the coefficients of metric (3.31) define solutions of the Einstein equations. We put $g_1 = -1, g_2 = -r^2(\xi)\, q(\xi)$ and arbitrary $h_3(\xi,\theta,\varphi)$ and $h_4(\xi,\theta)$ in order to find solutions the equations (3.17)–(3.19). If h_4 depends on anisotropic variable φ, the equation (3.18) may be solved if

$$
\sqrt{|\eta_3|} = \eta_0 \left(\sqrt{|\eta_4|}\right)^* \tag{3.39}
$$

for $\eta_0 = const$. Considering decompositions of type (3.32) we put $\eta_0 = \eta/\varepsilon$ where the constant η is taken as to have $\sqrt{|\eta_3|} = 1$ in the limits

$$
\frac{\left(\sqrt{|\eta_4|}\right)^* \to 0}{\varepsilon \to 0} \to \frac{1}{\eta} = const. \tag{3.40}
$$

These conditions are satisfied if the functions $\eta_{3[0]}$, $\lambda_{3,4}$ and $\gamma_{3,4}$ are related via relations

$$
\sqrt{|\eta_{3[0]}|} = \frac{\eta}{2}\lambda_4^*, \lambda_3 = \eta\sqrt{|\eta_{3[0]}|}\gamma_4^*
$$

for arbitrary $\gamma_3(r,\varphi)$. In this paper we select only such solutions which satisfy the conditions (3.39) and (3.40).

For linear infinitesimal extensions on ε of the Schwarzschild metric, we write the solution of (3.20) as

$$
n_1 = \varepsilon \widehat{n}_1(\xi,\varphi)
$$

where

$$
\begin{aligned}
\widehat{n}_1(\xi,\varphi) &= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi\ \eta_3(\xi,\varphi) / \left(\sqrt{|\eta_4(\xi,\varphi)|}\right)^3, \eta_4^* \neq 0; \\
&= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi\ \eta_3(\xi,\varphi), \eta_4^* = 0; \\
&= n_{1[1]}(\xi) + n_{1[2]}(\xi) \int d\varphi / \left(\sqrt{|\eta_4(\xi,\varphi)|}\right)^3, \eta_3^* = 0;
\end{aligned} \tag{3.41}
$$

with the functions $n_{k[1,2]}(\xi)$ to be stated by boundary conditions.

The data

$$\begin{aligned} g_1 &= -1, g_2 = -r^2(\xi)q(\xi), \\ h_3 &= -\eta_3(\xi,\varphi)r^2(\xi)\sin^2\theta,\ h_4 = 1 - \frac{2m}{r} + \varepsilon\frac{\Phi_4}{r^2}, \\ w_{1,2} &= 0, n_1 = \varepsilon\widehat{n}_1(\xi,\varphi), n_2 = 0, \end{aligned} \tag{3.42}$$

for the metric (3.31) define a class of solutions of the Einstein equations for the canonical distinguished connection (3.9), with non–trivial polarization function η_3 and extended on parameter ε up to the second order (the polarization functions being taken as to make zero the second order coefficients). Such solutions are generated by small deformations (in particular cases of rotation ellipsoid symmetry) of the Schwarschild metric.

We can relate our solutions with some small deformations of the Schwarzschild metric, as well we can satisfy the asymptotically flat condition, if we chose such functions $n_{k[1,2]}(x^i)$ as $n_k \to 0$ for $\varepsilon \to 0$ and $\eta_3 \to 1$. These functions have to be selected as to vanish far away from the horizon, for instance, like $\sim 1/r^{1+\tau}, \tau > 0$, for long distances $r \to \infty$.

3.4.2 Black ellipsoids and anistropic cosmological constants

We can generalize the gravitational field equations to the gravity with variable cosmological constants $\lambda_{[h]}(u^\alpha)$ and $\lambda_{[v]}(u^\alpha)$ which can be induced, for instance, from extra dimensions in string/brane gravity, when the non-trivial components of the Einstein equations are

$$R_{ij} = \lambda_{[h]}\left(x^1\right)g_{ij} \text{ and } R_{ab} = \lambda_{[v]}(x^k,v)g_{ab} \tag{3.43}$$

where Ricci tensor $R_{\mu\nu}$ with anholonomic variables has two nontrivial components R_{ij} and R_{ab}, and the indices take values $i,k = 1,2$ and $a,b = 3,4$ for $x^i = \xi$ and $y^3 = v = \varphi$ (see notations from the previous subsection). The equations (3.43) contain the equations (3.17) and (3.18) as particular cases when $\lambda_{[h]}(x^1) = \frac{\lambda_{[H]}^2}{4}$ and $\lambda_{[v]}(x^k,v) = \frac{\lambda_{[H]}^2}{4} + \Upsilon_2\left(x^k\right).$

For an ansatz of type (3.5)

$$\begin{aligned} \delta s^2 &= g_1(dx^1)^2 + g_2(dx^2)^2 + h_3\left(x^i,y^3\right)(\delta y^3)^2 + h_4\left(x^i,y^3\right)(\delta y^4)^2, \\ \delta y^3 &= dy^3 + w_i\left(x^k,y^3\right)dx^{i'}, \quad \delta y^4 = dy^4 + n_i\left(x^k,y^3\right)dx^i, \end{aligned} \tag{3.44}$$

the Einstein equations (3.43) are written (see [20] for details on computation)

$$R_1^1 = R_2^2 = -\frac{1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'} g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}] = \lambda_{[h]}\left(x^k\right), \quad (3.45)$$

$$R_3^3 = R_4^4 = -\frac{\beta}{2h_3 h_4} = \lambda_{[v]}(x^k, v), \quad (3.46)$$

$$R_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0, \quad (3.47)$$

$$R_{4i} = -\frac{h_4}{2h_3}\left[n_i^{**} + \gamma n_i^{*}\right] = 0. \quad (3.48)$$

The coefficients of equations (3.45) - (3.48) are given by

$$\alpha_i = \partial_i h_4^{*} - h_4^{*}\partial_i \ln\sqrt{|h_3 h_4|}, \qquad \beta = h_4^{**} - h_4^{*}[\ln\sqrt{|h_3 h_4|}]^{*}, \qquad \gamma = \frac{3h_4^{*}}{2h_4} - \frac{h_3^{*}}{h_3}. \quad (3.49)$$

The various partial derivatives are denoted as $a^{\bullet} = \partial a/\partial x^1, a^{'} = \partial a/\partial x^2, a^{*} = \partial a/\partial y^3$. This system of equations can be solved by choosing one of the ansatz functions (*e.g.* $g_1(x^i)$ or $g_2(x^i)$) and one of the ansatz functions (*e.g.* $h_3(x^i, y^3)$ or $h_4(x^i, y^3)$) to take some arbitrary, but physically interesting form. Then, the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way we can generate a lot of different solutions, but we impose the condition that the initial, arbitrary choice of the ansatz functions is "physically interesting" which means that one wants to make this original choice so that the generated final solution yield a well behaved metric.

In this subsection, we show that the data (3.42) can be extended as to generate exact black ellipsoid solutions with nontrivial polarized cosmological constant which can be imbedded in string theory. A complex generalization of the solution (3.42) is analyzed in Ref. [10] and the case locally isotropic cosmological constant was considered in Ref. [31].

At the first step, we find a class of solutions with $g_1 = -1$ and $g_2 = g_2(\xi)$ solving the equation (3.45), which under such parametrizations transforms to

$$g_2^{\bullet\bullet} - \frac{(g_2^{\bullet})^2}{2g_2} = 2g_2\lambda_{[h]}(\xi). \quad (3.50)$$

With respect to the variable $Z = (g_2)^2$ this equation is written as

$$Z^{\bullet\bullet} + 2\lambda_{[h]}(\xi)\, Z = 0$$

which can be integrated in explicit form if $\lambda_{[h]}(\xi) = \lambda_{[h]0} = const,$

$$Z = Z_{[0]} \sin\left(\sqrt{2\lambda_{[h]0}}\xi + \xi_{[0]}\right),$$

for some constants $Z_{[0]}$ and $\xi_{[0]}$ which means that

$$g_2 = -Z_{[0]}^2 \sin^2\left(\sqrt{2\lambda_{[h]0}}\xi + \xi_{[0]}\right) \tag{3.51}$$

parametrize in 'real' string gravity a class of solution of (3.45) for the signature $(-,-,-,+)$. For $\lambda_{[h]} \to 0$ we can approximate $g_2 = r^2(\xi)\, q(\xi) = -\xi^2$ and $Z_{[0]}^2 = 1$ which has compatibility with the data (3.42). The solution (3.51) with cosmological constant (of string or non–string origin) induces oscillations in the "horozontal" part of the metric written with respect to N–adapted frames. If we put $\lambda_{[h]}(\xi)$ in (3.50), we can search the solution as $g_2 = u^2$ where $u(\xi)$ solves the linear equation

$$u^{\bullet\bullet} + \frac{\lambda_{[h]}(\xi)}{4} u = 0.$$

The method of integration of such equations is given in Ref. [32]. The explicit forms of solutions depends on function $\lambda_{[h]}(\xi)$. In this case we have to write

$$g_2 = r^2(\xi)\, q^{[u]}(\xi) = u^2(\xi). \tag{3.52}$$

For a suitable smooth behavior of $\lambda_{[h]}(\xi)$, we can generate such $u(\xi)$ and $r(\xi)$ when the $r = r(\xi)$ is the inverse function after integration in (3.36).

The next step is to solve the equation (3.46),

$$h_4^{**} - h_4^{*}[\ln \sqrt{|h_3 h_4|}]^{*} = -2\lambda_{[v]}(x^k, v) h_3 h_4.$$

For $\lambda = 0$ a class of solution is given by any $\widehat{h}_3$ and $\widehat{h}_4$ related as

$$\widehat{h}_3 = \eta_0 \left[\left(\sqrt{|\hat{h}_4|}\right)^{*}\right]^2$$

for a constant η_0 chosen to be negative in order to generate the signature $(-,-,-,+)$. For non–trivial λ, we may search the solution as

$$h_3 = \widehat{h}_3(\xi,\varphi)\ \ f_3(\xi,\varphi) \text{ and } h_4 = \widehat{h}_4(\xi,\varphi), \tag{3.53}$$

which solves (3.46) if $f_3 = 1$ for $\lambda_{[v]} = 0$ and

$$f_3 = \frac{1}{4}\left[\int \frac{\lambda_{[v]}\hat{h}_3\hat{h}_4}{\hat{h}_4^*}d\varphi\right]^{-1} \text{ for } \lambda_{[v]} \neq 0.$$

Now it is easy to write down the solutions of equations (3.47) (being a linear equation for w_i) and (3.48) (after two integrations of n_i on φ),

$$w_i = \varepsilon\widehat{w}_i = -\alpha_i/\beta, \qquad (3.54)$$

were α_i and β are computed by putting (3.53) into corresponding values from (3.49) (we chose the initial conditions as $w_i \to 0$ for $\varepsilon \to 0$) and

$$n_1 = \varepsilon\widehat{n}_1(\xi,\varphi)$$

where the coefficients

$$\begin{aligned} \widehat{n}_1(\xi,\varphi) &= n_{1[1]}(\xi) + n_{1[2]}(\xi)\int d\varphi \;\; f_3(\xi,\varphi)\,\eta_3(\xi,\varphi)\,/\left(\sqrt{|\eta_4(\xi,\varphi)|}\right)^3, \eta_4^* \neq 0; \qquad (3.55)\\ &= n_{1[1]}(\xi) + n_{1[2]}(\xi)\int d\varphi \;\; f_3(\xi,\varphi)\,\eta_3(\xi,\varphi)\,, \eta_4^* = 0;\\ &= n_{1[1]}(\xi) + n_{1[2]}(\xi)\int d\varphi/\left(\sqrt{|\eta_4(\xi,\varphi)|}\right)^3, \eta_3^* = 0; \end{aligned}$$

with the functions $n_{k[1,2]}(\xi)$ to be stated by boundary conditions.

We conclude that the set of data $g_1 = -1$, with non–trivial $g_2(\xi), h_3, h_4, w_{i'}$ and n_1 stated respectively by (3.51), (3.53), (3.54), (3.55) we can define a black ellipsoid solution with explicit dependence on polarized cosmological "constants" $\lambda_{[h]}(x^1)$ and $\lambda_{[v]}(x^k, v)$, i. e. a metric (3.44).

Finally, we analyze the structure of noncommutative symmetries associated to the (anti) de Sitter black ellipsoid solutions. The metric (3.44) with real and/or complex coefficients defining the corresponding solutions and its analytic extensions also do not posses Killing symmetries being deformed by anholonomic transforms. For this solution, we can associate certain noncommutative symmetries following the same procedure as for the Einstein real/ complex gravity but with additional nontrivial coefficients of anholonomy and even with nonvanishing coefficients of the nonlinear connection curvature, $\Omega^3_{12} = \delta_1 N^3_2 - \delta_2 N^3_1$. Taking the data (3.54) and (3.55) and formulas (3.4), we compute the corresponding nontrivial anholonomy coefficients

$$\begin{aligned} w^{[N]4}_{\ \ 31} &= -w^{[N]4}_{\ \ 13} = \partial n_1(\xi,\varphi)/\partial\varphi = n_1^*(\xi,\varphi)\,, \qquad (3.56)\\ w^{[N]4}_{\ \ 12} &= -w^{[N]4}_{\ \ 21} = \delta_1(\alpha_2/\beta) - \delta_2(\alpha_1/\beta) \end{aligned}$$

for $\delta_1 = \partial/\partial\xi - w_1\partial/\partial\varphi$ and $\delta_2 = \partial/\partial\theta - w_2\partial/\partial\varphi$, with n_1 defined by (3.55) and $\alpha_{1,2}$ and β computed by using the formula (3.49) for the solutions (3.53). We have a 4D exact solution with nontrivial cosmological constant. So, for $n+m=4$, the condition $k^2-1=n+m$ can not satisfied by any integer numbers. We may trivially extend the dimensions like $n'=6$ and $m'=m=2$ and for $k=3$ to consider the Lie group $SL(3,\mathbb{C})$ noncommutativity with corresponding values of $Q_{\underline{\alpha\beta}}^{\underline{\gamma}}$ and structure constants $f_{\underline{\alpha\beta}}^{\underline{\gamma}}$, see (3.23). An extension $w^{[N]\gamma}_{\alpha\beta} \rightarrow W_{\underline{\alpha\beta}}^{\underline{\gamma}}$ may be performed by stating the N–deformed "structure" constants (3.26), $W_{\underline{\alpha\beta}}^{\underline{\gamma}} = f_{\underline{\alpha\beta}}^{\underline{\gamma}} + w^{[N]\underline{\gamma}}_{\underline{\alpha\beta}}$, with nontrivial values of $w^{[N]\underline{\gamma}}_{\underline{\alpha\beta}}$ given by (3.56). We note that the solutions with nontrivial cosmological constants are with induced torsion with the coefficients computed by using formulas (3.12) and the data (3.52), (3.53), (3.54) and (3.55).

3.4.3 Analytic extensions of black ellipsoid metrics

For the vacuum black ellipsoid metrics the method of analytic extension was considered in Ref. [29, 30]. The coefficients of the metric (3.31) (equivalently (3.38)) written with respect to the anholonomic frame (3.6) has a number of similarities with the Schwrzschild and Reissner–Nördstrom solutions. The cosmological "polarized" constants induce some additional factors like $q^{[u]}(\xi)$ and $f_3(\xi,\varphi)$ (see, respectively, formulas (3.52) and (3.53)) and modify the N–connection coefficients as in (3.54) and (3.55). For a corresponding class of smooth polarizations, the functions $q^{[u]}$ and f_3 do not change the singularity structure of the metric coefficients. If we identify ε with e^2, we get a static metric with effective "electric" charge induced by a small, quadratic on ε, off–diagonal metric extension. The coefficients of this metric are similar to those from the Reissner–Nördstrom solution but additionally to the mentioned frame anholonomy there are additional polarizations by the functions $q^{[u]}, h_{3[0]}, f_3, \eta_{3,4}, w_i$ and n_1. Another very important property is that the deformed metric was stated to define a vacuum, or with polarized cosmological constant, solution of the Einstein equations which differs substantially from the usual Reissner–Nördstrom metric being an exact static solution of the Einstein–Maxwell equations. For the limits $\varepsilon \rightarrow 0$ and $q, f_3, h_{3[0]} \rightarrow 1$ the metric (3.31) transforms into the usual Schwarzschild metric. A solution with ellipsoid symmetry can be selected by a corresponding condition of vanishing of the coefficient before the term δt which defines an ellipsoidal hypersurface like for the Kerr metric, but in our case the metric is non–rotating. In general, the space may be with frame induced torsion if we do not impose constraints on w_i and n_1 as to obtain vanishing nonlinear connection curvature and torsions.

The analytic extension of black ellipsoid solutions with cosmological constant can be performed similarly both for anholonomic frames with induced or trivial torsions. We note that the solutions in string theory may contain a frame induced torsion with the components (1.43) (in general, we can consider complex coefficients, see Ref. [10]) computed for nontrivial $N_{i'}^3 = -\alpha_{i'}/\beta$ (see (3.54)) and $N_1^4 = \varepsilon \widehat{n}_1(\xi, \varphi)$ (see (3.55)). This is an explicit example illustrating that the anholonomic frame method can be applied also for generating exact solutions in models of gravity with nontrivial torsion. For such solutions, we can perform corresponding analytic extensions and define Penrose diagram formalisms if the constructions are considered with respect to N–elongated vierbeins.

The metric (3.44) has a singular behavior for $r = r_\pm$, see (3.35). The aim of this subsection is to prove that this way we have constructed a solution of the Einstein equations with polarized cosmological constant. This solution possess an "anisotropic" horizon being a small deformation on parameter ε of the Schwarzschild's solution horizon. We may analyze the anisotropic horizon's properties for some fixed "direction" given in a smooth vicinity of any values $\varphi = \varphi_0$ and $r_+ = r_+(\varphi_0)$. The final conclusions will be some general ones for arbitrary φ when the explicit values of coefficients will have a parametric dependence on angular coordinate φ. The metrics (3.31), or (3.38), and (3.44) are regular in the regions I ($\infty > r > r_+^\Phi$), II ($r_+^\Phi > r > r_-^\Phi$) and III ($r_-^\Phi > r > 0$). As in the Schwarzschild, Reissner–Nördstrom and Kerr cases these singularities can be removed by introducing suitable coordinates and extending the manifold to obtain a maximal analytic extension [33, 34]. We have similar regions as in the Reissner–Nördstrom space–time, but with just only one possibility $\varepsilon < 1$ instead of three relations for static electro–vacuum cases ($e^2 < m^2, e^2 = m^2, e^2 > m^2$; where e and m are correspondingly the electric charge and mass of the point particle in the Reissner–Nördstrom metric). So, we may consider the usual Penrose's diagrams as for a particular case of the Reissner–Nördstrom space–time but keeping in mind that such diagrams and horizons have an additional polarizations and parametrization on an angular coordinate.

We can proceed in steps analogous to those in the Schwarzschild case (see details, for instance, in Ref. [37])) in order to construct the maximally extended manifold. The first step is to introduce a new coordinate

$$r^{\|} = \int dr \left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1}$$

for $r > r_+^1$ and find explicitly the coordinate

$$r^{\|} = r + \frac{(r_+^1)^2}{r_+^1 - r_-^1} \ln(r - r_+^1) - \frac{(r_-^1)^2}{r_+^1 - r_-^1} \ln(r - r_-^1), \tag{3.57}$$

where $r_{\pm}^{1}=r_{\pm}^{\Phi}$ with $\Phi=1$. If r is expressed as a function on ξ, than $r^{\parallel}$ can be also expressed as a function on ξ depending additionally on some parameters.

Defining the advanced and retarded coordinates, $v=t+r^{\parallel}$ and $w=t-r^{\parallel}$, with corresponding elongated differentials

$$\delta v=\delta t+dr^{\parallel} \text{ and } \delta w=\delta t-dr^{\parallel}$$

the metric (3.38) takes the form

$$\delta s^{2}=-r^{2}(\xi)q^{[u]}(\xi)d\theta^{2}-\eta_{3}(\xi,\varphi_{0})f_{3}(\xi,\varphi_{0})r^{2}(\xi)\sin^{2}\theta\delta\varphi^{2}+(1-\frac{2m}{r(\xi)}+\varepsilon\frac{\Phi_{4}(r,\varphi_{0})}{r^{2}(\xi)})\delta v\delta w, \quad (3.58)$$

where (in general, in non–explicit form) $r(\xi)$ is a function of type $r(\xi)=r\left(r^{\parallel}\right)=r\left(v,w\right).$ Introducing new coordinates $(v^{\prime\prime},w^{\prime\prime})$ by

$$v^{\prime\prime}=\arctan\left[\exp\left(\frac{r_{+}^{1}-r_{-}^{1}}{4(r_{+}^{1})^{2}}v\right)\right],w^{\prime\prime}=\arctan\left[-\exp\left(\frac{-r_{+}^{1}+r_{-}^{1}}{4(r_{+}^{1})^{2}}w\right)\right]$$

Defining r by

$$\tan v^{\prime\prime}\tan w^{\prime\prime}=-\exp\left[\frac{r_{+}^{1}-r_{-}^{1}}{2(r_{+}^{1})^{2}}r\right]\sqrt{\frac{r-r_{+}^{1}}{(r-r_{-}^{1})^{\chi}}},\chi=\left(\frac{r_{+}^{1}}{r_{-}^{1}}\right)^{2}$$

and multiplying (3.58) on the conformal factor we obtain

$$\begin{aligned}\delta s^{2} &= -r^{2}q^{[u]}(r)d\theta^{2}-\eta_{3}(r,\varphi_{0})f_{3}(r,\varphi_{0})r^{2}\sin^{2}\theta\delta\varphi^{2} \\ &+64\frac{(r_{+}^{1})^{4}}{(r_{+}^{1}-r_{-}^{1})^{2}}(1-\frac{2m}{r(\xi)}+\varepsilon\frac{\Phi_{4}(r,\varphi_{0})}{r^{2}(\xi)})\delta v^{\prime\prime}\delta w^{\prime\prime},\end{aligned} \quad (3.59)$$

As particular cases, we may chose $\eta_{3}\left(r,\varphi\right)$ as the condition of vanishing of the metric coefficient before $\delta v^{\prime\prime}\delta w^{\prime\prime}$ will describe a horizon parametrized by a resolution ellipsoid hypersurface. We emphasize that quadratic elements (3.58) and (3.59) have respective coefficients as the metrics investigated in Refs. [29, 30] but the polarized cosmological constants introduce not only additional polarizing factors $q^{[u]}(r)$ and $f_{3}(r,\varphi_{0})$ but also elongate the anholonomic frames in a different manner.

The maximal extension of the Schwarzschild metric deformed by a small parameter ε (for ellipsoid configurations treated as the eccentricity), i. e. the extension of the metric (3.44), is defined by taking (3.59) as the metric on the maximal manifold on which this metric is of smoothly class C^{2}. The Penrose diagram of this static but locally anisotropic space–time, for any fixed angular value φ_0 is similar to the Reissner–Nordstrom solution, for the case $e^2 \to \varepsilon$ and $e^2 < m^2$(see, for instance, Ref. [37])). There are an infinite number of asymptotically flat regions of type I, connected by intermediate regions II and III, where there is still an irremovable singularity at $r = 0$ for every region III. We may travel from a region I to another ones by passing through the 'wormholes' made by anisotropic deformations (ellipsoid off–diagonality of metrics, or anholonomy) like in the Reissner–Nordstrom universe because $\sqrt{\varepsilon}$ may model an effective electric charge. One can not turn back in a such travel. Of course, this interpretation holds true only for a corresponding smoothly class of polarization functions. For instance, if the cosmological constant is periodically polarized from a string model, see the formula (3.50), one could be additional resonances, aperiodicity and singularities.

It should be noted that the metric (3.59) can be analytic every were except at $r = r^1_-$. We may eliminate this coordinate degeneration by introducing another new coordinates

$$v''' = \arctan\left[\exp\left(\frac{r^1_+ - r^1_-}{2n_0(r^1_+)^2}v\right)\right], w''' = \arctan\left[-\exp\left(\frac{-r^1_+ + r^1_-}{2n_0(r^1_+)^2}w\right)\right],$$

where the integer $n_0 \geq (r^1_+)^2/(r^1_-)^2$. In these coordinates, the metric is analytic every were except at $r = r^1_+$ where it is degenerate. This way the space–time manifold can be covered by an analytic atlas by using coordinate carts defined by $(v'', w'', \theta, \varphi)$ and $(v''', w''', \theta, \varphi)$. Finally, we note that the analytic extensions of the deformed metrics were performed with respect to anholonomc frames which distinguish such constructions from those dealing only with holonomic coordinates, like for the usual Reissner–Nördstrom and Kerr metrics. We stated the conditions when on 'radial' like coordinates we preserve the main properties of the well know black hole solutions but in our case the metrics are generic off–diagonal and with vacuum gravitational polarizations.

3.4.4 Geodesics on static polarized ellipsoid backgrounds

We analyze the geodesic congruence of the metric (3.44) with the data (3.42) modified by polarized cosmological constant, for simplicity, being linear on ε,by introducing the effective Lagrangian (for instance, like in Ref. [16])

$$\begin{aligned} 2L &= g_{\alpha\beta}\frac{\delta u^\alpha}{ds}\frac{\delta u^\beta}{ds} = -\left(1 - \frac{2m}{r} + \frac{\varepsilon}{r^2}\right)^{-1}\left(\frac{dr}{ds}\right)^2 - r^2 q^{[u]}(r)\left(\frac{d\theta}{ds}\right)^2 \\ &-\eta_3(r,\varphi)f_3(r,\varphi)r^2\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2 + \left(1 - \frac{2m}{r} + \frac{\varepsilon\Phi_4}{r^2}\right)\left(\frac{dt}{ds} + \varepsilon\widehat{n}_1\frac{dr}{ds}\right)^2, \end{aligned} \tag{3.60}$$

for $r = r(\xi)$.

The corresponding Euler–Lagrange equations,

$$\frac{d}{ds}\frac{\partial L}{\partial \frac{\delta u^{\alpha}}{ds}} - \frac{\partial L}{\partial u^{\alpha}} = 0$$

are

$$\frac{d}{ds}\left[-r^2 q^{[u]}(r)\frac{d\theta}{ds}\right] = -\eta_3 f_3 r^2 \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2, \quad (3.61)$$
$$\frac{d}{ds}\left[-\eta_3 f_3 r^2\frac{d\varphi}{ds}\right] = -(\eta_3 f_3)^* \frac{r^2}{2}\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2$$
$$+\frac{\varepsilon}{2}\left(1-\frac{2m}{r}\right)\left[\frac{\Phi_4^*}{r^2}\left(\frac{dt}{ds}\right)^2 + \widehat{n}_1^*\frac{dt}{ds}\frac{d\xi}{ds}\right]$$

and

$$\frac{d}{ds}\left[(1-\frac{2m}{r}+\frac{\varepsilon\Phi_4}{r^2})\left(\frac{dt}{ds}+\varepsilon\widehat{n}_1\frac{d\xi}{ds}\right)\right] = 0, \quad (3.62)$$

where, for instance, $\Phi_4^* = \partial\,\Phi_4/\partial\varphi$ we have omitted the variations for $d\xi/ds$ which may be found from (3.60). The system of equations (3.60)–(3.62) transform into the usual system of geodesic equations for the Schwarzschild space–time if $\varepsilon \to 0$ and $q^{[u]}, \eta_3, f_3 \to 1$ which can be solved exactly [16]. For nontrivial values of the parameter ε and polarizations η_3, f_3 even to obtain some decompositions of solutions on ε for arbitrary η_3 and $n_{1[1,2]}$, see (3.41), is a cumbersome task. In spite of the fact that with respect to anholonomic frames the metrics (3.38) and/or (3.44) has their coefficients being very similar to the Reissner–Nordstom solution. The geodesic behavior, in our anisotropic cases, is more sophisticate because of anholonomy, polarization of constants and coefficients and "elongation" of partial derivatives. For instance, the equations (3.61) state that there is not any angular on φ, conservation law if $(\eta_3 f_3)^* \neq 0$, even for $\varepsilon \to 0$ (which holds both for the Schwarzschild and Reissner–Nordstom metrics). One follows from the equation (3.62) the existence of an energy like integral of motion, $E = E_0 + \varepsilon E_1$, with

$$E_0 = \left(1-\frac{2m}{r}\right)\frac{dt}{ds}$$
$$E_1 = \frac{\Phi_4}{r^2}\frac{dt}{ds} + \left(1-\frac{2m}{r}\right)\widehat{n}_1\frac{d\xi}{ds}.$$

The introduced anisotropic deformations of congruences of Schwarzschild's space–time geodesics maintain the known behavior in the vicinity of the horizon hypersurface defined by the condition of vanishing of the coefficient $(1 - 2m/r + \varepsilon\Phi_4/r^2)$ in (3.59). The simplest way to prove this is to consider radial null geodesics in the "equatorial plane", which satisfy the condition (3.60) with $\theta = \pi/2, d\theta/ds = 0, d^2\theta/ds^2 = 0$ and $d\varphi/ds = 0$, from which follows that

$$\frac{dr}{dt} = \pm\left(1 - \frac{2m}{r} + \frac{\varepsilon_0}{r^2}\right)\left[1 + \varepsilon\widehat{n}_1 d\varphi\right].$$

The integral of this equation, for every fixed value $\varphi = \varphi_0$ is

$$t = \pm r^{\|} + \varepsilon \int \left[\frac{\Phi_4(r,\varphi_0) - 1}{2\left(r^2 - 2mr\right)^2} - \widehat{n}_1(r,\varphi_0)\right] dr$$

where the coordinate $r^{\|}$ is defined in equation (3.57). In this formula the term proportional to ε can have non–singular behavior for a corresponding class of polarizations λ_4, see the formulas (3.33). Even the explicit form of the integral depends on the type of polarizations $\eta_3(r,\varphi_0)$, $f_3(r,\varphi_0)$ and values $n_{1[1,2]}(r)$, which results in some small deviations of the null–geodesics, we may conclude that for an in–going null–ray the coordinate time t increases from $-\infty$ to $+\infty$ as r decreases from $+\infty$ to r_+^1, decreases from $+\infty$ to $-\infty$ as r further decreases from r_+^1 to r_-^1, and increases again from $-\infty$ to a finite limit as r decreases from r_-^1 to zero. We have a similar behavior as for the Reissner–Nordstrom solution but with some additional anisotropic contributions being proportional to ε. Here we also note that as dt/ds tends to $+\infty$ for $r \to r_+^1 + 0$ and to $-\infty$ as $r_- + 0$, any radiation received from infinity appear to be infinitely red–shifted at the crossing of the event horizon and infinitely blue–shifted at the crossing of the Cauchy horizon.

The mentioned properties of null–geodesics allow us to conclude that the metric (3.31) (equivalently, (3.38)) with the data (3.42) and their maximal analytic extension (3.59) really define a black hole static solution which is obtained by anisotropic small deformations on ε and renormalization by $\eta_3 f_3$ of the Schwarzchild solution (for a corresponding type of deformations the horizon of such black holes is defined by ellipsoid hypersurfaces). We call such objects as black ellipsoids, or black rotoids. They exists in the framework of general relativity as certain solutions of the Einstein equations defined by static generic off–diagonal metrics and associated anholonomic frames or can be induced by polarized cosmological constants. This property disinguishes them from similar configurations of Reissner–Norstrom type (which are static electrovacuum solutions of the Einstein–Maxwell equations) and of Kerr type rotating configurations, with ellipsoid horizon, also defined by off–diagonal vacuum metrics (here we emphasized that the spherical coordinate system is associated to a holonomic frame which is a trivial case of anholonomic bases). By introducing the polarized cosmological constants, the anholonomic character of N–adapted frames allow to construct solutions being very different from the black hole solutions in (anti) de Sitter spacetimes. We selected here a class of solutions where cosmological factors correspond to some additional polarizations but do not change the singularity structure of black ellipsoid solutions.

The metric (3.31) and its analytic extensions do not posses Killing symmetries being deformed by anholonomic transforms. Nevertheless, we can associate to such solutions certain noncommutative symmetries [10]. Taking the data (3.42) and formulas (3.8), we compute the corresponding nontrivial anholonomy coefficients

$$w_{42}^{[N]5} = -w_{24}^{[N]5} = \partial n_2\left(\xi,\varphi\right)/\partial\varphi = n_2^{*}\left(\xi,\varphi\right) \tag{3.63}$$

with n_2 defined by (3.42). Our solutions are for 4D configuration. So for $n+m=4$, the condition $k^2-1=n+m$ can not satisfied in integer numbers. We may trivially extend the dimensions like $n'=6$ and $m'=m=2$ and for $k=3$ to consider the Lie group $SL\left(3,\mathbb{C}\right)$ noncommutativity with corresponding values of $Q_{\underline{\alpha\beta}}^{\gamma}$ and structure constants $f_{\underline{\alpha\beta}}^{\gamma}$, see (3.23). An extension $w_{\alpha\beta}^{[N]\gamma}\rightarrow W_{\underline{\alpha\beta}}^{\gamma}$ may be performed by stating the N–deformed "structure" constants (3.26), $W_{\underline{\alpha\beta}}^{\gamma}=f_{\underline{\alpha\beta}}^{\gamma}+w_{\underline{\alpha\beta}}^{[N]\gamma}$, with only two nontrivial values of $w_{\underline{\alpha\beta}}^{[N]\gamma}$ given by (3.63). In a similar manner we can compute the anholonomy coefficients for the black ellipsoid metric with cosmological constant contributions (3.44).

3.5 Perturbations of Anisotropic Black Holes

The stability of black ellipsoids was proven in Ref. [36]. A similar proof may hold true for a class of metrics with anholonomic noncommutative symmetry and possible complexification of some off–diagonal metric and tetradics coefficients [10]. In this section we reconsider the perturbation formalism and stability proofs for rotoid metrics defined by polarized cosmological constants.

3.5.1 Metrics describing anisotropic perturbations

We consider a four dimensional pseudo–Riemannian quadratic linear element

$$\begin{aligned} ds^2 = &\ \Omega(r,\varphi) \left[-\left(1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\right)^{-1} dr^2 - r^2 q^{[v]}(r) d\theta^2 - \eta_3^{[v]}(r,\theta,\varphi) r^2 \sin^2\theta \delta\varphi^2\right] \\ + &\ \left[1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\eta(r,\varphi)\right]\delta t^2, \end{aligned} \tag{3.64}$$

$$\eta_3^{[v]}(r,\theta,\varphi) = \eta_3(r,\theta,\varphi) f_3(r,\theta,\varphi)$$

with

$$\delta\varphi = d\varphi + \varepsilon w_1(r,\varphi)dr, \text{ and } \delta t = dt + \varepsilon n_1(r,\varphi)dr,$$

where the local coordinates are denoted $u = \{u^\alpha = (r,\theta,\varphi,t)\}$ (the Greek indices $\alpha, \beta, ...$ will run the values 1,2,3,4), ε is a small parameter satisfying the conditions $0 \leq \varepsilon \ll 1$ (for instance, an eccentricity for some ellipsoid deformations of the spherical symmetry) and the functions $\Omega(r,\varphi), q(r), \eta_3(r,\theta,\varphi)$ and $\eta(\theta,\varphi)$ are of necessary smooth class. The metric (3.64) is static, off–diagonal and transforms into the usual Schwarzschild solution if $\varepsilon \to 0$ and $\Omega, q^{[v]}, \eta_3^{[v]} \to 1$. For vanishing cosmological constants, it describes at least two classes of static black hole solutions generated as small anhlonomic deformations of the Schwarzschild solution [1, 2, 29, 30, 31, 36] but models also nontrivial vacuum polarized cosmological constants.

We can apply the perturbation theory for the metric (3.64) (not paying a special attention to some particular parametrization of coefficients for one or another class of anisotropic black hole solutions) and analyze its stability by using the results of Ref. [16] for a fixed anisotropic direction, i. e. by imposing certain anholonomic frame constraints for an angle $\varphi = \varphi_0$ but considering possible perturbations depending on three variables ($u^1 = x^1 = r, u^2 = x^2 = \theta,\ u^4 = t$). We suppose that if we prove that there is a stability on perturbations for a value φ_0, we can analyze in a similar manner another values of φ. A more general perturbative theory with variable anisotropy on coordinate φ, i. e. with dynamical anholonomic constraints, connects the approach with a two dimensional inverse problem which makes the analysis more sophisticate. There have been not elaborated such analytic methods in the theory of black holes.

It should be noted that in a study of perturbations of any spherically symmetric system and, for instance, of small ellipsoid deformations, without any loss of generality, we can restrict our considerations to axisymmetric modes of perturbations. Non–axisymmetric modes of perturbations with an $e^{in\varphi}$ dependence on the azimutal angle φ (n being an integer number) can be deduced from modes of axisymmetric perturbations with $n = 0$ by suitable rotations since there are not preferred axes in a spherically symmetric background. The ellipsoid like deformations may be included into the formalism as some low frequency and constrained perturbations.

We develop the black hole perturbation and stability theory as to include into consideration off–diagonal metrics with the coefficients polarized by cosmological constants. This is the main difference comparing to the paper [36]. For simplicity, in this section, we restrict our study only to fixed values of the coordinate φ assuming that anholonomic deformations are proportional to a small parameter ε; we shall investigate the stability of solutions only by applying the one dimensional inverse methods.

We state a quadratic metric element

$$\begin{aligned} ds^2 &= -e^{2\mu_1}(du^1)^2 - e^{2\mu_2}(du^2)^2 - e^{2\mu_3}(\delta u^3)^2 + e^{2\mu_4}(\delta u^4)^2, \\ \delta u^3 &= d\varphi - q_1 dx^1 - q_2 dx^2 - \omega dt, \\ \delta u^4 &= dt + n_1 dr \end{aligned} \quad (3.65)$$

where

$$\begin{aligned} \mu_\alpha(x^k,t) &= \mu_\alpha^{(\varepsilon)}(x^k,\varphi_0) + \delta\mu_\alpha^{(\varsigma)}(x^k,t), \\ q_i(x^k,t) &= q_i^{(\varepsilon)}(r,\varphi_0) + \delta q_i^{(\varsigma)}(x^k,t),\ \omega(x^k,t) = 0 + \delta\omega^{(\varsigma)}(x^k,t) \end{aligned} \quad (3.66)$$

with

$$\begin{aligned} e^{2\mu_1^{(\varepsilon)}} &= \Omega(r,\varphi_0)(1-\frac{2m}{r}+\frac{\varepsilon}{r^2})^{-1},\ e^{2\mu_2^{(\varepsilon)}} = \Omega(r,\varphi_0)q^{[v]}(r)r^2, \\ e^{2\mu_3^{(\varepsilon)}} &= \Omega(r,\varphi_0)r^2\sin^2\theta\,\eta_3^{[v]}(r,\varphi_0),\ e^{2\mu_4^{(\varepsilon)}} = 1-\frac{2m}{r}+\frac{\varepsilon}{r^2}\eta(r,\varphi_0), \end{aligned} \quad (3.67)$$

and some non–trivial values for $q_i^{(\varepsilon)}$ and εn_i,

$$\begin{aligned} q_i^{(\varepsilon)} &= \varepsilon w_i(r,\varphi_0), \\ n_1 &= \varepsilon\left(n_{1[1]}(r) + n_{1[2]}(r)\int_0^{\varphi_0}\eta_3(r,\varphi)d\varphi\right). \end{aligned}$$

We have to distinguish two types of small deformations from the spherical symmetry. The first type of deformations, labelled with the index (ε) are generated by some ε–terms which define a fixed ellipsoid like configuration and the second type ones, labelled with the index (ς), are some small linear fluctuations of the metric coefficients

The general formulas for the Ricci and Einstein tensors for metric elements of class (3.65) with $w_i, n_1 = 0$ are given in [16]. We compute similar values with respect to anholnomic frames, when, for a conventional splitting $u^\alpha = (x^i, y^a)$, the coordinates x^i and y^a are treated respectively as holonomic and anholonomic ones. In this case the partial derivatives $\partial/\partial x^i$ must be changed into certain 'elongated' ones

$$\begin{aligned} \frac{\partial}{\partial x^1} &\to \frac{\delta}{\partial x^1} = \frac{\partial}{\partial x^1} - w_1\frac{\partial}{\partial\varphi} - n_1\frac{\partial}{\partial t}, \\ \frac{\partial}{\partial x^2} &\to \frac{\delta}{\partial x^2} = \frac{\partial}{\partial x^2} - w_2\frac{\partial}{\partial\varphi}, \end{aligned}$$

see details in Refs [20, 3, 29, 30]. In the ansatz (3.65), the anholonomic contributions of w_i are included in the coefficients $q_i(x^k,t)$. For convenience, we give present bellow the necessary formulas for $R_{\alpha\beta}$ (the Ricci tensor) and $G_{\alpha\beta}$ (the Einstein tensor) computed for the ansatz (3.65) with three holonomic coordinates (r, θ, φ) and on anholonomic coordinate t (in our case, being time like), with the partial derivative operators

$$\partial_1 \to \delta_1 = \frac{\partial}{\partial r} w_1 \frac{\partial}{\partial \varphi} - n_1 \frac{\partial}{\partial t}, \delta_2 = \frac{\partial}{\partial \theta} - w_2 \frac{\partial}{\partial \varphi}, \partial_3 = \frac{\partial}{\partial \varphi},$$

and for a fixed value φ_0.

A general perturbation of an anisotropic black–hole described by a quadratic line element (3.65) results in some small quantities of the first order ω and q_i, inducing a dragging of frames and imparting rotations, and in some functions μ_α with small increments $\delta\mu_\alpha$, which do not impart rotations. Some coefficients contained in such values are proportional to ε, another ones are considered only as small quantities. The perturbations of metric are of two generic types: axial and polar one. We shall investigate them separately in the next two subsection after we shall have computed the coefficients of the Ricci tensor.

We compute the coefficients of the the Ricci tensor as

$$R^{\alpha}_{\beta\gamma\alpha} = R_{\beta\gamma}$$

and of the Einstein tensor as

$$G_{\beta\gamma} = R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} R$$

for $R = g^{\beta\gamma} R_{\beta\gamma}$. Straightforward computations for the quadratic line element (3.65) give

$$\begin{aligned} R_{11} &= -e^{-2\mu_1}[\delta^2_{11}(\mu_3 + \mu_4 + \mu_2) + \delta_1\mu_3\delta_1(\mu_3 - \mu_1) + \delta_1\mu_2\delta_1(\mu_2 - \mu_1) + \\ &\quad \delta_1\mu_4\delta_1(\mu_4 - \mu_1)] - e^{-2\mu_2}[\delta^2_{22}\mu_1 + \delta_2\mu_1\delta_2(\mu_3 + \mu_4 + \mu_1 - \mu_2)] + \\ &\quad e^{-2\mu_4}[\partial^2_{44}\mu_1 + \partial_4\mu_1\partial_4(\mu_3 - \mu_4 + \mu_1 + \mu_2)] - \frac{1}{2}e^{2(\mu_3 - \mu_1)}[e^{-2\mu_2}Q^2_{12} + e^{-2\mu_4}Q^2_{14}], \end{aligned} \quad (3.68)$$

$$\begin{aligned} R_{12} &= -e^{-\mu_1-\mu_2}[\delta_2\delta_1(\mu_3 + \mu_2) - \delta_2\mu_1\delta_1(\mu_3 + \mu_1) - \delta_1\mu_2\partial_4(\mu_3 + \mu_1) \\ &\quad +\delta_1\mu_3\delta_2\mu_3 + \delta_1\mu_4\delta_2\mu_4] + \frac{1}{2}e^{2\mu_3 - 2\mu_4 - \mu_1 - \mu_2}Q_{14}Q_{24}, \end{aligned}$$

$$R_{31} = -\frac{1}{2}e^{2\mu_3 - \mu_4 - \mu_2}[\delta_2(e^{3\mu_3 + \mu_4 - \mu_1 - \mu_2}Q_{21}) + \partial_4(e^{3\mu_3 - \mu_4 + \mu_2 - \mu_1}Q_{41})],$$

$$\begin{aligned} R_{33} &= -e^{-2\mu_1}[\delta^2_{11}\mu_3 + \delta_1\mu_3\delta_1(\mu_3 + \mu_4 + \mu_2 - \mu_1)] - \\ &\quad e^{-2\mu_2}[\delta^2_{22}\mu_3 + \partial_2\mu_3\partial_2(\mu_3 + \mu_4 - \mu_2 + \mu_1)] + \frac{1}{2}e^{2(\mu_3 - \mu_1 - \mu_2)}Q^2_{12} + \\ &\quad e^{-2\mu_4}[\partial^2_{44}\mu_3 + \partial_4\mu_3\partial_4(\mu_3 - \mu_4 + \mu_2 + \mu_1)] - \frac{1}{2}e^{2(\mu_3 - \mu_4)}[e^{-2\mu_2}Q^2_{24} + e^{-2\mu_1}Q^2_{14}], \end{aligned}$$

$$
\begin{aligned}
R_{41} &= -e^{-\mu_1-\mu_4}[\partial_4\delta_1(\mu_3+\mu_2)+\delta_1\mu_3\partial_4(\mu_3-\mu_1)+\delta_1\mu_2\partial_4(\mu_2-\mu_1) \\
&\quad -\delta_1\mu_4\partial_4(\mu_3+\mu_2)]+\frac{1}{2}e^{2\mu_3-\mu_4-\mu_1-2\mu_2}Q_{12}Q_{34},
\end{aligned}
$$

$$
R_{43}=-\frac{1}{2}e^{2\mu_3-\mu_1-\mu_2}[\delta_1(e^{3\mu_3-\mu_4-\mu_1+\mu_2}Q_{14})+\delta_2(e^{3\mu_3-\mu_4+\mu_1-\mu_2}Q_{24})],
$$

$$
\begin{aligned}
R_{44} &= -e^{-2\mu_4}[\partial_{44}^2(\mu_1+\mu_2+\mu_3)+\partial_4\mu_3\partial_4(\mu_3-\mu_4)+\partial_4\mu_1\partial_4(\mu_1-\mu_4)+ \\
&\quad \partial_4\mu_2\partial_4(\mu_2-\mu_4)]+e^{-2\mu_1}[\delta_{11}^2\mu_4+\delta_1\mu_4\delta_1(\mu_3+\mu_4-\mu_1+\mu_2)]+ \\
&\quad e^{-2\mu_2}[\delta_{22}^2\mu_4+\delta_2\mu_4\delta_2(\mu_3+\mu_4-\mu_1+\mu_2)]-\frac{1}{2}e^{2(\mu_3-\mu_4)}[e^{-2\mu_1}Q_{14}^2+e^{-2\mu_2}Q_{24}^2],
\end{aligned}
$$

where the rest of coefficients are defined by similar formulas with a corresponding changing of indices and partial derivative operators, R_{22}, R_{42} and R_{32} is like R_{11}, R_{41} and R_{31} with with changing the index $1 \to 2$. The values Q_{ij} and Q_{i4} are defined respectively

$$
Q_{ij}=\delta_j q_i-\delta_i q_j \text{ and } Q_{i4}=\partial_4 q_i-\delta_i\omega.
$$

The nontrivial coefficients of the Einstein tensor are

$$
\begin{aligned}
G_{11} &= e^{-2\mu_2}[\delta_{22}^2(\mu_3+\mu_4)+\delta_2(\mu_3+\mu_4)\delta_2(\mu_4-\mu_2)+\delta_2\mu_3\delta_2\mu_3]- \\
&\quad e^{-2\mu_4}[\partial_{44}^2(\mu_3+\mu_2)+\partial_4(\mu_3+\mu_2)\partial_4(\mu_2-\mu_4)+\partial_4\mu_3\partial_4\mu_3]+ \\
&\quad e^{-2\mu_1}[\delta_1\mu_4+\delta_1(\mu_3+\mu_2)+\delta_1\mu_3\delta_1\mu_2]- \\
&\quad \frac{1}{4}e^{2\mu_3}[e^{-2(\mu_1+\mu_2)}Q_{12}^2-e^{-2(\mu_1+\mu_4)}Q_{14}^2+e^{-2(\mu_2+\mu_3)}Q_{24}^2],
\end{aligned} \tag{3.69}
$$

$$
\begin{aligned}
G_{33} &= e^{-2\mu_1}[\delta_{11}^2(\mu_4+\mu_2)+\delta_1\mu_4\delta_1(\mu_4-\mu_1+\mu_2)+\delta_1\mu_2\delta_1(\mu_2-\mu_1)]+ \\
&\quad e^{-2\mu_2}[\delta_{22}^2(\mu_4+\mu_1)+\delta_2(\mu_4-\mu_2+\mu_1)+\delta_2\mu_1\partial_2(\mu_1-\mu_2)]- \\
&\quad e^{-2\mu_4}[\partial_{44}^2(\mu_1+\mu_2)+\partial_4\mu_1\partial_4(\mu_1-\mu_4)+\partial_4\mu_2\partial_4(\mu_2-\mu_4)+\partial_4\mu_1\partial_4\mu_2]+ \\
&\quad \frac{3}{4}e^{2\mu_3}[e^{-2(\mu_1+\mu_2)}Q_{12}^2-e^{-2(\mu_1+\mu_4)}Q_{14}^2-e^{-2(\mu_2+\mu_3)}Q_{24}^2],
\end{aligned}
$$

$$
\begin{aligned}
G_{44} &= e^{-2\mu_1}[\delta_{11}^2(\mu_3+\mu_2)+\delta_1\mu_3\delta_1(\mu_3-\mu_1+\mu_2)+\delta_1\mu_2\delta_1(\mu_2-\mu_1)]- \\
&\quad e^{-2\mu_2}[\delta_{22}^2(\mu_3+\mu_1)+\delta_2(\mu_3-\mu_2+\mu_1)+\delta_2\mu_1\partial_2(\mu_1-\mu_2)]-\frac{1}{4}e^{2(\mu_3-\mu_1-\mu_2)}Q_{12}^2 \\
&\quad +e^{-2\mu_4}[\partial_4\mu_3\partial_4(\mu_1+\mu_2)+\partial_4\mu_1\partial_4\mu_2]-\frac{1}{4}e^{2(\mu_3-\mu_4)}[e^{-2\mu_1}Q_{14}^2-e^{-2\mu_2}Q_{24}^2].
\end{aligned}
$$

The component G_{22} is to be found from G_{11} by changing the index $1 \to 2$. We note that the formulas (3.69) transform into similar ones from Ref. [36] if $\delta_2 \to \partial_2$.

3.5.2 Axial metric perturbations

Axial perturbations are characterized by non–vanishing ω and q_i which satisfy the equations

$$R_{3i} = 0,$$

see the explicit formulas for such coefficients of the Ricci tensor in (3.68). The resulting equations governing axial perturbations, $\delta R_{31} = 0$, $\delta R_{32} = 0$, are respectively

$$\begin{aligned} \delta_2 \left(e^{3\mu_3^{(\varepsilon)} + \mu_4^{(\varepsilon)} - \mu_1^{(\varepsilon)} - \mu_2^{(\varepsilon)}} Q_{12} \right) &= -e^{3\mu_3^{(\varepsilon)} - \mu_4^{(\varepsilon)} - \mu_1^{(\varepsilon)} + \mu_2^{(\varepsilon)}} \partial_4 Q_{14}, \\ \delta_1 \left(e^{3\mu_3^{(\varepsilon)} + \mu_4^{(\varepsilon)} - \mu_1^{(\varepsilon)} - \mu_2^{(\varepsilon)}} Q_{12} \right) &= e^{3\mu_3^{(\varepsilon)} - \mu_4^{(\varepsilon)} + \mu_1^{(\varepsilon)} - \mu_2^{(\varepsilon)}} \partial_4 Q_{24}, \end{aligned} \quad (3.70)$$

where

$$Q_{ij} = \delta_i q_j - \delta_j q_i, Q_{i4} = \partial_4 q_i - \delta_i \omega \quad (3.71)$$

and for μ_i there are considered unperturbed values $\mu_i^{(\varepsilon)}$. Introducing the values of coefficients (3.66) and (3.67) and assuming that the perturbations have a time dependence of type $\exp(i\sigma t)$ for a real constant σ, we rewrite the equations (3.70)

$$\frac{1 + \varepsilon\left(\Delta^{-1} + 3r^2\phi/2\right)}{r^4 \sin^3\theta (\eta_3^{[v]})^{3/2}} \delta_2 Q^{(\eta)} = -i\sigma\delta_r\omega - \sigma^2 q_1, \quad (3.72)$$

$$\frac{\Delta}{r^4 \sin^3\theta (\eta_3^{[v]})^{3/2}} \delta_1 \left\{ Q^{(\eta)} \left[1 + \frac{\varepsilon}{2} \left(\frac{\eta - 1}{\Delta} - r^2\phi \right) \right] \right\} = i\sigma\partial_\theta\omega + \sigma^2 q_2 \quad (3.73)$$

for

$$Q^{(\eta)}(r, \theta, \varphi_0, t) = \Delta Q_{12} \sin^3\theta = \Delta \sin^3\theta (\partial_2 q_1 - \delta_1 q_2), \Delta = r^2 - 2mr,$$

where $\phi = 0$ for solutions with $\Omega = 1$ and $\phi(r, \varphi) = \eta_3^{[v]}(r, \theta, \varphi) \sin^2\theta$, i. e. $\eta_3(r, \theta, \varphi) \sim \sin^{-2}\theta$ for solutions with $\Omega = 1 + \varepsilon$....

We can exclude the function ω and define an equation for $Q^{(\eta)}$ if we take the sum of the (3.72) subjected by the action of operator ∂_2 and of the (3.73) subjected by the action of operator δ_1. Using the relations (3.71), we write

$$\begin{aligned} r^4 \delta_1 \left\{ \frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}} \left[\delta_1 \left[Q^{(\eta)} + \frac{\varepsilon}{2} \left(\frac{\eta - 1}{\Delta} - r^2 \right) \phi \right] \right] \right\} + & \\ \sin^3\theta \partial_2 \left[\frac{1 + \varepsilon(\Delta^{-1} + 3r^2\phi/2)}{\sin^3\theta (\eta_3^{[v]})^{3/2}} \delta_2 Q^{(\eta)} \right] + \frac{\sigma^2 r^4}{\Delta \eta_3^{3/2}} Q^{(\eta)} &= 0. \end{aligned}$$

The solution of this equation is searched in the form $Q^{(\eta)} = Q + \varepsilon Q^{(1)}$ which results in

$$r^4\partial_1\left(\frac{\Delta}{r^4(\eta_3^{[v]})^{3/2}}\partial_1 Q\right) + \sin^3\theta\partial_2\left(\frac{1}{\sin^3\theta(\eta_3^{[v]})^{3/2}}\partial_2 Q\right) + \frac{\sigma^2 r^4}{\Delta(\eta_3^{[v]})^{3/2}}Q = \varepsilon A\left(r,\theta,\varphi_0\right), \tag{3.74}$$

where

$$\begin{aligned} A\left(r,\theta,\varphi_0\right) &= r^4\partial_1\left(\frac{\Delta}{r^4(\eta_3^{[v]})^{3/2}}n_1\right)\frac{\partial Q}{\partial t} - r^4\partial_1\left(\frac{\Delta}{r^4(\eta_3^{[v]})^{3/2}}\partial_1 Q^{(1)}\right) \\ & -\sin^3\theta\delta_2\left[\frac{1+\varepsilon(\Delta^{-1}+3r^2\phi/2)}{\sin^3\theta(\eta_3^{[v]})^{3/2}}\delta_2 Q^{(1)} - \frac{\sigma^2 r^4}{\Delta(\eta_3^{[v]})^{3/2}}Q^{(1)}\right], \end{aligned}$$

with a time dependence like $\exp[i\sigma t]$

It is possible to construct different classes of solutions of the equation (3.74). At the first step we find the solution for Q when $\varepsilon = 0$. Then, for a known value of $Q\left(r,\theta,\varphi_0\right)$ from

$$Q^{(\eta)} = Q + \varepsilon Q^{(1)},$$

we can define $Q^{(1)}$ from the equations (3.72) and (3.73) by considering the values proportional to ε which can be written

$$\begin{aligned} \partial_1 Q^{(1)} &= B_1\left(r,\theta,\varphi_0\right), \\ \partial_2 Q^{(1)} &= B_2\left(r,\theta,\varphi_0\right). \end{aligned} \tag{3.75}$$

The integrability condition of the system (3.75), $\partial_1 B_2 = \partial_2 B_1$ imposes a relation between the polarization functions $\eta_3, \eta, w_1$and n_1 (for a corresponding class of solutions). In order to prove that there are stable anisotropic configurations of anisotropic black hole solutions, we may consider a set of polarization functions when $A\left(r,\theta,\varphi_0\right) = 0$ and the solution with $Q^{(1)} = 0$ is admitted. This holds, for example, if

$$\Delta n_1 = n_0 r^4(\eta_3^{[v]})^{3/2},\ n_0 = const.$$

In this case the axial perturbations are described by the equation

$$(\eta_3^{[v]})^{3/2}r^4\partial_1\left(\frac{\Delta}{r^4(\eta_3^{[v]})^{3/2}}\partial_1 Q\right) + \sin^3\theta\delta_2\left(\frac{1}{\sin^3\theta}\delta_2 Q\right) + \frac{\sigma^2 r^4}{\Delta}Q = 0 \tag{3.76}$$

which is obtained from (3.74) for $\eta_3^{[v]} = \eta_3^{[v]}\left(r,\varphi_0\right)$, or for $\phi(r,\varphi_0) = \eta_3^{[v]}\left(r,\theta,\varphi_0\right)\sin^2\theta$.

In the limit $\eta_3^{[v]} \to 1$ the solution of equation (3.76) is investigated in details in Ref. [16]. Here, we prove that in a similar manner we can define exact solutions for non–trivial values of $\eta_3^{[v]}$. The variables r and θ can be separated if we substitute

$$Q(r, \theta, \varphi_0) = Q_0(r, \varphi_0) C_{l+2}^{-3/2}(\theta),$$

where C_n^{ν} are the Gegenbauer functions generated by the equation

$$\left[\frac{d}{d\theta} \sin^{2\nu}\theta \frac{d}{d\theta} + n\,(n + 2\nu) \sin^{2\nu}\theta\right] C_n^{\nu}(\theta) = 0.$$

The function $C_{l+2}^{-3/2}(\theta)$ is related to the second derivative of the Legendre function $P_l(\theta)$ by formulas

$$C_{l+2}^{-3/2}(\theta) = \sin^3\theta \frac{d}{d\theta}\left[\frac{1}{\sin\theta}\frac{dP_l(\theta)}{d\theta}\right].$$

The separated part of (3.76) depending on radial variable with a fixed value φ_0 transforms into the equation

$$(\eta_3^{[v]})^{3/2}\Delta \frac{d}{dr}\left(\frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}}\frac{dQ_0}{dr}\right) + \left(\sigma^2 - \frac{\mu^2 \Delta}{r^4}\right) Q_0 = 0, \qquad (3.77)$$

where $\mu^2 = (l-1)(l+2)$ for $l = 2, 3, ...$ A further simplification is possible for $\eta_3^{[v]} = \eta_3^{[v]}(r, \varphi_0)$ if we introduce in the equation (3.77) a new radial coordinate

$$r_{\#} = \int (\eta_3^{[v]})^{3/2}(r, \varphi_0) r^2 dr$$

and a new unknown function $Z^{(\eta)} = r^{-1} Q_0(r)$. The equation for $Z^{(\eta)}$ is an Schrodinger like one–dimensional wave equation

$$\left(\frac{d^2}{dr_{\#}^2} + \frac{\sigma^2}{(\eta_3^{[v]})^{3/2}}\right) Z^{(\eta)} = V^{(\eta)} Z^{(\eta)} \qquad (3.78)$$

with the potential

$$V^{(\eta)} = \frac{\Delta}{r^5 (\eta_3^{[v]})^{3/2}}\left[\mu^2 - r^4 \frac{d}{dr}\left(\frac{\Delta}{r^4 (\eta_3^{[v]})^{3/2}}\right)\right] \qquad (3.79)$$

and polarized parameter

$$\widetilde{\sigma}^2 = \sigma^2/(\eta_3^{[v]})^{3/2}.$$

This equation transforms into the so–called Regge–Wheeler equation if $\eta_3^{[v]} = 1$. For instance, for the Schwarzschild black hole such solutions are investigated and tabulated for different values of $l = 2, 3$ and 4 in Ref. [16].

We note that for static anisotropic black holes with nontrivial anisotropic conformal factor, $\Omega = 1 + \varepsilon...$, even η_3 may depend on angular variable θ because of condition that $\phi(r, \varphi_0) = \eta_3^{[v]}(r, \theta, \varphi_0)\sin^2\theta$ the equation (3.76) transforms directly in (3.78) with $\mu = 0$ without any separation of variables r and θ. It is not necessary in this case to consider the Gegenbauer functions because Q_0 does not depend on θ which corresponds to a solution with $l = 1$.

We may transform (3.78) into the usual form,

$$\left(\frac{d^2}{dr_\star^2} + \sigma^2\right) Z^{(\eta)} = \widetilde{V}^{(\eta)} Z^{(\eta)}$$

if we introduce the variable

$$r_\star = \int dr_\#(\eta_3^{[v]})^{-3/2}(r_\#, \varphi_0)$$

for $\widetilde{V}^{(\eta)} = (\eta_3^{[v]})^{3/2} V^{(\eta)}$. So, the polarization function $\eta_3^{[v]}$, describing static anholonomic deformations of the Schwarzshild black hole, "renormalizes" the potential in the one–dimensional Schrodinger wave–equation governing axial perturbations of such objects.

We conclude that small static "ellipsoid" like deformations and polarizations of constants of spherical black holes (the anisotropic configurations being described by generic off–diagonal metric ansatz) do not change the type of equations for axial petrubations: one modifies the potential barrier,

$$V^{(-)} = \frac{\Delta}{r^5}\left[\left(\mu^2 + 2\right) r - 6m\right] \longrightarrow \widetilde{V}^{(\eta)}$$

and re–defines the radial variables

$$r_* = r + 2m \ln\left(r/2m - 1\right) \longrightarrow r_\star(\varphi_0)$$

with a parametric dependence on anisotropic angular coordinate which is caused by the existence of a deformed static horizon.

3.5.3 Polar metric perturbations

The polar perturbations are described by non–trivial increments of the diagonal metric coefficients, $\delta\mu_\alpha = \delta\mu_\alpha^{(\varepsilon)} + \delta\mu_\alpha^{(\varsigma)}$, for

$$\mu_\alpha^{(\varepsilon)} = \nu_\alpha + \delta\mu_\alpha^{(\varepsilon)}$$

where $\delta\mu_\alpha^{(\varsigma)}(x^k, t)$ parametrize time depending fluctuations which are stated to be the same both for spherical and/or spheroid configurations and $\delta\mu_\alpha^{(\varepsilon)}$ is a static deformation from the spherical symmetry. Following notations (3.66) and (3.67) we write

$$e^{v_1} = r/\sqrt{|\Delta|}, e^{v_2} = r\sqrt{|q^{[v]}(r)|}, e^{v_3} = rh_3 \sin\theta, e^{v_4} = \Delta/r^2$$

and

$$\delta\mu_1^{(\varepsilon)} = -\frac{\varepsilon}{2}\left(\Delta^{-1} + r^2\phi\right), \delta\mu_2^{(\varepsilon)} = \delta\mu_3^{(\varepsilon)} = -\frac{\varepsilon}{2} r^2\phi, \delta\mu_4^{(\varepsilon)} = \frac{\varepsilon\eta}{2\Delta}$$

where $\phi = 0$ for the solutions with $\Omega = 1$.

Examining the expressions for $R_{4i}, R_{12,}R_{33}$ and G_{11} (see respectively (3.68) and (3.69)) we conclude that the values Q_{ij} appear quadratically which can be ignored in a linear perturbation theory. Thus the equations for the axial and the polar perturbations decouple. Considering only linearized expressions, both for static ε–terms and fluctuations depending on time about the Schwarzschild values we obtain the equations

$$\begin{aligned}
\delta_1\left(\delta\mu_2 + \delta\mu_3\right) + \left(r^{-1} - \delta_1\mu_4\right)\left(\delta\mu_2 + \delta\mu_3\right) - 2r^{-1}\delta\mu_1 &= 0 \quad (\delta R_{41} = 0), \\
\delta_2\left(\delta\mu_1 + \delta\mu_3\right) + \left(\delta\mu_2 - \delta\mu_3\right)\cot\theta &= 0 \quad (\delta R_{42} = 0), \\
\delta_2\delta_1\left(\delta\mu_3 + \delta\mu_4\right) - \delta_1\left(\delta\mu_2 - \delta\mu_3\right)\cot\theta - & \\
\left(r^{-1} - \delta_1\mu_4\right)\delta_2(\delta\mu_4) - \left(r^{-1} + \delta_1\mu_4\right)\delta_2(\delta\mu_1) &= 0 \quad (\delta R_{42} = 0), \\
e^{2\mu_4}\{2\left(r^{-1} + \delta_1\mu_4\right)\delta_1(\delta\mu_3) + r^{-1}\delta_1\left(\delta\mu_3 + \delta\mu_4 - \delta\mu_1 + \delta\mu_2\right) + & \qquad (3.80) \\
\delta_1\left[\delta_1(\delta\mu_3)\right] - 2r^{-1}\delta\mu_1\left(r^{-1} + 2\delta_1\mu_4\right)\} - 2e^{-2\mu_4}\partial_4[\partial_4(\delta\mu_3)] + & \\
r^{-2}\{\delta_2[\delta_2(\delta\mu_3)] + \delta_2\left(2\delta\mu_3 + \delta\mu_4 + \delta\mu_1 - \delta\mu_2\right)\cot\theta + 2\delta\mu_2\} &= 0 \quad (\delta R_{33} = 0), \\
e^{-2\mu_1}[r^{-1}\delta_1(\delta\mu_4) + \left(r^{-1} + \delta_1\mu_4\right)\delta_1\left(\delta\mu_2 + \delta\mu_3\right) - & \\
2r^{-1}\delta\mu_1\left(r^{-1} + 2\delta_1\mu_4\right)] - e^{-2\mu_4}\partial_4[\partial_4(\delta\mu_3 + \delta\mu_2)] & \\
+r^{-2}\{\delta_2[\delta_2(\delta\mu_3)] + \delta_2\left(2\delta\mu_3 + \delta\mu_4 - \delta\mu_2\right)\cot\theta + 2\delta\mu_2\} &= 0 \quad (\delta G_{11} = 0).
\end{aligned}$$

The values of type $\delta\mu_\alpha = \delta\mu_\alpha^{(\varepsilon)} + \delta\mu_\alpha^{(\varsigma)}$ from (3.80) contain two components: the first ones are static, proportional to ε, and the second ones may depend on time coordinate t. We shall assume that the perturbations $\delta\mu_\alpha^{(\varsigma)}$ have a time–dependence $\exp[\sigma t]$ so that the partial time derivative "∂_4" is replaced by the factor $i\sigma$. In order to treat both type of increments in a similar fashion we may consider that the values labelled with (ε) also oscillate in time like $\exp[\sigma^{(\varepsilon)}t]$ but with a very small (almost zero) frequency $\sigma^{(\varepsilon)} \to 0$. There are also actions of "elongated" partial derivative operators like

$$\delta_1 (\delta\mu_\alpha) = \partial_1 (\delta\mu_\alpha) - \varepsilon n_1 \partial_4 (\delta\mu_\alpha).$$

To avoid a calculus with complex values we associate the terms proportional $\varepsilon n_1 \partial_4$ to amplitudes of type $\varepsilon i n_1 \partial_4$ and write this operator as

$$\delta_1 (\delta\mu_\alpha) = \partial_1 (\delta\mu_\alpha) + \varepsilon n_1 \sigma (\delta\mu_\alpha).$$

For the "non-perturbed" Schwarzschild values, which are static, the operator δ_1 reduces to ∂_1, i.e. $\delta_1 v_\alpha = \partial_1 v_\alpha$. Hereafter we shall consider that the solution of the system (3.80) consists from a superposition of two linear solutions, $\delta\mu_\alpha = \delta\mu_\alpha^{(\varepsilon)} + \delta\mu_\alpha^{(\varsigma)}$; the first class of solutions for increments will be provided with index (ε), corresponding to the frequency $\sigma^{(\varepsilon)}$ and the second class will be for the increments with index (ς) and correspond to the frequency $\sigma^{(\varsigma)}$. We shall write this as $\delta\mu_\alpha^{(A)}$ and $\sigma_{(A)}$ for the labels $A = \varepsilon$ or ς and suppress the factors $\exp[\sigma^{(A)}t]$ in our subsequent considerations. The system of equations (3.80) will be considered for both type of increments.

We can separate the variables by substitutions (see the method in Refs. [13, 16])

$$\begin{aligned} \delta\mu_1^{(A)} &= L^{(A)}(r)P_l(\cos\theta), \quad \delta\mu_2^{(A)} = \left[T^{(A)}(r)P_l(\cos\theta) + V^{(A)}(r)\partial^2 P_l/\partial\theta^2\right], \quad (3.81) \\ \delta\mu_3^{(A)} &= \left[T^{(A)}(r)P_l(\cos\theta) + V^{(A)}(r)\cot\theta\,\partial P_l/\partial\theta\right], \quad \delta\mu_4^{(A)} = N^{(A)}(r)P_l(\cos\theta) \end{aligned}$$

and reduce the system of equations (3.80) to

$$\begin{aligned} \delta_1 \left(N^{(A)} - L^{(A)}\right) &= \left(r^{-1} - \partial_1\nu_4\right) N^{(A)} + \left(r^{-1} + \partial_1\nu_4\right) L^{(A)}, \\ \delta_1 L^{(A)} + \left(2r^{-1} - \partial_1\nu_4\right) N^{(A)} &= -\left[\delta_1 X^{(A)} + \left(r^{-1} - \partial_1\nu_4\right) X^{(A)}\right], \quad (3.82) \end{aligned}$$

and

$$\begin{aligned} & 2r^{-1}\delta_1 \left(N^{(A)}\right) - l(l+1)r^{-2}e^{-2v_4}N^{(A)} - 2r^{-1}(r^{-1} + 2\partial_1\nu_4)L^{(A)} - 2(r^{-1} + \quad (3.83) \\ & \partial_1\nu_4)\delta_1 \left[N^{(A)} + (l-1)(l+2)V^{(A)}/2\right] - (l-1)(l+2)r^{-2}e^{-2v_4}\left(V^{(A)} - L^{(A)}\right) - \\ & 2\sigma_{(A)}^2 e^{-4v_4}\left[L^{(A)} + (l-1)(l+2)V^{(A)}/2\right] = 0, \end{aligned}$$

where we have introduced new functions

$$X^{(A)} = \frac{1}{2}(l-1)(l+2)V^{(A)}$$

and considered the relation

$$T^{(A)} - V^{(A)} + L^{(A)} = 0 \quad (\delta R_{42} = 0).$$

We can introduce the functions

$$\begin{aligned} \widetilde{L}^{(A)} &= L^{(A)} + \varepsilon\sigma_{(A)} \int n_1 L^{(A)} dr, \quad \widetilde{N}^{(A)} = N^{(A)} + \varepsilon\sigma_{(A)} \int n_1 N^{(A)} dr, \\ \widetilde{T}^{(A)} &= N^{(A)} + \varepsilon\sigma_{(A)} \int n_1 N^{(A)} dr, \quad \widetilde{V}^{(A)} = V^{(A)} + \varepsilon\sigma_{(A)} \int n_1 V^{(A)} dr, \end{aligned} \tag{3.84}$$

for which

$$\partial_1 \widetilde{L}^{(A)} = \delta_1 \left(L^{(A)}\right), \partial_1 \widetilde{N}^{(A)} = \delta_1 \left(N^{(A)}\right), \partial_1 \widetilde{T}^{(A)} = \delta_1 \left(T^{(A)}\right), \partial_1 \widetilde{V}^{(A)} = \delta_1 \left(V^{(A)}\right),$$

and, this way it is possible to substitute in (3.82) and (3.83) the elongated partial derivative δ_1 by the usual one acting on "tilded" radial increments.

By straightforward calculations (see details in Ref. [16]) one can check that the functions

$$Z^{(+)}_{(A)} = r^2 \frac{6mX^{(A)}/r(l-1)(l+2) - L^{(A)}}{r(l-1)(l+2)/2 + 3m}$$

satisfy one–dimensional wave equations similar to (3.78) for $Z^{(\eta)}$ with $\eta_3 = 1$, when $r_\star = r_*$,

$$\begin{aligned} \left(\frac{d^2}{dr_*^2} + \sigma^2_{(A)}\right) \widetilde{Z}^{(+)}_{(A)} &= V^{(+)} Z^{(+)}_{(A)}, \\ \widetilde{Z}^{(+)}_{(A)} &= Z^{(+)}_{(A)} + \varepsilon\sigma_{(A)} \int n_1 Z^{(+)}_{(A)} dr, \end{aligned} \tag{3.85}$$

where

$$\begin{aligned} V^{(+)} &= \frac{2\Delta}{r^5[r(l-1)(l+2)/2 + 3m]^2} \times \{9m^2 \left[\frac{r}{2}(l-1)(l+2) + m\right] \\ &+ \frac{1}{4}(l-1)^2(l+2)^2 r^3 \left[1 + \frac{1}{2}(l-1)(l+2) + \frac{3m}{r}\right]\}. \end{aligned} \tag{3.86}$$

For $\varepsilon \to 0$, the equation (3.85) transforms in the usual Zerilli equation [40, 16].

To complete the solution we give the formulas for the "tilded" $L-$, $X-$ and N–factors,

$$
\begin{aligned}
\widetilde{L}^{(A)} &= \frac{3m}{r^2}\widetilde{\Phi}^{(A)} - \frac{(l-1)(l+2)}{2r}\widetilde{Z}^{(+)}_{(A)}, \qquad (3.87)\\
\widetilde{X}^{(A)} &= \frac{(l-1)(l+2)}{2r}(\widetilde{\Phi}^{(A)} + \widetilde{Z}^{(+)}_{(A)}),\\
\widetilde{N}^{(A)} &= \left(m - \frac{m^2 + r^4\sigma^2_{(A)}}{r-2m}\right)\frac{\widetilde{\Phi}^{(A)}}{r^2} - \frac{(l-1)(l+2)r}{2(l-1)(l+2)+12m}\frac{\partial \widetilde{Z}^{(+)}_{(A)}}{\partial r_{\#}}\\
&-\frac{(l-1)(l+2)}{[r(l-1)(l+2)+6m]^2}\times\\
&\left\{\frac{12m^2}{r} + 3m(l-1)(l+2) + \frac{r}{2}(l-1)(l+2)\left[(l-1)(l+2)+2\right]\right\},
\end{aligned}
$$

where

$$
\widetilde{\Phi}^{(A)} = (l-1)(l+2)e^{\nu_4}\int \frac{e^{-\nu_4}\widetilde{Z}^{(+)}_{(A)}}{(l-1)(l+2)r+6m}dr.
$$

Following the relations (3.84) we can compute the corresponding "untilded" values an put them in (3.81) in order to find the increments of fluctuations driven by the system of equations (3.80). For simplicity, we omit the rather cumbersome final expressions.

The formulas (3.87) together with a solution of the wave equation (3.85) complete the procedure of definition of formal solutions for polar perturbations. In Ref. [16] there are tabulated the data for the potential (3.86) for different values of l and $(l-1)(l+2)/2$. In the anisotropic case the explicit form of solutions is deformed by terms proportional to $\varepsilon n_1 \sigma$. The static ellipsoidal like deformations can be modelled by the formulas obtained in the limit $\sigma_{(\varepsilon)} \to 0$.

3.5.4 The stability of polarized black ellipsoids

The problem of stability of anholonomically deformed Schwarzschild metrics to external perturbation is very important to be solved in order to understand if such static black ellipsoid like objects may exist in general relativity and its cosmological constant generalizations. We address the question: Let be given any initial values for a static locally anisotropic configuration confined to a finite interval of $r_\star$, for axial perturbations, and r_*, for polar perturbations, will one remain bounded such perturbations at all times of evolution? The answer to this question is to obtained similarly to Refs. [16] and [36] with different type of definitions of functions g $Z^{(\eta)}$ and $Z^{(+)}_{(A)}$ for different type of black holes.

We have proved that even for anisotropic configurations every type of perturbations are governed by one dimensional wave equations of the form

$$\frac{d^2 Z}{d\rho} + \sigma^2 Z = VZ \tag{3.88}$$

where ρ is a radial type coordinate, Z is a corresponding $Z^{(\eta)}$ or $Z^{(+)}_{(A)}$ with respective smooth real, independent of $\sigma > 0$ potentials $\widetilde{V}^{(\eta)}$ or $V^{(-)}$ with bounded integrals. For such equations a solution $Z(\rho, \sigma, \varphi_0)$ satisfying the boundary conditions $Z \rightarrow e^{i\sigma\rho} + R(\sigma)e^{-i\sigma\rho} \quad (\rho \rightarrow +\infty)$ and $Z \rightarrow T(\sigma)e^{i\sigma\rho} \qquad (\rho \rightarrow -\infty)$ (the first expression corresponds to an incident wave of unit amplitude from $+\infty$ giving rise to a reflected wave of amplitude $R(\sigma)$ at $+\infty$ and the second expression is for a transmitted wave of amplitude $T(\sigma)$ at $-\infty$), provides a basic complete set of wave functions which allows to obtain a stable evolution. For any initial perturbation that is smooth and confined to finite interval of ρ, we can write the integral

$$\psi(\rho, 0) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \widehat{\psi}(\sigma, 0) Z(\rho, \sigma) d\sigma$$

and define the evolution of perturbations,

$$\psi(\rho, t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \widehat{\psi}(\sigma, 0) e^{i\sigma t} Z(\rho, \sigma) d\sigma.$$

The Schrodinger theory states the conditions

$$\int_{-\infty}^{+\infty} |\psi(\rho, 0)|^2 d\rho = \int_{-\infty}^{+\infty} |\widehat{\psi}(\sigma, 0)|^2 d\sigma = \int_{-\infty}^{+\infty} |\psi(\rho, 0)|^2 d\rho,$$

from which the boundedness of $\psi(\rho, t)$ follows for all $t > 0$.

In our consideration we have replaced the time partial derivative $\partial/\partial t$ by $i\sigma$, which was represented by the approximation of perturbations to be periodic like $e^{i\sigma t}$. This is connected with a time–depending variant of (3.88), like

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} - VZ.$$

Multiplying this equation on $\partial \overline{Z}/\partial t$, where $\overline{Z}$ denotes the complex conjugation, and integrating on parts, we obtain

$$\int_{-\infty}^{+\infty} \left(\frac{\partial \overline{Z}}{\partial t} \frac{\partial^2 Z}{\partial t^2} + \frac{\partial Z}{\partial \rho} \frac{\partial^2 \overline{Z}}{\partial t \partial \rho} + VZ \frac{\partial \overline{Z}}{\partial t} \right) d\rho = 0$$

providing the conditions of convergence of necessary integrals. This equation added to its complex conjugate results in a constant energy integral,

$$\int_{-\infty}^{+\infty} \left(\left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + V \left| Z \right|^2 \right) d\rho = const,$$

which bounds the expression $|\partial Z/\partial t|^2$ and excludes an exponential growth of any bounded solution of the equation (3.88). We note that this property holds for every type of "ellipsoidal" like deformation of the potential, $V \to V + \varepsilon V^{(1)}$, with possible dependencies on polarization functions as we considered in (3.79) and/or (3.86).

The general properties of the one–dimensional Schrodinger equations related to perturbations of holonomic and anholonomic solutions of the Einstein equations allow us to conclude that there are locally anisotropic static configurations which are stable under linear deformations.

In a similar manner we may analyze perturbations (axial or polar) governed by a two–dimensional Schrodinger waive equation like

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 Z}{\partial \rho^2} + A(\rho, \varphi, t) \frac{\partial^2 Z}{\partial \varphi^2} - V(\rho, \varphi, t) Z$$

for some functions of necessary smooth class. The stability in this case is proven if exists an (energy) integral

$$\int_0^{\pi} \int_{-\infty}^{+\infty} \left(\left| \frac{\partial Z}{\partial t} \right|^2 + \left| \frac{\partial Z}{\partial \rho} \right|^2 + \left| A \frac{\partial Z}{\partial \rho} \right|^2 + V \left| Z \right|^2 \right) d\rho d\varphi = const$$

which bounds $|\partial Z/\partial t|^2$ for two–dimensional perturbations. For simplicity, we omitted such calculus in this work.

We emphasize that this way we can also prove the stability of perturbations along "anisotropic" directions of arbitrary anholonomic deformations of the Schwarzschild solution which have non–spherical horizons and can be covered by a set of finite regions approximated as small, ellipsoid like, deformations of some spherical hypersurfaces. We may analyze the geodesic congruence on every deformed sub-region of necessary smoothly class and proof the stability as we have done for the resolution ellipsoid horizons. In general, we may consider horizons of with non–trivial topology, like vacuum black tori, or higher genus anisotropic configurations. This is not prohibited by the principles of topological censorship [41] if we are dealing with off–diagonal metrics and associated anholonomic frames [1]. The vacuum anholonomy in such cases may be treated as an effective matter which change the conditions of topological theorems.

3.6 Two Additional Examples of Off–Diagonal Exact Solutions

There are some classes of exact solutions which can be modelled by anholonomic frame transforms and generic off–diagonal metric ansatz and related to configurations constructed by using another methods [18, 19]. We analyze in this section two classes of such 4D spacetimes.

3.6.1 Anholonomic ellipsoidal shapes

The present status of ellipsoidal shapes in general relativity associated to some perfect–fluid bodies, rotating configurations or to some families of confocal ellipsoids in Riemannian spaces is examined in details in Ref. [18]. We shall illustrate in this subsection how such configurations may be modelled by generic off–diagonal metrics and/or as spacetimes with anisotropic cosmological constant. The off–diagonal coefficients will be subjected to certain anholonomy conditions resulting (roughly speaking) in effects similar to those of perfect–fluid bodies.

We consider a metric ansatz with conformal factor like in (3.64)

$$\begin{aligned} \delta s^2 &= \Omega(\theta,\nu)\left[g_1 d\theta^2 + g_2(\theta)d\varphi^2 + h_3(\theta,\nu)\delta\nu^2 + h_4(\theta,\nu)\delta t^2\right], \\ \delta\nu &= d\nu + w_1(\theta,\nu)d\theta + w_2(\theta,\nu)d\varphi, \\ \delta t &= dt + n_1(\theta,\nu)d\theta + n_2(\theta,\nu)d\varphi, \end{aligned} \tag{3.89}$$

where the coordinated $(x^1=\theta, x^2=\varphi)$ are holonomic and the coordinate $y^3=\nu$ and the timelike coordinate $y^4=t$ are 'anisotropic' ones. For a particular parametrization when

$$\begin{aligned} \Omega &= \Omega_{[0]}(\nu) = v(\rho)\,\rho^2,\ g_1 = 1, \\ g_2 &= g_{2[0]} = \sin^2\theta, h_3 = h_{3[0]} = 1, h_4 = h_{4[0]} = -1, \\ w_1 &= 0,\ w_2 = w_{2[0]}(\theta) = \sin^2\theta,\ n_1 = 0,\ n_2 = n_{2[0]}(\theta) = 2R_0\cos\theta \end{aligned} \tag{3.90}$$

and the coordinate ν is defined related to ρ as

$$d\nu = \int \left|\frac{f(\rho)}{v(\rho)}\right|^{1/2} \frac{d\rho}{\rho},$$

we obtain the metric element for a special case spacetimes with co–moving ellipsoidal symmetry defined by an axially symmetric, rigidly rotating perfect–fluid configuration with confocal inside ellipsoidal symmetry (see formula (4.21) and related discussion in Ref. [18], where the status of constant R_0 and functions $v(\rho)$ and $f(\rho)$ are explicitly defined).

By introducing nontrivial "polarization" functions $q^{[v]}(\theta)$ and $\eta_{3,4}(\theta,\nu)$ for which

$$g_2 = g_{2[0]} q^{[v]}(\theta), \; h_{3,4} = \eta_{3,4}(\theta,\nu) h_{3,4[0]}$$

we can state the conditions when the ansatz (3.89) defines a) an off–diagonal ellipsoidal shape or b) an ellipsoidal configuration induced by anisotropically polarized cosmological constant.

Let us consider the case a). The Theorem 2 from Ref. [20] and the formula (72) in Ref. [9] (see also the Appendix in [10]) states that any metric of type (3.89) is vacuum if $\Omega^{p_1/p_2} = h_3$ for some integers p_1 and p_2, the factor $\Omega = \Omega_{[1]}(\theta,\nu)\Omega_{[0]}(\nu)$ satisfies the condition

$$\partial_i \Omega - (w_i + \zeta_i)\, \partial_\nu \Omega = 0$$

for any additional deformation functions $\zeta_i(\theta,\nu)$ and the coefficients

$$g_1 = 1, g_2 = g_{2[0]} q^{[v]}(\theta), \; h_{3,4} = \eta_{3,4}(\theta,\nu) h_{3,4[0]}, w_i(\theta,\nu), n_i(\theta,\nu) \tag{3.91}$$

satisfy the equations (3.17)–(3.20). The procedure of constructing such exact solutions is very similar to the considered in subsection 4.1 for black ellipsoids. For anholonomic ellipsoidal shapes (they are characterized by nontrivial anholonomy coefficients (3.8) and respectively induced noncommutative symmetries) we have to put as "boundary" condition in integrals of type (3.41) just to have $n_1 = 0, \; n_2 = n_{2[0]}(\theta) = 2R_0 \cos\theta$ from data (3.90) in the limit when dependence on "anisotropic" variable ν vanishes. The functions $w_i(\theta,\nu)$ and $n_i(\theta,\nu)$ must be subjected to additional constraints if we wont to construct ellipsoidal shape configurations with zero anholonomically induced torsion (1.43) and N–connection curvature, $\Omega^a_{jk} = \delta_k N^a_j - \delta_j N^a_k = 0.$

b) The simplest way to construct an ellipsoidal shape configuration induced by anisotropic cosmological constant is to find data (3.91) solving the equations (3.43) following the procedure defined in subsection 4.2. We note that we can solve the equation (3.50) for $g_2 = g_2(\theta) = \sin^2\theta$ with $q^{[v]}(\theta) = 1$ if $\lambda_{[h]0} = 1/2$, see solution (3.51) with $\xi \rightarrow \theta$. For simplicity, we can consider that $\lambda_{[v]} = 0$. Such type configurations contain, in general, anholonomically induced torsion.

We conclude, that by using the anholonomic frame method we can generate ellipsoidal shapes (in general, with nontrivial polarized cosmological constants and induced torsions). Such solutions are similar to corresponding rotation configurations in general relativity with rigidly rotating perfect-fluid sources. The rough analogy consists in the fact that by certain frame constraints induced by off–diagonal metric terms we can model gravitational–matter like metrics. In previous section we proved the stability of black ellipsoids for small excentricities. Similar investigations for ellipsoidal shapes is a task for future (because the shapes could be with arbitrary excentricity). In Ref. [18], there were discussed points of matchings of locally rotationally symmetric spacetimes to Taub–NUT metrics. We emphasize that this topic was also specifically elaborated by using anholonomic frame transforms in Refs. [4].

3.6.2 Generalization of Canfora–Schmidt solutions

In general, the solutions generated by anholonomic transforms cannot be reduced to a diagonal transform only by coordinate transforms (this is stated in our previous works [1, 2, 3, 4, 10, 20, 29, 30, 31, 36, 10], see also Refs. [48] for modelling Finsler like geometries in (pseudo) Riemannian spacetimes). We discus here how 4D off–diagonal ansatz (3.2) generalize the solutions obtained in Ref. [19] by a corresponding parametrization of coordinates as $x^1 = x, x^2 = t, y^3 = \nu = y$ and $y^4 = p$. If we consider for (3.2) (equivalently, for (3.5)) the non-trivial data

$$\begin{aligned} g_1 &= g_{1[0]} = 1,\ g_2 = g_{2[0]}(x^1) = -B\,(x)\,P(x)^2 - C(x), \\ h_3 &= h_{3[0]}(x^1) = A\,(x) > 0,\ h_4 = h_{4[0]}(x^1) = B\,(x)\,, \\ w_i &= 0, n_1 = 0, n_2 = n_{2[0]}(x^1) = P(x)/B\,(x) \end{aligned} \tag{3.92}$$

we obtain just the ansatz (12) from Ref. [19] (in this subsection we use a different label for coordinates) which, for instance, for $B+C = 2, B-C = \ln|x|, P = -1/(B-C)$ with $e^{-1} < \sqrt{|x|} < e$ for a constant e, defines an exact 4D solution of the Einstein equation (see metric (27) from [19]). By introducing 'polarization' functions $\eta_k = \eta_k(x^i)$ [when $i, k, ... = 1, 2$] and $\eta_a = \eta_a(x^i, \nu)$ [when $a, b, ... = 3, 4$] we can generalize the data (3.92) as to have

$$g_k(x^i) = \eta_k(x^i) g_{k[0]},\ h_a(x^i, \nu) = \eta_a(x^i, \nu) h_{a[0]}$$

and certain nontrivial values $w_i = w_i(x^i, \nu)$ and $n_i = n_i(x^i, \nu)$ solving the Einstein equations with anholonomic variables (3.17)–(3.20). We can easy find new classes of exact solutions, for instance, for $\eta_1 = 1$ and $\eta_2 = \eta_2(x^1)$. In this case $g_1 = 1$ and the function $g_2(x^1)$ is any solution of the equation

$$g_2^{\bullet\bullet} - \frac{(g_2^{\bullet})^2}{2g_2} = 0 \tag{3.93}$$

(see equation (3.50) for $\lambda_{[v]} = 0$), $g_2^{\bullet} = \partial g_2/\partial x^1$ which is solved as a particular case if $g_2 = (x^1)^2$. This impose certain conditions on $\eta_2(x^1)$ if we wont to take $g_{2[0]}(x^1)$ just as in (3.92). For more general solutions with arbitrary $\eta_k(x^i)$, we have to take solutions of equation (3.17) and not of a particular case like (3.93).

We can generate solutions of (3.18) for any $\eta_a(x^i,\nu)$ satisfying the condition (3.39), $\sqrt{|\eta_3|} = \eta_0 \left(\sqrt{|\eta_4|}\right)^*, \eta_0 = const.$ For instance, we can take arbitrary η_4 and using elementary derivations with $\eta_4^* = \partial\eta_4/\partial\nu$ and a nonzero constant η_0, to define $\sqrt{|\eta_3|}$. For the vacuum solutions, we can put $w_i = 0$ because $\beta = \alpha_i = 0$ (see formulas (3.18) and (3.21)). In this case the solutions of (3.19) are trivial. Having defined $\eta_a(x^i,\nu)$ we can integrate directly the equation (3.20) and find $n_i(x^i,\nu)$ like in formula (3.41) with fixed value $\varepsilon = 1$ and considering dependence on all holonomic variables,

$$\begin{aligned} n_i\left(x^k,\nu\right) &= n_{i[1]}\left(x^k\right) + n_{i[2]}\left(x^k\right)\int d\nu \ \ \eta_3\left(x^k,\nu\right)/\left(\sqrt{|\eta_4\left(x^k,\nu\right)|}\right)^3, \eta_4^* \neq 0; \\ &= n_{i[1]}\left(x^k\right) + n_{i[2]}\left(x^k\right)\int d\nu \ \eta_3\left(x^k,\nu\right), \eta_4^* = 0; \\ &= n_{i[1]}\left(x^k\right) + n_{i[2]}\left(x^k\right)\int d\nu/\left(\sqrt{|\eta_4\left(x^k,\nu\right)|}\right)^3, \eta_3^* = 0. \end{aligned} \tag{3.94}$$

These values will generalize the data (3.92) if we identify $n_{1[1]}\left(x^k\right) = 0$ and $n_{1[1]}\left(x^k\right) = n_{2[0]}(x^1) = P(x)/B\left(x\right).$ The solutions with vanishing induced torsions and zero non-linear connection curvatures are to be selected by choosing $n_i\left(x^k,\nu\right)$ and $\eta_3\left(x^k,\nu\right)$ (or $\eta_4\left(x^k,\nu\right)$) as to reduce the canonical connection (3.9) to the Levi–Civita connection (as we discussed in the end of Section 2).

The solution defined by the data (3.92) is compared in Ref. [19] with the Kasner diagonal solution which define the simplest models of anisotropic cosmology. The metrics obtained by F. Canfora and H.-J. Schmidt (CS) is generic off–diagonal and can not written in diagonal form by coordinate transforms. We illustrated that the CS metrics can be effectively diagonalized with respect to N–adapted anholonomic frames (like a more general ansatz (3.2) can be reduced to (3.5)) and that by anholonomic frame transforms of the CS metric we can generate new classes of generic off–diagonal solutions. Such spacetimes may describe certain models of anisotropic and/or inhomogeneous cosmologies (see, for instance, Refs. [48] were we considered a model of Friedman–Robertson–Walker metric with ellipsoidal symmetry). The anholonomic generalizations of CS metrics are with nontrivial noncommutative symmetry because the anholonomy coefficients (3.8) (see also (3.26)) are not zero being defined by nontrivial values (3.94).

3.7 Outlook and Conclusions

The work is devoted to investigation of a new class of exact solutions in metric–affine and string gravity describing static back rotoid (ellipsoid) and shape configurations possessing hidden noncommutative symmetries. There are generated also certain generic off–diagonal cosmological metrics.

We consider small, with nonlinear gravitational polarization, static deformations of the Schwarschild black hole solution (in particular cases, to some resolution ellipsoid like configurations) preserving the horizon and geodesic behavior but slightly deforming the spherical constructions. It was proved that there are such parameters of the exact solutions of the Einstein equations defined by off–diagonal metrics with ellipsoid symmetry constructed in Refs. [1, 2, 20, 29, 30, 36] as the vacuum solutions positively define static ellipsoid black hole configurations.

We illustrate that the new class of static ellipsoidal black hole solutions posses some similarities with the Reissner–Nordstrom metric if the metric's coefficients are defined with respect to correspondingly adapted anholonomic frames. The parameter of ellipsoidal deformation results in an effective electromagnetic charge induced by off–diagonal vacuum gravitational interactions. We note that effective electromagnetic charges and Reissner–Nordstrom metrics induced by interactions in the bulk of extra dimension gravity were considered in brane gravity [42]. In our works we proved that such Reissner–Nordstrom like ellipsoid black hole configurations may be constructed even in the framework of vacuum Einstein gravity. It should be emphasized that the static ellipsoid black holes posses spherical topology and satisfy the principle of topological censorship [39]. Such solutions are also compatible with the black hole uniqueness theorems [43]. In the asymptotical limits at least for a very small eccentricity such black ellipsoid metrics transform into the usual Schwarzschild one. We have proved that the stability of static ellipsoid black holes can be proved similarly by considering small perturbations of the spherical black holes [29, 30] even the solutions are extended to certain classes of spacetimes with anisotropically polarized cosmological constants. (On the stability of the Schwarzschild solution see details in Ref. [16].)

The off–diagonal metric coefficients induce a specific spacetime distorsion comparing to the solutions with metrics diagonalizable by coordinate transforms. So, it is necessary to compare the off–diagonal ellipsoidal metrics with those describing the distorted diagonal black hole solutions (see the vacuum case in Refs. [44] and an extension to the case of non–vanishing electric fields [45]). For the ellipsoidal cases, the distorsion of spacetime can be of vacuum origin caused by some anisotropies (anholonomic constraints) related to off–diagonal terms. In the case of "pure diagonal" distorsions such effects follow from the fact that the vacuum Einstein equations are not satisfied in some regions because of presence of matter.

The off–diagonal gravity may model some gravity–matter like interactions like in Kaluza–Klein theory (for some particular configurations and topological compactifications) but, in general, the off–diagonal vacuum gravitational dynamics can not be associated to any effective matter dynamics. So, we may consider that the anholonomic ellipsoidal deformations of the Schwarzschild metric are some kind of anisotropic off–diagonal distorsions modelled by certain vacuum gravitational fields with the distorsion parameteres (equivalently, vacuum gravitational polarizations) depending both on radial and angular coordinates.

There is a common property that, in general, both classes of off–diagonal anisotropic and "pure" diagonal distorsions (like in Refs. [44]) result in solutions which are not asymptotically flat. However, it is possible to find asymptotically flat extensions even for ellipsoidal configurations by introducing the corresponding off–diagonal terms (the asymptotic conditions for the diagonal distorsions are discussed in Ref. [45]; to satisfy such conditions one has to include some additional matter fields in the exterior portion of spacetime).

We analyzed the conditions when the anholonomic frame method can model ellipsoid shape configurations. It was demonstrated that the off–diagonal metric terms and respectively associated nonlinear connection coefficients may model ellipsoidal shapes being similar to those derived from solutions with rotating perfect fluids (roughly speaking, a corresponding frame anholnomy/ anisotropy may result in modelling of specific matter interactions but with polarizations of constants, metric coefficients and related frames).

In order to point to some possible observable effects, we note that for the ellipsoidal metrics with the Schwarzschild asymptotic, the ellipsoidal character could result in some observational effects in the vicinity of the horizon (for instance, scattering of particles on a static ellipsoid; we can compute anisotropic matter accretion effects on an ellipsoidal black hole put in the center of a galactic being of ellipsoidal or another configuration). A point of further investigations could be the anisotropic ellipsoidal collapse when both the matter and spacetime are of ellipsoidal generic off–diagonal symmetry and/or shape configurations (former theoretical and computational investigations were performed only for rotoids with anisotropic matter and particular classes of perturbations of the Schwarzshild solutions [46]). For very small eccentricities, we may not have any observable effects like perihelion shift or light bending if we restrict our investigations only to the Schwarzshild–Newton asymptotic.

We present some discussion on mechanics and thermodynamics of ellipsoidal black holes. For the static black ellipsoids with flat asymptotic, we can compute the area of the ellipsoidal horizon, associate an entropy and develop a corresponding black ellipsoid thermodynamics. This can be done even for stable black torus configurations. But this is a very rough approximation because, in general, we are dealing with off–diagonal metrics depending anisotropically on two/three coordinates. Such solutions are with anholonomically deformed Killing horizons and should be described by a thermodynamics (in general, both non-equilibrium and irreversible) of black ellipsoids self–consistently embedded into an off–diagonal anisotropic gravitational vacuum. This is a ground for numerous new conceptual issues to be developed and related to anisotropic black holes and the anisotropic kinetics and thermodynamics [2] as well to a framework of isolated anisotropic horizons [47] which is a matter of our further investigations. As an example of a such new concept, we point to a noncommutative dynamics which can be associated to black ellipsoids.

We emphasize that it is a remarkable fact that, in spite of appearance complexity, the perturbations of static off–diagonal vacuum gravitational configurations are governed by similar types of equations as for diagonal holonomic solutions. Perhaps in a similar manner (as a future development of this work) by using locally adapted "N–elongated" partial derivatives we can prove stability of very different classes of exact solutions with ellipsoid, toroidal, dilaton and spinor–soliton symmetries constructed in Refs. [1, 2, 20, 29, 30, 36]. The origin of this mystery is located in the fact that by anholnomic transforms we effectively diagonalized the off–diagonal metrics by "elongating" some partial derivatives. This way the type of equations governing the perturbations is preserved but, for small deformations, the systems of linear equations for fluctuations became "slightly" nondiagonal and with certain tetradic modifications of partial derivatives and differentials.

It is known that in details the question of relating the particular integrals of such systems associated to systems of linear differential equations is investigated in Ref. [16]. For anholonomic configurations, one holds the same relations between the potentials $\widetilde{V}^{(\eta)}$ and $V^{(-)}$ and wave functions $Z^{(\eta)}$ and $Z^{(+)}_{(A)}$ with that difference that the physical values and formulas where polarized by some anisotropy functions $\eta_3(r,\theta,\varphi), \Omega(r,\varphi)$, $q(r), \eta(r,\varphi)$, $w_1(r,\varphi)$ and $n_1(r,\varphi)$ and deformed on a small parameter ε. It is not clear that a similar procedure could be applied in general for proofs of stability of ellipsoidal shapes but it would be true for small deformations from a supposed to be stable primordial configuration.

We conclude that there are static black ellipsoid vacuum configurations as well induced by nontrivially polarized cosmological constants which are stable with respect to one dimensional perturbations, axial and/or polar ones, governed by solutions of the corresponding one–dimensional Schrodinger equations. The problem of stability of such objects with respect to two, or three, dimensional perturbations, and the possibility of modelling such perturbations in the framework of a two–, or three–, dimensional inverse scattering problem is a topic of our further investigations. The most important problem to be solved is to find a geometrical interpretation for the anholonomic Schrodinger mechanics of stability to the anholonomic frame method and to see if we can extend the approach at least to the two dimensional scattering equations.

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Part II

Generic Off–Diagonal Exact Solutions

Chapter 4

Locally Anisotropic Black Holes in Einstein Gravity

Abstract [1]

By applying the method of moving frames modelling one and two dimensional local anisotropies we construct new solutions of Einstein equations on pseudo–Riemannian spacetimes. The first class of solutions describes non–trivial deformations of static spherically symmetric black holes to locally anisotropic ones which have elliptic (in three dimensions) and ellipsoidal, toroidal and elliptic and another forms of cylinder symmetries (in four dimensions). The second class consists from black holes with oscillating elliptic horizons.

4.1 Introduction

In recent years, there has been great interest in investigation of gravitational models with anisotropies and applications in modern cosmology and astrophysics. There are possible locally anisotropic inflational and black hole like solutions of Einstein equations in the framework of so–called generalized Finsler–Kaluza–Klein models [9] and in low–energy locally anisotropic limits of (super) string theories [10].

In this paper we shall restrict ourselves to a more limited problem of definition of black hole solutions with local anisotropy in the framework of the Einstein theory (in three and four dimensions). Our purpose is to construct solutions of gravitational field equations by imposing symmetries differing in appearance from the static spherical one (which uniquely results in the Schwarzschild solution) and search for solutions with configurations of event horizons like rotation ellipsoids, torus and ellipsoidal and cylinders. We shall proof that there are possible elliptic oscillations in time of horizons.

In order to simplify the procedure of solution and investigate more deeply the physical implications of general relativistic models with local anisotropy we shall transfer our analysis with respect to anholonomic frames which are equivalently characterized by nonlinear connection (N–connection) structures [2, 3, 8, 9, 10]. This geometric approach is very useful for construction of metrics with prescribed symmetries of horizons and definition of conditions when such type black hole like solutions could be selected from an integral variety of the Einstein field equations with a corresponding energy–momentum tensor. We argue that, in general, the symmetries of solutions are not completely determined by the field equations and coordinate conditions but there are also required some physical motivations for choosing of corresponding classes of systems of reference (prescribed type of local anisotropy and symmetries of horizons) with respect to which the 'picture' of interactions, symmetries and conservation laws is drawn in the simplest form.

The paper is organized as follows: In section 2 we introduce metrics and anholonomic frames with local anisotropies admitting equivalent N–connection structures. We write down the Einstein equations with respect to such locally anisotropic frames. In section 3 we analyze the general properties of the system of gravitational field equations for an ansatz for metrics with local anisotropy. In section 4 we generalize the three dimensional static black hole solution to the case with elliptic horizon and proof that there are possible elliptic oscillations in time of locally anisotropic black holes. The section 5 is devoted to four dimensional locally anisotropic static solutions with rotation ellipsoidal, toroidal and cylindrical like horizons and consider elliptic oscillations in time. In the last section we make some final remarks.

4.2 Anholonomic frames and N–connections

In this section we outline the necessary results on spacetime differential geometry [4] and anholonomic frames induced by N–connection structures [8, 9, 10]. We examine an ansatz for locally anisotropic (pseudo) Riemannian metrics with respect to coordinate bases and illustrate a substantial geometric simplification and reduction of the number of coefficients of geometric objects and field equations after linear transforms to anholonomic bases defined by coefficients of a corresponding N–connection. The Einstein equations are rewritten in an invariant form with respect to such locally anisotropic bases.

Consider a class of pseudo–Riemannian metrics

$$g = g_{\underline{\alpha}\underline{\beta}}\left(u^{\varepsilon}\right)\ du^{\underline{\alpha}} \otimes du^{\underline{\beta}}$$

in a $n+m$ dimensional spacetime $V^{(n+m)}$, $(n=2$ and $m=1,2)$, with components

$$g_{\underline{\alpha}\underline{\beta}} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right], \tag{4.1}$$

where $g_{ij} = g_{ij}\left(u^{\alpha}\right)$ and $h_{ab} = h_{ab}\left(u^{\alpha}\right)$ are respectively some symmetric $n\times n$ and $m\times m$ dimensional matrices, $N_j^e = N_j^e\left(u^{\beta}\right)$ is a $n\times m$ matrix, and the $n+m$ dimensional local coordinates are provide with general Greek indices and denoted $u^{\beta} = (x^i, y^a)$. The Latin indices $i,j,k,...$ in (4.1) run values $1,2$ and $a,b,c,..$ run values $3,4$ and we note that both type of isotropic, x^i, and the so–called anisotropic, y^a, coordinates could be space or time like ones. We underline indices in order to emphasize that components are given with respect to a coordinate (holonomic) basis

$$e_{\underline{\alpha}} = \partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}} \tag{4.2}$$

and/or its dual

$$e^{\underline{\alpha}} = du^{\underline{\alpha}}. \tag{4.3}$$

The class of metrics (4.1) transform into a $(n\times n)\oplus(m\times m)$ block form

$$g = g_{ij}\left(u^{\varepsilon}\right)\ dx^i \otimes dx^j + h_{ab}\left(u^{\varepsilon}\right)\left(\delta y^a\right)^2 \otimes \left(\delta y^a\right)^2 \tag{4.4}$$

if one chooses a frame of basis vectors

$$\delta_{\alpha} = \delta/\partial u^{\alpha} = \left(\delta/\partial x^i = \partial_i - N_i^a\left(u^{\varepsilon}\right)\partial_a, \partial_b\right), \tag{4.5}$$

where $\partial_i = \partial/x^i$ and $\partial_a = \partial/\partial y^a$, with the dual basis being

$$\delta^{\alpha} = \delta u^{\alpha} = \left(dx^i, \delta y^a = dy^a + N_i^a\left(u^{\varepsilon}\right)dx^i\right). \tag{4.6}$$

The set of coefficients $N = \{N_i^a\left(u^{\varepsilon}\right)\}$ from (4.5) and (4.6) could be associated to components of a nonlinear connection (in brief, N–connection) structure defining a local decomposition of spacetime into n isotropic directions x^i and one or two anisotropic directions y^a. The global definition of N–connection is due to W. Barthel [2] (the rigorous mathematical definition of N–connection is possible on the language of exact sequences of vector, or tangent, subbundles) and this concept is largely applied in Finsler geometry and its generalizations [3, 8]. It was concluded [9, 10] that N–connection structures are induced under non–trivial dynamical compactifications of higher dimensions in (super) string and (super) gravity theories and even in general relativity if we are dealing with anholonomic frames.

A N–connection is characterized by its curvature, N–curvature,

$$\Omega_{ij}^{a}=\partial_i N_j^a-\partial_j N_i^a+N_i^b\partial_b N_j^a-N_j^b\partial_b N_i^a. \tag{4.7}$$

As a particular case we obtain a linear connection field $\Gamma_{ib}^{a}\left(x^{i}\right)$ if $N_i^a(x^i,y^a)=\Gamma_{ib}^{a}\left(x^{i},y^{a}\right)$ [8, 9].

For nonvanishing values of Ω_{ij}^{a} the basis (4.5) is anholonomic and satisfies the conditions

$$\delta_\alpha\delta_\beta-\delta_\beta\delta_\alpha=w^{\gamma}{}_{\alpha\beta}\delta_\gamma,$$

where the anholonomy coefficients $w^{\gamma}{}_{\alpha\beta}$ are defined by the components of N–connection,

$$\begin{aligned} w^{k}{}_{ij} &= 0, w^{k}{}_{aj}=0, w^{k}{}_{ia}=0, w^{k}{}_{ab}=0, w^{c}{}_{ab}=0, \\ w^{a}{}_{ij} &= -\Omega_{ij}^{a}, w^{b}{}_{aj}=-\partial_a N_i^b, w^{b}{}_{ia}=\partial_a N_i^b. \end{aligned}$$

We emphasize that the elongated by N–connection operators (4.5) and (4.6) must be used, respectively, instead of local operators of partial derivation (4.2) and differentials (4.3) if some differential calculations are performed with respect to any anholonomic bases locally adapted to a fixed N–connection structure (in brief, we shall call such local frames as la–bases or la–frames, where, in brief, la– is from locally anisotropic).

The torsion, $T\left(\delta_\gamma,\delta_\beta\right)=T^{\alpha}{}_{\beta\gamma}\delta_\alpha$, and curvature, $R\left(\delta_\tau,\delta_\gamma\right)\delta_\beta=R_{\beta}{}^{\alpha}{}_{\gamma\tau}\delta_\alpha$, tensors of a linear connection $\Gamma^{\alpha}{}_{\beta\gamma}$ are introduced in a usual manner and, respectively, have the components

$$T^{\alpha}{}_{\beta\gamma}=\Gamma^{\alpha}{}_{\beta\gamma}-\Gamma^{\alpha}{}_{\gamma\beta}+w^{\alpha}{}_{\beta\gamma} \tag{4.8}$$

and

$$R_{\beta}{}^{\alpha}{}_{\gamma\tau}=\delta_\tau\Gamma^{\alpha}{}_{\beta\gamma}-\delta_\gamma\Gamma^{\alpha}{}_{\beta\delta}+\Gamma^{\varphi}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\varphi\tau}-\Gamma^{\varphi}{}_{\beta\tau}\Gamma^{\alpha}{}_{\varphi\gamma}+\Gamma^{\alpha}{}_{\beta\varphi}w^{\varphi}{}_{\gamma\tau}. \tag{4.9}$$

The Ricci tensor is defined

$$R_{\beta\gamma}=R_{\beta}{}^{\alpha}{}_{\gamma\alpha} \tag{4.10}$$

and the scalar curvature is

$$R=g^{\beta\gamma}R_{\beta\gamma}. \tag{4.11}$$

The Einstein equations with respect to a la–basis (4.6) are written

$$R_{\beta\gamma}-\frac{R}{2}g_{\beta\gamma}=k\Upsilon_{\beta\gamma}, \tag{4.12}$$

where the energy–momentum d–tensor $\Upsilon_{\beta\gamma}$ includes the cosmological constant terms and possible contributions of torsion (4.8) and matter and k is the coupling constant. For a symmetric linear connection the torsion field can be considered as induced by the anholonomy coefficients. For dynamical torsions there are necessary additional field equations, see, for instance, the case of locally anisotropic gauge like theories [11].

The geometrical objects with respect to a la–bases are distinguished by the corresponding N–connection structure and called (in brief) d–tensors, d–metrics (4.4), linear d–connections and so on [8, 9, 10].

A linear d–connection D on a spacetime V,

$$D_{\delta_\gamma}\delta_\beta = \Gamma^{\alpha}{}_{\beta\gamma}\left(x^k, y^a\right)\delta_\alpha,$$

is parametrized by non–trivial horizontal (isotropic) – vertical (anisotropic), in brief, h–v–components,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right). \tag{4.13}$$

Some d–connection and d–metric structures are compatible if there are satisfied the conditions

$$D_\alpha g_{\beta\gamma} = 0.$$

For instance, the canonical compatible d–connection

$${}^c\Gamma^{\alpha}{}_{\beta\gamma} = \left({}^cL^i{}_{jk}, {}^cL^a{}_{bk}, {}^cC^i{}_{jc}, {}^cC^a{}_{bc}\right)$$

is defined by the coefficients of d–metric (4.4), $g_{ij}\left(x^i, y^a\right)$ and $h_{ab}\left(x^i, y^a\right)$, and of N–connection, $N_i^a = N_i^a\left(x^i, y^b\right)$,

$$\begin{aligned}
{}^cL^i{}_{jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \\
{}^cL^a{}_{bk} &= \partial_b N_k^a + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N_i^d - h_{db}\partial_c N_i^d\right), \\
{}^cC^i{}_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk}, \\
{}^cC^a{}_{bc} &= \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right).
\end{aligned} \tag{4.14}$$

The coefficients of the canonical d–connection generalize with respect to la–bases the well known Christoffel symbols.

For a d–connection (4.13) we can compute the non–trivial components of d–torsion (4.8)

$$\begin{aligned}
T^i_{.jk} &= T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = C^i_{.ja}, T^i_{aj} = -C^i_{ja}, \\
T^i_{.ja} &= 0, \quad T^a_{.bc} = S^a_{.bc} = C^a_{bc} - C^a_{cb}, \\
T^a_{.ij} &= -\Omega^a_{ij}, \quad T^a_{.bi} = \partial_b N_i^a - L^a_{.bj}, \quad T^a_{.ib} = -T^a_{.bi}.
\end{aligned} \tag{4.15}$$

In a similar manner, putting non–vanishing coefficients (4.13) into the formula for curvature (4.9), we can compute the coefficients of d–curvature

$$R\left(\delta_\tau, \delta_\gamma\right)\delta_\beta = R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha,$$

split into h–, v–invariant components,

$$\begin{aligned}
R_{h.jk}^{.i} &= \delta_k L^i_{.hj} - \delta_j L^i_{.hk} + L^m_{.hj}L^i_{mk} - L^m_{.hk}L^i_{mj} - C^i_{.ha}\Omega^a_{.jk},\\
R_{b.jk}^{.a} &= \delta_k L^a_{.bj} - \delta_j L^a_{.bk} + L^c_{.bj}L^a_{.ck} - L^c_{.bk}L^a_{.cj} - C^a_{.bc}\Omega^c_{.jk},\\
P_{j.ka}^{.i} &= \partial_k L^i_{.jk} + C^i_{.jb}T^b_{.ka} - (\partial_k C^i_{.ja} + L^i_{.lk}C^l_{.ja} - L^l_{.jk}C^i_{.la} - L^c_{.ak}C^i_{.jc}),\\
P_{b.ka}^{.c} &= \partial_a L^c_{.bk} + C^c_{.bd}T^d_{.ka} - (\partial_k C^c_{.ba} + L^c_{.dk}C^d_{.ba} - L^d_{.bk}C^c_{.da} - L^d_{.ak}C^c_{.bd})\\
S_{j.bc}^{.i} &= \partial_c C^i_{.jb} - \partial_b C^i_{.jc} + C^h_{.jb}C^i_{.hc} - C^h_{.jc}C^i_{hb},\\
S_{b.cd}^{.a} &= \partial_d C^a_{.bc} - \partial_c C^a_{.bd} + C^e_{.bc}C^a_{.ed} - C^e_{.bd}C^a_{.ec}.
\end{aligned}$$

The components of the Ricci tensor (4.10) with respect to locally adapted frames (4.5) and (4.6) (in this case, d–tensor) are as follows:

$$\begin{aligned}
R_{ij} &= R_{i.jk}^{.k}, \quad R_{ia} = -{}^2P_{ia} = -P_{i.ka}^{.k},\\
R_{ai} &= {}^1P_{ai} = P_{a.ib}^{.b}, \quad R_{ab} = S_{a.bc}^{.c}.
\end{aligned} \tag{4.16}$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d–tensor is non symmetric. This is a consequence of anholonomy of la–bases.

Having defined a d-metric of type (4.4) on spacetime V we can compute the scalar curvature (4.11) of a d-connection D,

$$\overleftarrow{R} = G^{\alpha\beta}R_{\alpha\beta} = \widehat{R} + S, \tag{4.17}$$

where $\widehat{R} = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$.

Now, by introducing the values of (4.16) and (4.17) into equations (4.12), the Einstein equations with respect to a la–basis seen to be

$$\begin{aligned}
R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right)g_{ij} &= k\Upsilon_{ij},\\
S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right)h_{ab} &= k\Upsilon_{ab},\\
{}^1P_{ai} &= k\Upsilon_{ai},\\
{}^2P_{ia} &= -k\Upsilon_{ia},
\end{aligned} \tag{4.18}$$

where $\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}$ and Υ_{ia} are the components of the energy–momentum d–tensor field $\Upsilon_{\beta\gamma}$ (which includes possible cosmological constants, contributions of anholonomy d–torsions (4.15) and matter) and k is the coupling constant. For simplicity, we omitted the upper left index c pointing that for the Einstein theory the Ricci d–tensor and curvature scalar should be computed by applying the coefficients of canonical d–connection (4.14).

4.3 An ansatz for la–metrics

Let us consider a four dimensional (in brief, 4D) spacetime $V^{(2+2)}$ (with two isotropic plus two anisotropic local coordinates) provided with a metric (4.1) (of signature (-,+,+,+), or (+,+,+,-), (+,+,-,+)) parametrized by a symmetric matrix of type

$$\begin{bmatrix} g_1 + {q_1}^2 h_3 + {n_1}^2 h_4 & 0 & q_1 h_3 & n_1 h_4 \\ 0 & g_2 + {q_2}^2 h_3 + {n_2}^2 h_4 & q_2 h_3 & n_2 h_4 \\ q_1 h_3 & q_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix} \tag{4.1}$$

with components being some functions

$$g_i = g_i(x^j), q_i = q_i(x^j, z), n_i = n_i(x^j, z), h_a = h_a(x^j, z)$$

of necessary smoothly class. With respect to a la–basis (4.6) this ansatz results in diagonal 2×2 h– and v–metrics for a d–metric (4.4) (for simplicity, we shall consider only diagonal 2D nondegenerated metrics because for such dimensions every symmetric matrix can be diagonalized).

An equivalent diagonal d–metric (4.4) is obtained for the associated N–connection with coefficients being functions on three coordinates (x^i, z),

$$\begin{aligned} N_1^3 &= q_1(x^i, z), \ N_2^3 = q_2(x^i, z), \\ N_1^4 &= n_1(x^i, z), \ N_2^4 = n_2(x^i, z). \end{aligned} \tag{4.2}$$

For simplicity, we shall use brief denotations of partial derivatives, like $\dot{a} = \partial a/\partial x^1, a' = \partial a/\partial x^2$, $a^* = \partial a/\partial z$ $\dot{a}' = \partial^2 a/\partial x^1 \partial x^2$, $a^{**} = \partial^2 a/\partial z \partial z$.

The non–trivial components of the Ricci d–tensor (4.16) (for the ansatz (4.1)) when $R_1^1 = R_2^2$ and $S_3^3 = S_4^4$, are computed

$$R_1^1 = \frac{1}{2g_1 g_2}[-(g_1^{''} + \ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1^{\prime} g_2^{\prime}\right) + \frac{1}{2g_1}\left(g_1^{\prime\, 2} + \dot{g}_1 \dot{g}_2\right)], \tag{4.3}$$

$$S_3^3 = \frac{1}{h_3 h_4}[-h_4^{**} + \frac{1}{2h_4}(h_4^*)^2 + \frac{1}{2h_3}h_3^* h_4^*], \tag{4.4}$$

$$P_{3i} = \frac{q_i}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3} + \frac{h_4^*}{2{h_4}^2} - \frac{h_3^* h_4^*}{2h_3 h_4}] + \frac{1}{2h_4}[\frac{\dot{h}_4}{2h_4}h_4^* - \dot{h}_4^* + \frac{\dot{h}_3}{2h_3}h_4^*], \tag{4.5}$$

$$P_{4i} = -\frac{h_4}{2h_3}n_i^{**}. \tag{4.6}$$

The curvature scalar $\overleftarrow{R}$ (4.17) is defined by two non-trivial components $\widehat{R} = 2R_1^1$ and $S = 2S_3^3$.

The system of Einstein equations (4.18) transforms into

$$R_1^1 = -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, \tag{4.7}$$

$$S_3^3 = -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, \tag{4.8}$$

$$P_{3i} = \kappa \Upsilon_{3i}, \tag{4.9}$$

$$P_{4i} = \kappa \Upsilon_{4i}, \tag{4.10}$$

where the values of R_1^1, S_3^3, P_{ai}, are taken respectively from (4.3), (4.4), (4.5), (4.6).

We note that we can define the N–coefficients (4.2), $q_i(x^k, z)$ and $n_i(x^k, z)$, by solving the equations (4.9) and (4.10) if the functions $h_i(x^k, z)$ are known as solutions of the equations (4.8).

Let us analyze the basic properties of equations (4.8)–(4.10) (the h–equations will be considered for 3D and 4D in the next sections). The v–component of the Einstein equations (4.7)

$$\frac{\partial^2 h_4}{\partial z^2} - \frac{1}{2h_4}\left(\frac{\partial h_4}{\partial z}\right)^2 - \frac{1}{2h_3}\left(\frac{\partial h_3}{\partial z}\right)\left(\frac{\partial h_4}{\partial z}\right) - \frac{\kappa}{2}\Upsilon_1 h_3 h_4 = 0$$

(here we write down the partial derivatives on z in explicit form) follows from (4.4) and (4.8) and relates some first and second order partial on z derivatives of diagonal components $h_a(x^i,z)$ of a v–metric with a source $\kappa\Upsilon_1(x^i,z) = \kappa\Upsilon_1^1 = \kappa\Upsilon_2^2$ in the h–subspace. We can consider as unknown the function $h_3(x^i,z)$ (or, inversely, $h_4(x^i,z)$) for some compatible values of $h_4(x^i,z)$ (or $h_3(x^i,z)$) and source $\Upsilon_1(x^i,z)$.

By introducing a new variable $\beta = h_4^*/h_4$ the equation (4.11) transforms into

$$\beta^* + \frac{1}{2}\beta^2 - \frac{\beta h_3^*}{2h_3} - 2\kappa\Upsilon_1 h_3 = 0 \tag{4.11}$$

which relates two functions $\beta\left(x^i,z\right)$ and $h_3\left(x^i,z\right)$. There are two possibilities: 1) to define β (i. e. h_4) when $\kappa\Upsilon_1$ and h_3 are prescribed and, inversely 2) to find h_3 for given $\kappa\Upsilon_1$ and h_4 (i. e. β); in both cases one considers only "*" derivatives on z–variable (coordinates x^i are treated as parameters).

1. In the first case the explicit solutions of (4.11) have to be constructed by using the integral varieties of the general Riccati equation [6] which by a corresponding redefinition of variables, $z \to z\left(\varsigma\right)$ and $\beta\left(z\right) \to \eta\left(\varsigma\right)$ (for simplicity, we omit here the dependencies on x^i) could be written in the canonical form
$$\frac{\partial\eta}{\partial\varsigma} + \eta^2 + \Psi\left(\varsigma\right) = 0$$
where Ψ vanishes for vacuum gravitational fields. In vacuum cases the Riccati equation reduces to a Bernoulli equation which (we can use the former variables) for $s(z) = \beta^{-1}$ transforms into a linear differential (on z) equation,
$$s^* + \frac{h_3^*}{2h_3}s - \frac{1}{2} = 0. \tag{4.12}$$

2. In the second (inverse) case when h_3 is to be found for some prescribed $\kappa\Upsilon_1$ and β the equation (4.11) is to be treated as a Bernoulli type equation,
$$h_3^* = -\frac{4\kappa\Upsilon_1}{\beta}(h_3)^2 + \left(\frac{2\beta^*}{\beta} + \beta\right) h_3 \tag{4.13}$$
which can be solved by standard methods. In the vacuum case the squared on h_3 term vanishes and we obtain a linear differential (on z) equation.

A particular interest presents those solutions of the equation (4.11) which via 2D conformal transforms with a factor $\omega = \omega(x^i,z)$ are equivalent to a diagonal h–metric on x–variables, i.e. one holds the parametrization

$$h_3 = \omega(x^i,z)\ a_3\left(x^i\right) \text{ and } h_4 = \omega(x^i,z)\ a_4\left(x^i\right), \tag{4.14}$$

where $a_3(x^i)$ and $a_4(x^i)$ are some arbitrary functions (for instance, we can impose the condition that they describe some 2D soliton like or black hole solutions). In this case $\beta = \omega^*/\omega$ and for $\gamma = \omega^{-1}$ the equation (4.11) transforms into

$$\gamma\, \gamma^{**} = -2\kappa \Upsilon_1 a_3\left(x^i\right) \qquad (4.15)$$

with the integral variety determined by

$$z = \int \frac{d\gamma}{\sqrt{|-4k\Upsilon_1 a_3(x^i)\ln|\gamma| + C_1(x^i)|}} + C_2(x^i), \qquad (4.16)$$

where it is considered that the source Υ_1 does not depend on z.

Finally, we conclude that the v–metrics are defined by the integral varieties of corresponding Riccati and/or Bernoulli equations with respect to z–variables with the h–coordinates x^i treated as parameters.

4.4 3D black la–holes

Let us analyze some basic properties of 3D spacetimes $V^{(2+1)}$ (we emphasize that in approach $(2+1)$ points to a splitting into two isotropic and one anisotropic directions and not to usual 2D space plus one time like coordinates; in general anisotropies could be associate to both space and/or time like coordinates) provided with d–metrics of type

$$\delta s^2 = g_1\left(x^k\right)\left(dx^1\right)^2 + g_2\left(x^k\right)\left(dx^2\right)^2 + h_3(x^i,z)\left(\delta z\right)^2, \qquad (4.1)$$

where x^k are 2D coordinates, $y^3 = z$ is the anisotropic coordinate and

$$\delta z = dz + N_i^3(x^k,z)dx^i.$$

The N–connection coefficients are

$$N_1^3 = q_1(x^i,z),\ N_2^3 = q_2(x^i,z). \qquad (4.2)$$

The non–trivial components of the Ricci d–tensor (4.16), for the ansatz (4.1) with $h_4 = 1$ and $n_i = 0$, $R_1^1 = R_2^2$ and P_{3i}, are

$$R_1^1 = \frac{1}{2g_1g_2}[-(g_1^{''}+\ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1^{\prime}g_2^{\prime}\right) + \frac{1}{2g_1}\left(g_1^{\prime\,2} + \dot{g}_1\dot{g}_2\right)], \qquad (4.3)$$

$$P_{3i} = \frac{q_i}{2}[\left(\frac{h_3^*}{h_3}\right)^2 - \frac{h_3^{**}}{h_3}] \qquad (4.4)$$

(for 3D the component $S^3_3 \equiv 0$, see (4.4)).

The curvature scalar $\overleftarrow{R}$ (4.17) is $\overleftarrow{R} = \widehat{R} = 2R^1_1$.

The system of Einstein equations (4.18) transforms into

$$R^1_1 = -\kappa\Upsilon^3_3, \quad (4.5)$$
$$P_{3i} = \kappa\Upsilon_{3i}, \quad (4.6)$$

which is compatible for energy–momentum d–tensors with $\Upsilon^1_1 = \Upsilon^2_2 = 0$; the values of R^1_1 and P_{3i} are taken respectively from (4.3) and (4.4).

By using the equation (4.6) we can define the N–coefficients (4.2), $q_i(x^k, z)$, if the function $h_3(x^k, z)$ and the components Υ_{3i} of the energy–momentum d–tensor are given. We note that the equations (4.4) are solved for arbitrary functions $h_3 = h_3(x^k)$ and $q_i = q_i(x^k, z)$ if $\Upsilon_{3i} = 0$ and in this case the component of d–metric $h_3(x^k)$ is not contained in the system of 3D field equations.

4.4.1 Static elliptic horizons

Let us consider a class of 3D d-metrics which local anisotropy which are similar to Banados–Teitelboim–Zanelli (BTZ) black holes [1].

The d–metric is parametrized

$$\delta s^2 = g_1\left(\chi^1, \chi^2\right)(d\chi^1)^2 + \left(d\chi^2\right)^2 - h_3\left(\chi^1, \chi^2, t\right)\ (\delta t)^2, \quad (4.7)$$

where $\chi^1 = r/r_h$ for $r_h = const$, $\chi^2 = \theta/r_a$ if $r_a = \sqrt{|\kappa\Upsilon^3_3|} \neq 0$ and $\chi^2 = \theta$ if $\Upsilon^3_3 = 0$, $y^3 = z = t$, where t is the time like coordinate. The Einstein equations (4.5) and (4.6) transforms respectively into

$$\frac{\partial^2 g_1}{\partial(\chi^2)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial\chi^2}\right)^2 - 2\kappa\Upsilon^3_3 g_1 = 0 \quad (4.8)$$

and

$$\left[\frac{1}{h_3}\frac{\partial^2 h_3}{\partial z^2} - \left(\frac{1}{h_3}\frac{\partial h_3}{\partial z}\right)^2\right] q_i = -\kappa\Upsilon_{3i}. \quad (4.9)$$

By introducing new variables

$$p = g_1'/g_1 \text{ and } s = h_3^*/h_3 \quad (4.10)$$

where the 'prime' in this subsection denotes the partial derivative ∂/χ^2, the equations (4.8) and (4.9) transform into

$$p' + \frac{p^2}{2} + 2\epsilon = 0 \quad (4.11)$$

and

$$s^{*}q_i = \kappa \Upsilon_{3i}, \tag{4.12}$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = 1(-1)$ for the signature $1(-1)$ of the anisotropic coordinate.

A class of solutions of 3D Einstein equations for arbitrary $q_i = q_i(\chi^k, t)$ and $\Upsilon_{3i} = 0$ is obtained if $s = s(\chi^i)$. After integration of the second equation from (4.10), we find

$$h_3(\chi^k, t) = h_{3(0)}(\chi^k) \exp\left[s_{(0)}\left(\chi^k\right) t\right] \tag{4.13}$$

as a general solution of the system (4.12) with vanishing right part. Static solutions are stipulated by $q_i = q_i(\chi^k)$ and $s_{(0)}(\chi^k) = 0$.

The integral curve of (4.11), intersecting a point $\left(\chi^2_{(0)}, p_{(0)}\right)$, considered as a differential equation on χ^2 is defined by the functions [6]

$$p = \frac{p_{(0)}}{1 + \frac{p_{(0)}}{2}\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon = 0; \tag{4.14}$$

$$p = \frac{p_{(0)} - 2\tanh\left(\chi^2 - \chi^2_{(0)}\right)}{1 + \frac{p_{(0)}}{2}\tanh\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon > 0; \tag{4.15}$$

$$p = \frac{p_{(0)} - 2\tan\left(\chi^2 - \chi^2_{(0)}\right)}{1 + \frac{p_{(0)}}{2}\tan\left(\chi^2 - \chi^2_{(0)}\right)}, \qquad \epsilon < 0. \tag{4.16}$$

Because the function p depends also parametrically on variable χ^1 we must consider functions $\chi^2_{(0)} = \chi^2_{(0)}\left(\chi^1\right)$ and $p_{(0)} = p_{(0)}\left(\chi^1\right)$.

For simplicity, here we elucidate the case $\epsilon < 0$. The general formula for the nontrivial component of h–metric is to be obtained after integration on χ^1 of (4.16) (see formula (4.10))

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right)\left\{\sin[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] + \arctan\frac{2}{p_{(0)}\left(\chi^1\right)}\right\}^2,$$

for $p_{(0)}\left(\chi^1\right) \neq 0$, and

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right)\ \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] \tag{4.17}$$

for $p_{(0)}\left(\chi^1\right) = 0$, where $g_{1(0)}\left(\chi^1\right), \chi^2_{(0)}\left(\chi^1\right)$ and $p_{(0)}\left(\chi^1\right)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, when ε is the energy density. If we consider

$\Upsilon_{3i} = 0$ and a nontrivial diagonal components of energy–momentum d–tensor, $\Upsilon_{\beta}^{\alpha} = diag[0,0,-\varepsilon]$, the N–connection coefficients $q_i(\chi^i,t)$ could be arbitrary functions.

For simplicity, in our further considerations we shall apply the solution (4.17).

The d–metric (4.7) with the coefficients (4.17) and (4.13) gives a general description of a class of solutions with generic local anisotropy of the Einstein equations (4.18).

Let us construct static black la–hole solutions for $s_{(0)}\left(\chi^k\right) = 0$ in (4.13).

In order to construct an explicit la–solution we have to chose some coefficients $h_{3(0)}(\chi^k), g_{1(0)}\left(\chi^1\right)$ and $\chi_0\left(\chi^1\right)$ from some physical considerations. For instance, the Schwarzschild solution is selected from a general 4D metric with some general coefficients of static, spherical symmetry by relating the radial component of metric with the Newton gravitational potential. In this section, we construct a locally anisotropic BTZ like solution by supposing that it is conformally equivalent to the BTZ solution if one neglects anisotropies on angle θ),

$$g_{1(0)}\left(\chi^1\right) = \left[r\left(-M_0 + \frac{r^2}{l^2}\right)\right]^{-2},$$

where $M_0 = const > 0$ and $-1/l^2$ is a constant (which is to be considered the cosmological from the locally isotropic limit. The time–time coefficient of d–metric is chosen

$$h_3\left(\chi^1,\chi^2\right) = r^{-2}\lambda_3\left(\chi^1,\chi^2\right)\cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)]. \tag{4.18}$$

If we chose in (4.18)

$$\lambda_3 = (-M_0 + \frac{r^2}{l^2})^2,$$

when the constant

$$r_h = \sqrt{M_0}l$$

defines the radius of a circular horizon, the la–solution is conformally equivalent, with the factor $r^{-2}\cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)]$, to the BTZ solution embedded into a anholonomic background given by arbitrary functions $q_i(\chi^i,t)$ which are defined by some initial conditions of gravitational la–background polarization.

A more general class of la–solutions could be generated if we put, for instance,

$$\lambda_3\left(\chi^1,\chi^2\right) = (-M\left(\theta\right) + \frac{r^2}{l^2})^2,$$

with

$$M\left(\theta\right) = \frac{M_0}{(1+e\cos\theta)^2},$$

where $e < 1$. This solution has a horizon, $\lambda_3 = 0$, parametrized by an ellipse

$$r = \frac{r_h}{1 + e\cos\theta}$$

with parameter r_h and eccentricity e.

We note that our solution with elliptic horizon was constructed for a diagonal energy–momentum d-tensor with nontrivial energy density but without cosmological constant. On the other hand the BTZ solution was constructed for a generic 3D cosmological constant. There is not a contradiction here because the la–solutions can be considered for a d–tensor $\Upsilon^{\alpha}_{\beta} = diag[p_1 - 1/l^2, p_2 - 1/l^2, -\varepsilon - 1/l^2]$ with $p_{1,2} = 1/l^2$ and $\varepsilon_{(eff)} = \varepsilon + 1/l^2$ (for $\varepsilon = const$ the last expression defines the effective constant r_a). The locally isotropic limit to the BTZ black hole could be realized after multiplication on r^2 and by approximations $e \simeq 0$, $\cos[\theta - \theta_0(\chi^1)] \simeq 1$ and $q_i(x^k, t) \simeq 0$.

4.4.2 Oscillating elliptic horizons

The simplest way to construct 3D solutions of the Einstein equations with oscillating in time horizon is to consider matter states with constant nonvanishing values of $\Upsilon_{31} = const$. In this case the coefficient h_3 could depend on t–variable. For instance, we can chose such initial values when

$$h_3(\chi^1, \theta, t) = r^{-2}\left(-M(t) + \frac{r^2}{l^2}\right)\cos^2[\theta - \theta_0(\chi^1)] \tag{4.19}$$

with

$$M = M_0 \exp(-\widetilde{p}t)\sin\widetilde{\omega}t,$$

or, for an another type of anisotropy,

$$h_3(\chi^1, \theta, t) = r^{-2}\left(-M_0 + \frac{r^2}{l^2}\right)\cos^2\theta\ \sin^2[\theta - \theta_0(\chi^1, t)] \tag{4.20}$$

with

$$\cos\theta_0(\chi^1, t) = e^{-1}\left(\frac{r_a}{r}\cos\omega_1 t - 1\right),$$

when the horizon is given parametrically,

$$r = \frac{r_a}{1 + e\cos\theta}\cos\omega_1 t,$$

where the new constants (comparing with those from the previous subsection) are fixed by some initial and boundary conditions as to be $\widetilde{p} > 0$, and $\widetilde{\omega}$ and ω_1 are treated as some real numbers.

For a prescribed value of $h_3(\chi^1,\theta,t)$ with non–zero source Υ_{31}, in the equation (4.6), we obtain

$$q_1(\chi^1,\theta,t) = \kappa\Upsilon_{31}\left(\frac{\partial^2}{\partial t^2}\ln|h_3(\chi^1,\theta,t)|\right)^{-1}. \quad (4.21)$$

A solution (4.1) of the Einstein equations (4.5) and (4.6) with $g_2(\chi^i)=1$ and $g_1(\chi^1,\theta)$ and $h_3(\chi^1,\theta,t)$ given respectively by formulas (4.17) and (4.19) describe a 3D evaporating black la–hole solution with circular oscillating in time horizon. An another type of solution, with elliptic oscillating in time horizon, could be obtained if we choose (4.20). The non–trivial coefficient of the N–connection must be computed following the formula (4.21).

4.5 4D la–solutions

4.5.1 Basic properties

The purpose of this section is the construction of d–metrics which are conformally equivalent to some la–deformations of black hole, torus and cylinder like solutions in general relativity. We shall analyze 4D d-metrics of type

$$\delta s^2 = g_1\left(x^k\right)\left(dx^1\right)^2 + \left(dx^2\right)^2 + h_3(x^i,z)\left(\delta z\right)^2 + h_4(x^i,z)\left(\delta y^4\right)^2. \quad (4.1)$$

The Einstein equations (4.7) with the Ricci h–tensor (4.3) and diagonal energy momentum d–tensor transforms into

$$\frac{\partial^2 g_1}{\partial(x^2)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial x^2}\right)^2 - 2\kappa\Upsilon_3^3 g_1 = 0. \quad (4.2)$$

By introducing a dimensionless coordinate, $\chi^2 = x^2/\sqrt{|\kappa\Upsilon_3^3|}$, and the variable $p = g_1'/g_1$, where by 'prime' in this section is considered the partial derivative ∂/χ^2, the equation (4.2) transforms into

$$p' + \frac{p^2}{2} + 2\epsilon = 0, \quad (4.3)$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = 1(-1)$. The equations (4.2) and (4.3) are, correspondingly, equivalent to the equations (4.8) and (4.11) with that difference that in this section we are dealing with 4D coefficients and values. The solutions for the h–metric are parametrized like (4.14), (4.15), and (4.16) and the coefficient $g_1(\chi^i)$ is given by a similar to (4.17) formula (for simplicity, here we elucidate the case $\epsilon < 0$) which for $p_{(0)}(\chi^1) = 0$ transforms into

$$g_1\left(\chi^1, \chi^2\right) = g_{1(0)}\left(\chi^1\right) \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)], \tag{4.4}$$

where $g_1(\chi^1), \chi^2_{(0)}(\chi^1)$ and $p_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, ε is the energy density. The coefficients $g_1(\chi^1, \chi^2)$ (4.4) and $g_2(\chi^1, \chi^2) = 1$ define a h–metric. The next step is the construction of h–components of d–metrics, $h_a = h_a(\chi^i, z)$, for different classes of symmetries of anisotropies.

The system of equations (4.8) with the vertical Ricci d–tensor component (4.4) is satisfied by arbitrary functions

$$h_3 = a_3(\chi^i) \text{ and } h_4 = a_4(\chi^i). \tag{4.5}$$

For v–metrics depending on three coordinates (χ^i, z) the v–components of the Einstein equations transform into (4.11) which reduces to (4.11) for prescribed values of $h_3(\chi^i, z)$, and, inversely, to (4.13) if $h_4(\chi^i, z)$ is prescribed. For h–metrics being conformally equivalent to (4.5) (see transforms (4.14)) we are dealing to equations of type (4.15) with integral varieties (4.16).

4.5.2 Rotation Hypersurfaces Horizons

We proof that there are static black hole and cylindrical like solutions of the Einstein equations with horizons being some 3D rotation hypersurfaces. The space components of corresponding d–metrics are conformally equivalent to some locally anisotropic deformations of the spherical symmetric Schwarzschild and cylindrical Weyl solutions. We note that for some classes of solutions the local anisotropy is contained in non–perturbative anholonomic structures.

Rotation ellipsoid configuration

There two types of rotation ellipsoids, elongated and flattened ones. We examine both cases of such horizon configurations.

Elongated rotation ellipsoid coordinates:
An elongated rotation ellipsoid hypersurface is given by the formula [7]

$$\frac{\widetilde{x}^2 + \widetilde{y}^2}{\sigma^2 - 1} + \frac{\widetilde{z}^2}{\sigma^2} = \widetilde{\rho}^2, \tag{4.6}$$

where $\sigma \geq 1$ and $\widetilde{\rho}$ is similar to the radial coordinate in the spherical symmetric case.

The space 3D coordinate system is defined

$$\widetilde{x} = \widetilde{\rho} \sinh u \sin v \cos \varphi, \ \widetilde{y} = \widetilde{\rho} \sinh u \sin v \sin \varphi, \ \widetilde{z} = \widetilde{\rho} \cosh u \cos v,$$

where $\sigma = \cosh u, (0 \leq u < \infty, \ 0 \leq v \leq \pi, \ 0 \leq \varphi < 2\pi)$. The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv} = \widetilde{\rho}^2 \left(\sinh^2 u + \sin^2 v \right), \\ g_{\varphi\varphi} &= \widetilde{\rho}^2 \sinh^2 u \sin^2 v. \end{aligned} \tag{4.7}$$

Let us introduce a d–metric

$$\delta s^2 = g_1(u,v) du^2 + dv^2 + h_3 (u,v,\varphi) (\delta t)^2 + h_4 (u,v,\varphi) (\delta \varphi)^2 , \tag{4.8}$$

where δt and $\delta \varphi$ are N–elongated differentials.

As a particular solution (4.4) for the h–metric we choose the coefficient

$$g_1(u,v) = \cos^2 v. \tag{4.9}$$

The $h_3(u,v,\varphi) = h_3(u,v,\widetilde{\rho}(u,v,\varphi))$ is considered as

$$h_3(u,v,\widetilde{\rho}) = \frac{1}{\sinh^2 u + \sin^2 v} \frac{\left[1 - \frac{r_g}{4\widetilde{\rho}}\right]^2}{\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^6}. \tag{4.10}$$

In order to define the h_4 coefficient solving the Einstein equations, for simplicity with a diagonal energy–momentum d–tensor for vanishing pressure we must solve the equation (4.11) which transforms into a linear equation (4.12) if $\Upsilon_1 = 0$. In our case $s(u,v,\varphi) = \beta^{-1}(u,v,\varphi)$, where $\beta = (\partial h_4 / \partial \varphi) / h_4$, must be a solution of

$$\frac{\partial s}{\partial \varphi} + \frac{\partial \ln \sqrt{|h_3|}}{\partial \varphi} s = \frac{1}{2}.$$

After two integrations (see [6]) the general solution for $h_4(u,v,\varphi)$, is

$$h_4(u,v,\varphi) = a_4(u,v) \exp \left[- \int_0^{\varphi} F(u,v,z) \, dz \right], \tag{4.11}$$

where

$$F(u,v,z) = 1/\{ \sqrt{|h_3(u,v,z)|} [s_{1(0)}(u,v) + \frac{1}{2} \int_{z_0(u,v)}^{z} \sqrt{|h_3(u,v,z)|} dz] \},$$

$s_{1(0)}(u,v)$ and $z_0(u,v)$ are some functions of necessary smooth class. We note that if we put $h_4 = a_4(u,v)$ the equations (4.8) are satisfied for every $h_3 = h_3(u,v,\varphi)$.

Every d–metric (4.8) with coefficients of type (4.9), (4.10) and (4.11) solves the Einstein equations (4.7)–(4.10) with the diagonal momentum d–tensor

$$\Upsilon^{\alpha}_{\beta} = diag\left[0, 0, -\varepsilon = -m_0, 0\right],$$

when $r_g = 2\kappa m_0$; we set the light constant $c = 1$. If we choose

$$a_4(u,v) = \frac{\sinh^2 u \ \sin^2 v}{\sinh^2 u + \sin^2 v}$$

our solution is conformally equivalent (if not considering the time–time component) to the hypersurface metric (4.7). The condition of vanishing of the coefficient (4.10) parametrizes the rotation ellipsoid for the horizon

$$\frac{\widetilde{x}^2 + \widetilde{y}^2}{\sigma^2 - 1} + \frac{\widetilde{z}^2}{\sigma^2} = \left(\frac{r_g}{4}\right)^2,$$

where the radial coordinate is redefined via relation $\widetilde{r} = \widetilde{\rho}\left(1 + \frac{r_g}{4\widetilde{\rho}}\right)^2$. After multiplication on the conformal factor

$$\left(\sinh^2 u + \sin^2 v\right)\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4,$$

approximating $g_1(u,v) = \cos^2 v \approx 1$, in the limit of locally isotropic spherical symmetry,

$$\widetilde{x}^2 + \widetilde{y}^2 + \widetilde{z}^2 = r_g^2,$$

the d–metric (4.8) reduces to

$$ds^2 = \left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4 \left(d\widetilde{x}^2 + d\widetilde{y}^2 + d\widetilde{z}^2\right) - \frac{\left[1 - \frac{r_g}{4\widetilde{\rho}}\right]^2}{\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^2} dt^2$$

which is just the Schwarzschild solution with the redefined radial coordinate when the space component becomes conformally Euclidean.

So, the d–metric (4.8), the coefficients of N–connection being solutions of (4.9) and (4.10), describe a static 4D solution of the Einstein equations when instead of a spherical symmetric horizon one considers a locally anisotropic deformation to the hypersurface of rotation elongated ellipsoid.

Flattened rotation ellipsoid coordinates

In a similar fashion we can construct a static 4D black hole solution with the horizon parametrized by a flattened rotation ellipsoid [7],

$$\frac{\widetilde{x}^2+\widetilde{y}^2}{1+\sigma^2}+\frac{\widetilde{z}^2}{\sigma^2}=\widetilde{\rho}^2,$$

where $\sigma \geq 0$ and $\sigma = \sinh u$.

The space 3D special coordinate system is defined

$$\widetilde{x}=\widetilde{\rho}\cosh u \sin v \cos\varphi,\ \widetilde{y}=\widetilde{\rho}\cosh u \sin v \sin\varphi,\ \widetilde{z}=\widetilde{\rho}\sinh u \cos v,$$

where $0\leq u<\infty,\ 0\leq v\leq\pi,\ 0\leq\varphi<2\pi.$

The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv}=\widetilde{\rho}^2\left(\sinh^2 u+\cos^2 v\right), \\ g_{\varphi\varphi} &= \widetilde{\rho}^2\sinh^2 u\cos^2 v. \end{aligned}$$

In the rest the black hole solution is described by the same formulas as in the previous subsection but with respect to new canonical coordinates for flattened rotation ellipsoid.

Cylindrical, Bipolar and Toroidal Configurations

We consider a d–metric of type (4.1). As a coefficient for h–metric we choose $g_1(\chi^1,\chi^2)=(\cos\chi^2)^2$ which solves the Einstein equations (4.7). The energy momentum d–tensor is chosen to be diagonal, $\Upsilon^\alpha_\beta = diag[0,0,-\varepsilon,0]$ with $\varepsilon\simeq m_0=\int m_{(lin)}dl$, where $\varepsilon_{(lin)}$ is the linear 'mass' density. The coefficient $h_3\left(\chi^i,z\right)$ will be chosen in a form similar to (4.10),

$$h_3\simeq\left[1-\frac{r_g}{4\widetilde{\rho}}\right]^2/\left[1+\frac{r_g}{4\widetilde{\rho}}\right]^6$$

for a cylindrical elliptic horizon. We parametrize the second v–component as $h_4=a_4(\chi^1,\chi^2)$ when the equations (4.8) are satisfied for every $h_3=h_3(\chi^1,\chi^2,z)$.

Cylindrical coordinates:

Let us construct a solution of the Einstein equation with the horizon having the symmetry of ellipsoidal cylinder given by hypersurface formula [7]

$$\frac{\widetilde{x}^2}{\sigma^2}+\frac{\widetilde{y}^2}{\sigma^2-1}=\rho_*^2,\ \widetilde{z}=\widetilde{z},$$

where $\sigma \geq 1$. The 3D radial coordinate $\widetilde{r}$ is to be computed from $\widetilde{\rho}^2 = \rho_*^2 + \widetilde{z}^2$.

The 3D space coordinate system is defined

$$\widetilde{x} = \rho_* \cosh u \cos v,\ \widetilde{y} = \rho_* \sinh u \sin v \sin,\ \widetilde{z} = \widetilde{z},$$

where $\sigma = \cosh u,\ (0 \leq u < \infty,\ 0 \leq v \leq \pi)$.

The hypersurface metric is

$$g_{uu} = g_{vv} = \rho_*^2 \left(\sinh^2 u + \sin^2 v\right), g_{zz} = 1. \tag{4.12}$$

A solution of the Einstein equations with singularity on an ellipse is given by

$$\begin{aligned} h_3 &= \frac{1}{\rho_*^2 \left(\sinh^2 u + \sin^2 v\right)} \times \frac{\left[1 - \frac{r_g}{4\widetilde{\rho}}\right]^2}{\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^6}, \\ h_4 &= a_4 = \frac{1}{\rho_*^2 \left(\sinh^2 u + \sin^2 v\right)}, \end{aligned}$$

where $\widetilde{r} = \widetilde{\rho}\left(1 + \frac{r_g}{4\widetilde{\rho}}\right)^2$. The condition of vanishing of the time–time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\frac{\widetilde{x}^2}{\sigma^2} + \frac{\widetilde{y}^2}{\sigma^2 - 1} = \left(\frac{\rho_{*(g)}}{4}\right)^2,\ \widetilde{z} = \widetilde{z},$$

where $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d–metric on the conformal factor

$$\rho_*^2 \left(\sinh^2 u + \sin^2 v\right) \left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4,$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the ellipse), for $\rho_* \to 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (4.12).

Bipolar coordinates:

Let us construct 4D solutions of the Einstein equation with the horizon having the symmetry of the bipolar hypersurface given by the formula [7]

$$\left(\sqrt{\widetilde{x}^2 + \widetilde{y}^2} - \frac{\widetilde{\rho}}{\tan \sigma}\right)^2 + \widetilde{z}^2 = \frac{\widetilde{\rho}^2}{\sin^2 \sigma},$$

which describes a hypersurface obtained under the rotation of the circles

$$\left(\widetilde{y} - \frac{\widetilde{\rho}}{\tan\sigma}\right)^2 + \widetilde{z}^2 = \frac{\widetilde{\rho}^2}{\sin^2\sigma}$$

around the axes Oz; because $|c\tan\sigma| < |\sin\sigma|^{-1}$, the circles intersect the axes Oz. The 3D space coordinate system is defined

$$\begin{aligned}
\widetilde{x} &= \frac{\widetilde{\rho}\sin\sigma\cos\varphi}{\cosh\tau - \cos\sigma}, \qquad \widetilde{y} = \frac{\widetilde{\rho}\sin\sigma\sin\varphi}{\cosh\tau - \cos\sigma}, \\
\widetilde{z} &= \frac{\widetilde{r}\sinh\tau}{\cosh\tau - \cos\sigma} \quad (-\infty < \tau < \infty, 0 \leq \sigma < \pi, 0 \leq \varphi < 2\pi).
\end{aligned}$$

The hypersurface metric is

$$g_{\tau\tau} = g_{\sigma\sigma} = \frac{\widetilde{\rho}^2}{(\cosh\tau - \cos\sigma)^2}, g_{\varphi\varphi} = \frac{\widetilde{\rho}^2\sin^2\sigma}{(\cosh\tau - \cos\sigma)^2}. \tag{4.13}$$

A solution of the Einstein equations with singularity on a circle is given by

$$h_3 = \left[1 - \frac{r_g}{4\widetilde{\rho}}\right]^2 / \left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^6 \text{ and } h_4 = a_4 = \sin^2\sigma,$$

where $\widetilde{r} = \widetilde{\rho}\left(1 + \frac{r_g}{4\widetilde{\rho}}\right)^2$. The condition of vanishing of the time–time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\widetilde{x}^2 + \widetilde{y}^2} - \frac{r_g}{2}\ c\tan\sigma\right)^2 + \widetilde{z}^2 = \frac{r_g^2}{4\sin^2\sigma},$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the circle), $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d–metric on the conformal factor

$$\frac{1}{(\cosh\tau - \cos\sigma)^2}\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4, \tag{4.14}$$

for $\rho_* \to 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (4.13).

Toroidal coordinates:

Let us consider solutions of the Einstein equations with toroidal symmetry of horizons. The hypersurface formula of a torus is [7]

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\widetilde{\rho}\ c\tanh\sigma\right)^2+\widetilde{z}^2=\frac{\widetilde{\rho}^2}{\sinh^2\sigma}.$$

The 3D space coordinate system is defined

$$\begin{aligned}
\widetilde{x} &= \frac{\widetilde{\rho}\sinh\tau\cos\varphi}{\cosh\tau-\cos\sigma}, \qquad \widetilde{y}=\frac{\widetilde{\rho}\sin\sigma\sin\varphi}{\cosh\tau-\cos\sigma},\\
\widetilde{z} &= \frac{\widetilde{\rho}\sinh\sigma}{\cosh\tau-\cos\sigma}\ \left(-\pi<\sigma<\pi, 0\leq\tau<\infty, 0\leq\varphi<2\pi\right).
\end{aligned}$$

The hypersurface metric is

$$g_{\sigma\sigma}=g_{\tau\tau}=\frac{\widetilde{\rho}^2}{\left(\cosh\tau-\cos\sigma\right)^2}, g_{\varphi\varphi}=\frac{\widetilde{\rho}^2\sin^2\sigma}{\left(\cosh\tau-\cos\sigma\right)^2}. \tag{4.15}$$

This, another type of solution of the Einstein equations with singularity on a circle, is given by

$$h_3=\left[1-\frac{r_g}{4\widetilde{\rho}}\right]^2/\left[1+\frac{r_g}{4\widetilde{\rho}}\right]^6 \text{ and } h_4=a_4=\sinh^2\sigma,$$

where $\widetilde{r}=\widetilde{\rho}\left(1+\frac{r_g}{4\widetilde{\rho}}\right)^2$. The condition of vanishing of the time–time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\frac{r_g}{2\tanh\sigma}c\right)^2+\widetilde{z}^2=\frac{r_g^2}{4\sinh^2\sigma},$$

where $r_g=\int\rho_{*(g)}dl$ (the integration is taken along the circle), $\rho_{*(g)}=2\kappa m_{(lin)}$.

By multiplying the d–metric on the conformal factor (4.14), for $\rho_*\to 1$, in the local isotropic limit, $\sin v\approx 0$, the space component transforms into (4.15).

4.5.3 A Schwarzschild like la–solution

The d–metric of type (4.8) is taken

$$\delta s^2=g_1(\chi^1,\theta)d(\chi^1)^2+d\theta^2+h_3\left(\chi^1,\theta,\varphi\right)(\delta t)^2+h_4\left(\chi^1,\theta,\varphi\right)(\delta\varphi)^2, \tag{4.16}$$

where on the horizontal subspace $\chi^1=\rho/r_a$ is the dimensionless radial coordinate (the constant r_a will be defined below), $\chi^2=\theta$ and in the vertical subspace $y^3=z=t$ and

$y^4 = \varphi$. The energy–momentum d–tensor is taken to be diagonal $\Upsilon^\alpha_\beta = diag[0, 0, -\varepsilon, 0]$. The coefficient g_1 is chosen to be a solution of type (4.4)

$$g_1\left(\chi^1, \theta\right) = \cos^2\theta.$$

For

$$h_4 = \sin^2\theta \text{ and } h_3(\rho) = -\frac{\left[1 - r_a/4\rho\right]^2}{\left[1 + r_a/4\rho\right]^6},$$

where $r = \rho\left(1 + \frac{r_g}{4\rho}\right)^2, r^2 = x^2 + y^2 + z^2$, $r_a \doteq r_g$ is the Schwarzschild gravitational radius, the d–metric (4.16) describes a la–solution of the Einstein equations which is conformally equivalent, with the factor $\rho^2\left(1 + r_g/4\rho\right)^2$, to the Schwarzschild solution (written in coordinates $(\rho, \theta, \varphi, t)$), embedded into a la–background given by non–trivial values of $q_i(\rho, \theta, t)$ and $n_i(\rho, \theta, t)$. In the anisotropic case we can extend the solution for anisotropic (on angle θ) gravitational polarizations of point particles masses, $m = m(\theta)$, for instance in elliptic form, when

$$r_a(\theta) = \frac{r_g}{(1 + e\cos\theta)}$$

induces an ellipsoidal dependence on θ of the radial coordinate,

$$\rho = \frac{r_g}{4(1 + e\cos\theta)}.$$

We can also consider arbitrary solutions with $r_a = r_a(\theta, t)$ of oscillation type, $r_a \simeq \sin(\omega_1 t)$, or modelling the mass evaporation, $r_a \simeq \exp[-st], s = const > 0$.

So, fixing a physical solution for $h_3(\rho, \theta, t)$, for instance,

$$h_3(\rho, \theta, t) = -\frac{\left[1 - r_a \exp[-st]/4\rho\,(1 + e\cos\theta)\right]^2}{\left[1 + r_a \exp[-st]/4\rho\,(1 + e\cos\theta)\right]^6},$$

where $e = const < 1$, and computing the values of $q_i(\rho, \theta, t)$ and $n_i(\rho, \theta, t)$ from (4.9) and (4.10), corresponding to given h_3 and h_4, we obtain a la–generalization of the Schwarzschild metric.

We note that fixing this type of anisotropy, in the locally isotropic limit we obtain not just the Schwarzschild metric but a conformally transformed one, multiplied on the factor $1/\rho^2\left(1 + r_g/4\rho\right)^4$.

4.6 Final remarks

We have presented new classes of three and four dimensional black hole solutions with local anisotropy which are given both with respect to a coordinate basis or to an anholonomic frame defined by a N–connection structure. We proved that for a corresponding ansatz such type of solutions can be imbedded into the usual (three or four dimensional) Einstein gravity. It was demonstrated that in general relativity there are admitted static, but anisotropic (with nonspheric symmetry), and elliptic oscillating in time black hole like configurations with horizons of events being elliptic (in three dimensions) and rotation ellipsoidal, elliptic cylinder, toroidal and another type of closed hypersurfaces or cylinders.

From the results obtained, it appears that the components of metrics with generic local anisotropy are somehow undetermined from field equations if the type of symmetry and a correspondence with locally isotropic limits are not imposed. This is the consequence of the fact that in general relativity only a part of components of the metric field (six from ten in four dimensions and three from six in three dimensions) can be treated as dynamical variables. This is caused by the Bianchi identities which hold on (pseudo) Riemannian spaces. The rest of components of metric should be defined from some symmetry prescriptions on the type of locally anisotropic solutions and corresponding anholonomic frames and, if existing, compatibility with the locally isotropic limits when some physically motivated coordinate and/or boundary conditions are enough to state and solve the Cauchy problem.

Some of the problems discussed so far might be solved by considering theories containing non–trivial torsion fields like metric–affine and gauge gravity and for so–called generalized Finsler–Kaluza–Klein models. More general solutions connected with locally anisotropic low energy limits in string/M–theory and supergravity could be also generated by applying the method of computation with respect to anholonomic (super) frames adapted to a N–connection structure. This topic is currently under study.

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Chapter 5

Anholonomic Triads and New Classes of (2+1)-Dimensional Black Hole Solutions

Abstract [1]

We apply the method of moving anholonomic frames in order to construct new classes of solutions of the Einstein equations on (2+1)–dimensional pseudo–Riemannian spaces. The anholonomy associated to a class of off–diagonal metrics results in alternative classes of black hole solutions which are constructed following distinguished (by nonlinear connection structure) linear connections and metrics. There are investigated black holes with deformed horizons and renormalized locally anisotropic constants. We speculate on properties of such anisotropic black holes with characteristics defined by anholonomic frames and anisotropic interactions of matter and gravity. The thermodynamics of locally anisotropic black holes is discussed in connection with a possible statistical mechanics background based on locally anisotropic variants of Chern–Simons theories.

5.1 Introduction

In recent years there has occurred a substantial interest to the (2+1)–dimensional gravity and black holes and possible connections of such objects with string/M–theory. Since the first works of Deser, Jackiv and 't Hooft [10] and Witten [29] on three dimensional gravity and the seminal solution for (2+1)–black holes constructed by Bañados, Teitelboim, and Zanelli (BTZ) [3] the gravitational models in three dimensions have become a very powerful tool for exploring the foundations of classical and quantum gravity, black hole physics, as well the geometrical properties of the spaces on which the low–dimensional physics takes place [5].

On the other hand, the low–dimensional geometries could be considered as an arena for elaboration of new methods of solution of gravitational field equations. One of peculiar features of general relativity in 2+1 dimensions is that the bulk of physical solutions of Einstein equations are constructed for a negative cosmological constant and on a space of constant curvature. There are not such limitations if anholonomic frames modelling locally anisotropic (la) interactions of gravity and matter are considered.

In our recent works [25] we emphasized the importance of definition of frames of reference in general relativity in connection with new methods of construction of solutions of the Einstein equations. The former priority given to holonomic frames holds good for the 'simplest' spherical symmetries and is less suitable for construction of solutions with 'deformed' symmetries, for instance, of static black holes with elliptic (or ellipsoidal and/or torus) configurations of horizons. Such type of 'deformed', locally anisotropc, solutions of the Einstein equations are easily to be derived from the ansatz of metrics diagonalized with respect to some classes of anholonomic frames induced by locally anisotropic 'elongations' of partial derivatives. After the task has been solved in anholonomic variables it can be removed with respect to usual coordinate bases when the metric becomes off–diagonal and the (for instance, elliptic) symmetry is hidden in some nonlinear dependencies of the metric components.

The specific goal of the present work is to formulate the (2+1)–dimensional gravity theory with respect to anholonomic frames with associated nonlinear connection (N–connection) structure and to construct and investigate some new classes of solutions of Einstein equations on locally anisotropic spacetimes (modelled as usual pseudo–Riemannian spaces provided with an anholonomic frame structure). A material of interest are the properties of the locally anisotropic elastic media and rotating null fluid and anisotropic collapse described by gravitational field equations with locally anisotropic matter. We investigate black hole solutions that arise from coupling in a self–consistent manner the three dimensional (3D) pseudo–Riemannian geometry and its anholonomic deformations to the physics of locally anisotropic fluids formulated with respect to anholonomic frames of reference. For certain special cases the locally anisotropic matter gives the BTZ black holes with/or not rotation and electrical charge and variants of their anisotropic generalizations. For other cases, the resulting solutions are generic black holes with "locally anisotropic hair".

It should be emphasized, that general anholonomic frame transforms with associated N–connection structure result in deformation of both metric and linear connection structures. One can be generated spaces with nontrivial nonmetricity and torsion fields. There are subclasses of deformations when the condition of metric and connection compatibility is preserved and the torsion fields are effectively induced by the anholonomic frame structure. On such spaces we can work with the torsionless Levi–Civita connection or (equivalently, but in a more general geometric form) with certain linear connections with effective torsion. With respect to anholonomic frames (this can be naturally adapted to the nonlinear connection structure), the Ricci tensor can be nonsymmetric and the conservation lows are to be formulated in more sophisticate form. This is similar to the nonholonomic mechanics with various types of constraints on dynamics (in modern literature, one uses two equivalent terms, nonholonomic and/or anholonomic). Such geometric constructions are largely used in generalized Lagrange–Hamilton and Finsler–Cartan geometry [19], but for nonholonomic manifolds (i.e. manifolds provided with nonintegrable distributions, in the simplest case defining a nonlinear connection) such generalized geometries can be modelled by nonholnomic frames and their deformations on (pseudo) Riemannian spaces, see details in Refs. [27]. This work is devoted to a study of such 3D nonholonomic frame deformations and their possible physical implications in lower–dimensional gravity.

We note that the anisotropic gravitational field has very unusual properties. For instance, the vacuum solutions of Einstein anisotropic gravitational field equations could describe anisotropic black holes with elliptic symmetry. Some subclasses of such locally anisotropic spaces are teleparallel (with non–zero induced torsion but with vanishing curvature tensor) another are characterized by nontrivial, induced from general relativity on anholonomic frame bundle, N–connection and Riemannian curvature and anholonomy induced torsion. In a more general approach the N–connection and torsion are induced also from the condition that the metric and nonlinear connection must solve the Einstein equations.

The paper is organized as follows: In the next section we briefly review the locally anisotropic gravity in (2+1)–dimensions. Conformal transforms with anisotropic factors and corresponding classes of solutions of Einstein equations with dynamical equations for N–connection coefficients are examined in Sec. 3. In Sec. 4 we derive the energy–momentum tensors for locally anisotropic elastic media and rotating null fluids. Sec. 5 is devoted to the local anisotropy of (2+1)–dimensional solutions of Einstein equations with anisotropic matter. The nonlinear self–polarization of anisotropic vacuum gravitational fields and matter induced polarizations and related topics on anisotropic black hole solutions are considered in Sec. 6. We derive some basic formulas for thermodynamics of anisotropic black holes in Sec. 7. The next Sec. 8 provides a statistical mechanics background for locally anisotropic thermodynamics starting from the locally anisotropic variants of Chern–Simons and Wess–Zumino–Witten models of locally anisotropic gravity. Finally, in Sec. 9 we conclude and discuss the obtained results.

5.2 Anholonomic Frames and 3D Gravity

In this Section we wish to briefly review and reformulate the Cartan's method of moving frames [8] for investigation of gravitational and matter field interactions with mixed subsets of holonomic (unconstrained) and anholonomic (constrained, equivalently, locally anisotropic, in brief, la) variables [25]. Usual tetradic (frame, or vielbein) approaches to general relativity, see, for instance, [20, 12], consider 'non–mixed' cases when all basic vectors are anholonomic or transformed into coordinate (holonomic) ones. We note that a more general geometric background for locally anisotropic interactions and locally anisotropic spacetimes, with applications in physics, was elaborated by Miron and Anastasiei [19] in their generalized Finsler and Lagrange geometry; further developments for locally anisotropic spinor bundles and locally anisotropic superspaces are contained in Refs [23, 24]. Here we restrict our constructions only to three dimensional (3D) pseudo–Riemannian spacetimes provided with a global splitting characterized by two holonomic and one anholonomic coordinates.

5.2.1 Anholonomic frames and nonlinear connections

We model the low dimensional spacetimes as a smooth (i. e. class C^∞) 3D (pseudo) Riemannian manifolds $V^{(3)}$ being Hausdorff, paracompact and connected and enabled with the fundamental structures of symmetric metric $g_{\alpha\beta}$, with signature $(-,+,+)$ and of linear, in general nonsymmetric (if we consider anholonomic frames), metric connection $\Gamma^{\alpha}{}_{\beta\gamma}$ defining the covariant derivation D_α. The indices of geometrical objects on $V^{(3)}$ are stated with respect to a frame vector field (triad, or dreibien) e_α and its dual e^α. A holonomic frame structure on 3D spacetime could be given by a local coordinate base

$$\partial_\alpha = \partial/\partial u^\alpha, \tag{5.1}$$

consisting from usual partial derivatives on local coordinates $u = \{u^\alpha\}$ and the dual basis

$$d^\alpha = du^\alpha, \tag{5.2}$$

consisting from usual coordinate differentials du^α.

An arbitrary holonomic frame e_α could be related to a coordinate one by a local linear transform $e_\alpha = A_\alpha^{\ \beta}(u)\partial_\beta$, for which the matrix $A_\alpha^{\ \beta}$ is nondegenerate and there are satisfied the holonomy conditions

$$e_\alpha e_\beta - e_\beta e_\alpha = 0.$$

Let us consider a 3D metric parametrized into $(2+1)$ components

$$g_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^{\bullet} N_j^{\bullet} h_{\bullet\bullet} & N_j^{\bullet} h_{\bullet\bullet} \\ N_i^{\bullet} h_{\bullet\bullet} & h_{\bullet\bullet} \end{array} \right] \quad (5.3)$$

given with respect to a local coordinate basis (5.2), $du^{\alpha} = (dx^i, dy)$, where the Greek indices run values $1,2,3$, the Latin indices $i,j,k,...$ from the middle of the alphabet run values for $n = 1,2,...$ and the Latin indices from the beginning of the alphabet, $a,b,c,...$, run values for $m = 3,4,....$ if we wont to consider imbedding of 3D spaces into higher dimension ones. The coordinates x^i are treated as isotropic ones and the coordinate $y^{\bullet} = y$ is considered anholonomic (anisotropic). For 3D we denote that $a,b,c,... = \bullet$, $y^{\bullet} \to y$, $h_{ab} \to h_{\bullet\bullet} = h$ and $N_i^a \to N_i^{\bullet} = w_i$. The coefficients $g_{ij} = g_{ij}(u), h_{\bullet\bullet} = h(u)$ and $N_i^{\bullet} = N_i(u)$ are supposed to solve the 3D Einstein gravitational field equations. The metric (5.3) can be rewritten in a block $(2\times 2)\oplus 1$ form

$$g_{\alpha\beta} = \left(\begin{array}{cc} g_{ij}(u) & 0 \\ 0 & h(u) \end{array} \right) \quad (5.4)$$

with respect to the anholonomic basis (frame, anisotropic basis)

$$\delta_{\alpha} = (\delta_i, \partial_{\bullet}) = \frac{\delta}{\partial u^{\alpha}} = \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^{\bullet}(u)\frac{\partial}{\partial y}, \partial_{\bullet} = \frac{\partial}{\partial y} \right) \quad (5.5)$$

and its dual anholonomic frame

$$\delta^{\beta} = \left(d^i, \delta^{\bullet}\right) = \delta u^{\beta} = \left(d^i = dx^i, \delta^{\bullet} = \delta y = dy + N_k^{\bullet}(u)\, dx^k\right).$$

where the coefficients $N_j^{\bullet}(u)$ from (5.5) and (5.6) could be treated as the components of an associated nonlinear connection (N–connection) structure [2, 19, 23, 24] which was considered in Finsler and generalized Lagrange geometries and applied in general relativity and Kaluza–Klein gravity for construction of new classes of solutions of Einstein equations by using the method of moving anholonomic frames [25]. On 3D (pseudo)–Riemannian spaces the coefficients $N_j^{\bullet}$ define a triad of basis vectors (dreibein) with respect to which the geometrical objects (tensors, connections and spinors) are decomposed into holonomic (with indices $i,j,...$) and anholonomic (provided with $\bullet$–index) components.

A local frame (local basis) structure δ_{α} on $V^{(3)} \to V^{(2+1)}$ (by $(2+1)$ we denote the N–connection splitting into 2 holonomic and 1 anholonomic variables in explicit form; this decomposition differs from the usual two space and one time–like parametrizations) is characterized by its anholonomy coefficients $w^{\alpha}{}_{\beta\gamma}$ defined from relations

$$\delta_{\alpha}\delta_{\beta} - \delta_{\beta}\delta_{\alpha} = w^{\gamma}{}_{\alpha\beta}\delta_{\gamma}. \quad (5.6)$$

The rigorous mathematical definition of N–connection is based on the formalism of horizontal and vertical subbundles and on exact sequences in vector bundles [2, 19]. In this work we introduce a N–connection as a distribution which for every point $u=(x,y)\in V^{(2+1)}$ defines a local decomposition of the tangent space

$$T_u V^{(2+1)} = H_u V^{(2)} \oplus V_u V^{(1)}.$$

into horizontal subspace, $H_u V^{(2)}$, and vertical (anisotropy) subspace, $V_u V^{(1)}$, which is given by a set of coefficients $N_j^{\bullet}(u^{\alpha})$. A N–connection is characterized by its curvature

$$\Omega_{ij}^{\bullet} = \frac{\partial N_i^{\bullet}}{\partial x^j} - \frac{\partial N_j^{\bullet}}{\partial x^i} + N_i^{\bullet}\frac{\partial N_j^{\bullet}}{\partial y} - N_j^{\bullet}\frac{\partial N_i^{\bullet}}{\partial y}. \quad (5.7)$$

The class of usual linear connections can be considered as a particular case of N–connections when

$$N_j^{\bullet}(x,y) = \Gamma^{\bullet}_{\bullet j}(x) y^{\bullet}.$$

The elongation (by N–connection) of partial derivatives and differentials in the adapted to the N–connection operators (5.5) and (5.6) reflects the fact that on the (pseudo) Riemannian spacetime $V^{(2+1)}$ it is modelled a generic local anisotropy characterized by anholonomy relations (5.6) when the anholonomy coefficients are computed as follows

$$\begin{aligned} w^k{}_{ij} &= 0, w^k{}_{\bullet j} = 0, w^k{}_{i\bullet} = 0, w^k{}_{\bullet\bullet} = 0, w^{\bullet}{}_{\bullet\bullet} = 0, \\ w^{\bullet}{}_{ij} &= -\Omega^{\bullet}_{ij}, w^{\bullet}{}_{\bullet j} = -\partial_{\bullet} N_i^{\bullet}, w^{\bullet}{}_{i\bullet} = \partial_{\bullet} N_i^{\bullet}. \end{aligned} \quad (5.8)$$

The frames (5.5) and (5.6) are locally adapted to the N–connection structure, define a local anisotropy and, in brief, are called anholonomic bases. A N–connection structure distinguishes (d) the geometrical objects into horizontal and vertical components, i. e. transform them into d–objects which are briefly called d–tensors, d–metrics and d–connections. Their components are defined with respect to an anholonomic basis of type (5.5), its dual (5.6), or their tensor products (d–linear or d–affine transforms of such frames could also be considered). For instance, a covariant and contravariant d–tensor Q, is expressed

$$Q = Q^{\alpha}{}_{\beta}\delta_{\alpha}\otimes\delta^{\beta} = Q^i{}_j\delta_i\otimes d^j + Q^i{}_{\bullet}\delta_i\otimes\delta^{\bullet} + Q^{\bullet}{}_j\partial_{\bullet}\otimes d^j + Q^{\bullet}{}_{\bullet}\partial_{\bullet}\otimes\delta^{\bullet}.$$

Similar decompositions on holonomic–anholonomic, conventionally on horizontal (h) and vertical (v) components, hold for connection, torsion and curvature components adapted to the N–connection structure.

5.2.2 Compatible N- and d–connections and metrics

A linear d–connection D on a locally anisotropic spacetime $V^{(2+1)}$, $D_{\delta_\gamma}\delta_\beta = \Gamma^{\alpha}{}_{\beta\gamma}(x,y)\,\delta_\alpha$, is given by its h–v–components,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \left(L^{i}{}_{jk}, L^{\bullet}{}_{\bullet k}, C^{i}{}_{j\bullet}, C^{\bullet}{}_{\bullet\bullet}\right)$$

where

$$D_{\delta_k}\delta_j = L^{i}{}_{jk}\delta_i, D_{\delta_k}\partial_\bullet = L^{\bullet}_{\bullet k}\partial_\bullet, D_{\partial_\bullet}\delta_j = C^{i}{}_{j\bullet}\delta_i, D_{\delta_\bullet}\partial_\bullet = C^{\bullet}_{\bullet\bullet}\partial_\bullet. \qquad (5.9)$$

A metric on $V^{(2+1)}$ with its coefficients parametrized as (5.3) can be written in distinguished form (5.4), as a metric d–tensor (in brief, d–metric), with respect to an anholonomic base (5.6), i. e.

$$\delta s^2 = g_{\alpha\beta}(u)\,\delta^\alpha \otimes \delta^\beta = g_{ij}(x,y)dx^i dx^j + h(x,y)(\delta y)^2. \qquad (5.10)$$

Some N–connection, d–connection and d–metric structures are compatible if there are satisfied the conditions

$$D_\alpha g_{\beta\gamma} = 0.$$

For instance, a canonical compatible d–connection

$${}^{c}\Gamma^{\alpha}{}_{\beta\gamma} = \left({}^{c}L^{i}{}_{jk}, {}^{c}L^{\bullet}{}_{\bullet k}, {}^{c}C^{i}{}_{j\bullet}, {}^{c}C^{\bullet}{}_{\bullet\bullet}\right)$$

is defined by the coefficients of d–metric (5.10), $g_{ij}(x,y)$ and $h(x,y)$, and by the coefficients of N–connection,

$$\begin{aligned} {}^{c}L^{i}{}_{jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \\ {}^{c}L^{\bullet}{}_{\bullet k} &= \partial_\bullet N_k^\bullet + \frac{1}{2}h^{-1}\left(\delta_k h - 2h\partial_\bullet N_i^\bullet\right), \\ {}^{c}C^{i}{}_{j\bullet} &= \frac{1}{2}g^{ik}\partial_\bullet g_{jk}, \\ {}^{c}C^{\bullet}{}_{\bullet\bullet} &= \frac{1}{2}h^{-1}\left(\partial_\bullet h\right). \end{aligned} \qquad (5.11)$$

The coefficients of the canonical d–connection generalize for locally anisotropic spacetimes the well known Christoffel symbols. By a local linear non–degenerate transform to a coordinate frame we obtain the coefficients of the usual (pseudo) Riemannian metric connection. For a canonical d–connection (5.11), hereafter we shall omit the left–up index "c", the components of canonical torsion,

$$\begin{aligned} T(\delta_\gamma, \delta_\beta) &= T^{\alpha}_{\beta\gamma}\delta_\alpha, \\ T^{\alpha}_{\beta\gamma} &= \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} + w^{\alpha}{}_{\beta\gamma} \end{aligned}$$

are expressed via d–torsions

$$\begin{aligned} T^i_{.jk} &= T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{j\bullet} = C^i_{.j\bullet}, T^i_{\bullet j} = -C^i_{j\bullet}, \\ T^a_{.bc} &= S^a_{.bc} = C^a_{bc} - C^a_{cb} \to S^{\bullet}_{.\bullet\bullet} \equiv 0, \\ T^{\bullet}_{.ij} &= -\Omega^{\bullet}_{ij}, \quad T^{\bullet}_{.\bullet i} = \partial_\bullet N^{\bullet}_i - L^{\bullet}_{.\bullet i}, \quad T^{\bullet}_{.i\bullet} = -T^{\bullet}_{.\bullet i} \end{aligned} \tag{5.12}$$

which reflects the anholonomy of the corresponding locally anisotropic frame of reference on $V^{(2+1)}$; they are induced effectively. With respect to holonomic frames the d–torsions vanishes. Putting the non–vanishing coefficients (5.11) into the formula for curvature

$$\begin{aligned} R(\delta_\tau, \delta_\gamma)\,\delta_\beta &= R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha, \\ R_{\beta\ \gamma\tau}^{\ \alpha} &= \delta_\tau\Gamma^\alpha_{\ \beta\gamma} - \delta_\gamma\Gamma^\alpha_{\ \beta\delta} + \Gamma^\varphi_{\ \beta\gamma}\Gamma^\alpha_{\ \varphi\tau} - \Gamma^\varphi_{\ \beta\tau}\Gamma^\alpha_{\ \varphi\gamma} + \Gamma^\alpha_{\ \beta\varphi}w^\varphi_{\ \gamma\tau} \end{aligned}$$

we compute the components of canonical d–curvatures

$$\begin{aligned} R^{.i}_{h.jk} &= \delta_k L^i_{.hj} - \delta_j L^i_{.hk} + L^m_{.hj}L^i_{mk} - L^m_{.hk}L^i_{mj} - C^i_{.h\bullet}\Omega^{\bullet}_{.jk}, \\ R^{.\bullet}_{\bullet.jk} &= \delta_k L^{\bullet}_{.\bullet j} - \delta_j L^{\bullet}_{.\bullet k} - C^{\bullet}_{.\bullet\bullet}\Omega^{\bullet}_{.jk}, \\ P^{.i}_{j.k\bullet} &= \delta_k L^i_{.jk} + C^i_{.j\bullet}T^{\bullet}_{.k\bullet} - (\delta_k C^i_{.j\bullet} + L^i_{.lk}C^l_{.j\bullet} - L^l_{.jk}C^i_{.l\bullet} - L^{\bullet}_{.\bullet k}C^i_{.j\bullet}), \\ P^{.\bullet}_{\bullet.k\bullet} &= \partial_\bullet L^{\bullet}_{.\bullet k} + C^{\bullet}_{.\bullet\bullet}T^{\bullet}_{.k\bullet} - (\delta_k C^{\bullet}_{.\bullet\bullet} - L^{\bullet}_{.\bullet k}C^{\bullet}_{.\bullet\bullet}), \\ S^{.i}_{j.bc} &= \partial_c C^i_{.jb} - \partial_b C^i_{.jc} + C^h_{.jb}C^i_{.hc} - C^h_{.jc}C^i_{hb} \to S^{.i}_{j.\bullet\bullet} \equiv 0, \\ S^{.a}_{b.cd} &= \partial_d C^a_{.bc} - \partial_c C^a_{.bd} + C^e_{.bc}C^a_{.ed} - C^e_{.bd}C^a_{.ec} \to S^{.\bullet}_{\bullet.\bullet\bullet} \equiv 0. \end{aligned} \tag{5.13}$$

The h–v–decompositions for the torsion, (5.12), and curvature, (5.13), are invariant under local coordinate transforms adapted to a prescribed N–connection structure.

5.2.3 Anholonomic constraints and Einstein equations

The Ricci d–tensor $R_{\beta\gamma} = R_{\beta\ \gamma\alpha}^{\ \alpha}$ has the components

$$\begin{aligned} R_{ij} &= R^{.k}_{i.jk}, \quad R_{i\bullet} = -{}^2P_{ia} = -P^{.k}_{i.k\bullet}, \\ R_{\bullet i} &= {}^1P_{\bullet i} = P^{.\bullet}_{\bullet.i\bullet}, \quad R_{ab} = S^{.c}_{a.bc} \to S_{\bullet\bullet} \equiv 0 \end{aligned} \tag{5.14}$$

and, in general, this d–tensor is non symmetric. We can compute the scalar curvature $\overleftarrow{R} = g^{\beta\gamma}R_{\beta\gamma}$ of a d-connection D,

$$\overleftarrow{R} = \widehat{R} + S, \tag{5.15}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab} \equiv 0$ for one dimensional anisotropies. By introducing the values (5.14) and (5.15) into the usual Einstein equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = k \Upsilon_{\beta\gamma}, \tag{5.16}$$

where

$$G_{\alpha\beta} = R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} R \tag{5.17}$$

is the Einstein tensor, written with respect to an anholonomic frame of reference, we obtain the system of field equations for locally anisotropic gravity with N–connection structure [19]:

$$R_{ij} - \frac{1}{2}\left(\widehat{R} - 2\Lambda\right) g_{ij} = k\Upsilon_{ij}, \tag{5.18}$$

$$-\frac{1}{2}\left(\widehat{R} - 2\Lambda\right) h_{\bullet\bullet} = k\Upsilon_{\bullet\bullet}, \tag{5.19}$$

$$^{1}P_{\bullet i} = k\Upsilon_{\bullet i}, \tag{5.20}$$

$$^{2}P_{i\bullet} = -k\Upsilon_{i\bullet}, \tag{5.21}$$

where $\Upsilon_{ij}, \Upsilon_{\bullet\bullet}, \Upsilon_{\bullet i}$ and $\Upsilon_{i\bullet}$ are the components of the energy–momentum d–tensor field $\Upsilon_{\beta\gamma}$ which includes the cosmological constant terms and possible contributions of d–torsions and matter, and k is the coupling constant.

The bulk of nontrivial locally isotropic solutions in 3D gravity were constructed by considering a cosmological constant $\Lambda = -1/l^2$, with and equivalent vacuum energy–momentum $\Upsilon^{(\Lambda)}_{\beta\gamma} = -\Lambda g_{\beta\gamma}$.

5.2.4 Some ansatz for d–metrics

Diagonal d-metrics

Let us introduce on 3D locally anisotropic spacetime $V^{(2+1)}$ the local coordinates (x^1, x^2, y), where y is considered as the anisotropy coordinate, and parametrize the d–metric (5.10) in the form

$$\delta s^2 = a\left(x^i\right)\left(dx^1\right)^2 + b\left(x^i\right)(dx^2)^2 + h\left(x^i, y\right)(\delta y)^2, \tag{5.22}$$

where

$$\delta y = dy + w_1(x^i, y)dx^1 + w_2(x^i, y)dx^2,$$

i. e. $N_i^{\bullet} = w_i(x^i, y)$.

With respect to the coordinate base (5.1) the d–metric (5.10) transforms into the ansatz

$$g_{\alpha\beta} = \left[\begin{array}{ccc} a + {w_1}^2 h & w_1 w_2 h & w_1 h \\ w_1 w_2 h & b + {w_2}^2 h & w_2 h \\ w_1 h & w_2 h & h \end{array} \right]. \qquad (5.23)$$

The nontrivial components of the Ricci d–tensor (5.14) are computed

$$2abR_1^1 = 2abR_2^2 - \ddot{b} + \frac{1}{2b}\dot{b}^2 + \frac{1}{2a}\dot{a}\dot{b} + \frac{1}{2b}a'b' - a'' + \frac{1}{2a}(a')^2$$

where the partial derivatives are denoted, for instance, $\dot{h} = \partial h/\partial x^1, h' = \partial h/\partial x^2$ and $h^* = \partial h/\partial y$. The scalar curvature is $R = 2R_1^1$.

The Einstein d–tensor has a nontrivial component

$$G_3^3 = -hR_1^1.$$

In the vacuum case with $\Lambda = 0$, the Einstein equations (5.18)–(5.21) are satisfied by arbitrary functions $a\left(x^i\right), b\left(x^i\right)$ solving the equation

$$-\ddot{b} + \frac{1}{2b}\dot{b}^2 + \frac{1}{2a}\dot{a}\dot{b} + \frac{1}{2b}a'b' - a'' + \frac{1}{2a}(a')^2 = 0 \qquad (5.24)$$

and arbitrary function $h\left(x^i, y\right)$. Such functions should be defined following some boundary conditions in a manner as to have compatibility with the locally isotropic limit.

Off–diagonal d–metrics

For our further investigations it is convenient to consider d–metrics of type

$$\delta s^2 = g\left(x^i\right)\left(dx^1\right)^2 + 2dx^1 dx^2 + h\left(x^i, y\right)(\delta y)^2. \qquad (5.25)$$

The nontrivial components of the Ricci d–tensor are

$$R_{11} = \frac{1}{2} g \frac{\partial^2 g}{\partial (x^2)^2}, \qquad R_{12} = R_{21} = \frac{1}{2}\frac{\partial^2 g}{\partial (x^2)^2}, \qquad (5.26)$$

when the scalar curvature is $R = 2R_{12}$ and the nontrivial component of the Einstein d–tensor is

$$G_{33} = -\frac{h}{2}\frac{\partial^2 g}{\partial (x^2)^2}.$$

We note that for the both d–metric ansatz (5.22) and (5.25) and corresponding coefficients of Ricci d–tensor, (5.24) and (5.26), the h–components of the Einstein d–tensor vanishes for arbitrary values of metric coefficients, i. e. $G_{ij} = 0$. In absence of matter such ansatz admit arbitrary nontrivial anholonomy (N–connection and N–curvature) coefficients (5.8) because the values w_i are not contained in the 3D vacuum Einstein equations. The h–component of the d–metric, $h(x^k, y)$, and the coefficients of d–connection, $w_i(x^k, y)$, are to be defined by some boundary conditions (for instance, by a compatibility with the locally isotropic limit) and compatibility conditions between nontrivial values of the cosmological constant and energy–momentum d–tensor.

5.3 Conformal Transforms with Anisotropic Factors

One of pecular proprieties of the d–metric ansatz (5.22) and (5.25) is that there is only one non–trivial component of the Einstein d–tensor, G_{33}. Because the values P_{3i} and P_{i3} for the equations (5.19) and (5.20) vanish identically the coefficients of N–connection, w_i, are not contained in the Einstein equations and could take arbitrary values. For static anisotropic configurations the solutions constructed in Sections IV and V can be considered as 3D black hole like objects embedded in a locally anisotropic background with prescribed anholonomic frame (N–connection) structure.

In this Section we shall proof that there are d–metrics for which the Einstein equations reduce to some dynamical equations for the N–connection coefficients.

5.3.1 Conformal transforms of d–metrics

A conformal transform of a d–metric

$$(g_{ij}, h_{ab}) \longrightarrow \left(\widetilde{g}_{ij} = \Omega^2\left(x^i, y\right) g_{ij}, \widetilde{h}_{ab} = \Omega^2\left(x^i, y\right) h_{ab}\right) \tag{5.27}$$

with fixed N–connection structure, $\widetilde{N}_i^a = N_i^a$, deforms the coefficients of canonical d–connection,

$$\widetilde{\Gamma}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \widehat{\Gamma}^{\alpha}{}_{\beta\gamma},$$

where the coefficients of deformation d–tensor $\widehat{\Gamma}^{\alpha}{}_{\beta\gamma} = \{\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\}$ are computed by introducing the values (5.27) into (5.11),

$$\begin{aligned} \widehat{L}^i_{jk} &= \delta^i_j \psi_k + \delta^i_k \psi_j - g_{jk} g^{in} \psi_n, \qquad \widehat{L}^a_{bk} = \delta^a_b \psi_k, \\ \widehat{C}^i_{jc} &= \delta^i_j \psi_c,\ \widehat{C}^a_{bc} = \delta^a_b \psi_c + \delta^a_c \psi_b - h_{bc} h^{ae} \psi_e \end{aligned} \tag{5.28}$$

with δ_j^i and δ_b^a being corresponding Kronecker symbols in h– and v–subspaces and

$$\psi_i = \delta_i \ln \Omega \text{ and } \psi_a = \partial_a \ln \Omega.$$

In this subsection we present the general formulas for a n–dimensional h–subspace, with indices $i, j, k... = 1, 2, ...n$, and m–dimensional v–subspace, with indices $a, b, c, ... = 1, 2, ...m$.

The d–connection deformations (5.28) induce conformal deformations of the Ricci d–tensor (5.14),

$$\begin{aligned}
\widetilde{R}_{hj} &= R_{hj} + \widehat{R}_{[1]hj} + \widehat{R}_{[2]hj},\ \widetilde{R}_{ja} = R_{ja} + \widehat{R}_{ja}, \\
\widetilde{R}_{bk} &= R_{bk} + \widehat{R}_{bk},\ \widetilde{S}_{bc} = S_{bc} + \widehat{S}_{bc},
\end{aligned}$$

where the deformation Ricci d–tensors are

$$\begin{aligned}
\widehat{R}_{[1]hj} &= \partial_i \widehat{L}^i_{hj} - \partial_j \widehat{L}_h + \widehat{L}^m{}_{hj} L_m + L^m{}_{hj} \widehat{L}_m + \widehat{L}^m{}_{hj} \widehat{L}_m - \widehat{L}^m{}_{hi} L^i_{mj} - L^m{}_{hi} \widehat{L}^i_{mj} - \widehat{L}^m{}_{hi} \widehat{L}^i_{mj}, \\
\widehat{R}_{[2]hj} &= N_i^a \partial_a \widehat{L}^i_{hj} - N_j^a \partial_a \widehat{L}_h + \widehat{C}^i_{ha} R^a{}_{ji}; \qquad (5.29)\\
\widehat{R}_{ja} &= -\partial_a \widehat{L}_j + \delta_i \widehat{C}^i_{ja} + L^i_{ki} \widehat{C}^k_{ja} - L^k_{ji} \widehat{C}^i_{ka} - L^b_{ai} \widehat{C}^i_{jb} - \widehat{C}^i{}_{jb} P^b{}_{ia} - C^i{}_{jb} \widehat{P}^b{}_{ia} - \widehat{C}^i{}_{jb} \widehat{P}^b{}_{ia}, \\
\widehat{R}_{bk} &= \partial_a \widehat{L}^a{}_{bk} - \delta_k \widehat{C}_b + L^a{}_{bk} \widehat{C}_a + \widehat{C}^a{}_{bd} P^d{}_{ka} + C^a{}_{bd} \widehat{P}^d{}_{ka} + \widehat{C}^a{}_{bd} \widehat{P}^d{}_{ka}, \\
\widehat{S}_{bc} &= \partial_a \widehat{C}^a{}_{bc} - \partial_c \widehat{C}_b + \widehat{C}^e{}_{bc} C_e + C^e{}_{bc} \widehat{C}_e + \widehat{C}^e{}_{bc} \widehat{C}_e - \widehat{C}^e{}_{ba} C^a{}_{ec} - C^e{}_{ba} \widehat{C}^a{}_{ec} - \widehat{C}^e{}_{ba} \widehat{C}^a{}_{ec},
\end{aligned}$$

when $\widehat{L}_h = \widehat{L}^i_{hi}$ and $\widehat{C}_b = \widehat{C}^e{}_{be}$.

5.3.2 An ansatz with adapted conformal factor and N–connection

We consider a 3D metric

$$g_{\alpha\beta} = \begin{bmatrix} \Omega^2(a - w_1{}^2 h) & -w_1 w_2 h \Omega^2 & -w_1 h \Omega^2 \\ -w_1 w_2 h \Omega^2 & \Omega^2(b - w_2{}^2 h) & -w_2 h \Omega^2 \\ -w_1 h \Omega^2 & -w_2 h \Omega^2 & -h\Omega^2 \end{bmatrix} \qquad (5.30)$$

where $a = a(x^i), b = b\left(x^i\right), w_i = w_i(x^k, y), \Omega = \Omega\left(x^k, y\right) \geq 0$ and $h = h\left(x^k, y\right)$ when the conditions

$$\psi_i = \delta_i \ln \Omega = \frac{\partial}{\partial x^i} \ln \Omega - w_i \ln \Omega = 0$$

are satisfied. With respect to anholonomic bases (5.6) the (5.30) transforms into the d–metric

$$\delta s^2 = \Omega^2(x^k, y)[a(x^k)(dx^1)^2 + b(x^k)(dx^1)^2 + h\left(x^k, y\right)(\delta y)^2]. \qquad (5.31)$$

By straightforward calculus, by applying consequently the formulas (5.13)–(5.21) we find that there is a non–trivial coefficient of the Ricci d–tensor (5.14), of the deformation d–tensor (5.29),

$$\widehat{R}_{j3} = \psi_3 \cdot \delta_j \ln \sqrt{|h|},$$

which results in non–trivial components of the Einstein d-tensor (5.17),

$$G_3^3 = -hR_1^1 \text{ and } P_{\bullet i} = -\psi_3 \cdot \delta_j \ln \sqrt{|h|},$$

where R_1^1 is given by the formula (5.24).

We can select a class of solutions of 3D Einstein equations with $P_{\bullet j} = 0$ but with the horizontal components of metric depending on anisotropic coordinate y, via conformal factor $\Omega(x^k, y)$, and dynamical components of the N–connection, w_i, if we choose

$$h(x^k, y) = \pm\Omega^2(x^k, y)$$

and state

$$w_i(x^k, y) = \partial_i \ln |\ln \Omega|. \qquad (5.32)$$

Finally, we not that for the ansatz (5.30) (equivalently (5.31)) the coefficients of N–connection have to be found as dynamical values by solving the Einstein equations.

5.4 Matter Energy Momentum D–Tensors

5.4.1 Variational definition of energy-momentum d–tensors

For locally isotropic spacetimes the symmetric energy momentum tensor is to be computed by varying on the metric (see, for instance, Refs. [12, 20]) the matter action

$$S = \frac{1}{c}\int \mathcal{L}\sqrt{|g|}dV,$$

where $\mathcal{L}$ is the Lagrangian of matter fields, c is the light velocity and dV is the infinitesimal volume, with respect to the inverse metric $g^{\alpha\beta}$. By definition one states that the value

$$\frac{1}{2}\sqrt{|g|}T_{\alpha\beta} = \frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial u^\tau}\frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial g^{\alpha\beta}/\partial u^\tau} \qquad (5.33)$$

is the symmetric energy–momentum tensor of matter fields. With respect to anholonomic frames (5.5) and (5.6) there are imposed constraints of type

$$g_{ib} - N_i^\bullet h = 0$$

in order to obtain the block representation for d–metric (5.4). Such constraints, as well the substitution of partial derivatives into N–elongated, could result in nonsymmetric energy–momentum d–tensors $\Upsilon_{\alpha\beta}$ which is compatible with the fact that on a locally anisotropic spacetime the Ricci d–tensor could be nonsymmetric.

The gravitational–matter field interactions on locally anisotropic spacetimes are described by dynamical models with imposed constraints (a generalization of anholonomic analytic mechanics for gravitational field theory). The physics of systems with mixed holonomic and anholonomic variables states additional tasks connected with the definition of conservation laws, interpretation of non–symmetric energy–momentum tensors $\Upsilon_{\alpha\beta}$ on locally anisotropic spacetimes and relation of such values with, for instance, the non–symmetric Ricci d–tensor. In this work we adopt the convention that for locally anisotropic gravitational matter field interactions the non–symmetric Ricci d–tensor induces a non–symmetric Einstein d–tensor which has as a source a corresponding non–symmetric matter energy–momentum tensor. The values $\Upsilon_{\alpha\beta}$ should be computed by a variational calculus on locally anisotropic spacetime as well by imposing some constraints following the symmetry of anisotropic interactions and boundary conditions.

In the next subsection we shall investigate in explicit form some cases of definition of energy momentum tensor for locally anisotropic matter on locally anisotropic spacetime.

5.4.2 Energy–Momentum D–Tensors for Anisotropic Media

Following DeWitt approach [28] and recent results on dynamical collapse and hair of black holes of Husain and Brown [13], we set up a formalism for deriving energy–momentum d–tensors for locally anisotropic matter.

Our basic idea for introducing a local anisotropy of matter is to rewrite the energy–momentum tensors with respect to locally adapted frames and to change the usual partial derivations and differentials into corresponding operators (5.5) and (5.6), "elongated" by N–connection. The energy conditions (weak, dominant, or strong) in a locally anisotropic background have to be analyzed with respect to a locally anisotropic basis.

We start with DeWitt's action written in locally anisotropic spacetime,

$$S\left[g_{\alpha\beta}, z^{\underline{i}}\right] = -\int\limits_{V} \delta^3 u \sqrt{-g}\rho\left(z^{\underline{i}}, q_{\underline{j}\underline{k}}\right),$$

as a functional on region V, of the locally anisotropic metric $g_{\alpha\beta}$ and the Lagrangian coordinates $z^{\underline{i}} = z^{\underline{i}}\left(u^{\alpha}\right)$ (we use underlined indices $\underline{i}, \underline{j}, ... = 1,2$ in order to point out that the 2–dimensional matter space could be different from the locally anisotropic spacetime). The functions $z^{\underline{i}} = z^{\underline{i}}\left(u^{\alpha}\right)$ are two scalar locally anisotropic fields whose locally anisotropic gradients (with partial derivations substituted by operators (5.1)) are orthogonal to the matter world lines and label which particle passes through the point u^{α}. The action $S\left[g_{\alpha\beta}, z^{\underline{i}}\right]$ is the proper volume integral of the proper energy density ρ in the rest anholonomic frame of matter. The locally anisotropic density $\rho\left(z^{\underline{i}}, q_{\underline{j}\underline{k}}\right)$ depends explicitly on $z^{\underline{i}}$ and on matter space d–metric $q^{\underline{i}\underline{j}} = \left(\delta_{\alpha} z^{\underline{i}}\right) g^{\alpha\beta}\left(\delta_{\beta} z^{\underline{j}}\right)$, which is interpreted as the inverse d–metric in the rest anholonomic frame of the matter.

Using the d–metric $q^{\underline{i}\underline{j}}$ and locally anisotropic fluid velocity V^{α}, defined as the future pointing unit d–vector orthogonal to d–gradients $\delta_{\alpha} z^{\underline{i}}$, the locally anisotropic spacetime d–metric (5.10) of signature (–,+,+) may be written in the form

$$g_{\alpha\beta} = -V_{\alpha}V_{\beta} + q_{\underline{j}\underline{k}}\delta_{\alpha}z^{\underline{j}}\delta_{\beta}z^{\underline{k}}$$

which allow us to define the energy–momentum d–tensor for elastic locally anisotropic medium as

$$\Upsilon_{\beta\gamma} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\beta\gamma}}\rho V_{\beta}V_{\gamma} + t_{\underline{j}\underline{k}}\delta_{\beta}z^{\underline{j}}\delta_{\gamma}z^{\underline{k}},$$

where the locally anisotropic matter stress d–tensor $t_{\underline{j}\underline{k}}$ is expressed as

$$t_{\underline{j}\underline{k}} = 2\frac{\delta\rho}{\partial q^{\underline{j}\underline{k}}} - \rho q_{\underline{j}\underline{k}} = \frac{2}{\sqrt{q}}\frac{\delta\left(\sqrt{q}\rho\right)}{\partial q^{\underline{j}\underline{k}}}. \qquad (5.34)$$

Here one should be noted that on locally anisotropic spaces

$$D_{\alpha}\Upsilon^{\alpha\beta} = D_{\alpha}\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = J^{\beta} \neq 0$$

and this expression must be treated as a generalized type of conservation law with a geometric source J^{β} for the divergence of locally anisotropic matter d–tensor [19].

The stress–energy–momentum d–tensor for locally anisotropic elastic medium is defined by applying N–elongated operators δ_{α} of partial derivatives (5.1),

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = -\rho g_{\alpha\beta} + 2\frac{\partial\rho}{\partial q^{\underline{i}\underline{j}}}\delta_{\alpha}z^{\underline{j}}\delta_{\beta}z^{\underline{k}} = -V_{\alpha}V_{\beta} + \tau_{\underline{i}\underline{j}}\delta_{\alpha}z^{\underline{i}}\delta_{\beta}z^{\underline{j}},$$

where we introduce the matter stress d–tensor

$$\tau_{\underline{i}\underline{j}} = 2\frac{\partial\rho}{\partial q^{\underline{i}\underline{j}}} - \rho q_{\underline{i}\underline{j}} = \frac{2}{\sqrt{q}}\frac{\partial\left(\sqrt{q}\rho\right)}{\partial q^{\underline{i}\underline{j}}}.$$

The obtained formulas generalize for spaces with nontrivial N–connection structures the results on isotropic and anisotropic media on locally isotropic spacetimes.

5.4.3 Isotropic and anisotropic media

The *isotropic elastic*, but in general locally anisotropic medium is introduced as one having equal all principal pressures with stress d–tensor being for a perfect fluid and the density $\rho = \rho(n)$, where the proper density (the number of particles per unit proper volume in the material rest anholonomic frame) is $n = \underline{n}(z^{\underline{i}})/\sqrt{q}$; the value $\underline{n}(z^{\underline{i}})$ is the number of particles per unit coordinate cell $\delta^3 z$. With respect to a locally anisotropic frame, using the identity

$$\frac{\partial \rho(n)}{\partial q^{\underline{j}\underline{k}}} = \frac{n}{2}\frac{\partial \rho}{\partial n} q_{\underline{j}\underline{k}}$$

in (5.34), the energy–momentum d–tensor (5.34) for a isotropic elastic locally anisotropic medium becomes

$$\Upsilon_{\beta\gamma} = \rho V_\beta V_\gamma + \left(n \frac{\partial \rho}{\partial n} - \rho \right) (g_{\beta\gamma} + V_\beta V_\gamma).$$

This medium looks like isotropic with respect to anholonomic frames but, in general, it is locally anisotropic.

The *anisotropic elastic* and locally anisotropic medium has not equal principal pressures. In this case we have to introduce (1+1) decompositions of locally anisotropic matter d–tensor $q_{\underline{j}\underline{k}}$

$$q_{\underline{j}\underline{k}} = \begin{pmatrix} \alpha^2 + \beta^2 & \beta \\ \beta & \sigma \end{pmatrix},$$

and consider densities $\rho(n_{\underline{1}}, n_{\underline{2}})$, where $n_{\underline{1}}$ and $n_{\underline{2}}$ are respectively the particle numbers per unit length in the directions given by bi–vectors $v^{1}_{\underline{j}}$ and $v^{2}_{\underline{j}}$. Substituting

$$\frac{\partial \rho(n_{\underline{1}}, n_{\underline{2}})}{\partial h^{\underline{j}\underline{k}}} = \frac{n_{\underline{1}}}{2}\frac{\partial \rho}{\partial n_{\underline{1}}} v^{1}_{\underline{j}} v^{1}_{\underline{k}} + \frac{n_{\underline{2}}}{2}\frac{\partial \rho}{\partial n_{\underline{2}}} v^{2}_{\underline{j}} v^{2}_{\underline{k}}$$

into (5.34), which gives

$$t_{\underline{j}\underline{k}} = \left(n_{\underline{1}} \frac{\partial \rho}{\partial n_{\underline{1}}} - \rho \right) v^{1}_{\underline{j}} v^{1}_{\underline{k}} + \left(n_{\underline{2}} \frac{\partial \rho}{\partial n_{\underline{2}}} - \rho \right) v^{2}_{\underline{j}} v^{2}_{\underline{k}},$$

we obtain from (5.34) the energy–momentum d–tensor for the anisotropic locally anisotropic matter

$$\Upsilon_{\beta\gamma} = \rho V_\beta V_\gamma + \left(n_{\underline{1}} \frac{\partial \rho}{\partial n_{\underline{1}}} - \rho \right) v^{1}_{\underline{j}} v^{1}_{\underline{k}} + \left(n_{\underline{2}} \frac{\partial \rho}{\partial n_{\underline{2}}} - \rho \right) v^{2}_{\underline{j}} v^{2}_{\underline{k}}.$$

So, the pressure $P_1 = \left(n_{\underline{1}} \frac{\partial \rho}{\partial n_{\underline{1}}} - \rho \right)$ in the direction $v_{\underline{j}}^{\underline{1}}$ differs from the pressure $P_2 = \left(n_{\underline{2}} \frac{\partial \rho}{\partial n_{\underline{2}}} - \rho \right)$ in the direction $v_{\underline{j}}^{\underline{2}}$. For instance, if for the (2+1)–dimensional locally anisotropic spacetime we impose the conditions $\Upsilon_1^1 = \Upsilon_2^2 \neq \Upsilon_3^3$, when

$$\rho = \rho \left(n_{\underline{1}} \right), z^{\underline{1}} \left(u^{\alpha} \right) = r, z^{\underline{2}} \left(u^{\alpha} \right) = \theta,$$

r and θ are correspondingly radial and angle coordinates on locally anisotropic spacetime, we have

$$\Upsilon_1^1 = \Upsilon_2^2 = \rho, \Upsilon_3^3 = \left(n_{\underline{1}} \frac{\partial \rho}{\partial n_{\underline{1}}} - \rho \right). \tag{5.35}$$

We shall also consider the variant when the coordinated θ is anisotropic (t and r being isotropic). In this case we shall impose the conditions $\Upsilon_1^1 \neq \Upsilon_2^2 = \Upsilon_3^3$ for

$$\rho = \rho \left(n_{\underline{1}} \right), z^{\underline{1}} \left(u^{\alpha} \right) = t, z^{\underline{2}} \left(u^{\alpha} \right) = r$$

and

$$\Upsilon_1^1 = \left(n_{\underline{1}} \frac{\partial \rho}{\partial n_{\underline{1}}} - \rho \right), \Upsilon_2^2 = \Upsilon_3^3 = \rho, . \tag{5.36}$$

The anisotropic elastic locally anisotropic medium described here satisfies respectively weak, dominant, or strong energy conditions only if the corresponding restrictions are placed on the equation of state considered with respect to an anholonomic frame (see Ref. [13] for similar details in locally isotropic cases). For example, the weak energy condition is characterized by the inequalities $\rho \geq 0$ and $\partial \rho / \partial n_{\underline{1}} \geq 0$.

5.4.4 Spherical symmetry with respect to holonomic and anholonomic frames

In radial coordinates (t, r, θ) (with $-\infty \leq t < \infty$, $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$) for a spherically symmetric 3D metric (5.23)

$$ds^2 = -f(r)\, dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2, \tag{5.37}$$

with the energy–momentum tensor (5.33) written

$$T_{\alpha\beta} = \rho(r) \left(v_{\alpha} w_{\beta} + v_{\beta} w_{\alpha} \right) + P(r) \left(g_{\alpha\beta} + v_{\alpha} w_{\beta} + v_{\beta} w_{\alpha} \right),$$

where the null d–vectors v_α and w_β are defined by

$$\begin{aligned} V_\alpha &= \left(\sqrt{f}, -\frac{1}{\sqrt{f}}, 0\right) = \frac{1}{\sqrt{2}}\left(v_\alpha + w_\alpha\right), \\ q_\alpha &= \left(0, \frac{1}{\sqrt{f}}, 0\right) = \frac{1}{\sqrt{2}}\left(v_\alpha - w_\alpha\right). \end{aligned}$$

In order to investigate the dynamical spherically symmetric [5] collapse solutions it is more convenient to use the coordinates (v, r, θ), where the advanced time coordinate v is defined by $dv = dt + (1/f)\, dr$. The metric (5.37) may be written

$$ds^2 = -e^{2\psi(v,r)} F(v,r)\, dv^2 + 2e^{\psi(v,r)} dv dr + r^2\theta^2, \tag{5.38}$$

where the mass function $m(v,r)$ is defined by $F(v,r) = 1 - 2m(v,r)/r$. Usually, one considers the case $\psi(v,r) = 0$ for the type II [12] energy–momentum d–tensor

$$T_{\alpha\beta} = \frac{1}{2\pi r^2}\frac{\delta m}{\partial v} v_\alpha v_\beta + \rho(v,r)\left(v_\alpha w_\beta + v_\beta w_\alpha\right) + P(v,r)\left(g_{\alpha\beta} + v_\alpha w_\beta + v_\beta w_\alpha\right)$$

with the eigen d–vectors $v_\alpha = (1,0,0)$ and $w_\alpha = (F/2, -1, 0)$ and the non–vanishing components

$$\begin{aligned} T_{vv} &= \rho(v,r)\left(1 - \frac{2m(v,r)}{r}\right) + \frac{1}{2\pi r^2}\frac{\delta m(v,r)}{\partial v}, \\ T_{vr} &= -\rho(v,r), \quad T_{\theta\theta} = P(v,r)\, g_{\theta\theta}. \end{aligned} \tag{5.39}$$

To describe a locally isotropic collapsing pulse of radiation one may use the metric

$$ds^2 = \left[\Lambda r^2 + m(v)\right] dv^2 + 2dvdr - j(v)\, dvd\theta + r^2 d\theta^2, \tag{5.40}$$

with the Einstein field equations (5.16) reduced to

$$\frac{\partial m(v)}{dv} = 2\pi\rho(v), \frac{\partial j(v)}{dv} = 2\pi\omega(v)$$

having non–vanishing components of the energy–momentum d–tensor (for a rotating null locally anisotropic fluid),

$$T_{vv} = \frac{\rho(v)}{r} + \frac{j(v)\,\omega(v)}{2r^3}, \quad T_{v\theta} = -\frac{\omega(v)}{r}, \tag{5.41}$$

where $\rho(v)$ and $\omega(v)$ are arbitrary functions.

In a similar manner we can define energy–momentum d–tensors for various systems of locally anisotropic distributed matter fields; all values have to be re–defined with respect to anholonomic bases of type (5.5) and (5.6). For instance, let us consider the angle θ as the anisotropic variable. In this case we have to 'elongate' the differentials,

$$d\theta \rightarrow \delta\theta = d\theta + w_i\,(v,r,\theta)\,dx^i,$$

for the metric (5.38) (or (5.40)), by transforming it into a d–metric, substitute all partial derivatives into N–elongated ones,

$$\partial_i \rightarrow \delta_i = \partial_i - w_i\,(v,r,\theta)\,\frac{\partial}{\partial\theta},$$

and 'N–extend' the operators defining the Riemannian, Ricci, Einstein and energy–momentum tensors $T_{\alpha\beta}$, transforming them into respective d–tensors. We compute the components of the energy–momentum d–tensor for elastic media as the coefficients of usual energy–momentum tensor redefined with respect to locally anisotropic frames,

$$\begin{aligned} \Upsilon_{11} &= T_{11} + (w_1)^2\,T_{33}, \Upsilon_{33} = T_{33} \\ \Upsilon_{22} &= T_{22} + (w_2)^2\,T_{33}, \Upsilon_{12} = \Upsilon_{21} = T_{21} + w_2 w_1 T_{33}, \\ \Upsilon_{i3} &= T_{i3} + w_i T_{33}, \Upsilon_{3i} = T_{3i} + w_i T_{33}, \end{aligned} \quad (5.42)$$

where the $T_{\alpha\beta}$ are given by the coefficients (5.39) (or (5.41)). If the isotropic energy–momentum tensor does not contain partial derivatives, the corresponding d–tensor is also symmetric which is less correlated with the possible antisymmetry of the Ricci tensor (for such configurations we shall search solutions with vanishing antisymmetric components).

5.5 3D Solutions Induced by Anisotropic Matter

We investigate a new class of solutions of (2+1)–dimensional Einstein equations coupled with anisotropic matter [5, 13, 3, 9, 22] which describe locally anisotropic collapsing configurations.

Let us consider the locally isotropic metric

$$\widehat{g}_{\alpha\beta} = \begin{bmatrix} g\,(v,r) & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & r^2 \end{bmatrix} \quad (5.43)$$

which solves the locally isotopic variant of Einstein equations (5.16) if

$$g\,(v,r) = -[1 - 2g(v) - 2h(v)r^{1-k} - \Lambda r^2], \quad (5.44)$$

where the functions $g(v)$ and $h(v)$ define the mass function

$$m(r,v) = g(v)r + h(v)r^{2-k} + \frac{\Lambda}{2}r^2$$

satisfying the dominant energy conditions

$$P \geq 0, \rho \geq P, T_{ab}w^a w^b > 0$$

if

$$\frac{dm}{dv} = \frac{dg}{dv}r + \frac{dh}{dv}r^{2-k} > 0.$$

Such solutions of the Vaidya type with locally isotropic null fluids have been considered in Ref. [13].

5.5.1 Solutions with generic local anisotropy in spherical coordinates

By introducing a new time–like variable

$$t = v + \int \frac{dr}{g\left(v,r\right)}$$

the metric(5.43) can be transformed in diagonal form

$$ds^2 = -g\left(t,r\right)dt^2 + \frac{1}{g\left(t,r\right)}dr^2 + r^2 d\theta^2 \tag{5.45}$$

which describe the locally isotropic collapse of null fluid matter.

A variant of locally anisotropic inhomogeneous collapse could be modelled, for instance, by the N–elongation of the variable θ in (5.45) and considering solutions of vacuum Einstein equations for the d–metric (a particular case of (5.46))

$$ds^2 = -g\left(t,r\right)dt^2 + \frac{1}{g\left(t,r\right)}dr^2 + r^2 \delta\theta^2, \tag{5.46}$$

where

$$\delta\theta = d\theta + w_1\left(t,r,\theta\right)dt + w_2\left(t,r,\theta\right)dr.$$

The coefficients $g\left(t,r\right), 1/g\left(t,r\right)$ and r^2 of the d–metric were chosen with the aim that in the locally isotropic limit, when $w_i \to 0$, we shall obtain the 3D metric (5.45). We note that the gravitational degrees of freedom are contained in nonvanishing values of the Ricci d–tensor (5.14),

$$R_1^1 = R_2^2 = \frac{1}{2g^3}[(\frac{\partial g}{\partial r})^2 - g^3\frac{\partial^2 g}{\partial t^2} - g\frac{\partial^2 g}{\partial r^2}], \tag{5.47}$$

of the N–curvature (5.7),

$$\Omega_{12}^3 = -\Omega_{21}^3 = \frac{\partial w_1}{\partial r} - \frac{\partial w_2}{\partial t} - w_2\frac{\partial w_1}{\partial \theta} + w_1\frac{\partial w_2}{\partial \theta},$$

and d–torsion (5.12)

$$P_{13}^3 = \frac{1}{2}\left(1+r^4\right)\frac{\partial w_1}{\partial \theta},\ P_{23}^3 = \frac{1}{2}\left(1+r^4\right)\frac{\partial w_2}{\partial \theta} - r^3.$$

We can construct a solution of 3D Einstein equations with cosmological constant Λ (5.16) and energy momentum d–tensor $\Upsilon_{\alpha\beta}$, when $\Upsilon_{ij} = T_{ij} + w_i w_j T_{33}$, $\Upsilon_{3j} = T_{3j} + w_j T_{33}$ and $\Upsilon_{33} = T_{33}$ when $T_{\alpha\beta}$ is given by a d–tensor of type (5.36), $T_{\alpha\beta} = \{n_{\underline{1}}\frac{\partial\rho}{\partial n_{\underline{1}}}, P, 0\}$ with anisotropic matter pressure P. A self–consistent solution is given by

$$\Lambda = \kappa n_{\underline{1}}\frac{\partial \rho}{\partial n_{\underline{1}}} = \kappa P,\ \text{and}\ h = \frac{\kappa\rho}{R_1^1 + \Lambda} \tag{5.48}$$

where R_1^1 is computed by the formula (5.47) for arbitrary values $g\,(t,r)$. For instance, we can take the $g(\nu,r)$ from (5.44) with $\nu \to t = \nu + \int g^{-1}(\nu,r)dr$.

For $h = r^2$, the relation (5.48) results in an equation for $g(t,r)$,

$$(\frac{\partial g}{\partial r})^2 - g^3\frac{\partial^2 g}{\partial t^2} - g\frac{\partial^2 g}{\partial r^2} = 2g^3\left(\frac{\kappa\rho}{r^2} - \Lambda\right).$$

The static configurations are described by the equation

$$gg'' - (g')^2 + \varpi(r)g^3 = 0, \tag{5.49}$$

where

$$\varpi(r) = 2\left(\frac{\kappa\rho\,(r)}{r^2} - \Lambda\right)$$

and the prime denote the partial derivative $\partial/\partial r$. There are four classes (see ([17])) of solutions of the equation (5.49), which depends on constants of the relation

$$(\ln|g|)' = \pm\sqrt{2|\varpi(r)|(C_1 \mp g)},$$

where the minus (plus) sign under square root is taken for $\varpi(r) > 0$ ($\varpi(r) < 0$) and the constant C_1 can be negative, $C_1 = -c^2$, or positive, $C_1 = c^2$. In explicit form the solutions are

$$g(r) = \begin{cases} c^{-2}\cosh^{-2}\left[\frac{c}{2}\sqrt{2|\varpi(r)|}\,(r-C_2)\right] & , \\ \qquad \text{for } \varpi(r) > 0, C_1 = c^2 & ; \\ c^{-2}\sinh^{-2}\left[\frac{c}{2}\sqrt{2|\varpi(r)|}\,(r-C_2)\right] & , \\ \qquad \text{for } \varpi(r) < 0, C_1 = c^2 & ; \\ c^{-2}\sin^{-2}\left[\frac{c}{2}\sqrt{2|\varpi(r)|}\,(r-C_2)\right] \neq 0 & , \\ \qquad \text{for } \varpi(r) < 0, C_1 = -c^2 & ; \\ -2\varpi(r)^{-1}(r-C_2)^{-2} & , \\ \qquad \text{for } \varpi(r) < 0, C_1 = 0, & \end{cases} \tag{5.50}$$

where $C_2 = const$. The values of constants are to be found from boundary conditions. In dependence of prescribed type of matter density distribution and of values of cosmological constant one could fix one of the four classes of obtained solutions with generic local anisotropy of 3D Einstein equations.

The constructed in this section static solutions of 3D Einstein equations are locally anisotropic alternatives (with proper phases of anisotropic polarizations of gravitational field) to the well know BTZ solution. Such configurations are possible if anholonomic frames with associated N–connection structures are introduced into consideration.

5.5.2 An anisotropic solution in (ν, r, θ)–coordinates

For modelling a spherical collapse with generic local anisotropy we use the d–metric (5.25) by stating the coordinates $x^1 = v, x^2 = r$ and $y = \theta$. The equations (5.16) are solved if

$$\kappa\rho(v,r) = \Lambda \text{ and } \kappa P(v,r) = -\Lambda - \frac{1}{2}\frac{\partial^2 g}{\partial r^2}$$

for

$$g = \frac{\kappa}{\Lambda}\left[\rho(v,r)\left(1 - \frac{2m(v,r)}{r}\right) + \frac{1}{2\pi r^2}\frac{\delta m(v,r)}{\delta v}\right].$$

Such metrics depend on classes of functions. They can be extended ellipsoidal configurations and additional polarizations.

5.5.3 A solution for rotating two locally anisotropic fluids

The anisotropic configuration from the previous subsection admits a generalization to a two fluid elastic media, one of the fluids being of locally anisotropic rotating configuration. For this model we consider an anisotropic extension of the metric (5.40) and of the sum of energy–momentum tensors (5.39) and (5.41). The coordinates are parametrized $x^1 = v, x^2 = r, y = \theta$ and the d–metric is given by the ansatz

$$g_{ij} = \begin{pmatrix} g(v,r) & 1 \\ 1 & 0 \end{pmatrix} \text{ and } h = h(v,r,\theta).$$

The nontrivial components of the Einstein d–tensor is

$$G_{33} = -\frac{1}{2} h \frac{\partial^2 g}{\partial r^2}.$$

We consider a non–rotating fluid component with nontrivial energy–momentum components

$$^{(1)}T_{vv} = {}^{(1)}\rho\,(v,r)\left(1 - \frac{2^{(1)}m\,(v,r)}{r}\right) + \frac{1}{2\pi r^2}\frac{\delta^{(1)}m\,(v,r)}{\partial v},\ \ ^{(1)}T_{vr} = -^{(1)}\rho\,(v,r)\,.$$

and a rotating null locally anisotropic fluid with energy–momentum components

$$^{(2)}T_{vv} = \frac{^{(2)}\rho\,(v)}{r} + \frac{^{(2)}j\,(v)\ \ ^{(2)}\omega\,(v)}{2r^3},\ \ ^{(2)}T_{v\theta} = -\frac{^{(2)}\omega\,(v)}{r}.$$

The nontrivial components of energy momentum d–tensor $\Upsilon_{\alpha\beta} = {}^{(1)}\Upsilon_{\alpha\beta} + {}^{(2)}\Upsilon_{\alpha\beta}$ (associated in the locally anisotropic limit to (5.41) and/or (5.39)) are computed by using the formulas (5.51), (5.51) and (5.42).

The Einstein equations are solved by the set of functions

$$g(v,r), {}^{(1)}\rho\,(v,r)\,, {}^{(1)}m\,(v,r)\,, {}^{(2)}\rho\,(v)\,, {}^{(2)}j\,(v)\,, {}^{(2)}\omega\,(v)$$

satisfying the conditions

$$g(v,r) = \frac{\kappa}{\Lambda}\left[{}^{(1)}T_{vv} + \ {}^{(2)}T_{vv}\right],\text{ and } \Lambda = \frac{1}{2}\frac{\partial^2 g}{\partial r^2} = \kappa\ {}^{(1)}T_{vr},$$

where $h\,(v,r,\theta)$ is an arbitrary function which results in nontrivial solutions for the N–connection coefficients $w_i\,(v,r,\theta)$ if $\Lambda \neq 0$. In the locally isotropic limit, for $^{(1)}\rho,^{(1)}m = 0$, we could take $g(v,r) = g_1(v) + \Lambda r^2, w_1 = -j(v)/(2r^2)$ and $w_2 = 0$ which results in a solution of the Vaidya type with locally isotropic null fluids [9].

The main conclusion of this subsection is that we can model the 3D collapse of inhomogeneous null fluid by using vacuum locally anisotropic configurations polarized by an anholonomic frame in a manner as to reproduce in the locally isotropic limit the usual BTZ geometry.

We end this section with the remark that the locally isotropic collapse of dust without pressure was analyzed in details in Ref. [22].

5.6 Gravitational Anisotropic Polarizations and Black Holes

If we introduce in consideration anholonomic frames, locally anisotropic black hole configurations are possible even for vacuum locally anisotropic spacetimes without matter. Such solutions could have horizons with deformed circular symmetries (for instance, elliptic one) and a number of unusual properties comparing with locally isotropic black hole solutions. In this Section we shall analyze two classes of such solutions. Then we shall consider the possibility to introduces matter sources and analyze such configurations of matter energy density distribution when the gravitational locally anisotropic polarization results into constant renormalization of constants of BTZ solution.

5.6.1 Non–rotating black holes with ellipsoidal horizon

We consider a metric (5.30) for local coordinates $(x^1 = r, x^2 = \theta, y = t)$, where t is the time–like coordinate and the coefficients are parametrized

$$a(x^i) = a\,(r)\,, b(x^i) = b(r,\theta) \tag{5.51}$$

and

$$h(x^i, y) = h\,(r,\theta)\,. \tag{5.52}$$

The functions $a(r)$ and $b\,(r,\theta)$ and the coefficients of nonlinear connection $w_i(r,\theta,t)$ will be found as to satisfy the vacuum Einstein equations (5.24) with arbitrary function $h(x^i, y)$ (5.52) stated in the form in order to have compatibility with the BTZ solution in the locally isotropic limit.

We consider a particular case of d-metrics (5.31) with coefficients like (5.51) and (5.52) when

$$h(r,\theta) = 4\Lambda^3(\theta)\left(1 - \frac{r_+^2(\theta)}{r^2}\right)^3 \tag{5.53}$$

where, for instance,

$$r_+^2(\theta) = \frac{p^2}{\left[1 + \varepsilon\cos\theta\right]^2} \tag{5.54}$$

is taken as to construct a 3D solution of vacuum Einstein equations with generic local anisotropy having the horizon given by the parametric equation

$$r^2 = r_+^2(\theta)$$

describing a ellipse with parameter p and eccentricity ε. We have to identify

$$p^2 = r_{+[0]}^2 = -M_0/\Lambda_0,$$

where $r_{+[0]}, M_0$ and Λ_0 are respectively the horizon radius, mass parameter and cosmological constant of the non–rotating BTZ solution [3] if we wont to have a connection with locally isotropic limit with $\varepsilon \to 0$. We can consider that the elliptic horizon (5.54) is modelled by the anisotropic mass

$$M(\theta) = M_0/\left[1 + \varepsilon\cos\theta\right]^2.$$

For the coefficients (5.51) the equations (5.24) simplifies into

$$-\ddot{b} + \frac{1}{2b}\dot{b}^2 + \frac{1}{2a}\dot{a}\dot{b} = 0, \quad (5.55)$$

where (in this subsection) $\dot{b} = \partial b/\partial r$. The general solution of (5.55), for a given function $a(r)$ is defined by two arbitrary functions $b_{[0]}(\theta)$ and $b_{[1]}(\theta)$ (see [17]),

$$b(r,\theta) = \left[b_{[0]}(\theta) + b_{[1]}(\theta)\int\sqrt{|a(r)|}dr\right]^2.$$

If we identify

$$b_{[0]}(\theta) = 2\frac{\Lambda(\theta)}{\sqrt{|\Lambda_0|}}r_+^2(\theta) \text{ and } b_{[1]}(\theta) = -2\frac{\Lambda(\theta)}{\Lambda_0},$$

we construct a d–metric locally anisotropic solution of vacuum Einstein equations

$$\delta s^2 = \Omega^2(r,\theta) \quad (5.56)$$
$$\left[4r^2|\Lambda_0|dr^2 + \frac{4}{|\Lambda_0|}\Lambda^2(\theta)\left[r_+^2(\theta) - r^2\right]^2 d\theta^2 - \frac{4}{|\Lambda_0|r^2}\Lambda^3(\theta)\left[r_+^2(\theta) - r^2\right]^3 \delta t^2\right],$$

where

$$\delta t = dt + w_1(r,\theta)dr + w_2(r,\theta)d\theta$$

is to be associated to a N–connection structure

$$w_r = \partial_r \ln|\ln\Omega| \text{ and } w_\theta = \partial_\theta \ln|\ln\Omega|$$

with $\Omega^2 = \pm h(r,\theta)$, where $h(r,\theta)$ is taken from (5.53). In the simplest case we can consider a constant effective cosmological constant $\Lambda(\theta) \simeq \Lambda_0$.

The matrix

$$g_{\alpha\beta} = \Omega^2 \left[\begin{array}{ccc} a - {w_1}^2 h & -w_1 w_2 h & -w_1 h \\ -w_1 w_2 h & b - {w_2}^2 h & -w_2 h \\ -w_1 h & -w_2 h & -h \end{array} \right].$$

parametrizes a class of solutions of 3D vacuum Einstein equations with generic local anisotropy and nontrivial N–connection curvature (5.7), which describes black holes with variable mass parameter $M(\theta)$ and elliptic horizon. As a matter of principle, by fixing necessary functions $b_{[0]}(\theta)$ and $b_{[1]}(\theta)$ we can construct solutions with effective (polarized by the vacuum anisotropic gravitational field) variable cosmological constant $\Lambda(\theta)$. We emphasize that this type of anisotropic black hole solutions have been constructed by solving the vacuum Einstein equations without cosmological constant. Such type of constants or varying on θ parameters were introduced as some values characterizing anisotropic polarizations of vacuum gravitational field and this approach can be developed if we are considering anholonomic frames on (pseudo) Riemannian spaces. For the examined anisotropic model the cosmological constant is induced effectively in locally isotropic limit via specific gravitational field vacuum polarizations.

5.6.2 Rotating black holes with running in time constants

A new class of solutions of vacuum Einstein equations is generated by a d–metric (5.22) written for local coordinates $(x^1 = r, x^2 = t, y = \theta)$, where as the anisotropic coordinate is considered the angle variable θ and the coefficients are parametrized

$$a(x^i) = a(r), b(x^i) = b(r,t) \tag{5.57}$$

and

$$h(x^i, y) = h(r,t). \tag{5.58}$$

Let us consider a 3D metric

$$ds^2 = 4\frac{\psi^2}{r^2}dr^2 - \frac{N^4_{[s]}\psi^4}{r^4}dt^2 + \frac{N^2_{(s)}\psi^6}{r^4}\left[d\theta + N_{[\theta]}dt\right]^2$$

which is conformally equivalent (if multiplied to the conformal factor $4N^2_{(s)}\psi^4/r^4$) to the rotating BTZ solution with

$$
\begin{aligned}
N_{[s]}^2(r) &= -\Lambda_0 \frac{r^2}{\psi^2}\left(r^2 - r_{+[0]}^2\right), N_{[\theta]}(r) = -\frac{J_0}{2\psi}, \\
\psi^2(r) &= r^2 - \frac{1}{2}\left(\frac{M_0}{\Lambda_0} + r_{+[0]}^2\right), \\
r_{+[0]}^2 &= -\frac{M_0}{\Lambda_0}\sqrt{1 + \Lambda_0\left(\frac{J_0}{M_0}\right)^2},
\end{aligned}
$$

where J_0 is the rotation moment and Λ_0 and M_0 are respectively the cosmological and mass BTZ constants.

A d–metric (5.22) defines a locally anisotropic extension of (5.59) if the solution of (5.55), in variables $(x^1 = r, x^2 = t)$, with coefficients (5.57) and (5.58), is written

$$
b(r,t) = -\left[b_{[0]}(t) + b_{[1]}(t)\int\sqrt{|a(r)|}dr\right]^2 = -\Lambda^2(t)\left[r_+^2(t) - r^2\right]^2,
$$

for

$$
a(r) = 4\Lambda_0 r^2, b_{[0]}(t) = \Lambda(t) r_+^2(t), b_{[1]}(t) = 2\Lambda(t)/\sqrt{|\Lambda_0|}
$$

with $\Lambda(t) \sim \Lambda_0$ and $r_+(t) \sim r_{+[0]}$ being some running in time values.

The functions $a(r)$ and $b(r,t)$ and the coefficients of nonlinear connection $w_i(r,t,\theta)$ must solve the vacuum Einstein equations (5.24) with arbitrary function $h(x^i, y)$ (5.52) stated in the form in order to have a relation with the BTZ solution for rotating black holes in the locally isotropic limit. This is possible if we choose

$$
w_1(r,t) = -\frac{J(t)}{2\psi(r,t)}, \qquad h(r,t) = \frac{4N_{[s]}^2(r,t)\psi^6(r,t)}{r^4},
$$

for an arbitrary function $w_2(r,t,\theta)$ with $N_{[s]}(r,t)$ and $\psi(r,t)$ computed by the same formulas (5.59) with the constant substituted into running values,

$$
\Lambda_0 \rightarrow \Lambda(t), M_0 \rightarrow M(t), J_0 \rightarrow J(t).
$$

We can model a dissipation of 3D black holes, by anisotropic gravitational vacuum polarizations if for instance,

$$
r_+^2(t) \simeq r_{+[0]}^2 \exp[-\lambda t]
$$

for $M(t) = M_0 \exp[-\lambda t]$ with M_0 and λ being some constants defined from some "experimental" data or a quantum model for 3D gravity. The gravitational vacuum admits also polarizations with exponential and/or oscillations in time for $\Lambda(t)$ and/or of $M(t)$.

5.6.3 Anisotropic Renormalization of Constants

The BTZ black hole [3] in "Schwarzschild" coordinates is described by the metric

$$ds^2 = -(N^{\perp})^2 dt^2 + f^{-2} dr^2 + r^2 \left(d\phi + N^{\phi} dt\right)^2 \tag{5.59}$$

with lapse and shift functions and radial metric

$$\begin{aligned} N^{\perp} &= f = \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{1/2}, \\ N^{\phi} &= -\frac{J}{2r^2} \qquad (|J| \leq M\ell). \end{aligned} \tag{5.60}$$

which satisfies the ordinary vacuum field equations of (2+1)-dimensional general relativity (5.16) with a cosmological constant $\Lambda = -1/\ell^2$.

If we are considering anholonomic frames, the matter fields "deform" such solutions not only by presence of a energy–momentum tensor in the right part of the Einstein equations but also via anisotropic polarizations of the frame fields. In this Section we shall construct a subclass of d–metrics (5.46) selecting by some particular distributions of matter energy density $\rho(r)$ and pressure $P(r)$ solutions of type (5.59) but with renormalized constants in (5.60),

$$M \to \overline{M} = \alpha^{(M)} M, J \to \overline{J} = \alpha^{(J)} J, \Lambda \to \overline{\Lambda} = \alpha^{(\Lambda)} \Lambda, \tag{5.61}$$

where the receptivities $\alpha^{(M)}, \alpha^{(J)}$ and $\alpha^{(\Lambda)}$ are considered, for simplicity, to be constant (and defined "experimentally" or computed from a more general model of quantum 3D gravity) and tending to a trivial unity value in the locally isotropic limit. The d–metric generalizing (5.59) is stated in the from

$$\delta s^2 = -F(r)^{-1} dt^2 + F(r) dr^2 + r^2 \delta\theta^2 \tag{5.62}$$

where

$$F(r) = \left(-\overline{M} - \overline{\Lambda} r^2 + \frac{J^2}{4r^2}\right), \ \delta\theta = d\theta + w_1 dt \text{ and } w_1 = -\frac{\overline{J}}{2r^2}.$$

The d–metric (5.62) is a static variant of d–metric (5.46) when the solution (5.50) is constructed for a particular function

$$\overline{\varpi}(r) = 2\left(\frac{\kappa \overline{\rho}(r)}{r^2} - \overline{\Lambda}\right)$$

is defined by corresponding matter distribution $\overline{\rho}(r)$ when the function $F(r)$ is the solution of equations (5.49) with coefficient $\overline{\varpi}(r)$ before F^3, i. e.

$$FF'' - (F')^2 + \varpi(r)F^3 = 0.$$

The d–metric (5.62) is singular when $r = \bar{r}_{\pm}$, where

$$\overline{r}_{\pm}^2 = -\frac{\overline{M}}{2\overline{\overline{\Lambda}}}\left\{1 \pm \left[1 + \overline{\Lambda}\left(\frac{\overline{J}}{\overline{\overline{M}}}\right)^2\right]^{1/2}\right\}, \tag{5.63}$$

i.e.,

$$\overline{M} = -\overline{\Lambda}(\overline{r}_+^2 + \overline{r}_-^2), \quad \overline{J} = \frac{2\overline{r}_+\overline{r}_-}{\overline{\ell}}, \overline{\Lambda} = -1/\overline{\ell}^2.$$

In locally isotropic gravity the surface gravity was computed [16]

$$\sigma^2 = -\frac{1}{2}D^\alpha\chi^\beta D_\alpha\chi_\beta = \frac{r_+^2 - r_-^2}{\ell^2 r_+},$$

where the vector $\chi = \partial_v - N^\theta(r_+)\partial_\theta$ is orthogonal to the Killing horizon defined by the surface equation $r = r_+$. For locally anisotropic renormalized (overlined) values we have

$$\overline{\chi} = \delta_\nu = \partial_\nu - w_1(\overline{r}_+)\partial_\theta$$

and

$$\overline{\sigma}^2 = -\frac{1}{2}D^\alpha\overline{\chi}^\beta D_\alpha\overline{\chi}_\beta = \overline{\Lambda}\frac{\overline{r}_-^2 - \overline{r}_+^2}{\overline{r}_+}.$$

The renormalized values allow us to define a corresponding thermodynamics of locally anisotropic black holes.

5.6.4 Ellipsoidal black holes with running in time constants

The anisotropic black hole solution of 3D vacuum Einstein equations (5.56) with elliptic horizon can be generalized for a case with varying in time cosmological constant $\Lambda_0(t)$. For this class of solutions we choose the local coordinates $(x^1 = r, x^2 = \theta, y = t)$ and a d–metric of type (5.31),

$$\delta s^2 = \Omega_{(el)}^2(r,\theta,t)[a(r)(dr)^2 + b(r,\theta)(d\theta)^2 + h(r,\theta,t)(\delta t)^2], \tag{5.64}$$

where

$$h(r,\theta,t) = -\Omega_{(el)}^2(r,\theta,t) = -\frac{4\Lambda^3(\theta)}{|\Lambda_0(t)|r^2}\left[r_+^2(\theta,t) - r^2\right]^3,$$

for

$$r_+^2(\theta,t) = \frac{p(t)}{(1+\varepsilon\cos\theta)^2}, \text{ and } p(t) = r_{+(0)}^2(\theta,t) = -M_0/\Lambda_0(t)$$

and it is considered that $\Lambda_0(t) \simeq \Lambda_0$ for static configurations.

The d–metric (5.64) is a solution of 3D vacuum Einstein equations if the 'elongated' differential

$$\delta t = dt + w_r(r,\theta,t)dr + w_\theta(r,\theta,t)d\theta$$

has the N–connection coefficients are computed following the condition (5.32),

$$w_r = \partial_r \ln|\ln\Omega_{(el)}| \text{ and } w_\theta = \partial_\theta \ln|\ln\Omega_{(el)}|.$$

The functions $a(r)$ and $b(r,\theta)$ from (5.64) are arbitrary ones of type (5.51) satisfying the equations (5.55) which in the static limit could be fixed to transform into static locally anisotropic elliptic configurations. The time dependence of $\Lambda_0(t)$ has to be computed, for instance, from a higher dimension theory or from experimental data.

5.7 On the Thermodynamics of Anisotropic Black Holes

A general approach to the anisotropic black holes should be based on a kind of nonequilibrium thermodynamics of such objects imbedded into locally anisotropic gravitational (locally anisotropic ether) continuous, which is a matter of further investigations (see the first works on the theory of locally anisotropic kinetic processes and thermodynamics in curved spaces [26]).

In this Section, we explore the simplest type of locally anisotropic black holes with anisotropically renormalized constants being in thermodynamic equilibrium with the locally anisotropic spacetime "bath" for suitable choices of N–connection coefficients. We do not yet understand the detailed thermodynamic behavior of locally anisotropic black holes but believe one could define their thermodynamics in the neighborhoods of some equilibrium states when the horizons are locally anisotropically deformed and constant with respect to an anholonomic frame.

In particular, for a class of BTZ like locally anisotropic spacetimes with horizons radii (5.63) we can still use the first law of thermodynamics to determine an entropy with respect to some fixed anholonomic bases (5.6) and (5.5) (here we note that there are developed some approaches even to the thermodynamics of usual BTZ black holes and that uncertainty is to be transferred in our considerations, see discussions and references in [5]).

In the approximation that the locally anisotropic spacetime receptivities $\alpha^{(m)}, \alpha^{(J)}$ and $\alpha^{(\Lambda)}$ do not depend on coordinates we have similar formulas as in locally isotropic gravity for the locally anisotropic black hole temperature at the boundary of a cavity of radius r_H,

$$\overline{T} = -\frac{\overline{\sigma}}{2\pi\left(\overline{M} + \overline{\Lambda} r_H^2\right)^{1/2}}, \tag{5.65}$$

and entropy

$$\overline{S} = 4\pi \overline{r}_+ \tag{5.66}$$

in Plank units.

For a elliptically deformed locally anisotropic black hole with the outer horizon $r_+(\theta)$ given by the formula (5.54) the Bekenstein–Hawking entropy,

$$S^{(a)} = \frac{L_+}{4G_{(gr)}^{(a)}},$$

were

$$L_+ = 4\int_0^{\pi/2} r_+(\theta)\, d\theta$$

is the length of ellipse's perimeter and $G_{(gr)}^{(a)}$ is the three dimensional gravitational coupling constant in locally anisotropic media, has the value

$$S^{(a)} = \frac{2p}{G_{(gr)}^{(a)}\sqrt{1-\varepsilon^2}} arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}.$$

If the eccentricity vanishes, $\varepsilon = 0$, we obtain the locally isotropic formula with p being the radius of the horizon circumference, but the constant $G_{(gr)}^{(a)}$ could be locally anisotropically renormalized.

In dependence of dispersive or amplification character of locally anisotropic ether with $\alpha^{(m)}, \alpha^{(J)}$ and $\alpha^{(\Lambda)}$ being less or greater than unity we can obtain temperatures of locally anisotropic black holes less or greater than that for the locally isotropic limit. For example, we get anisotropic temperatures $T^{(a)}(\theta)$ if locally anisotropic black holes with horizons of type (5.54) are considered.

If we adapt the Euclidean path integral formalism of Gibbons and Hawking [14] to locally anisotropic spacetimes, by performing calculations with respect to an anholonomic frame, we develop a general approach to the locally anisotropic black hole irreversible thermodynamics. For locally anisotropic backgrounds with constant receptivities we obtain similar to [4, 7, 5] but anisotropically renormalized formulas.

Let us consider the Euclidean variant of the d–metric (5.62)

$$\delta s_E^2 = (F_E)\, d\tau^2 + (F_E)^{-1}\, dr^2 + r^2 \delta\theta^2 \tag{5.67}$$

where $t = i\tau$ and the Euclidean lapse function is taken with locally anisotropically renormalized constants, as in (5.61) (for simplicity, there is analyzed a non–rotating locally anisotropic black hole), $F = \left(-\overline{M} - \overline{\Lambda} r^2\right)$, which leads to the root $\overline{r}_+ = \left[-\overline{M}/\overline{\Lambda}\right]^{1/2}$. By applying the coordinate transforms

$$\begin{aligned} x &= \left(1 - \left(\frac{\overline{r}_+}{r}\right)^2\right)^{1/2} \cos\left(-\overline{\Lambda}\overline{r}_+\tau\right) \exp\left(\sqrt{|\overline{\Lambda}|}\overline{r}_+\theta\right), \\ y &= \left(1 - \left(\frac{\overline{r}_+}{r}\right)^2\right)^{1/2} \sin\left(-\overline{\Lambda}\overline{r}_+\tau\right) \exp\left(\sqrt{|\overline{\Lambda}|}\overline{r}_+\theta\right), \\ z &= \left(\left(\frac{\overline{r}_+}{r}\right)^2 - 1\right)^{1/2} \exp\left(\sqrt{|\overline{\Lambda}|}\overline{r}_+\theta\right), \end{aligned}$$

the d–metric (5.67) is rewritten in a standard upper half–space $(z > 0)$ representation of locally anisotropic hyperbolic 3–space,

$$\delta s_E^2 = -\frac{1}{\overline{\Lambda}}(z^2 dz^2 + dy^2 + \delta z^2).$$

The coordinate transform (5.68) is non–singular at the z–axis $r = \overline{r}_+$ if we require the periodicity

$$(\theta, \tau) \sim \left(\theta, \tau + \overline{\beta}_0\right)$$

where

$$\overline{\beta}_0 = \frac{1}{\overline{T}_0} = -\frac{2\pi}{\overline{\Lambda}\ \overline{r}_+} \tag{5.68}$$

is the inverse locally anisotropically renormalized temperature, see (5.65).

To get the locally anisotropically renormalized entropy from the Euclidean locally anisotropic path integral we must define a locally anisotropic extension of the grand canonical partition function

$$\overline{Z} = \int [dg]\, e^{\overline{I}_E[g]}, \tag{5.69}$$

where $\overline{I}_E$ is the Euclidean locally anisotropic action. We consider as for usual locally isotropic spaces the classical approximation $\overline{Z} \sim \exp\{\overline{I}_E[\overline{g}]\}$, where as the extremal d–metric $\overline{g}$ is taken (5.67). In (5.69) there are included boundary terms at $\overline{r}_+$ and ∞ (see the basic conclusions and detailed discussions for the locally isotropic case [4, 7, 5] which are also true with respect to anholonomic bases).

For an inverse locally anisotropic temperature $\overline{\beta}_0$ the action from (5.69) is

$$\overline{I}_E[\overline{g}] = 4\pi\overline{r}_+ - \overline{\beta}_0 M$$

which corresponds to the locally anisotropic entropy (5.66) being a locally anisotropic renormalization of the standard Bekenstein entropy.

5.8 Chern–Simons Theories and Locally Anisotropic Gravity

In order to compute the first quantum corrections to the locally anisotropic path integral (5.69), inverse locally anisotropic temperature (5.68) and locally anisotropic entropy (5.66) we take the advantage of the Chern–Simons formalism generalized for (2+1)–dimensional locally anisotropic spacetimes.

By using the locally anisotropically renormalized cosmological constant $\overline{\Lambda}$ and adapting the Achucarro and Townsend [1] construction to anholonomic frames we can define two SO(2,1) gauge locally anisotropic fields

$$A^{\underline{a}} = \omega^{\underline{a}} + \frac{1}{\sqrt{|\overline{\Lambda}|}} e^{\underline{a}} \text{ and } \widetilde{A}^{\underline{a}} = \omega^{\underline{a}} - \frac{1}{\sqrt{|\overline{\Lambda}|}} e^{\underline{a}}$$

where the index $\underline{a}$ enumerates an anholonomic triad $e^{\underline{a}} = e^{\underline{a}}_{\mu}\delta x^{\mu}$ and $\omega^{\underline{a}} = \frac{1}{2}\epsilon^{\underline{abc}}\omega_{\mu\underline{bc}}\delta x^{\mu}$ is a spin d–connection (d–spinor calculus is developed in [23]). The first–order action for locally anisotropic gravity is written

$$\overline{I}_{grav} = \overline{I}_{CS}[A] - \overline{I}_{CS}[\widetilde{A}] \tag{5.70}$$

with the Chern–Simons action for a (2+1)–dimensional vector bundle $\widetilde{E}$ provided with N–connection structure,

$$\overline{I}_{CS}[A] = \frac{\overline{k}}{4\pi} \int_{\widetilde{E}} Tr\left(A \wedge \delta A + \frac{2}{3} A \wedge A \wedge A\right) \tag{5.71}$$

where the coupling constant $\overline{k} = \sqrt{|\overline{\Lambda}|}/(4\sqrt{2}G_{(gr)})$ and $G_{(gr)}$ is the gravitational constant. The one d–form from (5.71) $A = A_{\mu}^{\underline{a}} T_{\underline{a}} \delta x^{\mu}$ is a gauge d–field for a Lie algebra with generators $\{T_{\underline{a}}\}$. Following [6] we choose

$$(T_{\underline{a}})_{\underline{b}}^{\ \ \underline{c}} = -\epsilon_{\underline{abd}}\eta^{\underline{dc}},\ \eta_{\underline{ab}} = diag\,(-1,1,1)\,,\ \epsilon_{\underline{012}} = 1$$

and considering Tr as the ordinary matrix trace we write

$$[T_{\underline{a}}, T_{\underline{b}}] = f_{\underline{ab}}^{\ \ \underline{c}} T_{\underline{c}} = \epsilon_{\underline{abd}}\eta^{\underline{dc}} T_{\underline{c}},\ TrT_{\underline{a}}T_{\underline{b}} = 2\eta_{\underline{ab}},$$
$$g_{\mu\nu} = 2\eta_{\underline{ab}} e_{\mu}^{\underline{a}} e_{\nu}^{\underline{b}},\ \eta^{\underline{ad}}\eta^{\underline{be}} f_{\underline{ab}}^{\ \ \underline{c}} f_{\underline{de}}^{\ \ \underline{s}} = -2\eta^{\underline{cs}}.$$

If the manifold $\widetilde{E}$ is closed the action (5.70) is invariant under locally anisotropic gauge transforms

$$\widetilde{A} \to A = q^{-1}\widetilde{A}q + q^{-1}\delta q.$$

This invariance is broken if $\widetilde{E}$ has a boundary $\partial\widetilde{E}$. In this case we must add to (5.71) a boundary term, written in (v,θ)–coordinates as

$$\overline{I}_{CS}' = -\frac{\overline{k}}{4\pi}\int_{\partial\widetilde{E}} TrA_{\theta}A_{v}, \tag{5.72}$$

which results in a term proportional to the standard chiral Wess–Zumino–Witten (WZW) action [21, 11]:

$$\left(\overline{I}_{CS} + \overline{I}_{CS}'\right)[A] = \left(\overline{I}_{CS} + \overline{I}_{CS}'\right)[\overline{A}] - \overline{k}\ \overline{I}_{WZW}^{+}[q,\overline{A}]$$

where

$$\overline{I}_{WZW}^{+}[q,\overline{A}] = \frac{1}{4\pi}\int_{\partial\widetilde{E}} Tr\left(q^{-1}\delta_{\theta}q\right)\left(q^{-1}\delta_{v}q\right) \tag{5.73}$$
$$+\frac{1}{2\pi}\int_{\partial\widetilde{E}} Tr\left(q^{-1}\delta_{v}q\right)\left(q^{-1}\overline{A}_{\theta}q\right) + \frac{1}{12\pi}\int_{\widetilde{E}} Tr\left(q^{-1}\delta q\right)^{3}.$$

With respect to a locally anisotropic base the gauge locally anisotropic field satisfies standard boundary conditions

$$A_{\theta}^{+} = A_{v}^{+} = \widetilde{A}_{\theta}^{+} = \widetilde{A}_{v}^{+} = 0.$$

By applying the action (5.70) with boundary terms (5.72) and (5.73) we can formulate a statistical mechanics approach to the (2+1)–dimensional locally anisotropic black holes with locally anisotropically renormalized constants when the locally anisotropic entropy of the black hole can be computed as the logarithm of microscopic states at the anisotropically deformed horizon. In this case the Carlip's results [6, 15] could be generalized for locally anisotropic black holes. We present here the formulas for one–loop corrected locally anisotropic temperature (5.65) and locally anisotropic entropy (5.66)

$$\overline{\beta}_0 = -\frac{\pi}{4\overline{\Lambda}\hbar G_{(gr)}\ \overline{r}_+}\left(1+\frac{8\hbar G_{(gr)}}{\sqrt{|\overline{\Lambda}|}}\right) \text{ and } \overline{S}^{(a)} = \frac{\pi\overline{r}_+}{2\hbar G_{(gr)}}\left(1+\frac{8\hbar G_{(gr)}}{\sqrt{|\overline{\Lambda}|}}\right).$$

We do not yet have a general accepted approach even to the thermodynamics and its statistical mechanics foundation of locally isotropic black holes and this problem is not solved for locally anisotropic black holes for which one should be associated a model of nonequilibrium thermodynamics. Nevertheless, the formulas presented in this section allows us a calculation of basic locally anisotropic thermodynamical values for equilibrium locally anisotropic configurations by using locally anisotropically renormalized constants.

5.9 Conclusions and Discussion

In this paper we have aimed to justify the use of moving frame method in construction of metrics with generic local anisotropy, in general relativity and its modifications for higher and lower dimension models [25, 26].

We argued that the 3D gravity reformulated with respect to anholonomic frames (with two holonomic and one anholonomic coordinates) admits new classes of solutions of Einstein equations, in general, with nonvanishing cosmological constants. Such black hole like and another type ones, with deformed horizons, variation of constants and locally anisotropic gravitational polarizations in the vacuum case induced by anholonomic moving triads with associated nonlinear connection structure, or (in the presence of 3D matter) by self–consistent distributions of matter energy density and pressure and dreibein (3D moving frame) fields.

The solutions considered in the present paper have the following properties: 1) they are exact solutions of 3D Einstein equations; 2) the integration constants are to be found from boundary conditions and compatibility with locally isotropic limits; 3) having been rewritten in 'pure' holonomic variables the 3D metrics are off–diagonal; 4) it is induced a nontrivial torsion structure which vanishes in holonomic coordinates; for vacuum solutions the 3D gravity is transformed into a teleparallel theory; 5) such solutions are characterized by nontrivial nonlinear connection curvature.

The arguments in this paper extend the results in the literature on the black hole thermodynamics by elucidating the fundamental questions of formulation of this theory for anholonomic gravitational systems with local frame anisotropy. We computed the entropy and temperature of black holes with elliptic horizons and/or with anisotropic variation and renormalizations of constants.

We also showed that how the 3D gravity models with anholonomic constraints can be transformed into effective Chern–Simons theories and following this priority we computed the locally anisotropic quantum corrections for the entropy and temperature of black holes.

Our results indicate that there exists a kind of universality of inducing locally anisotropic interactions in physical theories formulated in mixed holonomic–anholonomic variables: the spacetime geometry and gravitational field are effectively polarized by imposed constraints which could result in effective renormalization and running of interaction constants.

Finally, we conclude that problem of definition of adequate systems of reference for a prescribed type of symmetries of interactions could be of nondynamical nature if we fix at the very beginning the class of admissible frames and symmetries of solutions, but could be transformed into a dynamical task if we deform symmetries (for instance, a circular horizon into a elliptic one) and try to find self–consistently a corresponding anholomic frame for which the metric is diagonal but with generic anisotropic structure).

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Chapter 6

Anholonomic Frames and Thermodynamic Geometry of 3D Black Holes

Abstract [1]

We study new classes three dimensional black hole solutions of Einstein equations written in two holonomic and one anholonomic variables with respect to anholonomic frames. Thermodynamic properties of such $(2+1)$–black holes with generic local anisotropy (for instance, with elliptic horizons) are studied by applying geometric methods. The corresponding thermodynamic systems are three dimensional with entropy S being a hypersurface function on mass M, anisotropy angle θ and eccentricity of elliptic deformations ε. Two–dimensional curved thermodynamic geometries for locally anistropic deformed black holes are constructed after integration on anisotropic parameter θ. Two approaches, the first one based on two–dimensional hypersurface parametric geometry and the second one developed in a Ruppeiner–Mrugala–Janyszek fashion, are analyzed. The thermodynamic curvatures are computed and the critical points of curvature vanishing are defined.

6.1 Introduction

This is the second paper in a series in which we examine black holes for spacetimes with generic local anisotropy. Such spacetimes are usual pseudo–Riemannian spaces for which an anholonomic frame structure By using moving anholonomic frames one can construct solutions of Einstein equations with deformed spherical symmetries (for instance, black holes with elliptic horizons (in three dimensions, 3D), black tori and another type configurations) which are locally anisotropic [25, 27].

In the first paper [26] (hereafter referred to as Paper I) we analyzed the low–dimensional locally anisotropic gravity (we shall use terms like locally anisotropic gravity, locally anisotropic spacetime, locally anisotropic geometry, locally anisotropic black holes and so on) and constructed new classes of locally anisotropic $(2+1)$–dimensional black hole solutions. We emphasize that in this work the splitting $(2+1)$ points not to a space–time decomposition, but to a spacetime distribution in two isotropic and one anisotropic coordinate.

In particular, it was shown following [24] how black holes can recast in a new fashion in generalized Kaluza–Klein spaces and emphasized that such type solutions can be considered in the framework of usual Einstein gravity on anholonomic manifolds. We discussed the physical properties of $(2+1)$–dimensional black holes with locally anisotropic matter, induced by a rotating null fluid and by an inhomogeneous and non–static collapsing null fluid, and examined the vacuum polarization of locally anisotropic spacetime by non–rotating black holes with ellipsoidal horizon and by rotating locally anisotropic black holes with time oscillating and ellipsoidal horizons. It was concluded that a general approach to the locally anisotropic black holes should be based on a kind of nonequilibrium thermodynamics of such objects imbedded into locally anisotropic spacetime background. Nevertheless, we proved that for the simplest type of locally anisotropic black holes theirs thermodynamics could be defined in the neighborhoods of some equilibrium states when the horizons are deformed but constant with respect to a frame base locally adapted to a nonlinear connection structure which model a locally anisotropic configuration.

In this paper we will specialize to the geometric thermodynamics of, for simplicity non–rotating, locally anisotropic black holes with elliptical horizons. We follow the notations and results from the Paper I which are reestablished in a manner compatible in the locally isotropic thermodynamic [7] and spacetime [1] limits with the Banados–Teitelboim–Zanelli (BTZ) black hole. This new approach (to black hole physics) is possible for locally anisotropic spacetimes and is based on classical results [10, 15, 16, 22].

Since the seminal works of Bekenstein [4], Bardeen, Carter and Hawking [2] and Hawking [12], black holes were shown to have properties very similar to those of ordinary thermodynamics. One was treated the surface gravity on the event horizon as the temperature of the black hole and proved that a quarter of the event horizon area corresponds to the entropy of black holes. At present time it is widely believed that a black hole is a thermodynamic system (in spite of the fact that one have been developed a number of realizations of thermodynamics involving radiation) and the problem of statistical interpretation of the black hole entropy is one of the most fascinating subjects of modern investigations in gravitational and string theories.

In parallel to the 'thermodynamilazation' of black hole physics one have developed a new approach to the classical thermodynamics based of Riemannian geometry and its generalizations (a review on this subject is contained in Ref. [20]). Here is to be emphasized that geometrical methods have always played an important role in thermodynamics (see, for instance, a work by Blaschke [5] from 1923). Buchdahl used in 1966 a Euclidean metric in thermodynamics [6] and then Weinhold considered a sort of Riemannian metric [28]. It is considered that the Weinhold's metric has not physical interpretation in the context of purely equilibrium thermodynamics [19, 20] and Ruppeiner introduced a new metric (related via the temperature T as the conformal factor with the Weinhold's metric).

The thermodynamical geometry was generalized in various directions, for instance, by Janyszek and Mrugala [13, 14, 18] even to discussions of applications of Finsler geometry in thermodynamic fluctuation theory and for nonequilibrium thermodynamics [22].

Our goal will be to provide a characterization of thermodynamics of $(2+1)$–dimensional locally anisotropic black holes with elliptical (constant in time) horizon obtained in [25, 26]. From one point of view we shall consider the thermodynamic space of such objects (locally anisotropic black holes in local equilibrium with locally anisotropic spacetime ether) to depend on parameter of anisotropy, the angle θ, and on deformation parameter, the eccentricity ε. From another point, after we shall integrate the formulas on θ, the thermodynamic geometry will be considered in a usual two–dimensional Ruppeiner–Mrugala–Janyszek fashion. The main result of this work are the computation of thermodynamic curvatures and the proof that constant in time elliptic locally anisotropic black holes have critical points of vanishing of curvatures (under both approaches to two–dimensional thermodynamic geometry) for some values of eccentricity, i. e. for under corresponding deformations of locally anisotropic spacetimes.

The paper is organized as follows. In Sec. 2, we briefly review the geometry pseudo–Riemannian spaces provided with anholonimic frame and associated nonlinear connection structure and present the $(2+1)$–dimensional constant in time elliptic black hole solution. In Sec. 3, we state the thermodynamics of nearly equilibrium stationary locally anisotropic black holes and establish the basic thermodynamic law and relations. In Sec. 4 we develop two approaches to the thermodynamic geometry of locally anisotropic black holes, compute thermodynamic curvatures and the equations for critical points of vanishing of curvatures for some values of eccentricity. In Sec. 5, we draw a discussion and conclusions.

6.2 Locally Anisotropic Spacetimes and Black Holes

In this section we outline for further applications the basic results on (2 + 1)–dimensional locally anisotropic spacetimes and locally anisotropic black hole solutions [25, 26].

6.2.1 Anholonomic frames and nonlinear connections in $(2+1)$–dimensional spacetimes

A (2+1)–dimensional locally anisotropic spacetime is defined as a 3D pseudo–Riemannian space provided with a structure of anholonomic frame with two holonomic coordinates $x^i, i = 1,2$ and one anholonomic coordinate y, for which $u = (x,y) = \{u^\alpha = (x^i, y)\}$, the Greek indices run values $\alpha = 1,2,3$, when $u^3 = y$. We shall use also underlined indices, for instance $\underline{\alpha}, \underline{i}$, in order to emphasize that some tensors are given with respect to a local coordinate base $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$.

An anholonomic frame structure of triads (dreibein) is given by a set of three independent basis fields

$$e_\alpha(u) = e_\alpha^{\underline{\alpha}}(u)\partial_{\underline{\alpha}}$$

which satisfy the relations

$$e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma{}_{\alpha\beta} e_\gamma,$$

where $w^\gamma{}_{\alpha\beta} = w^\gamma{}_{\alpha\beta}(u)$ are called anholonomy coefficients.

We investigate anholonomic structures with mixed holonomic and anholonomic triads when

$$e_\alpha^{\underline{\alpha}}(u) = \{e_j^{\underline{i}} = \delta_j^i, e_j^{\underline{3}} = N_j^3(u) = w_i(u), e_3^{\underline{3}} = 1\}.$$

In this case we have to apply 'elongated' by N–coefficients operators instead of usual local coordinate basis $\partial_\alpha = \partial/\partial u^\alpha$ and $d^\alpha = du^\alpha$, (for simplicity we shall omit underling of indices if this does not result in ambiguities):

$$\begin{aligned} \delta_\alpha &= (\delta_i, \partial_{(y)}) = \frac{\delta}{\partial u^\alpha} \\ &= \left(\frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - w_i\left(x^j, y\right)\frac{\partial}{\partial y}, \partial_{(y)} = \frac{\partial}{\partial y}\right) \end{aligned} \tag{6.1}$$

and their duals

$$\begin{aligned} \delta^\beta &= \left(d^i, \delta^{(y)}\right) = \delta u^\beta \\ &= \left(d^i = dx^i, \delta^{(y)} = \delta y = dy + w_k\left(x^j, y\right)dx^k\right). \end{aligned} \tag{6.2}$$

The coefficients $N=\{N_i^3(x,y)=w_i(x^j,y)\}$, are associated to a nonlinear connection (in brief, N–connection, see [3]) structure which on pseudo-Riemannian spaces defines a locally anisotropic, or equivalently, mixed holonomic–anholonomic structure. The geometry of N–connection was investigated for vector bundles and generalized Finsler geometry [17] and for superspaces and locally anisotropic (super)gravity and string theory [24] with applications in general relativity, extra dimension gravity and formulation of locally anisotropic kinetics and thermodynamics on curved spacetimes [25, 26, 27]. In this paper (following the Paper I) we restrict our considerations to the simplest case with one anholonomic (anisotropic) coordinate when the N–connection is associated to a subclass of anholonomic triads (6.1), and/or (6.2), defining some locally anisotropic frames (in brief, anholonomic basis, anholonomic frames).

With respect to a fixed structure of locally anisotropic bases and their tensor products we can construct distinguished, by N–connection, tensor algebras and various geometric objects (in brief, one writes d–tensors, d–metrics, d–connections and so on).

A symmetrical locally anisotropic metric, or d–metric, could be written with respect to an anhlonomic basis (6.2) as

$$\begin{aligned}\delta s^2 &= g_{\alpha\beta}(u^\tau)\,\delta u^\alpha \delta u^\beta \\ &= g_{ij}\left(x^k,y\right)dx^i dx^j + h\left(x^k,y\right)(\delta y)^2 .\end{aligned} \tag{6.3}$$

We note that the anisotropic coordinate y could be both type time–like ($y=t$, or space–like coordinate, for instance, $y=r$, radial coordinate, or $y=\theta$, angular coordinate).

6.2.2 Non–rotating black holes with ellipsoidal horizon

Let us consider a 3D locally anisotropic spacetime provided with local space coordinates $x^1=r,\ x^2=\theta$ when as the anisotropic direction is chosen the time like coordinate, $y=t$. We proved (see the Paper I) that a d–metric of type (6.3),

$$\delta s^2=\Omega^2(r,\theta)\left[a(r)dr^2+b(r,\theta)\,d\theta^2+h(r,\theta)\delta t^2\right], \tag{6.4}$$

where

$$\begin{aligned}\delta t &= dt+w_1(r,\theta)\,dr+w_2(r,\theta)d\theta, \\ w_1 &= \partial_r \ln|\ln\Omega|\,, w_2=\partial_\theta \ln|\ln\Omega|\,,\end{aligned}$$

for $\Omega^2=\pm h(r,\theta)$, satisfies the system of vacuum locally anisotropic gravitational equations with cosmological constant $\Lambda_{[0]}$,

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R-\Lambda_{[0]}g_{\alpha\beta}=0$$

if

$$a\,(r) = 4r^2|\Lambda_0|, b(r,\theta) = \frac{4}{|\Lambda_0|}\Lambda^2(\theta)\left[r_+^2(\theta) - r^2\right]^2$$

and

$$h\,(r,\theta) = -\frac{4}{|\Lambda_0| r^2}\Lambda^3(\theta)\left[r_+^2(\theta) - r^2\right]^3. \quad (6.5)$$

The functions $a(r), b\,(r,\theta)$ and $h(x^i, y)$ and the coefficients of nonlinear connection $w_i(r,\theta,t)$ (for this class of solutions being arbitrary prescribed functions) were defined as to have compatibility with the locally isotropic limit.

We construct a black hole like solution with elliptical horizon $r^2 = r_+^2(\theta)$, on which the function (6.5) vanishes if we chose

$$r_+^2(\theta) = \frac{p^2}{\left[1 + \varepsilon\cos\theta\right]^2}. \quad (6.6)$$

where p is the ellipse parameter and $\dot{\varepsilon}$ is the eccentricity. We have to identify

$$p^2 = r_{+[0]}^2 = -M_0/\Lambda_0,$$

where $r_{+[0]}, M_0$ and Λ_0 are respectively the horizon radius, mass parameter and the cosmological constant of the non–rotating BTZ solution [1] if we wont to have a connection with locally isotropic limit with $\varepsilon \to 0$. In the simplest case we can consider that the elliptic horizon (6.6) is modelled by an anisotropic mass

$$M\,(\theta,\varepsilon) = \frac{M_0}{2\pi\left(1 + \varepsilon\cos\left(\theta - \theta_0\right)\right)^2} = \frac{r_+^2}{2\pi} \quad (6.7)$$

and constant effective cosmological constant, $\Lambda(\theta) \simeq \Lambda_0$. The coefficient 2π was introduced in order to have the limit

$$\lim_{\varepsilon\to 0} 2\int_0^{\pi} M\,(\theta,\varepsilon)\,d\theta = M_0. \quad (6.8)$$

Throughout this paper, the units $c = \hbar = k_B = 1$ will be used, but we shall consider that for an locally anisotropically renormalized gravitational constant $8G_{(gr)}^{(a)} \neq 1$, see [26].

6.3 On the Thermodynamics of Elliptical Black Holes

In this paper we will be interested in thermodynamics of locally anisotropic black holes defined by a d–metric (6.4).

The Hawking temperature $T(\theta,\varepsilon)$ of a locally anisotropic black hole is anisotropic and is computed by using the anisotropic mass (6.7):

$$T(\theta,\varepsilon) = \frac{M(\theta,\varepsilon)}{2\pi r_+(\theta,\varepsilon)} = \frac{r_+(\theta,\varepsilon)}{4\pi^2} > 0. \tag{6.9}$$

The two parametric analog of the Bekenstein–Hawking entropy is to be defined as

$$S(\theta,\varepsilon) = 4\pi r_+ = \sqrt{32\pi^3}\sqrt{M(\theta,\varepsilon)} \tag{6.10}$$

The introduced thermodynamic quantities obey the first law of thermodynamics (under the supposition that the system is in local equilibrium under the variation of parameters (θ,ε))

$$\Delta M(\theta,\varepsilon) = T(\theta,\varepsilon)\,\Delta S, \tag{6.11}$$

where the variation of entropy is

$$\Delta S = 4\pi\Delta r_+ = 4\pi\frac{1}{\sqrt{M(\theta,\varepsilon)}}\left(\frac{\partial M}{\partial\theta}\Delta\theta + \frac{\partial M}{\partial\varepsilon}\Delta\varepsilon\right).$$

According to the formula $C = (\partial m/\partial T)$ we can compute the heat capacity

$$C = 2\pi r_+(\theta,\varepsilon) = 2\pi\sqrt{M(\theta,\varepsilon)}.$$

Because of $C > 0$ always holds the temperature is increasing with the mass.

The formulas (6.7)–(6.11) can be integrated on angular variable θ in order to obtain some thermodynamic relations for black holes with elliptic horizon depending only on deformation parameter, the eccentricity ε.

For a elliptically deformed black hole with the outer horizon r_+ given by formula (6.10) the depending on eccentricity[26] Bekenstein–Hawking entropy is computed as

$$S^{(a)}(\varepsilon) = \frac{L_+}{4G^{(a)}_{(gr)}},$$

were

$$L_+(\varepsilon) = 4\int_0^{\pi/2} r_+(\theta,\varepsilon)\,d\theta$$

is the length of ellipse's perimeter and $G_{(gr)}^{(a)}$ is the three dimensional gravitational coupling constant in locally anisotropic media (the index (a) points to locally anisotropic renormalizations), and has the value

$$S^{(a)}(\varepsilon) = \frac{2p}{G_{(gr)}^{(a)}\sqrt{1-\varepsilon^2}} arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}. \tag{6.12}$$

If the eccentricity vanishes, $\varepsilon = 0$, we obtain the locally isotropic formula with p being the radius of the horizon circumference, but the constant $G_{(gr)}^{(a)}$ could be locally anisotropic renormalized.

The total mass of a locally anisotropic black hole of eccentricity ε is found by integrating (6.7) on angle θ :

$$M(\varepsilon) = \frac{M_0}{(1-\varepsilon^2)^{3/2}} \tag{6.13}$$

which satisfies the condition (6.8).

The integrated on angular variable θ temperature $T(\varepsilon)$ is to by defined by using $T(\theta,\varepsilon)$ from (6.9),

$$T(\varepsilon) = 4\int_0^{\pi/2} T(\theta,\varepsilon)\,d\theta = \frac{2\sqrt{M_0}}{\pi^2\sqrt{1-\varepsilon^2}} arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}. \tag{6.14}$$

Formulas (6.12)–(6.14) describes the thermodynamics of ε–deformed black holes.

Finally, in this section, we note that a black hole with elliptic horizon is to be considered as a thermodynamic subsystem placed into the anisotropic ether bath of spacetime. To the locally anisotropic ether one associates a continuous locally anisotropic medium assumed to be in local equilibrium. The locally anisotropic black hole subsystem is considered as a subsystem described by thermodynamic variables which are continuous field on variables (θ,ε), or in the simplest case when one have integrated on θ, on ε. It will be our first task to establish some parametric thermodynamic relations between the mass $m(\theta,\varepsilon)$ (equivalently, the internal locally anisotropic black hole energy), temperature $T(\theta,\varepsilon)$ and entropy $S(\theta,\varepsilon)$.

6.4 Thermodynamic Metrics and Curvatures of Anisotropic Black Holes

We emphasize in this paper two approaches to the thermodynamic geometry of nearly equilibrium locally anisotropic black holes based on their thermodynamics. The first one is to consider the thermodynamic space as depending locally on two parameters θ and ε and to compute the corresponding metric and curvature following standard formulas from curved bi–dimensional hypersurface Riemannian geometry. The second possibility is to take as basic the Ruppeiner metric in the thermodynamic space with coordinates (M, ε), in a manner proposed in Ref. [7] with that difference that as the extensive coordinate is taken the black hole eccentricity ε (instead of the usual angular momentum J for isotropic $(2+1)$–black holes). Of course, in this case we shall background our thermodynamic geometric constructions starting from the relations (6.12)–(6.14).

6.4.1 The thermodynamic parametric geometry

Let us consider the thermodynamic parametric geometry of the elliptic (2+1)–dimensional black hole based on its thermodynamics given by formulas (6.7)–(6.11).

Rewriting equations (6.11, we have

$$\Delta S = \beta\left(\theta, \varepsilon\right) \Delta M\left(\theta, \varepsilon\right),$$

where $\beta\left(\theta, \varepsilon\right) = 1/T\left(\theta, \varepsilon\right)$ is the inverse to temperature (6.9). This case is quite different from that from [7, 9] where there are considered, respectively, BTZ and dilaton black holes (by introducing Ruppeiner and Weinhold thermodynamic metrics). Our thermodynamic space is defined by a hypersurface given by parametric dependencies of mass and entropy. Having chosen as basic the relative entropy function,

$$\varsigma = \frac{S\left(\theta, \varepsilon\right)}{4\pi\sqrt{M_0}} = \frac{1}{1 + \varepsilon\cos\theta},$$

in the vicinity of a point $P = (0, 0)$, when, for simplicity, $\theta_0 = 0$, our hypersurface is given locally by conditions

$$\varsigma = \varsigma\left(\theta, \varepsilon\right) \text{ and } grad_{|P}\varsigma = 0.$$

For the components of bi–dimensional metric on the hypersurface we have

$$\begin{aligned} g_{11} &= 1 + \left(\frac{\partial\varsigma}{\partial\theta}\right)^2, \; g_{12} = \left(\frac{\partial\varsigma}{\partial\theta}\right)\left(\frac{\partial\varsigma}{\partial\varepsilon}\right), \\ g_{22} &= 1 + \left(\frac{\partial\varsigma}{\partial\varsigma}\right)^2, \end{aligned}$$

The nonvanishing component of curvature tensor in the vicinity of the point $P = (0, 0)$ is

$$R_{1212} = \frac{\partial^2\varsigma}{\partial\theta^2}\frac{\partial^2\zeta}{\partial\varepsilon^2} - \left(\frac{\partial^2\varsigma}{\partial\varepsilon\partial\theta}\right)^2$$

and the curvature scalar is

$$R = 2R_{1212}. \tag{6.15}$$

By straightforward calculations we can find the condition of vanishing of the curvature (6.15) when

$$\varepsilon_{\pm} = \frac{-1 \pm (2 - \cos^2\theta)}{\cos\theta\,(3 - \cos^2\theta)}. \tag{6.16}$$

So, the parametric space is separated in subregions with elliptic eccentricities $0 < \varepsilon_{\pm} < 0$ and θ satisfying conditions (6.16).

Ruppeiner suggested that the curvature of thermodynamic space is a measure of the smallest volume where classical thermodynamic theory based on the assumption of a uniform environment could conceivably work and that near the critical point it is expected this volume to be proportional to the scalar curvature [20]. There were also proposed geometric equations relating the thermodynamic curvature via inverse relations to free energy. Our definition of thermodynamic metric and curvature in parametric spaces differs from that of Ruppeiner or Weinhold and it is obvious that relations of type (6.16) (stating the conditions of vanishing of curvature) could be related with some conditions for stability of thermodynamic space under variations of eccentricity ε and anisotropy angle θ. This interpretation is very similar to that proposed by Janyszek and Mrugala [13] and supports the viewpoint that the first law of thermodynamics makes a statement about the first derivatives of the entropy, the second law is for the second derivatives and the curvature is a statement about the third derivatives. This treatment holds good also for the parametric thermodynamic spaces for locally anisotropic black holes.

6.4.2 Thermodynamic Metrics and Eccentricity of Black Hole

A variant of thermodynamic geometry of locally anisotropic black holes could be grounded on integrated on anisotropy angle θ formulas (6.12)–(6.14). The Ruppeiner metric of elliptic black holes in coordinates (M, ε) is

$$ds_R^2 = -\left(\frac{\partial^2 S}{\partial M^2}\right)_{\varepsilon} dM^2 - \left(\frac{\partial^2 S}{\partial \varepsilon^2}\right)_{M} d\varepsilon^2. \tag{6.17}$$

For our further analysis we shall use dimensionless values $\mu = M(\varepsilon)/M_0$ and $\zeta = S^{(a)} G_{gr}^{(a)}/2p$ and consider instead of (6.17) the thermodynamic diagonal metrics $g_{ij}(a^1, a^2)$ $= g_{ij}(\mu, \varepsilon)$ with components

$$g_{11} = -\frac{\partial^2 \zeta}{\partial \mu^2} = -\zeta_{,11} \text{ and } g_{22} = -\frac{\partial^2 \zeta}{\partial \varepsilon^2} = -\zeta_{,22}, \tag{6.18}$$

where by comas we have denoted partial derivatives.

The expressions (6.12) and (6.12) are correspondingly rewritten as

$$\zeta = \frac{1}{\sqrt{1-\varepsilon^2}} arctg \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$$

and

$$\mu = \left(1-\varepsilon^2\right)^{-3/2}.$$

By straightforward calculations we obtain

$$\begin{aligned}\zeta_{,11} &= -\frac{1}{9}\left(1-\varepsilon^2\right)^{5/2} arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} \\ &+ \frac{1}{9\varepsilon}\left(1-\varepsilon^2\right)^3 + \frac{1}{18\varepsilon^4}\left(1-\varepsilon^2\right)^4\end{aligned}$$

and

$$\zeta_{,22} = \frac{1+2\varepsilon^2}{\left(1-\varepsilon^2\right)^{5/2}} arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} - \frac{3\varepsilon}{\left(1-\varepsilon^2\right)^2}.$$

The thermodynamic curvature of metrics of type (6.18) can be written in terms of second and third derivatives [13] by using third and second order determinants:

$$\begin{aligned}R &= \frac{1}{2}\begin{vmatrix} -\zeta_{,11} & 0 & -\zeta_{,22} \\ -\zeta_{,111} & -\zeta_{,112} & 0 \\ -\zeta_{,112} & 0 & -\zeta_{,222}\end{vmatrix} \times \begin{vmatrix} -\zeta_{,11} & 0 \\ 0 & -\zeta_{,22}\end{vmatrix}^{-2} \\ &= -\frac{1}{2}\left(\frac{1}{\zeta_{,11}}\right)_{,2} \times \left(\frac{\zeta_{,11}}{\zeta_{,22}}\right)_{,2}. \end{aligned} \tag{6.19}$$

The conditions of vanishing of thermodynamic curvature (6.19) are as follows

$$\zeta_{,112}\left(\varepsilon_1\right) = 0 \text{ or } \left(\frac{\zeta_{,11}}{\zeta_{,22}}\right)_{,2}\left(\varepsilon_2\right) = 0 \tag{6.20}$$

for some values of eccentricity, $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$, satisfying conditions $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1$. For small deformations of black holes, i.e. for small values of eccentricity, we can approximate $\varepsilon_1 \approx 1/\sqrt{5.5}$ and $\varepsilon_2 \approx 1/(18\lambda)$, where λ is a constant for which $\zeta_{,11} = \lambda\zeta_{,22}$ and the condition $0 < \varepsilon_2 < 1$ is satisfied. We omit general formulas for curvature (6.19) and conditions (6.20), when the critical points ε_1 and/or ε_2 must be defined from nonlinear equations containing $arctg\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}$ and powers of $(1-\varepsilon^2)$ and ε.

6.5 Discussion and Conclusions

In closing, we would like to discuss the meaning of geometric thermodynamics following from locally anisotropic black holes.

(1) *Nonequilibrium thermodynamics of locally anisotropic black holes in locally anisotropic spacetimes.* In this paper and in the Paper I [26] we concluded that the thermodynamics in locally anisotropic spacetimes has a generic nonequilibrium character and could be developed in a geometric fashion following the approach proposed by S. Sieniutycz, P. Salamon and R. S. Berry [22, 21]. This is a new branch of black hole thermodynamics which should be based on locally anisotropic nonequilibrium thermodynamics and kinetics [27].

(2) *Locally Anisotropic Black holes thermodynamics in vicinity of equilibrium points.* The usual thermodynamical approach in the Bekenstein–Hawking manner is valid for anisotropic black holes for a subclass of such physical systems when the hypothesis of local equilibrium is physically motivated and corresponding renormalizations, by locally anisotropic spacetime parameters, of thermodynamical values are defined.

(3) *The geometric thermodynamics of locally anisotropic black holes with constant in time elliptic horizon* was formulated following two approaches: for a parametric thermodynamic space depending on anisotropy angle θ and eccentricity ε and in a standard Ruppeiner–Mrugala–Janyszek fashion, after integration on anisotropy θ but maintaining locally anisotropic spacetime deformations on ε.

(4) *The thermodynamic curvatures of locally anisotropic black holes* were shown to have critical values of eccentricity when the scalar curvature vanishes. Such type of thermodynamical systems are rather unusual and a corresponding statistical model is not that for ordinary systems composed by classical or quantum like gases.

(5) *Thermodynamic systems with constraints* require a new geometric structure in addition to the thermodynamical metrics which is that of nonlinear connection. We note this object must be introduced both in spacetime geometry and in thermodynamic geometry if generic anisotropies and constrained field and/or thermodynamic behavior are analyzed.

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Chapter 7

Off–Diagonal 5D Metrics and Mass Hierarchies with Anisotropies and Running Constants

Abstract [1]

The gravitational equations of the three dimensional (3D) brane world are investigated for both off–diagonal and warped 5D metrics which can be diagonalized with respect to some anholonomic frames when the gravitational and matter fields dynamics are described by mixed sets of holonomic and anholonomic variables. We construct two new classes of exact solutions of Kaluza–Klein gravity which generalize the Randall–Sundrum metrics to configurations with running on the 5th coordinate gravitational constant and anisotropic dependencies of effective 4D constants on time and/or space variables. We conclude that by introducing gauge fields as off–diagonal components of 5D metrics, or by considering anholonomic frames modelling some anisotropies in extra dimension spacetime, we induce anisotropic tensions (gravitational polarizations) and running of constants on the branes. This way we can generate the TeV scale as a hierarchically suppressed anisotropic mass scale and the Newtonian and general relativistic gravity are reproduced with adequate precisions but with corrections which depend anisotropically on some coordinates.

Recent approaches to String/M–theory and particle physics are based on the idea that our universe is realized as a three brane, modelling a four dimensional, 4D, pseudo–Riemannian spacetime, embedded in the 5D anti–de Sitter (AdS_5) bulk spacetime. In such models the extra dimension need not be small (they could be even infinite) if a nontrivial warped geometric configuration, being essential for solving the mass hierarchy problem and localization of gravity, can "bound" the matter fields on a 3D subspace on which we live at low energies, the gravity propagating, in general, in a higher dimension spacetime (see Refs.: [1] for string gravity papers; [2] for extra dimension particle fields and gravity phenomenology with effective Plank scale; [3] for the simplest and comprehensive models proposed by Randall and Sundrum; here we also point the early works [4] in this line and cite [5] as some further developments with supresymmetry, black hole solutions and cosmological scenarios).

In higher dimensional gravity much attention has been paid to off–diagonal metrics beginning the Salam, Strathee and Perracci work [6] which showed that including off–diagonal components in higher dimensional metrics is equivalent to including $U(1), SU(2)$ and $SU(3)$ gauge fields. Recently, the off–diagonal metrics were considered in a new fashion by applying the method of anholonomic frames with associated nonlinear connections [7] which allowed us to construct new classes of solutions of Einstein's equations in three (3D), four (4D) and five (5D) dimensions, with generic local anisotropy (*e.g.* static black hole and cosmological solutions with ellipsoidal or torus symmetry, soliton–dilaton 2D and 3D configurations in 4D gravity, and wormhole and flux tubes with anisotropic polarizations and/or running on the 5th coordinate constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and torus symmetry and anisotropy.

The point of this paper is to argue that if the 5D gravitational interactions are parametrized by off–diagonal metrics with a warped factor, which could be related with an anholonomic higher dimensional gravitational dynamics and/or with the fact that the gauge fields are included into a Salam–Strathee–Peracci manner, the fundamental Plank scale M_{4+d} in $4+d$ dimensions can be not only considerably smaller than the the effective Plank scale, as in the usual locally isotropic Randall–Sundrum (in brief, RS) scenarios, but the effective Plank constant is also anisotropically polarized which could have profound consequences for elaboration of gravitational experiments and for models of the very early universes.

We will give two examples with one additional dimension ($d=1$) when an extra dimension gravitational anisotropic polarization on a space coordinate is emphasized or, in the second case, a running of constants in time is modelled. We will show that effective gravitational Plank scale is determined by the higher-dimensional curvature and anholonomy of pentad (funfbein, of frame basis) fields rather than the size of the extra dimension. Such curvatures and anholonomies are not in conflict with the local four-dimensional Poincare invariance.

We will present a higher dimensional scenario which provides a new RS like approach generating anisotropic mass hierarchies. We consider that the 5D metric is both not factorizable and off–diagonal when the four-dimensional metric is multiplied by a "warp" factor which is a rapidly changing function of an additional dimension and depend anisotropically on a space direction and runs in the 5-th coordinate.

Let us consider a 5D pseudo–Riemannian spacetime provided with local coordinates $u^{\alpha} = (x^i, y^a) = (x^1 = x, x^2 = f, x^3 = y, y^4 = s, y^5 = p)$, where $(s,p) = (z,t)$ (Case I) or, inversely, $(s,p) = (t,z)$ (Case II) – or more compactly $u = (x,y)$ – where the Greek indices are conventionally split into two subsets x^i and y^a labelled respectively by Latin indices of type $i,j,k,... = 1,2,3$ and $a,b,... = 4,5$. The local coordinate bases, $\partial_{\alpha} = (\partial_i, \partial_a)$, and their duals, $d^{\alpha} = (d^i, d^a)$, are defined respectively as

$$\partial_\alpha \equiv \frac{\partial}{du^\alpha} = (\partial_i = \frac{\partial}{dx^i}, \partial_a = \frac{\partial}{dy^a}) \text{ and } d^\alpha \equiv du^\alpha = (d^i = dx^i, d^a = dy^a). \tag{7.1}$$

For the 5D (pseudo) Riemannian interval $dl^2 = G_{\alpha\beta}du^\alpha du^\beta$ we choose the metric coefficients $G_{\alpha\beta}$ (with respect to the coordinate frame (7.1)) to be parametrized by a off–diagonal matrix (ansatz)

$$\begin{bmatrix} g + {w_1}^2 h_4 + {n_1}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & 1 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_3 w_2 h_4 + n_2 n_3 h_5 & g + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix} \tag{7.2}$$

where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g &= g(f,y) = a(f)\ b(y), h_4 = h_4(f,y,s) = \eta_4(f,y)g(f,y)q_4(s), \\ h_5 &= h_5(f,y,s) = g(f,y)q_5(s), w_i = w_i(f,y,s), n_i = n_i(f,y,s). \end{aligned}$$

The metric (7.2) can be equivalently rewritten in the form

$$\delta l^2 = g_{ij}\,(f,y)\,dx^i dx^i + h_{ab}\,(f,y,s)\,\delta y^a \delta y^b, \tag{7.3}$$

with diagonal coefficients

$$g_{ij} = \begin{bmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{bmatrix} \text{ and } h_{ab} = \begin{bmatrix} h_4 & 0 \\ 0 & h_5 \end{bmatrix} \tag{7.4}$$

if instead the coordinate bases (7.1) one introduce the anholonomic frames (anisotropic bases)

$$\delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_i = \partial_i - N_i^b(u)\ \partial_b, \partial_a = \frac{\partial}{dy^a}), \delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_k^a(u)\ dx^k) \tag{7.5}$$

where the N–coefficients are parametrized $N_i^4 = w_i$ and $N_i^5 = n_i$.

In this paper we consider a slice of AdS_5 provided with an anholnomic frame structure (7.5) satisfying the relations $\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = W_{\alpha\beta}^\gamma \delta_\gamma$, with nontrivial anholonomy coefficients

$$\begin{aligned} W_{ij}^k &= 0, W_{aj}^k = 0, W_{ia}^k = 0, W_{ab}^k = 0, W_{ab}^c = 0, \\ W_{ij}^a &= \delta_i N_j^a - \delta_j N_i^a, W_{bj}^a = -\partial_b N_j^a, W_{ia}^b = \partial_a N_j^b. \end{aligned}$$

We assume there exists a solution of 5D Einstein equations with 3D brane configuration that effectively respects the local 4D Poincare invariance with respect to anholonomic frames (7.5) and that the metric ansatz (7.2) (equivalently, (7.4)) transforms into the usual RS solutions

$$ds^2 = e^{-2k|f|}\eta_{\underline{\mu}\underline{\nu}}dx^{\underline{\mu}}dx^{\underline{\nu}} + df^2 \tag{7.6}$$

for the data: $a(f) = e^{-2k|f|},\ k = const,\ b(y) = 1, \eta_4(f,y) = 1, q_4(s) = q_5(s) = 1, w_i = 0, n_i = 0$, where $\eta_{\underline{\mu}\underline{\nu}}$ and $x^{\underline{\mu}}$ are correspondingly the diagonal metric and Cartesian coordinates in 4D Minkowski spacetime and the extra-dimensional coordinate f is to be identified $f = r_c\phi$, ($r_c = const$ is the compactification radius, $0 \leq f \leq \pi r_c$) like in the first work [3] (or 'f" is just the coordinate 'y' in the second work [3]).

The set-up for our model is a single 3D brane with positive tension, subjected to some anholonomic constraints, embedded in a 5D bulk spacetime provided with a off–diagonal metric (7.2). In order to carefully quantize the system, and treat the non–normalizable modes which will appear in the Kaluza-Klein reduction, it is useful to work with respect to anholonomic frames were the metric is diagonalized by corresponding anholonomic transforms and is necessary to work in a finite volume by introducing another brane at a distance πr_c from the brane of interest, and taking the branes to be the boundaries of a finite 5th dimension. We can remove the second brane from the physical set-up by taking the second brane to infinity.

The action for our anholonomic funfbein (pentadic) system is

$$\begin{aligned} S &= S_{gravity} + S_{brane} + S_{brane'} \\ S_{gravity} &= \int \delta^4 x \int \delta f \sqrt{-G}\{-\Lambda(f) + 2M^3 R\}, \\ S_{brane} &= \int \delta^4 x \sqrt{-g_{brane}}\{V_{brane} + \mathcal{L}_{brane}\}, \end{aligned} \tag{7.7}$$

where R is the 5D Ricci scalar made out of the 5D metric, $G_{\alpha\beta}$, and Λ and V_{brane} are cosmological terms in the bulk and boundary respectively. We write down $\delta^4 x$ and δf, instead of usual differentials d^4x and df, in order to emphasize that the variational calculus should be performed by using N–elongated partial derivatives and differentials (7.5). The coupling to the branes and their fields and the related orbifold boundary conditions for vanishing N–coefficients are described in Refs. [3] and [8].

The Einstein equations,

$$R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} R = \Upsilon^{\alpha}_{\beta},$$

for a diagonal energy–momentum tensor $\Upsilon^{\beta}_{\alpha} = [\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5]$ and following from the action (7.7) and for the ansatz (7.2) (equivalently, (7.4)) with $g = a(f)b(y)$ transform into

$$\begin{aligned}
\frac{1}{a}\left[a_1'' - \frac{(a')^2}{2a}\right] + \frac{\beta}{h_4 h_5} &= 2\Upsilon_1,\ \frac{(a')^2}{2a} + \frac{P(y)}{a} + \frac{\beta}{h_4 h_5} = 2\Upsilon_2(f), \qquad (7.8)\\
\frac{1}{a}\left[a'' - \frac{(a')^2}{2a}\right] + \frac{P(y)}{a} &= 2\Upsilon_4,\ w_i\beta + \alpha_i = 0,\ n_i^{**} + \gamma n_i^* = 0,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1 &= h_5^{*\bullet} - \frac{h_5^*}{2}\left(\frac{h_4^\bullet}{h_4} + \frac{h_5^\bullet}{h_5}\right),\ \alpha_2 = h_5^{*\prime} - \frac{h_5^*}{2}\left(\frac{h_4^{\prime}}{h_4} + \frac{h_5^{\prime}}{h_5}\right),\\
\alpha_3 &= h_5^{*\#} - \frac{h_5^*}{2}\left(\frac{h_4^{\#}}{h_4} + \frac{h_5^{\#}}{h_5}\right),\\
\beta &= h_5^{**} - \frac{(h_5^*)^2}{2h_5} - \frac{h_5^* h_4^*}{2h_4}, P = \frac{1}{b^2}\left[b^{\#\#} - \frac{(b^{\#})^2}{b}\right],\ \gamma = \frac{3}{2}\frac{h_5{}^*}{h_5} - \frac{h_4{}^*}{h_4}, \qquad (7.9)
\end{aligned}$$

the partial derivatives are denoted: $h^\bullet = \partial h/\partial x^1, h' = \partial h/\partial x^2, h^{\#} = \partial h/\partial x^3,\ h^* = \partial h/\partial s$.

Our aim is to construct a metric

$$\delta s^2 = g(f,y)\left[dx^2 + dy^2 + \eta_4(f,y)\,\delta s^2 + q_5(s)\delta p^2\right] + df^2, \qquad (7.10)$$

with the anholonomic frame components defined by 'elongation' of differentials, $\delta s = ds + w_2 df + w_3 dy$, $\delta p = dp + n_1 dx + n_2 df + n_3 dy$, and the "warp" factor being written in a form similar to the RS solution

$$g(f,y) = a(f)b(y) = \exp[-2k_f|f| - 2k_y|y|], \qquad (7.11)$$

which defines anisotropic RS like solutions of 5D Einstein equations with variation on the 5th coordinate cosmological constant in the bulk and possible variations of induced on the brane cosmological constants.

By straightforward calculations we can verify that a class of exact solutions of the system of equations (7.8) for $P(y) = 0$ (see (7.9)) :

$$h_4 = g(f,y), h_5 = g(f,y)\rho^2(f,y,s),$$

were

$$\begin{aligned}
\rho(f,y,s) &= |\cos\tau_+(f,y)|,\ \tau_+ = \sqrt{(\Upsilon_4 - \Upsilon_2)\,g(f,y)}, \Upsilon_4 > \Upsilon_2;\\
&= \exp[-\tau_-(f,y)\,s], \tau_- = \sqrt{(\Upsilon_2 - \Upsilon_4)\,g(f,y)}, \Upsilon_4 < \Upsilon_2;\\
&= |c_1(f,y) + sc_2(f,y)|^2, \Upsilon_4 = \Upsilon_2,
\end{aligned}$$

and

$$w_i = -\partial_i(\ln|\rho^*|)/(\ln|\rho^*|)^*,$$
$$n_i = n_{i[0]}(f,y) + n_{i[1]}(f,y)\int \exp[-3\rho]ds,$$

with functions $c_{1,2}(f,y)$ and $n_{i[0,1]}(f,y)$ to be stated by some boundary conditions. We emphasize that the constants k_f and k_y have to be defined from some experimental data.

The solution (7.10) transforms into the usual RS solution (7.6) if $k_y = 0, n_{i[0,1]}(f,y) = 0, \Lambda = \Lambda_0 = const$ and $\Upsilon_2 \rightarrow \Upsilon_{2[0]} = -\frac{\Lambda_0}{4M^3}$; $\Upsilon_1, \Upsilon_3, \Upsilon_4, \Upsilon_5 \rightarrow \Upsilon_{[0]} = \frac{V_{brane}}{4M^3}\delta(f) + \frac{V_{brane'}}{4M^3}\delta(f - \pi r_c)$, which holds only when the boundary and bulk cosmological terms are related by formulas $V_{brane} = -V_{brane'} = 24M^3 k_f, \quad \Lambda_0 = -24M^3 k_f^2;$ we use values with the index [0] in order to emphasize that they belong to the usual (holonomic) RS solutions. In the anholonomic case with "variation of constants" we shall not impose such relations.

We note that using the metric (7.10) with anisotropic warp factor (7.11) it is easy to identify the massless gravitational fluctuations about our classical solutions like in the usual RS cases but performing (in this work) all computations with respect to anholonomic frames. All off–diagonal fluctuations of the anholonomic diagonal metric are massive and excluded from the low-energy effective theory.

We see that the physical mass scales are set by an anisotropic symmetry–breaking scale, $v(y) \equiv e^{-k_y|y|}e^{-k_f r_c \pi}v_0$. This result the conclusion: any mass parameter m_0 on the visible 3-brane in the fundamental higher-dimensional theory with Salam–Strathee –Peracci gauge interactions and/or effective anholonomic frames will correspond to an anisotropic dependence on coordinate y of the physical mass $m(y) \equiv e^{-k_y|y|}e^{-kr_c\pi}m_0$ when measured with the metric $\overline{g}_{\mu\nu}$ that appears in the effective Einstein action, since all operators get re-scaled according to their four-dimensional conformal weight. If $e^{kr_c\pi}$ is of order 10^{15}, this mechanism can produces TeV physical mass scales from fundamental mass parameters not far from the Planck scale, 10^{19} GeV. Because this geometric factor is an exponential, we clearly do not require very large hierarchies among the fundamental parameters, v_0, k, M, and $\mu_c \equiv 1/r_c$; in fact, we only require $kr_c \approx 50$. These conclusions were made in Refs. [3] with respect to diagonal (isotropic) metrics. But the physical consequences could radically change if the off–diagonal metrics with effective anholonomic frames and gauge fields are considered. In this case we have additional dependencies on variable y which make the fundamental spacetime geometry to be locally anisotropic, polarized via dependencies both on coordinate y receptivity k_y. We emphasize that our y coordinate is not that from [3].

The phenomenological implications of these anisotropic scenarios for future collider searches could be very distinctive: the geometry of experiments will play a very important role. In such anisotropic models we also have a roughly weak scale splitting with a relatively small number of excitations which can be kinematically accessible at accelerators.

We also reconsider in an anisotropic fashion the derivation of the 4D effective Planck scale M_{Pl} given in Ref. [3]. The 4D graviton zero mode follows from the solution, Eq. (7.10), by replacing the Minkowski metric by a effective 4D metric $\overline{g}_{\mu\nu}$ which it is described by an effective action following from substitution into Eq. (7.7),

$$S_{eff} \supset \int \delta^4 x \int_0^{\pi r_c} df \; 2M^3 r_c e^{-2k_f|f|} e^{-2k_y|y|} \sqrt{\overline{g}} \; \overline{R}, \tag{7.12}$$

where $\overline{R}$ denotes the four-dimensional Ricci scalar made out of $\overline{g}_{\mu\nu}(x)$, in contrast to the five-dimensional Ricci scalar, R, made out of $G_{MN}(x,f)$. We use the symbol $\delta^4 x$ in (7.7) in order to emphasize that our integration is adapted to the anholonomic structu5e stated by the differentials (7.5). We also can explicitly perform the f integral in (7.12) to obtain a purely 4D action and to derive

$$M_{Pl}^2 = 2M^3 \int_0^{\pi r_c} df e^{-2k_f|f|} = \frac{M^3}{k} e^{-2k_y|y|}[1 - e^{-2k_f r_c \pi}]. \tag{7.13}$$

We see that there is a well-defined value for M_{Pl}, even in the $r_c \to \infty$ limit, but which may have an anisotropic dependence on one of the 4D coordinates, in the stated parametrizations denoted by y. Nevertheless, we can get a sensible effective anisotropic 4D theory, with the usual Newtonian force law, even in the infinite radius limit, in contrast to the product–space expectation that $M_{Pl}^2 = M^3 r_c \pi$.

In consequence of (7.13), the gravitational potential behaves anisotropically as

$$V(r) = G_N \frac{m_1 m_2}{r} \left(1 + \frac{e^{-2k_y|y|}}{r^2 k_f^2} \right)$$

i.e. our models produce effective 4D theories of gravity with local anisotropy. The leading term due to the bound state mode is the usual Newtonian potential; the Kaluza Klein anholonomic modes generate an extremely anisotropically suppressed correction term, for k_f taking the expected value of order the fundamental Planck scale and r of the size tested with gravity.

Let us conclude the paper: It is known that we can consistently exist with an infinite 5th dimension, without violating known tests of gravity [3]. The scenarios consist of two or a single 3–brane, (a piece of) AdS_5 in the bulk, and an appropriately tuned tension on the brane. But if we consider off–diagonal 5D metrics like in Ref. [6], which was used for including of $U(1), SU(2)$ and $SU(3)$ gauge fields, or, in a different but similar fashion, for construction of generic anisotropic, partially anholonomic, solutions (like static black holes with ellipsoidal horizons, static black tori and anisotropic wormholes) in Einstein and extra dimension gravity, [7] the RS theories become substantially locally anisotropic. One obtains variations of constants on the 5th coordinate and possible anisotropic oscillations in time (in the first our model), or on space coordinate (in the second our model). Here it should be emphasized that the anisotropic oscillations (in time or in a space coordinate) are defined by the constant component of the cosmological constant (which in our model can generally run on the 5th coordinate). This sure is related to the the cosmological constant problem which in this work is taken as a given one, with an approximation of linear dependence on the 5th coordinate, and not solved. In the other hand a new, anisotropic, solution to the hierarchy problem is supposed to be subjected to experimental verification.

Finally, we note that many interesting questions remain to be investigated. Having constructed another, anisotropic, valid alternative to conventional 4D gravity, it is important to analyze the astrophysical and cosmological implications. These anisotropic scenarios might even provide a new perspective for solving unsolved issues in string/M-theory, quantum gravity and cosmology.

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Chapter 8

Anisotropic Black Holes in Einstein and Brane Gravity

Abstract [1]

We consider exact solutions of Einstein equations defining static black holes parametrized by off–diagonal metrics which by anholonomic mappings can be equivalently transformed into some diagonal metrics with coefficients being very similar to those from the Schwarzschild and/or Reissner-Nördstrom solutions with anisotropic renormalizations of constants. We emphasize that such classes of solutions, for instance, with ellipsoidal symmetry of horizons, can be constructed even in general relativity theory if off–diagonal metrics and anholonomic frames are introduced into considerations. Such solutions do not violate the Israel's uniqueness theorems on static black hole configurations [1] because at long radial distances one holds the usual Schwarzschild limit. We show that anisotropic deformations of the Reissner-Nördstrom metric can be an exact solution on the brane, re-interpreted as a black hole with an effective electromagnetic like charge anisotropically induced and polarized by higher dimension gravitational interactions.

The idea of extra–dimension is gone through a renewal in connection to string/M–theory [2] which in low energy limits results in models of brane gravity and/or high energy physics. It was proven that the matter fields could be localized on a 3–brane in $1+3+n$ dimensions, while gravity can propagate in the n extra dimensions which can be large (see, e. g., [3]) and even not compact, as in the 5-dimensional (in brief, 5D) warped space models of Randall and Sundrum [4] (in brief RS, see also early versions

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[5]).

The bulk of solutions of 5D Einstein equations and their reductions to 4D were constructed by using static diagonal metrics and extensions to solutions with rotations given with respect to holonomic coordinate frames of references. On the other hand much attention has been paid to off–diagonal metrics in higher dimensional gravity beginning the Salam, Strathee and Petracci work [6] which proved that including off–diagonal components in higher dimensional metrics is equivalent to including of $U(1), SU(2)$ and $SU(3)$ gauge fields. Recently, it was shown in Ref. [7] that if we consider off–diagonal metrics which can be equivalently diagonalized to some corresponding anholonomic frames, the RS theories become substantially locally anisotropic with variations of constants on extra dimension coordinate or with anisotropic angular polarizations of effective 4D constants, induced by higher dimension gravitational interactions.

If matter on a such anisotropic 3D branes collapses under gravity without rotating to form a black hole, then the metric on the brane-world should be close to some anisotropic deformations of the Schwarzschild metric at astrophysical scales in order to preserve the observationally tested predictions of general relativity. We emphasize that it is possible to construct anisotropic deformations of spherical symmetric black hole solutions to some static configurations with ellipsoidal or toroidal horizons even in the framework of 4D and in 5D Einstein theory if off–diagonal metrics and associated anholonomic frames and nonlinear connections are introduced into consideration [8].

Collapse to locally isotropic black holes in the Randall-Sundrum brane-world scenario was studied by Chamblin et al. [11] (see also [12, 14] and a review on the subject [13]). The item of definition of black hole solutions have to be reconsidered if we are dealing with off–diagonal metrics, anholonomic frames both in general relativity and on anisotropic branes.

In this Letter, we give four classes of exact black hole solutions which describes ellipsoidal static deformations with anisotropic polarizations and running of constants of the Schwarzschild and Reissner-Nördstrom solutions. We analyze the conditions when such type anisotropic solutions defined on 3D branes have their analogous in general relativity.

The 5D pseudo–Riemannian spacetime is provided with local coordinates $u^{\alpha} = (x^i, y^a) = (x^1 = f, x^2, x^3, y^4 = s, y^5 = p)$, where f is the extra dimension coordinate, (x^2, x^3) are some space coordinates and $(s = \varphi, p = t)$ (or inversely, $(s = t, p = \varphi)$) are correspondingly some angular and time like coordinates (or inversely). We suppose that indices run corresponding values: $i, j, k, ... = 1, 2, 3$ and $a, b, c, ... = 4, 5$. The local coordinate bases $\partial_{\alpha} = (\partial_i, \partial_a)$, and their duals, $d^{\alpha} = (d^i, d^a)$, are defined respectively as

$$\partial_\alpha \equiv \frac{\partial}{du^\alpha} = (\partial_i = \frac{\partial}{dx^i}, \partial_a = \frac{\partial}{dy^a}) \text{ and } d^\alpha \equiv du^\alpha = (d^i = dx^i, d^a = dy^a). \tag{8.1}$$

For the 5D line element $dl^2 = G_{\alpha\beta} du^\alpha du^\beta$ we choose the metric coefficients $G_{\alpha\beta}$ (with respect to the coordinate frame (8.1)) to be parametrized by a off–diagonal matrix (ansatz)

$$\begin{bmatrix} 1 + w_1{}^2 h_4 + n_1{}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + w_2{}^2 h_4 + n_2{}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_3 w_2 h_4 + n_2 n_3 h_5 & g_3 + w_3{}^2 h_4 + n_3{}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix} \tag{8.2}$$

where the coefficients are some necessary smoothly class functions of type:

$$g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^1, x^2, x^3, s), w_i = w_i(x^1, x^2, x^3, s), n_i = n_i(x^1, x^2, x^3, s).$$

The line element (8.2) can be equivalently rewritten in the form

$$\delta l^2 = g_{ij}\left(x^2, x^3\right) dx^i dx^i + h_{ab}\left(x^1, x^2, x^3, s\right) \delta y^a \delta y^b, \tag{8.3}$$

with diagonal coefficients $g_{ij} = diag[1, g_2, g_3]$ and $h_{ab} = diag[h_4, h_5]$ if instead the coordinate bases (8.1) one introduce the anholonomic frames (anisotropic bases)

$$\delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_i = \partial_i - N_i^b(u)\, \partial_b, \partial_a = \frac{\partial}{dy^a}),\ \delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_k^a(u)\, dx^k) \tag{8.4}$$

where the N–coefficients are parametrized $N_i^4 = w_i$ and $N_i^5 = n_i$ (on anholonomic frame method see details in [7]).

The nontrivial components of the 5D vacuum Einstein equations, $R_\alpha^\beta = 0$, for the ansatz (8.3) given with respect to anholonomic frames (8.4) are

$$R_2^2 = R_3^3 = -\frac{1}{2g_2 g_3}[g_3^{\bullet\bullet} - \frac{g_2^\bullet g_3^\bullet}{2g_2} - \frac{(g_3^\bullet)^2}{2g_3} + g_2^{''} - \frac{g_2^{'} g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = 0, \tag{8.5}$$

$$R_4^4 = R_5^5 = -\frac{\beta}{2h_4 h_5} = 0, \tag{8.6}$$

$$R_{4i} = -w_i \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \tag{8.7}$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^*\right] = 0, \tag{8.8}$$

where

$$\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \beta = h_5^{**} - h_5^* [\ln \sqrt{|h_4 h_5|}]^*, \gamma = (3h_5/2h_4) - h_4^*/h_4,$$

the partial derivatives are denoted like $a^{\hat{}} = \partial a/\partial x^1, h^\bullet = \partial h/\partial x^2, f' = \partial f/\partial x^2$ and $f^* = \partial f/\partial s$.

The system of second order nonlinear partial equations (8.5)–(8.8) can be solved in general form:

The equation (8.5) relates two functions $g_2(x^2,x^3)$ and $g_3(x^2,x^3)$. It is solved, for instance, by arbitrary two functions $g_2(x^2)$ and $g_3(x^3)$, or by $g_2 = g_3 = g_{[0]} \exp[a_2 x^2 + a_3 x^3]$, were $g_{[0]}, a_2$ and a_3 are some constants. For a given parametrization of $g_2 = b_2(x^2)c_2(x^3)$ we can find a decomposition in series for $g_3 = b_3(x^2)c_3(x^3)$ (in the inverse case a multiple parametrization is given for g_3 and we try to find g_2); for simplicity we omit such cumbersome formulas. We emphasize that we can always redefine the variables (x^2,x^3), or (equivalently) we can perform a 2D conformal transform to the flat 2D line element

$$g_2(x^2,x^3)(dx^2)^2 + g_3(x^2,x^3)(dx^3)^2 \to (dx^2)^2 + (dx^3)^2,$$

for which the solution of (8.5) becomes trivial.

The next step is to find solutions of the equation (8.6) which relates two functions $h_4(x^i,s)$ and $h_5(x^i,s)$. This equation is satisfied by arbitrary pairs of coefficients $h_4(x^i,s)$ and $h_{5[0]}(x^i)$. If dependencies of h_5 on anisotropic variable s are considered, there are two possibilities:

a) to compute

$$\begin{aligned} \sqrt{|h_5|} &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right) \int \sqrt{|h_4\left(x^i,s\right)|} ds, \ h_4^*\left(x^i,s\right) \neq 0; \\ &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right) s, h_4^*\left(x^i,s\right) = 0, \end{aligned}$$

for some functions $h_{5[1,2]}(x^i)$ stated by boundary conditions;

b) or, inversely, to compute h_4 for a given $h_5(x^i,s), h_5^* \neq 0$,

$$\sqrt{|h_4|} = h_{[0]}\left(x^i\right) (\sqrt{|h_5(x^i,s)|})^*, \tag{8.9}$$

with $h_{[0]}(x^i)$ given by boundary conditions.

Having the values of functions h_4 and h_5, we can define the coefficients $w_i(x^i,s)$ and $n_i(x^i,s)$:

The exact solutions of (8.7) is found by solving linear algebraic equation on w_k,

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_s \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \tag{8.10}$$

for $\partial_s = \partial/\partial s$ and $h_5^* \neq 0$. If $h_5^* = 0$ the coefficients w_k could be arbitrary functions on (x^i, s).

Integrating two times on variable s we find the exact solution of (8.8),

$$\begin{aligned} n_k &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int [h_4/(\sqrt{|h_5|})^3]ds,\ h_5^* \neq 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int h_4 ds,\ h_5^* = 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right)\int [1/(\sqrt{|h_5|})^3]ds,\ h_4^* \neq 0, \end{aligned} \tag{8.11}$$

for some functions $n_{k[1,2]}\left(x^i\right)$ stated by boundary conditions.

We shall construct some classes of exact solutions of 5D and 4D vacuum Einstein equations describing anholonomic deformations of black hole solutions of the Reissner-Nördstrom and Schwarzschild metrics. We consider two systems of 3D space coordinates:

a) The isotropic spherical coordinates (ρ, θ, φ), where the isotropic radial coordinate ρ is related with the usual radial coordinate r via relation $r = \rho\left(1 + r_g/4\rho\right)^2$ for $r_g = 2G_{[4]}m_0/c^2$ being the 4D gravitational radius of point particle of mass m_0, $G_{[4]} = 1/M_{P[4]}^2$ is the 4D Newton constant expressed via Plank mass $M_{P[4]}$ which following modern string/brane theories can considered as a value induced from extra dimensions, we shall put the light speed constant $c = 1$ (this system of coordinates is considered, for instance, for the so–called isotropic representation of the Schwarzschild solution [9]).

b) The rotation ellipsoid coordinates (in our case isotropic, in brief re–coordinates) [10] (u, v, φ) with $0 \leq u < \infty, 0 \leq v \leq \pi, 0 \leq \varphi \leq 2\pi$, where $\sigma = \cosh u = 4\rho/r_g \geq 1$ are related with the isotropic 3D Cartezian coordinates $(\tilde{x} = \sinh u \sin v \cos\varphi, \tilde{y} = \sinh u \sin v \sin\varphi, \tilde{z} = \cosh u \cos v)$ and define an elongated rotation ellipsoid hypersurface $\left(\tilde{x}^2 + \tilde{y}^2\right)/(\sigma^2 - 1) + \tilde{z}^2/\sigma^2 = 1.$

By straightforward calculations we can verify that we can generate from the ansatz (8.2) four classes of exact solutions of the system (8.5)–(8.8):

1. The anisotropic Reissner-Nördstrom black hole solutions with polarizations on extra dimension and 3D space coordinates are parametrized by the data

$$g_2 = \left(\frac{1 - \frac{r_g}{4\rho}}{1 + \frac{r_g}{4\rho}}\right)\frac{1}{\left[\rho^2 + a\rho/(1 + \frac{r_g}{4\rho})^2 + b/(1 + \frac{r_g}{4\rho})^4\right]}, g_3 = 1; \tag{8.12}$$

$$h_5 = -\frac{1}{\rho^2\left(1 + \frac{r_g}{4\rho}\right)^4}[1 + \frac{a\sigma_m\left(f, \rho, \theta, \varphi\right)}{\rho\left(1 + \frac{r_g}{4\rho}\right)^2} + \frac{b\sigma_q\left(f, \rho, \theta, \varphi\right)}{\rho^2\left(1 + \frac{r_g}{4\rho}\right)^4}],$$

$$h_4 = \sin^2\theta \left[\left(\sqrt{|h_5(f,\rho,\theta,\varphi)|}\right)\right]^2 \quad \text{(see (8.9))};$$

where a, b are constants and $\sigma_m(f,\rho,\theta,\varphi)$ and $\sigma_q(f,\rho,\theta,\varphi)$ are called respectively mass and charge polarizations and the coordinates are $(x^i, y^a) = (f,\rho,\theta,t,\varphi)$.

2. The anisotropic Reissner-Nördstrom black hole solutions with extra dimension and time running of constants are parametrized by the data

$$g_2 = \left(\frac{1-\frac{r_g}{4\rho}}{1+\frac{r_g}{4\rho}}\right)\frac{1}{\left[\rho^2 + a\rho/(1+\frac{r_g}{4\rho})^2 + b/(1+\frac{r_g}{4\rho})^4\right]}, g_3 = 1; \tag{8.13}$$
$$h_4 = -\frac{1}{\rho^2\left(1+\frac{r_g}{4\rho}\right)^4}[1+\frac{a\sigma_m(f,\rho,\theta,t)}{\rho\left(1+\frac{r_g}{4\rho}\right)^2} + \frac{b\sigma_q(f,\rho,\theta,t)}{\rho^2\left(1+\frac{r_g}{4\rho}\right)^4}], h_5 = \sin^2\theta,$$

where a, b are constants and $\sigma_m(f,\rho,\theta,\varphi)$ and $\sigma_q(f,\rho,\theta,\varphi)$ are called respectively mass and charge polarizations and the coordinates are $(x^i, y^a) = (f,\rho,\theta,\varphi,t)$.

3. The ellipsoidal Schwarzschild like black hole solutions with polarizations on extra dimension and 3D space coordinates are parametrized by the data $g_2 = g_3 = 1$ and

$$h_5 = -\frac{r_g^2}{16}\frac{\cosh^2 u}{(1+\cosh u)^4}\left(\frac{\cosh u_m(f,u,v,\varphi) - \cosh u}{\cosh u_m(f,u,v,\varphi)+\cosh u}\right)^2,$$
$$h_4 = \frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v}\left[\left(\sqrt{|h_5(f,u,v,\varphi)|}\right)\right]^2, \tag{8.14}$$

where $\sigma_m = \cosh u_m$ and the coordinates are $(x^i, y^a) = (f,u,v,\varphi,t)$.

4. The ellipsoidal Schwarzschild like black hole solutions with extra dimension and time running of constants are parametrized by the data $g_2 = g_3 = 1$ and

$$h_4 = -\frac{r_g^2}{16}\frac{\cosh^2 u}{(1+\cosh u)^4}\left(\frac{\cosh u_m(f,\rho,\theta,t) - \cosh u}{\cosh u_m(f,\rho,\theta,t)+\cosh u}\right)^2, h_5 = \frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v}, \tag{8.15}$$

where $\sigma_m = \cosh u_m$ and the coordinates are $(x^i, y^a) = (f,u,v,t,\varphi)$.

The N–coefficients w_i and n_i for the solutions (8.12)–(8.15) are computed respectively following formulas (8.10) and (2.88) (we omit the final expressions in this paper).

The mathematical form of the solutions (8.12) and (8.13), with constants $a = -2m/M_p^2$ and $b = Q$, is very similar to that of the Reissner-Nördstrom solution from RS gravity [13], but multiplied on a conformal factor $\left(1+\frac{r_g}{4\rho}\right)^{-4}\rho^{-2}$, with renormalized factors σ_m and σ_q and *without electric charge* being present. The induced 4D gravitational "receptivities" σ_m and σ_q in (8.12) emphasize dependencies on coordinates (f,ρ,θ,φ), where $s=\varphi$ is the anisotropic coordinate. In a similar fashion one induces running on time and the 5th coordinate, and anisotropic polarizations on ρ and θ, of constants for the solution (8.13).

Instead the Reissner-Nördstrom-type correction to the Schwarzschild potential the mentioned polarizations can be thought as defined by some nonlinear higher dimension gravitational interactions and anholonomic frame constraints for anisotropic Reissner-Nördstrom black hole configurations with a *'tidal charge' Q* arising from the projection onto the brane of free gravitational field effects in the bulk. These effects are transmitted via the bulk Weyl tensor, off–diagonal components of the metric and by anholonomic frames. The Schwarzschild potential $\Phi=-M/(M_{\rm p}^2 r)$, where $M_{\rm p}$ is the effective Planck mass on the brane, is modified to

$$\Phi=-\frac{M\sigma_m}{M_{\rm p}^2 r}+\frac{Q\sigma_q}{2r^2}\,, \tag{8.16}$$

where the 'tidal charge' parameter Q may be positive or negative. The possibility to modify anisotropically the Newton law via effective anisotropic masses $M\sigma_m$, or by anisotropic effective 4D Plank constants, renormalized like $\sigma_m/M_{\rm p}^2$, was recently emphasized in Ref. [7]. In this paper we state that there are possible additional renormalizations of the "effective" electric charge, $Q\sigma_q$. For diagonal metrics we put $\sigma_m=\sigma_q=1$ and by multiplication on corresponding conformal factors and with respect to holonomic frames we recover the locally isotropic results from Refs. [13]. We must also impose the condition that the 5D spacetime is modelled as a AdS_5 slice provided with an anholonomic frame structure.

The renormalized tidal charge $Q\sigma_q$ affects the geodesics and the gravitational potential, so that indirect limits may be placed on it by observations. Nevertheless, current observational limits on $|Q\sigma_q|$ are rather weak, since the correction term in Eq. (8.16) decreases off rapidly with increasing r, and astrophysical measurements (lensing and perihelion precession) probe mostly (weak-field) solar scales.

Now we analyze the properties of solutions (8.14) and (8.15). They describe Schwarzschild like solutions with the horizon forming a rotation ellipsoid horizon. For the general relativity such solutions were constructed in Refs. [8]. Here, it should be emphasized that static anisotropic deformations of the Schwarzschild metric are described by off–diagonal metrics and corresponding conformal transforms. At large radial distances from the horizon the anisotropic configurations transform into the usual one with spherical symmetry. That why the solutions with anisotropic rotation ellipsoidal horizons do not contradict the well known Israel and Carter theorems [1] which were proved in the assumption of spherical symmetry at asymptotic. Anisotropic 4D black hole solutions follow from the data (8.14) and (8.15) if you state some polarizations depending only on 3D space coordinates (u, v, φ), or on some of them. In this paper we show that in 5D there are warped to 4D static ellipsoidal like solutions with constants renormalized anisotropically on some 3D space coordinates and on extra dimension coordinate (in the class of solutions (8.14)) and running of constants on time and the 5th coordinate, with possible additional polarizations on some 3D coordinates (in the class of solutions (8.15)).

A geometric approach to the Randall-Sundrum scenario has been developed by Shiromizu et al. [15] (see also [16]), and proves to be a useful starting point for formulating the problem and seeing clear lines of approach. In this work we considered a variant of anholonomic RS geometry. The vacuum solutions (8.12)–(8.15) localized on the brane must satisfy the 5D equation in the Shiromizu et al. representation if in 4D some sources are considered as to be induced from extra dimension gravity.

The method of anholonomic frames covers the results on linear extensionss of the Schwarzschild horizon into the bulk [17]. The solutions presented in this paper are nonlinearly induced, are based on very general method of construction exact solutions in extra dimension gravity and generalize also the Reissner-Nördstrom solution from RS gravity. The obtained solutions are locally anisotopic but, nevertheless, they posses local 4D Lorentz symmetry, which is explicitly emphasized with respect to anholonomic frames. There are possible constructions with broken Lorentz symmetry as in [18] (if we impose not a locally isotropic limit of our solutions, but an anisotropic static one). We omit such considerations here.

In conclusion we formulate a prescription for mapping 4D general relativity solutions with diagonal metrics to 4D and 5D solutions of brane world: *a general relativity vacuum solution gives rise to a vacuum brane–world solution in 5D gravity given with similar coefficients of metrics but defined with respect to some anholonomic frames and with anisotropic renormalization of fundamental constants; such type of solutions are parametrized by off–diagonal metrics if of type (8.2) if they are re–defined with respect to coordinate frames* .

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Chapter 9

Off–Diagonal Metrics and Anisotropic Brane Inflation

Abstract [1]

We study anisotropic reheating (entropy production) on 3D brane from a decaying bulk scalar field in the brane–world picture of the Universe by considering off–diagonal metrics and anholonomic frames. We show that a significant amount of, in general, anisotropic dark radiation is produced in this process unless only the modes which satisfy a specific relation are excited. We conclude that subsequent entropy production within the brane is required in general before primordial nucleosynthesis and that the presence of off–diagonal components (like in the Salam, Strathee and Petracci works [1]) of the bulk metric modifies the processes of entropy production which could substantially change the brane–world picture of the Universe.

The brane world picture of the Universe [2] resulted in a number of works on brane world cosmology [3, 4] and inflationary solutions and scenaria [5, 6, 7, 8, 9]. Such solutions have been constructed by using diagonal cosmological metrics with respect to holonomic coordinate frames.

In Kaluza–Klein gravity there were also used off–diagonal five dimensional (in brief, 5D) metrics beginning Salam and Strathee and Perrachi works [1] which suggested to treat the off–diagonal components as some coefficients including $U(1), SU(2)$ and $SU(3)$ gauge fields. Recently, the off–diagonal metrics were considered in a new fashion both in Einstein and brane gravity [10, 11], by applying the method of anholonomic frames with associated nonlinear connection, which resulted in a new method of construction

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of exact solutions of Einstein equations describing, for instance, static black hole and cosmological solutions with ellipsoidal or torus symmetry, soliton–dilaton and wormhole–flux tube configurations with anisotropic polarizations and/or running of constants.

The aim of this paper is to investigate reheating after anisotropic inflation in the brane world with generic local anisotropy induced by off–diagonal metrics in the bulk [11]. In this scenario, our locally anisotropic Universe is described on a 4D boundary (3D anisotropic brane) of Z_2–symmetric 5D space–time with a gravitational vacuum polarization constant and its computed renormalized effective value. In the locally isotropic limit the constant of gravitational vacuum polarization results in a negative cosmological constant $\Lambda_5 \equiv -6k^2$, where k is a positive constant. Our approach is in the spirit of Horava–Witten theory [12, 13] and recovers the Einstein gravity around the brane with positive tension [2, 14, 15], the considerations being extended with respect to anholonomic frames.

Theories of gravity and/or high energy physics must satisfy a number of cosmological tests including cosmological inflation [16] which for brane models could directed by anisotropic renormalizations of parameters [11]. We shall develop a model of anisotropic inflation scenarios satisfying the next three requirements: 1) it is characterized by a sufficiently long quasi–exponential expansion driven by vacuum–like energy density of the potential energy of a scalar field; 2) the termination of accelerated anisotropic expansion is associated with an entropy production or reheating to satisfy the conditions for the initial state of the classical hot Big Bang cosmology, slightly anisotropically deformed, before the primordial nucleosynthesis [17] and 3) generation of primordial fluctuations with desired amplitude and spectrum [18].

We assume the 5D vacuum Einstein equations written with respect to anholonomic frames which for diagonal metrics with respect to holonomic frames contains a negative cosmological constant Λ_5 and a 3D brane at the 5th coordinate $w = 0$ about which the space–time is Z_2 symmetric and consider a quadratic line interval

$$\delta s^2 = \Omega^2(t,w,y)[dx^2 + g_2\,(t,w)\,dt^2 + g_3\,(t,w)\,dw^2 + h_4(t,w,y)\delta y^2 + h_5\,(t,w)\,dz^2], \quad (9.1)$$

where the 'elongated' differential $\delta y = dy + \zeta_2(t,w,y)dt + \zeta_3(t,w,y)dw$, together with dx, dt and dw define an anholonomic co–frame basis $(dx, dt, dw, \delta y,\ \delta z = dz)$ which is dual to the anholonomic frame basis [10, 11] $(\delta_1 = \frac{\partial}{\partial x}, \delta_2 = \frac{\partial}{\partial t} - \zeta_2 \frac{\partial}{\partial y}, \delta_3 = \frac{\partial}{\partial t} - \zeta_3 \frac{\partial}{\partial y}, \partial_4 = \frac{\partial}{\partial y}, \partial_5 = \frac{\partial}{\partial z})$; we denote the 4D space–time coordinates as (x,t,w,y,z) with t being the time like variable. The metric ansatz for the interval (9.1) is off–diagonal with respect to the usual coordinate basis (dx, dt, dw, dy, dz).

As a particular case we can parametrize from (9.1) the metric near an locally isotropic brane like a flat Robertson–Walker metric with the scale factor $a(t)$ [9] if we state the values

$$\Omega^2 = (aQ)^2, g_2 = -(aQ)^{-2}N^2, g_3 = (aQ)^{-2}, h_4 = 1, h_5 = 1, \zeta_{2,3} = 0,$$

for

$$\begin{aligned} N^2(t,w) &= Q^{-2}(t,w)[\cosh(2kw) + \frac{1}{2}k^{-2}\left(H^2 + \dot{H}\right)(\cosh(2kw) - 1) \\ &\quad - \frac{1 + \frac{1}{2}k^{-2}\left(2H^2 + \dot{H}\right)}{\sqrt{1 + k^{-2}H^2 + Ca^{-4}}}\sinh(2k|w|)] \\ Q^2(t,w) &= \cosh(2kw) + \frac{1}{2}k^{-2}H^2(\cosh(2kw) - 1) - \sqrt{1 + k^{-2}H^2 + Ca^{-4}}, \quad (9.2) \end{aligned}$$

when the bulk is in a vacuum state with a negative cosmological constant Λ_5, C is an integration constant [19]. One takes $N = Q = 1$ on the brane $w = 0$. The function $H(t)$ and constants k and C from (9.2) are related with the evolution equation on the brane in this case is given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa_5^4\sigma}{18}\rho_{\rm tot} + \frac{\Lambda_4}{3} + \frac{\kappa_5^4}{36}\rho_{\rm tot}^2 - \frac{k^2C}{a^4}, \ \Lambda_4 \equiv \frac{1}{2}\left(\Lambda_5 + \frac{\kappa_5^2}{6}\sigma^2\right), \quad (9.3)$$

where κ_5^2 is the 5D gravitational constant related with the 5D reduced Planck scale, M_5, by $\kappa_5^2 = M_5^{-3}$; σ is the brane tension, the total energy density on the brane is denoted by $\rho_{\rm tot}$, and the last term of (9.3) represents the dark radiation with C being an integration constant [15, 20, 19]. We recover the standard Friedmann equation with a vanishing cosmological constant at low energy scales if $\sigma = 6k/\kappa_5^2$ and $\kappa_4^2 = \kappa_5^4\sigma/6 = \kappa_5^2 k$, where κ_4^2 is the 4D gravitational constant related with the 4D reduced Planck scale, M_4, as $\kappa_4^2 = M_4^{-2}$. We find that $M_4^2 = M_5^3/k$. If we take $k = M_4$, all the fundamental scales in the theory take the same value, i. e. $k = M_4 = M_5$. The the scale above is stated by constant k which the nonstandard term quadratic in $\rho_{\rm tot}$ is effective in (9.3). One suppose [9] that k is much larger than the scale of inflation so that such quadratic corrections are negligible.

For simplicity, in locally isotropic cases one assumes that the bulk metric is governed by Λ_5 and neglect terms suppressed by k^{-1} and Ca^{-4} and writes

$$ds_5^2 = -e^{-2k|w|}dt^2 + e^{-2k|w|}a^2(t)\left(dx^2 + dy^2 + dz^2\right) + dw^2. \quad (9.4)$$

The conclusion of Refs [11] is that the presence of off–diagonal components in the bulk 5D metric results in locally anisotropic renormalizations of fundamental constants and modification of the Newton low on the brane. The purpose of this paper is to analyze the basic properties of models of anisotropic inflation on 3D brane with induced from the bulk local anisotropy of metrics of type (9.1) which define cosmological solutions of 5D vacuum Einstein equations depending on variables (w, t, y), being anisotropic on coordinate y (see details on construction of various classes of solutions by applying the method of moving anholonomic frames in Ref [10, 11]).

For simplicity, we shall develop a model of inflation on 3D brane with induced from the bulk local anisotropy by considering the ansatz

$$\delta s_5^2 = e^{-2(k|w|+k_y|y|)}a^2(t)[dx^2 - dt^2] + e^{-2k_y|y|}dw^2 + e^{-2(k|w|+k_y|y|)}a^2(t)(\delta y^2 + z^2) \quad (9.5)$$

which is a particular case of the metric (9.1) with $\Omega^2(w,t,y) = e^{-2(k|w|+k_y|y|)}a^2(t), g_2 = -1, g_3 = e^{2k|w|}, h_4 = 1, h_5 = 1$ and $\zeta_2 = k/k_y, \zeta_3 = (da/dt)/k_y a$ taken as the ansatz (9.5) would be an exact solution of 5D vacuum Einstein equations. The constants k and k_y have to be established experimentally. We emphasize that the metric (9.5) is induced alternatively on the brane from the 5D anholonomic gravitational vacuum with off–diagonal metrics. With respect to anholonomic frames it has some diagonal coefficients being similar to those from (9.4) but these metrics are very different in nature and describes two types of branes: the first one is with generic off–diagonal metrics and induced local anisotropy, the second one is locally isotropic defined by a brane configuration and the bulk cosmological constant. For anisotropic models, the respective constants can be treated as some 'receptivities' of the bulk gravitational vacuum polarization.

The next step is to investigate a scenarios of anisotropic inflation driven by a bulk scalar field ϕ with a 4D potential $V[\phi]$ [21, 22]. We shall study the evolution of ϕ after anisotropic brane inflation expecting that reheating is to proceed in the same way as in 4D theory with anholonomic modification (a similar idea is proposed in Ref. [23] but for locally isotropic branes). We suppose that the scalar field is homogenized in 3D space as a result of inflation, it depends only on t and w and anisotropically on y and consider a situation when ϕ rapidly oscillates around $\phi = 0$ by expressing $V[\phi] = m^2\phi^2/2$. The field $\phi(t, w, y)$ is non–homogeneous because of induced space–time anisotropy. Under such assumptions he Klein–Gordon equation in the background of metric (9.5) is written

$$\begin{aligned} \square_5\phi(f,t,y) - V'[\phi(f,t,y)] &= \frac{1}{\sqrt{|g|}}[\delta_t\left(\sqrt{|g|}g^{22}\delta_t\phi\right) + \delta_w\left(\sqrt{|g|}g^{33}\delta_f\phi\right) \\ &+\partial_y\left(\sqrt{|g|}h^{44}\partial_y\phi\right)] - V'[\phi] = 0, \end{aligned} \quad (9.6)$$

where $\delta_w = \frac{\partial}{\partial w} - \zeta_2\frac{\partial}{\partial y}, \delta_t = \frac{\partial}{\partial t} - \zeta_3\frac{\partial}{\partial y},$, $\square_5$ is the d'Alambert operator and $|g|$ is the determinant of the matrix of coefficients of metric given with respect to the anholonomic frame (in Ref. [9] the operator $\square_5$ is alternatively constructed by using the metric (9.4)).

The energy release of ϕ is modelled by introducing phenomenologically a dissipation terms defined by some constants Γ_D^w, Γ_D^y and Γ_B representing the energy release to the brane and to the entire space,

$$\Box_5\phi(w,t,y) - V'[\phi(w,t,y)] = \frac{\Gamma_D^w}{2k}\delta(w)\frac{1}{N}\delta_t\phi + \frac{\Gamma_D^y}{2k_y}\delta(y)\frac{1}{N}\delta_t\phi + \Gamma_B\frac{1}{N}\delta_t\phi. \qquad (9.7)$$

Following (9.6) and (9.7) together with the Z_2 symmetries on coordinates w and y, we have

$$\delta_w\phi^+ = -\delta_w\phi^- = \frac{\Gamma_D^w}{4k}\delta_t\phi(0,y,t), \partial_y\phi^+ = -\partial_y\phi^- = \frac{\Gamma_D^y}{4k_y}\delta_t\phi(w,0,t), \qquad (9.8)$$

where superscripts $+$ and $-$ imply values at $w, y \longrightarrow +0$ and -0, respectively. In this model we have two types of warping factors, on coordinates w and y. The constant k_y characterize the gravitational anisotropic polarization in the direction y.

Comparing the formulas (9.7) and (9.8) with similar ones from Ref. [9] we conclude the the induced from the bulk brane anisotropy could result in additional dissipation terms like that proportional to Γ_D^y. This modifies the divergence of divergence $T^{(\phi)C}_{\ \ A;C}$ of the energy–momentum tensor $T^{(\phi)}_{MN}$ of the scalar field ϕ: Taking

$$T^{(\phi)}_{MN} = \delta_M\phi\delta_N\phi - g_{MN}\left(\frac{1}{2}g^{PQ}\delta_P\phi\delta_Q\phi + V[\phi]\right),$$

with five dimensional indices, $M, N, ... = 1, 2, ..., 5$ and anholonomic partial derivative operators δ_P being dual to δ^P we compute

$$\begin{aligned} T^{(\phi)C}_{\ \ A;C} &= \{\Box_5\phi(w,t,y) - V'[\phi(w,t,y)]\}\,\phi_{,A} \\ &= \left[\frac{\Gamma_D^w}{2k}\delta(w)\frac{1}{N}\delta_t\phi + \frac{\Gamma_D^y}{2k_y}\delta(y)\frac{1}{N}\delta_t\phi + \Gamma_B\frac{1}{N}\delta_t\phi\right]\delta_A\phi. \end{aligned} \qquad (9.9)$$

We can integrate the $A = 0$ component of (9.9) from $w = -\epsilon$ to $w = +\epsilon$ near the brane, than we integrate from $y = -\epsilon_1$ to $y = +\epsilon_1$, in the zero order in ϵ and ϵ_1, we find from (9.8) that

$$\frac{\delta\rho_\phi(0,0,t)}{\partial t} = -(3H + \Gamma_B)(\delta_t\phi)^2(0,0,t) - J_\phi(0,0,t), \qquad (9.10)$$

with $\rho_\phi \equiv \frac{1}{2}(\delta_t\phi)^2 + V[\phi]$, $\quad J_\phi \equiv -\frac{\delta_t\phi}{\sqrt{|g|}}\delta_f\left(\sqrt{|g|}\delta_f\phi\right) - \frac{\delta_t\phi}{\sqrt{|g|}}\delta_y\left(\sqrt{|g|}\delta_y\phi\right)$, which states that the energy dissipated by the Γ_D^f and Γ_D^y terms on anisotropic brane is entirely compensated by the energy flows (locally isotropic and anisotropic) onto the brane. In this paper we shall model anisotropic inflation by considering that ϕ looks like homogeneous with respect to anholonomic frames; the local anisotropy and induced non–homogeneous effects are modelled by additional terms like Γ^y_D and elongated partial operators with a further integration on variable y.

Now we analyze how both the isotropic and anisotropic energy released from ϕ affects evolution of our brane Universe by analyzing gravitational field equations [15, 22] written with respect to anholonomic frames. We consider that the total energy–momentum tensor has a similar structure as in holonomic coordinates but with the some anholonomic variables, including the contribution of bulk cosmological constant,

$$T_{MN} = -\kappa_5^{-2}\Lambda_5 g_{MN} + T^{(\phi)}_{MN} + S_{MN}\delta(w),$$

where S_{MN} is the stress tensor on the brane and the capital Latin indices $M, N, ...$ run values $1, 2, ...5$ (we follow the denotations from [9] with that difference that the coordinates are reordered and stated with respect to anholonomic frames). One introduces a further decomposition as $S_{\mu\nu} = -\sigma q_{\mu\nu} + \tau_{\mu\nu}$, where $\tau_{\mu\nu}$ represents the energy–momentum tensor of the radiation fields produced by the decay of ϕ and it is of the form $\tau^\mu_\nu = \mathrm{diag}(2p_r, -\rho_r, p_r, 0)$ with $p_r = \rho_r/3$ which defines an anisotropic distribution of matter because of anholonomy of the frame of reference.

We can remove the considerations on an anisotropic brane (hypersurface) by using a unit vector $n_M = (0, 0, 1, 0, 0)$ normal to the brane for which the extrinsic curvature of a $w = const$ hypersurface is given by $K_{MN} = q^P_M q^Q_N n_{Q;P}$ with $q_{MN} = g_{MN} - n_M n_N$. Applying the Codazzi equation and the 5D Einstein equations with anholonomic variables [10, 11], we find

$$D_\nu K^\nu_\mu - D_\mu K = \kappa_5^2 T_{MN} n^N q^M_\mu = \kappa_5^2 T_{\mu w} = \kappa_5^2 (\delta_t\phi)(\delta_w\phi)\delta^2_\mu, \quad (9.11)$$

where δ^1_μ is the Kronecker symbol, small Greek indices parametrize coodinates on the brane, D_ν is the 4D covariant derivative with respect to the metric $q_{\mu\nu}$. The above equation reads

$$D_\nu K^{\nu+}_0 - D_0 K^+ = \kappa_5^2 \left[\frac{\Gamma_D}{4k}(\delta_t\phi)^2(0,t,0) + \frac{\Gamma_D}{4k_y}(\partial_y\phi)^2(0,t,0)\right], \quad (9.12)$$

near the brane $w \longrightarrow +0$ and neglecting non–homogeneous behavior, by putting $y = 0$. We have

$$D_\nu K^{\nu+}_\mu - D_\mu K^+ = -\frac{\kappa_5^2}{2} D_\nu S^\nu_\mu = -\frac{\kappa_5^2}{2} D_\nu \tau^\nu_\mu. \quad (9.13)$$

which follows from the junction condition and Z_2–symmetry with $K^{+}_{\mu\nu} = -\frac{\kappa_5^2}{2}\left(S_{\mu\nu} - \frac{1}{3}q_{\mu\nu}S\right)$. Using (9.12) and (9.13), we get

$$D_\nu \tau^\nu_\mu = -\frac{\Gamma_D}{2k}(\delta_t\phi)^2\delta^2_\mu - \frac{\Gamma^y_D}{2k_y}(\partial_y\phi)^2\delta^3_\mu,$$

i. e.,

$$\delta_t\rho_r = -3H(\rho_r + p_r) + \frac{\Gamma_D}{2k}(\delta_t\phi)^2 + \frac{\Gamma^y_D}{2k_y}(\partial_y\phi)^2 = -4H\rho_r + \frac{\Gamma_D}{2k}(\delta_t\phi)^2 + \frac{\Gamma^y_D}{2k_y}(\partial_y\phi)^2,$$

on the anisotropic brane. This equation describe the reheating in an anisotropic perturbation theory (for inflation in 4D theory see [24]).

The 4D Einstein equations with the Einstein tensor $G^{(4)\nu}_{\ \mu}$ were proven [22] to have the form

$$G^{(4)\nu}_{\ \mu} = \kappa_4^2\left(T^{(s)\nu}_{\ \mu} + \tau^\nu_\mu\right) + \kappa_5^4\pi^\nu_\mu - E^\nu_\mu,$$

with $T^{(s)\nu}_{\ \mu} \equiv \frac{1}{6k}\left[4q^{\nu\zeta}(\delta_\mu\phi)(\delta_\zeta\phi) + \left(\frac{3}{2}(\delta_\zeta\phi)^2 - \frac{5}{2}q^{\xi\zeta}(\delta_\xi\phi)(\delta_\zeta\phi) - \frac{3}{2}m^2\phi^2\right)q^\nu_\mu\right]$, where π^ν_μ contains terms quadratic in τ^β_α which are higher order in $\rho_r/(kM_4)^2$ and are consistently neglected in our analysis. $E^\nu_\mu \equiv C^{w\nu}_{\mu w}$ is a component of the 5D Weyl tensor C^{MN}_{PQ} treated as a source of dark radiation [19].

With respect to anholonomic frames the 4D Bianchi identities are written in the usual manner,

$$D_\nu G^{(4)\nu}_{\ \mu} = \kappa_4^2\left(D_\nu T^{(s)\nu}_{\ \mu} + D_\nu\tau^\nu_\mu\right) - D_\nu E^\nu_\mu = 0, \qquad (9.14)$$

with that difference that $D_\nu G^{(4)\nu}_{\ \mu} = 0$ only for holonomic frames but in the anholonomic cases, for general constraints one could be $D_\nu G^{(4)\nu}_{\ \mu} \neq 0$ [10, 11]. In this paper we shall consider such constraints for which the equalities (9.14) hold which yield

$$\begin{aligned} D_\nu E^\nu_2 &= -\frac{\kappa_4^2}{4k}\frac{\delta}{\partial t}\left[(\delta_t\phi)^2 - (\delta_w\phi)^2 - (\partial_y\phi)^2 + m^2\phi^2\right] \\ &\quad -\frac{2\kappa_4^2 H}{k}(\delta_t\phi)^2 - \frac{\kappa_4^2}{2k}\Gamma_D(\delta_t\phi)^2 - \frac{\kappa_4^2}{2k_y}\Gamma^y_D(\partial_y\phi)^2. \end{aligned}$$

Putting on the anisotropic brane $(\delta_w\phi)^2 = \Gamma^2_D(\delta_t\phi)^2/(16k^2)$ and $(\partial_y\phi)^2 = (\Gamma^y_D)^2(\delta_t\phi)^2/(16k^2_y)$ similarly to Ref. [9], by substituting usual partial derivatives into 'elongated' ones and introducing $\varphi(t) \equiv \phi(0, t.y)/\sqrt{2k}$, $b = a(t)e^{-k_y|y|}$ and $\varepsilon \equiv E^2_2/\kappa^2_4$ we prove the evolution equations in the brane universe $w = 0$

$$
\begin{aligned}
H^2 &= \left(\frac{\delta_t b}{b}\right)^2 = \frac{\kappa_4^2}{3}\left(\rho_\varphi + \rho_r + \varepsilon\right), \quad \rho_\varphi \equiv \frac{1}{2}(\delta_t\varphi)^2 + \frac{1}{2}m^2\varphi^2 = \frac{\rho_\phi}{2k}, \\
\delta_t\rho_\varphi &= -(3H + \Gamma_B)(\delta_t\varphi)^2 - J_\varphi, \quad J_\varphi \equiv \frac{J_\phi}{2k}, \\
\delta_t\rho_r &= -4H\rho_r + \Gamma_D(\delta_t\varphi)^2 + \Gamma_D^y(\delta_t\varphi)^2, \\
\delta_t\varepsilon &= -4H\varepsilon - (H + \Gamma_D + \Gamma_D^y - \Gamma_B)(\delta_t\varphi)^2 + J_\varphi.
\end{aligned}
$$

We find the solution of (9.6) in the background (9.5) in the way suggested by [25, 9] by introducing an additional factor depending on anisotropic variable y,

$$
\phi(t, w) = \sum_{n+n'} c_{n+n'} T_{n+n'}(t) Y_n(w) Y_{n'}(y) + H.C.
$$

with

$$
\begin{aligned}
T_{n+n'}(t) &\cong a^{-\frac{3}{2}}(t) e^{-i(m_n + m_{n'})t}, \\
Y_n(w) &= e^{2k|w|}\left[J_\nu\left(\frac{m_n}{k}e^{k|w|}\right) + b_n N_\nu\left(\frac{m_n}{k}e^{k|w|}\right)\right], \\
Y_{n'}(y) &= e^{2k_y|y|}\left[J_\nu\left(\frac{m_{n'}}{k}e^{k_y|y|}\right) + b_{n'} N_\nu\left(\frac{m_n}{k}e^{k_y|y|}\right)\right],
\end{aligned}
$$

for $\nu = 2\sqrt{1 + \frac{m^2}{4k^2}} \cong 2 + \frac{m^2}{4k^2}$, and considering that the field oscillates rapidly in cosmic expansion time scale. The values m_n and $m_{n'}$ are some constants which may take continuous values in the case of a single brane and b_n and $b_{n'}$ are some constants determined by the boundary conditions, $\delta_w\phi = 0$ at $w = 0$ and $\partial_y\phi = 0$ at $y = 0$. We write

$$
\begin{aligned}
b_n &= \left[2J_\nu\left(\frac{m_n}{k}\right) + \frac{m_n}{k}J'_\nu\left(\frac{m_n}{k}\right)\right]\left[2N_\nu\left(\frac{m_n}{k}\right) + \frac{m_n}{k}N'_\nu\left(\frac{m_n}{k}\right)\right]^{-1}, \\
b_{n'} &= \left[2J_\nu\left(\frac{m_{n'}}{k_y}\right) + \frac{m_{n'}}{k_y}J'_\nu\left(\frac{m_{n'}}{k_y}\right)\right]\left[2N_\nu\left(\frac{m_{n'}}{k_y}\right) + \frac{m_{n'}}{k_y}N'_\nu\left(\frac{m_{n'}}{k_y}\right)\right]^{-1}.
\end{aligned}
$$

The effect of dissipation on the boundary conditions is given by

$$
\begin{aligned}
b_n &\cong \left[\left(2+\frac{im_n\Gamma_D}{2k^2}\right)J_\nu\left(\frac{m_n}{k}\right)+\frac{m_n}{k}J'_\nu\left(\frac{m_n}{k}\right)\right]\times \\
&\quad \left[\left(2+\frac{im_n\Gamma_D}{2k^2}\right)N_\nu\left(\frac{m_n}{k}\right)+\frac{m_n}{k}N'_\nu\left(\frac{m_n}{k}\right)\right]^{-1}, \\
b_{n'} &\cong \left[\left(2+\frac{im_{n'}\Gamma_D^y}{2k_y^2}\right)J_\nu\left(\frac{m_{n'}}{k_y}\right)+\frac{m_{n'}}{k_y}J'_\nu\left(\frac{m_{n'}}{k_y}\right)\right]\times \\
&\quad \left[\left(2+\frac{im_{n'}\Gamma_D^y}{2k_y^2}\right)N_\nu\left(\frac{m_{n'}}{k_y}\right)+\frac{m_{n'}}{k_y}N'_\nu\left(\frac{m_{n'}}{k_y}\right)\right]^{-1},
\end{aligned}
$$

where use has been made of $\partial_t \dot{T}_{n+n'}(t) \cong -im_{n+n'}T_{n+n'}(t)$.

For simplicity, let analyze the case when a single oscillation mode exists, neglect explicit dependence of φ on variable y (the effect of anisotropy being modelled by terms like $m_{n'}$, k_y and $\Gamma_D^y$and compare our results with those for isotropic inflation [9]. In this case we approximate $\delta\varphi \simeq \dot{\varphi}$, where dot is used for the partial derivative ∂_t. We find

$$
J_\varphi = (m_n^2 + m_{n'}^2 - m^2)\varphi\dot{\varphi}. \tag{9.15}
$$

The evolution of the dark radiation is approximated in the regime when $\varphi(t)$ oscillates rapidly, parametrized as

$$
\varphi(t) = \varphi_i\left(\frac{a(t)}{a(t_i)}\right)^{-3/2} e^{-i\lambda_{n+n'}(t-t_i)},\ \lambda_{n+n'} \equiv m_n + m_{n'} - \frac{i}{2}\Gamma_B,
$$

with $m_n + m_{n'} \gg H$ and Γ_B being positive constants which assumes that only a single oscillation mode exists.

Then the evolution equation of $\varepsilon(t) \equiv \kappa_4^{-2}E_2^2$ is given by

$$
\frac{\partial\varepsilon}{\partial t} = -4H\varepsilon - (\Gamma_D + \Gamma_D^y + H - \Gamma_B)\dot{\varphi}^2 - (m^2 - m_n^2 - m_{n'}^2)\varphi\dot{\varphi}.
$$

The next approximation is to consider φ as oscillating rapidly in the expansion time scale by averaging the right-hand-side of evolution equations over an oscillation period. Using $\overline{\dot{\varphi}^2}(t) = (m_n^2 + m_{n'}^2)\overline{\varphi^2}(t)$ and $\overline{\varphi\dot{\varphi}}(t) = -(3H+\Gamma_B)\overline{\varphi^2}(t)/2$, we obtain the following set of evolution equations in the anisotropic brane universe $w=0$, for small non–homogeneities on y, where the bar denotes average over the oscillation period.

$$
\begin{aligned}
H^2 &= \left(\frac{\dot{a}}{a}\right)^2 \cong \frac{\kappa_4^2}{3}\left(\rho_\varphi + \rho_r + \varepsilon\right), \\
\frac{\partial \rho_\varphi}{\partial t} &= -\frac{1}{2}(3H + \Gamma_B)(m^2 + m_n^2 + m_{n'}^2)\overline{\varphi^2}, \\
\frac{\partial \rho_r}{\partial t} &= -4H\rho_r + \left(\Gamma_D m_n^2 + \Gamma_D^y m_{n'}^2\right)\overline{\varphi^2}, \qquad (9.16) \\
\frac{\partial \varepsilon}{\partial t} &= -4H\varepsilon - (\Gamma_D + H - \Gamma_B)m_n^2\overline{\varphi^2} - (\Gamma_D^y + H - \Gamma_B)m_{n'}^2\overline{\varphi^2} \\
&\quad + \frac{1}{2}(3H + \Gamma_B)(m^2 - m_n^2 - m_{n'}^2)\overline{\varphi^2}, \qquad (9.17)
\end{aligned}
$$

with $\overline{\varphi^2}(t) \equiv \overline{\varphi_i^2}\left(\frac{a(t)}{a(t_i)}\right)^{-3} e^{-\Gamma_B(t-t_i)}$.

The system (9.16) and (9.17) was analyzed in Ref. [9] for the case when $m_{n'}$ and Γ_D^y vanishes:

It was concluded that if $m_n \geq m$, we do not recover standard cosmology on the brane after inflation. The same holds true in the presence of anisotropic terms.

In the locally isotropic case it was proven that if $m_n \ll n$ the last term of (9.17) is dominant and we find more dark radiation than ordinary radiation unless Γ_B is extremely small with $\Gamma_B/\Gamma_D < m_n^2/m^2 \ll 1$. The presence of anisotropic values $m_{n'}$ and Γ_D^y can violate this condition.

The cases $m_n, m_{n'} \lesssim m$ are the most delicate cases because the final amount of dark radiation can be either positive or negative depending on the details of the model parameters and the type of anisotropy. The amount of extra radiation–like matter have to be hardly constrained [17] if we wont a successful primordial nucleosynthesis. In order to have sufficiently small ε compared with ρ_r after reheating without resorting to subsequent both isotropic and anisotropic entropy production within the brane, the magnitude of creation terms of ε should be vanishingly small at the reheating epoch $H \simeq \Gamma_B$. This hold if only there are presented isotropic and/or anisotropic modes which satisfies the inequality

$$
\left|2\Gamma_B(m^2 - m_n^2 - m_{n'}^2) - \Gamma_D m_n^2 - \Gamma_D^y m_{n'}^2\right| \ll \Gamma_D m_n^2 + \Gamma_D^y m_{n'}^2. \qquad (9.18)
$$

We conclude that the relation (9.18) should be satisfied for the graceful exit of anisotropic brane inflation driven by a bulk scalar field ϕ. The presence of off–diagonal components of the metric in the bulk which induces brane anisotropies could modify the process of nucleosynthesis.

In summary we have analyzed a model of anisotropic inflation generated by a 5D off–diagonal metric in the bulk. We studied entropy production on the anisotropic 3D brane from a decaying bulk scalar field ϕ by considering anholonomic frames and introducing dissipation terms to its equation of motion phenomenologically. We illustrated that the dark radiation is significantly produced at the same time unless the inequality (9.18) is satisfied. Comparing our results with a similar model in locally isotropic background we found that off–diagonal metric components and anisotropy results in additional dissipation terms and coefficients which could substantially modify the scenario of inflation but could not to fall the qualitative isotropic possibilities for well defined cases with specific form of isotropic and anisotropic dissipation. Although we have analyzed only a case of anisotropic metric with a specific form of the dissipation, we expect our conclusion is generic and applicable to other forms of anisotropies and dissipation, because it is essentially an outcome of the anholonomic frame method and Bianchi identities (9.14). We therefore conclude that in the brane–world picture of the Universe it is very important what type of metrics and frames we consider, respectively, diagonal or off–diagonal and holonomic or anholonomic (i. e. locally isotropic or anisotropic). In all cases there are conditions to be imposed on anisotropic parameters and polarizations when the dominant part of the entropy we observe experimentally originates within the brane rather than in the locally anisotropic bulk. Such extra dimensional vacuum gravitational anisotropic polarizations of cosmological inflation parameters may be observed experimentally.

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Chapter 10

A New Method of Constructing Black Hole Solutions in Einstein and 5D Gravity

Abstract [1]

It is formulated a new 'anholonomic frame' method of constructing exact solutions of Einstein equations with off–diagonal metrics in 4D and 5D gravity. The previous approaches and results [1, 2, 3, 4] are summarized and generalized as three theorems which state the conditions when two types of ansatz result in integrable gravitational field equations. There are constructed and analyzed different classes of anisotropic and/or warped vacuum 5D and 4D metrics describing ellipsoidal black holes with static anisotropic horizons and possible anisotropic gravitational polarizations and/or running constants. We conclude that warped metrics can be defined in 5D vacuum gravity without postulating any brane configurations with specific energy momentum tensors. Finally, the 5D and 4D anisotropic Einstein spaces with cosmological constant are investigated.

10.1 Introduction

During the last three years large extra dimensions and brane worlds attract a lot of attention as possible new paradigms for gravity, particle physics and string/M–theory. As basic references there are considered Refs. [5], for string gravity papers, the Refs. [6], for extra dimension particle fields, and gravity phenomenology with effective Plank scale and [7], for the simplest and comprehensive models proposed by Randall and Sundrum (in brief, RS; one could also find in the same line some early works [8] as well to cite, for instance, [9] for further developments with supersymmetry, black hole solutions and cosmological scenaria).

The new ideas are based on the assumption that our Universe is realized as a three dimensional (in brief, 3D) brane, modelling a 4D pseudo–Riemannian spacetime, embedded in the 5D anti–de Sitter (AdS_5) bulk spacetime. It was proved in the RS papers [7] that in such models the extra dimensions could be not compactified (being even infinite) if a nontrivial warped geometric configuration is defined. Some warped factors are essential for solving the mass hierarchy problem and localization of gravity which at low energies can "bound" the matter fields on a 3D subspace. In general, the gravity may propagate in extra dimensions.

In connection to modern string and brane gravity it is very important to develop new methods of constructing exact solutions of gravitational field equations in the bulk of extra dimension spacetime and to develop new applications in particle physics, astrophysics and cosmology. This paper is devoted to elaboration of a such method and investigation of new classes of anisotropic black hole solutions.

In higher dimensional gravity much attention has been paid to the off–diagonal metrics beginning the Salam, Strathee and Peracci works [10] which showed that including off–diagonal components in higher dimensional metrics is equivalent to including $U(1), SU(2)$ and $SU(3)$ gauge fields. They considered a parametrization of metrics of type

$$g_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right] \tag{10.1}$$

where the Greek indices run values $1, 2, ..., n+m$, the Latin indices $i, j, k, ...$ from the middle of the alphabet run values $1, 2, ..., n$ (usually, in Kaluza–Klein theories one put $n = 4$) and the Latin indices from the beginning of the alphabet, $a, b, c, ...$, run values $n+1, n+2, ..., n+m$ taken for extra dimensions. The local coordinates on higher dimensional spacetime are denoted $u^\alpha = (x^i, y^a)$ which defines respectively the local coordinate frame (basis), co–frame (co–basis, or dual basis)

$$\partial_\alpha = \frac{\partial}{\partial u^\alpha} = \left(\partial_i = \frac{\partial}{\partial x^i}, \partial_a = \frac{\partial}{\partial y^a} \right), \tag{10.2}$$

$$d^\alpha = du^\alpha = \left(d^i = dx^i, d^a = dy^a \right). \tag{10.3}$$

The coefficients $g_{ij} = g_{ij}(u^\alpha), h_{ab} = h_{ab}(u^\alpha)$ and $N_i^a = N_i^a(u^\alpha)$ should be defined by a solution of the Einstein equations (in some models of Kaluza–Klein gravity [11] one considers the Einstein–Yang–Mills fields) for extra dimension gravity.

The metric (10.1) can be rewritten in a block $(n \times n) \oplus (m \times m)$ form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ab} \end{pmatrix} \quad (10.4)$$

with respect to some anholonomic frames (N–elongated basis), co–frame (N–elongated co–basis),

$$\delta_\alpha = \frac{\delta}{\partial u^\alpha} = \left(\delta_i = \partial_i - N_i^b \partial_b, \delta_a = \partial_a\right), \quad (10.5)$$

$$\delta^\alpha = \delta u^\alpha = \left(\delta^i = d^i = dx^i, \delta^a = dy^a + N_i^a dx^i\right), \quad (10.6)$$

which satisfy the anholonomy relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w_{\alpha\beta}^\gamma \delta_\gamma$$

with the anholonomy coefficients computed as

$$w_{ij}^k = 0, w_{aj}^k = 0, w_{ab}^k = 0, w_{ab}^c = 0, w_{ij}^a = \delta_i N_j^a - \delta_j N_i^a, w_{ja}^b = -w_{aj}^b = \partial_a N_j^b. \quad (10.7)$$

In Refs. [10] the coefficients N_i^a (hereafter, N–coefficients) were treated as some $U(1)$, $SU(2)$ or $SU(3)$ gauge fields (depending on the extra dimension m). There are another classes of gravity models which are constructed on vector (or tangent) bundles generalizing the Finsler geometry [12]. In such approaches the set of functions N_i^a were stated to define a structure of nonlinear connection and the variables y^a were taken to parametrize fibers in some bundles. In the theory of locally anisotropic (super) strings and supergravity, and gauge generalizations of the so–called Finsler–Kaluza–Klein gravity the coefficients N_i^a were suggested to be found from some alternative string models in low energy limits or from gauge and spinor variants of gravitational field equations with anholonomic frames and generic local anisotropy [3].

The Salam, Strathee and Peracci [10] idea on a gauge field like status of the coefficients of off–diagonal metrics in extra dimension gravity was developed in a new fashion by applying the method of anholonomic frames with associated nonlinear connections just on the (pseudo) Riemannian spaces [1, 2]. The approach allowed to construct new classes of solutions of Einstein's equations in three (3D), four (4D) and five (5D) dimensions with generic local anisotropy (*e.g.* static black hole and cosmological solutions with ellipsoidal or toroidal symmetry, various soliton–dilaton 2D and 3D configurations in 4D gravity, and wormhole and flux tubes with anisotropic polarizations and/or running on the 5th coordinate constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy).

Recently, it was shown in Refs. [4] that if we consider off–diagonal metrics which can be equivalently diagonalized with respect to corresponding anholonomic frames, the RS theories become substantially locally anisotropic with variations of constants on extra dimension coordinate or with anisotropic angular polarizations of effective 4D constants, induced by higher dimension and/or anholonomic gravitational interactions.

The basic idea on the application of the anholonomic frame method for constructing exact solutions of the Einstein equations is to define such N–coefficients when a given type of off–diagonal metric is diagonalized with respect to some anholonomic frames (10.5) and the Einstein equations, re–written in mixed holonomic and anholonomic variables, transform into a system of partial differential equations with separation of variables which admit exact solutions. This approach differs from the usual tetradic method where the differential forms and frame bases are all 'pure' holonomic or 'pure" anholonomic. In our case the N–coefficients and associated N–elongated partial derivatives (10.5) are chosen as to be some undefined values which at the final step are fixed as to separate variables and satisfy the Einstein equations.

The first aim of this paper is to formulate three theorems (and to suggest the way of their proof) for two off–diagonal metric ansatz which admit anholonomic transforms resulting in a substantial simplification of the system of Einstein equations in 5D and 4D gravity. The second aim is to consider four applications of the anholonomic frame method in order to construct new classes of exact solutions describing ellipsoidal black holes with anisotropies and running of constants. We emphasize that is possible to define classes of warped on the extra dimension coordinate metrics which are exact solutions of 5D vacuum gravity. We analyze basic physical properties of such solutions. We also investigate 5D spacetimes with anisotropy and cosmological constants.

We use the term 'locally anisotropic' spacetime (or 'anisotropic' spacetime) for a 5D (4D) pseudo-Riemannian spacetime provided with an anholonomic frame structure with mixed holonomic and anholonomic variables. The anisotropy of gravitational interactions is modelled by off–diagonal metrics, or, equivalently, by theirs diagonalized analogs given with respect to anholonomic frames.

The paper is organized as follow: In Sec. II we formulate three theorems for two types of off–diagonal metric ansatz, construct the corresponding exact solutions of 5D vacuum Einstein equations and illustrate the possibility of extension by introducing matter fields (the necessary geometric background and some proofs are presented in the Appendix). We also consider the conditions when the method generates 4D metrics. In Sec. III we construct two classes of 5D anisotropic black hole solutions with rotation ellipsoid horizon and consider subclasses and reparemetization of such solutions in order to generate new ones. Sec. IV is devoted to 4D ellipsoidal black hole solutions. In Sec. V we extend the method for anisotropic 5D and 4D spacetimes with cosmological constant, formulate two theorems on basic properties of the system of field equations and theirs solutions, and give an example of 5D anisotropic black solution with cosmological constant. Finally, in Sec. VI, we conclude and discuss the obtained results.

10.2 Off–Diagonal Metrics in Extra Dimension Gravity

The bulk of solutions of 5D Einstein equations and their reductions to 4D (like the Schwarzshild solution and brane generalizations [13], metrics with cylindrical and toroidal symmetry [14], the Friedman–Robertson–Walker metric and brane generalizations [15]) were constructed by using diagonal metrics and extensions to solutions with rotation, all given with respect to holonomic coordinate frames of references. This Section is devoted to a geometrical and nonlinear partial derivation equations formalism which deals with more general, generic off–diagonal metrics with respect to coordinate frames, and anholonomic frames. It summarizes and generalizes various particular cases and ansatz used for construction of exact solutions of the Einstein gravitational field equations in 3D, 4D and 5D gravity [1, 2, 4].

10.2.1 The first ansatz for vacuum Einstein equations

Let us consider a 5D pseudo–Riemannian spacetime provided with local coordinates $u^\alpha = (x^i, y^4 = v, y^5)$, for $i = 1, 2, 3$. Our aim is to prove that a metric ansatz of type (10.1) can be diagonalized by some anholonomic transforms with the N–coefficients $N^i_a = N^i_a(x^i, v)$ depending on variables (x^i, v) and to define the corresponding system of vacuum Einstein equations in the bulk. The exact solutions of the Einstein equations to be constructed will depend on the so–called holonomic variables x^i and on one anholonomic (equivalently, anisotropic) variable $y^4 = v$. In our further considerations every coordinate from a set u^α can be stated to be time like, 3D space like or extra dimensional.

For simplicity, the partial derivatives will be denoted like $a^\times = \partial a/\partial x^1, a^\bullet = \partial a/\partial x^2$, $a' = \partial a/\partial x^3, a^* = \partial a/\partial v$.

We begin our approach by considering a 5D quadratic line element

$$ds^2 = g_{\alpha\beta}\left(x^i, v\right) du^\alpha du^\beta \tag{10.8}$$

with the metric coefficients $g_{\alpha\beta}$ parametrized (with respect to the coordinate frame (10.3)) by an off–diagonal matrix (ansatz)

$$\begin{bmatrix} g_1 + {w_1}^2 h_4 + {n_1}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix}, \quad (10.9)$$

where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g_1 &= \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \\ w_i &= w_i(x^i, v), n_i = n_i(x^i, v). \end{aligned}$$

Lemma 10.2.1. *The quadratic line element (10.8) with metric coefficients (10.9) can be diagonalized,*

$$\delta s^2 = [g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2], \quad (10.10)$$

with respect to the anholonomic co–frame $(dx^i, \delta v, \delta y^5)$, *where*

$$\delta v = dv + w_i dx^i \text{ and } \delta y^5 = dy^5 + n_i dx^i \quad (10.11)$$

which is dual to the frame $(\delta_i, \partial_4, \partial_5)$, *where*

$$\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5. \quad (10.12)$$

In the Lemma 1 the N–coefficients from (10.5) and (10.6) are parametrized like $N_i^4 = w_i$ and $N_i^5 = n_i$.

The proof of the Lemma 1 is a trivial computation if we substitute the values of (10.11) into the quadratic line element (10.10). Re-writing the metric coefficients with respect to the coordinate basis (10.3) we obtain just the quadratic line element (10.8) with the ansatz (10.9).

In the Appendix A we outline the basic formulas from the geometry of anholonomic frames with mixed holonomic and anholonomic variables and associated nonlinear connections on (pseudo) Riemannian spaces.

Now we can formulate the

Theorem 10.2.1. *The nontrivial components of the 5D vacuum Einstein equations,* $R_\alpha^\beta = 0$, *(see (10.103) in the Appendix) for the metric (10.10) given with respect to an-*

holonomic frames (10.11) and (10.12) are written in a form with separation of variables:

$$R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = 0, \quad (10.13)$$

$$S_4^4 = S_5^5 = -\frac{\beta}{2h_4h_5} = 0, \quad (10.14)$$

$$R_{4i} = -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \quad (10.15)$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right] = 0, \quad (10.16)$$

where

$$\alpha_i = \partial_i h_5^* - h_5^*\partial_i \ln\sqrt{|h_4h_5|}, \beta = h_5^{**} - h_5^*[\ln\sqrt{|h_4h_5|}]^*, \gamma = 3h_5^*/2h_5 - h_4^*/h_4. \quad (10.17)$$

Here the separation of variables means: 1) we can define a function $g_2(x^2,x^3)$ for a given $g_3(x^2,x^3)$, or inversely, to define a function $g_2(x^2,x^3)$ for a given $g_3(x^2,x^3)$, from equation (10.13); 2) we can define a function $h_4(x^1,x^2,x^3,v)$ for a given $h_5(x^1,x^2,x^3,v)$, or inversely, to define a function $h_5(x^1,x^2,x^3,v)$ for a given $h_4(x^1,x^2,x^3,v)$, from equation (10.14); 3-4) having the values of h_4 and h_5, we can compute the coefficients (10.17) which allow to solve the algebraic equations (10.15) and to integrate two times on v the equations (10.16) which allow to find respectively the coefficients $w_i(x^k,v)$ and $n_i(x^k,v)$.

The proof of Theorem 1 is a straightforward tensorial and differential calculus for the components of Ricci tensor (5.14) as it is outlined in the Appendix A. We omit such cumbersome calculations in this paper.

10.2.2 The second ansatz for vacuum Einstein equations

We can consider a generalization of the constructions from the previous subsection by introducing a conformal factor $\Omega(x^i,v)$ and additional deformations of the metric via coefficients $\zeta_{\hat{i}}(x^i,v)$ (indices with 'hat' take values like $\hat{i} = 1,2,3,5$). The new metric is written like

$$ds^2 = \Omega^2(x^i,v)\hat{g}_{\alpha\beta}\left(x^i,v\right)du^\alpha du^\beta, \quad (10.18)$$

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz

$$\begin{bmatrix} g_1 + (w_1{}^2 + \zeta_1{}^2)h_4 + n_1{}^2h_5 & (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_1 + \zeta_1)h_4 & n_1h_5 \\ (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & g_2 + (w_2{}^2 + \zeta_2{}^2)h_4 + n_2{}^2h_5 & (w_2w_3 + \zeta_2\zeta_3)h_4 + n_2n_3h_5 & (w_2 + \zeta_2)h_4 & n_2h_5 \\ (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_2w_3 + \zeta_2\zeta_3)h_4 + n_2n_3h_5 & g_3 + (w_3{}^2 + \zeta_3{}^2)h_4 + n_3{}^2h_5 & (w_3 + \zeta_3)h_4 & n_3h_5 \\ (w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & h_4 & 0 \\ n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5 + \zeta_5h_4 \end{bmatrix} \quad (10.19)$$

Such 5D pseudo–Riemannian metrics are considered to have second order anisotropy [3, 12]. For trivial values $\Omega = 1$ and $\zeta_{\hat{i}} = 0$, the squared line interval (10.18) transforms into (10.8).

Lemma 10.2.2. *The quadratic line element (10.18) with metric coefficients (10.19) can be diagonalized,*

$$\delta s^2 = \Omega^2(x^i, v)[g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\delta y^5)^2], \qquad (10.20)$$

with respect to the anholonomic co–frame $\left(dx^i, \hat{\delta} v, \delta y^5\right)$, *where*

$$\delta v = dv + (w_i + \zeta_i)dx^i + \zeta_5 \delta y^5 \text{ and } \delta y^5 = dy^5 + n_i dx^i \qquad (10.21)$$

which is dual to the frame $\left(\hat{\delta}_i, \partial_4, \hat{\partial}_5\right)$, *where*

$$\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i \partial_5, \hat{\partial}_5 = \partial_5 - \zeta_5 \partial_4. \qquad (10.22)$$

In the Lemma 2 the N–coefficients from (10.2) and (10.5) are parametrized in the first order anisotropy (with three anholonomic, x^i, and two anholonomic, y^4 and y^5, coordinates) like $N_i^4 = w_i$ and $N_i^5 = n_i$ and in the second order anisotropy (on the second 'shell', with four anholonomic, (x^i, y^5), and one anholonomic,y^4, coordinates) with $N_{\hat{i}}^5 = \zeta_{\hat{i}}$, in this work we state, for simplicity, $\zeta_{\hat{i}} = 0$.

The Theorem 1 can be extended as to include the generalization to the second ansatz:

Theorem 10.2.2. *The nontrivial components of the 5D vacuum Einstein equations,* $R_\alpha^\beta = 0$, *(see (10.103) in the Appendix) for the metric (10.20) given with respect to anholonomic frames (10.21) and (10.22) are written in the same form as in the system (10.13)–(10.16) with the additional conditions that*

$$\hat{\delta}_i h_4 = 0 \text{ and } \hat{\delta}_i \Omega = 0 \qquad (10.23)$$

and the values $\zeta_{\hat{i}} = (\zeta_i, \zeta_5 = 0)$ *are found as to be a unique solution of (10.23); for instance, if*

$$\Omega^{q_1/q_2} = h_4 \text{ } (q_1 \text{ and } q_2 \text{ are integers}), \qquad (10.24)$$

ζ_i *satisfy the equations*

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0. \qquad (10.25)$$

The proof of Theorem 2 consists from a straightforward calculation of the components of the Ricci tensor (5.14) as it is outlined in the Appendix. The simplest way is to use the calculus for Theorem 1 and then to compute deformations of the canonical d–connection (5.11). Such deformations induce corresponding deformations of the Ricci tensor (5.14). The condition that we have the same values of the Ricci tensor for the (10.9) and (10.19) results in equations (10.23) and (10.25) which are compatible, for instance, if $\Omega^{q_1/q_2} = h_4$. There are also another possibilities to satisfy the condition (10.23), for instance, if $\Omega = \Omega_1$ Ω_2, we can consider that $h_4 = \Omega_1^{q_1/q_2}\ \Omega_2^{q_3/q_4}$ for some integers q_1, q_2, q_3 and q_4.

10.2.3 General solutions

The surprising result is that we are able to construct exact solutions of the 5D vacuum Einstein equations for both types of the ansatz (10.9) and (10.19):

Theorem 10.2.3. *The system of second order nonlinear partial differential equations (10.13)–(10.16) and (10.25) can be solved in general form if there are given some values of functions $g_2(x^2,x^3)$ (or $g_3(x^2,x^3)$), $h_4(x^i,v)$ (or $h_5(x^i,v)$) and $\Omega(x^i,v)$:*

- *The general solution of equation (10.13) can be written in the form*
$$\varpi = g_{[0]} \exp[a_2 \widetilde{x}^2 \left(x^2,x^3\right) + a_3 \widetilde{x}^3 \left(x^2,x^3\right)], \tag{10.26}$$
were $g_{[0]}, a_2$ and a_3 are some constants and the functions $\widetilde{x}^{2,3}\left(x^2,x^3\right)$ define coordinate transforms $x^{2,3} \rightarrow \widetilde{x}^{2,3}$ for which the 2D line element becomes conformally flat, i. e.
$$g_2(x^2,x^3)(dx^2)^2 + g_3(x^2,x^3)(dx^3)^2 \rightarrow \varpi\left[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\right]. \tag{10.27}$$

- *The equation (10.14) relates two functions $h_4(x^i,v)$ and $h_5(x^i,v)$. There are two possibilities:*

  a) to compute
$$\begin{aligned}\sqrt{|h_5|} &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right)\int\sqrt{|h_4(x^i,v)|}dv,\ \ h_4^*\left(x^i,v\right)\neq 0;\\ &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right)v, h_4^*\left(x^i,v\right) = 0,\end{aligned}\tag{10.28}$$
  for some functions $h_{5[1,2]}\left(x^i\right)$ stated by boundary conditions;

  b) or, inversely, to compute h_4 for a given $h_5(x^i,v)$, $h_5^*\neq 0$,
$$\sqrt{|h_4|} = h_{[0]}\left(x^i\right)(\sqrt{|h_5(x^i,v)|})^*, \tag{10.29}$$
  with $h_{[0]}\left(x^i\right)$ given by boundary conditions.

- *The exact solutions of (10.15) for* $\beta \neq 0$ *is*

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \tag{10.30}$$

  with $\partial_v = \partial/\partial v$ *and* $h_5^* \neq 0$. *If* $h_5^* = 0$, *or even* $h_5^* \neq 0$ *but* $\beta = 0$, *the coefficients* w_k *could be arbitrary functions on* (x^i, v). *For vacuum Einstein equations this is a degenerated case which imposes the the compatibility conditions* $\beta = \alpha_i = 0$, *which are satisfied, for instance, if the* h_4 *and* h_5 *are related as in the formula (10.29) but with* $h_{[0]}(x^i) = const.$

- *The exact solution of (10.16) is*

$$\begin{aligned} n_k &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [h_4/(\sqrt{|h_5|})^3] dv, \; h_5^* \neq 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int h_4 dv, \qquad h_5^* = 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [1/(\sqrt{|h_5|})^3] dv, \; h_4^* = 0, \end{aligned} \tag{10.31}$$

  for some functions $n_{k[1,2]}(x^i)$ *stated by boundary conditions.*

- *The exact solution of (10.25) is given by some arbitrary functions* $\zeta_i = \zeta_i(x^i, v)$ *if both* $\partial_i \Omega = 0$ *and* $\Omega^* = 0$, *we chose* $\zeta_i = 0$ *for* $\Omega = const$, *and*

$$\begin{aligned} \zeta_i &= -w_i + (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0, \\ &= (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0, \; \textit{for vacuum solutions.} \end{aligned} \tag{10.32}$$

We note that a transform (10.27) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case the equation (10.13) is solved by arbitrary two functions $g_2(x^3)$ and $g_3(x^2)$. The equation (10.14) is satisfied by arbitrary pairs of coefficients $h_4(x^i, v)$ and $h_{5[0]}(x^i)$.

The proof of Theorem 3 is given in the Appendix B.

10.2.4 Consequences of Theorems 1–3

We consider three important consequences of the Lemmas and Theorems formulated in this Section:

Corollary 10.2.1. *The non–trivial diagonal components of the Einstein tensor,* $G^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \frac{1}{2}R \quad \delta^{\alpha}_{\beta}$, *for the metric (10.10), given with respect to anholonomic N–bases, are*

$$G^1_1 = -\left(R^2_2 + S^4_4\right), G^2_2 = G^3_3 = -S^4_4, G^4_4 = G^5_5 = -R^2_2. \tag{10.33}$$

So, the dynamics of the system is defined by two values R^2_2 *and* S^4_4. *The rest of non–diagonal components of the Ricci (Einstein tensor) are compensated by fixing corresponding values of N–coefficients.*

The formulas (10.33) are obtained following the relations for the Ricci tensor (10.13)–(10.16).

Corollary 10.2.2. *We can extend the system of 5D vacuum Einstein equations (10.13)–(10.16) by introducing matter fields for which the energy–momentum tensor* $\Upsilon_{\alpha\beta}$ *given with respect to anholonomic frames satisfy the conditions*

$$\Upsilon^1_1 = \Upsilon^2_2 + \Upsilon^4_4, \Upsilon^2_2 = \Upsilon^3_3, \Upsilon^4_4 = \Upsilon^5_5. \tag{10.34}$$

We note that, in general, the tensor $\Upsilon_{\alpha\beta}$ for the non–vacuum Einstein equations,

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa\Upsilon_{\alpha\beta},$$

is not symmetric because with respect to anholonomic frames there are imposed constraints which makes non symmetric the Ricci and Einstein tensors (the symmetry conditions hold only with respect to holonomic, coordinate frames; for details see the Appendix and the formulas (4.18)).

For simplicity, in our further investigations we shall consider only diagonal matter sources, given with respect to anholonomic frames, satisfying the conditions

$$\kappa\Upsilon^2_2 = \kappa\Upsilon^3_3 = \Upsilon_2, \kappa\Upsilon^4_4 = \kappa\Upsilon^5_5 = \Upsilon_4, \text{ and } \Upsilon_1 = \Upsilon_2 + \Upsilon_4, \tag{10.35}$$

where κ is the gravitational coupling constant. In this case the equations (10.13) and (10.14) are respectively generalized to

$$R^2_2 = R^3_3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = -\Upsilon_4 \tag{10.36}$$

and

$$S^4_4 = S^5_5 = -\frac{\beta}{2h_4h_5} = -\Upsilon_2. \tag{10.37}$$

Corollary 10.2.3. *The class of metrics (10.18) satisfying vacuum Einstein equations (10.13)–(10.16) and (10.25) contains as particular cases some solutions when the Schwarzschild potential* $\Phi = -M/(M_{\rm p}^2 r)$*, where* $M_{\rm p}$ *is the effective Planck mass on the brane, is modified to*

$$\Phi = -\frac{M\sigma_m}{M_{\rm p}^2 r} + \frac{Q\sigma_q}{2r^2}\,,$$

where the 'tidal charge' parameter Q *may be positive or negative.*

As proofs of this corollary we can consider the Refs [4] where the possibility to modify anisotropically the Newton law via effective anisotropic masses $M\sigma_m$, or by anisotropic effective 4D Plank constants, renormalized like $\sigma_m/M_{\rm p}^2$, and with "effective" electric charge, $Q\sigma_q$ was recently emphasized (see also the end of Section III in this paper). For diagonal metrics, in the locally isotropic limit, we put the effective polarizations $\sigma_m = \sigma_q = 1$.

10.2.5 Reduction from 5D to 4D gravity

The above presented results are for generic off–diagonal metrics of gravitational fields, anholonomic transforms and nonlinear field equations. Reductions to a lower dimensional theory are not trivial in such cases. We give a detailed analysis of this procedure.

The simplest way to construct a $5D \rightarrow 4D$ reduction for the ansatz (10.9) and (10.19) is to eliminate from formulas the variable x^1 and to consider a 4D space (parametrized by local coordinates (x^2, x^3, v, y^5)) being trivially embedded into 5D space (parametrized by local coordinates (x^1, x^2, x^3, v, y^5) with $g_{11} = \pm 1, g_{1\underline{\alpha}} = 0, \underline{\alpha} = 2, 3, 4, 5)$ with further possible conformal and anholonomic transforms depending only on variables (x^2, x^3, v). We admit that the 4D metric $g_{\underline{\alpha}\underline{\beta}}$ could be of arbitrary signature. In order to emphasize that some coordinates are stated just for a such 4D space we underline the Greek indices, $\underline{\alpha}, \underline{\beta}, ...$ and the Latin indices from the middle of alphabet, $\underline{i}, \underline{j}, ... = 2, 3$, where $u^{\underline{\alpha}} = (x^{\underline{i}}, y^a) = (x^2, x^3, y^4, y^5)$.

In result, the analogs of Lemmas 1and 2, Theorems 1-3 and Corollaries 1-3 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of a such reduction.

- The line element (10.8) with ansatz (10.9) and the line element (10.8) with (10.19) are respectively transformed on 4D space to the values:

  The first type 4D quadratic line element is taken

$$ds^2 = g_{\underline{\alpha}\underline{\beta}}\left(x^{\underline{i}}, v\right) du^{\underline{\alpha}} du^{\underline{\beta}} \qquad (10.38)$$

with the metric coefficients $g_{\alpha\beta}$ parametrized (with respect to the coordinate frame (10.3) in 4D) by an off–diagonal matrix (ansatz)

$$\left[\begin{array}{cccc} g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 \end{array}\right], \quad (10.39)$$

where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g_{2,3} &= g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^{\underline{k}}, v), \\ w_{\underline{i}} &= w_{\underline{i}}(x^{\underline{k}}, v), n_{\underline{i}} = n_{\underline{i}}(x^{\underline{k}}, v);\ \underline{i}, \underline{k} = 2, 3. \end{aligned}$$

The anholonomically and conformally transformed 4D line element is

$$ds^2 = \Omega^2(x^{\underline{i}}, v)\hat{g}_{\underline{\alpha\beta}}\left(x^{\underline{i}}, v\right) du^{\underline{\alpha}} du^{\underline{\beta}}, \quad (10.40)$$

were the coefficients $\hat{g}_{\underline{\alpha\beta}}$ are parametrized by the ansatz

$$\left[\begin{array}{cccc} g_2 + ({w_2}^2 + {\zeta_2}^2) h_4 + {n_2}^2 h_5 & (w_2 w_3 + + \zeta_2\zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\ (w_2 w_3 + + \zeta_2\zeta_3) h_4 + n_2 n_3 h_5 & g_3 + ({w_3}^2 + {\zeta_3}^2) h_4 + {n_3}^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\ (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4 \end{array}\right]. \quad (10.41)$$

where $\zeta_{\underline{i}} = \zeta_{\underline{i}}\left(x^{\underline{k}}, v\right)$ and we shall restrict our considerations for $\zeta_5 = 0$.

- In the 4D analog of Lemma 1 we have

$$\delta s^2 = [g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2], \quad (10.42)$$

with respect to the anholonomic co–frame $\left(dx^{\underline{i}}, \delta v, \delta y^5\right)$, where

$$\delta v = dv + w_{\underline{i}} dx^{\underline{i}} \text{ and } \delta y^5 = dy^5 + n_{\underline{i}} dx^{\underline{i}} \quad (10.43)$$

which is dual to the frame $\left(\delta_{\underline{i}}, \partial_4, \partial_5\right)$, where

$$\delta_{\underline{i}} = \partial_{\underline{i}} + w_{\underline{i}}\partial_4 + n_{\underline{i}}\partial_5. \quad (10.44)$$

- In the conditions of the 4D variant of Theorem 1 we have the same equations (10.13)–(10.16) were we must put $h_4 = h_4\left(x^{\underline{k}}, v\right)$ and $h_5 = h_5\left(x^{\underline{k}}, v\right)$. As a consequence we have that $\alpha_i\left(x^k, v\right) \to \alpha_{\underline{i}}\left(x^{\underline{k}}, v\right), \beta = \beta\left(x^{\underline{k}}, v\right)$ and $\gamma = \gamma\left(x^{\underline{k}}, v\right)$ which result that $w_{\underline{i}} = w_{\underline{i}}\left(x^{\underline{k}}, v\right)$ and $n_{\underline{i}} = n_{\underline{i}}\left(x^{\underline{k}}, v\right)$.

- The respective formulas from Lemma 2, for $\zeta_5 = 0$, transform into

$$\delta s^2 = \Omega^2(x^{\underline{i}}, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\delta y^5)^2], \qquad (10.45)$$

with respect to the anholonomic co–frame $\left(dx^{\underline{i}}, \hat{\delta} v, \delta y^5\right)$, where

$$\delta v = dv + (w_{\underline{i}} + \zeta_{\underline{i}})dx^{\underline{i}} \text{ and } \delta y^5 = dy^5 + n_{\underline{i}} dx^{\underline{i}} \qquad (10.46)$$

which is dual to the frame $\left(\hat{\delta}_{\underline{i}}, \partial_4, \hat{\partial}_5\right)$, where

$$\hat{\delta}_{\underline{i}} = \partial_{\underline{i}} - (w_{\underline{i}} + \zeta_{\underline{i}})\partial_4 + n_{\underline{i}}\partial_5, \hat{\partial}_5 = \partial_5. \qquad (10.47)$$

- The formulas (10.23) and (10.25) from Theorem 2 must be modified into a 4D form

$$\hat{\delta}_{\underline{i}} h_4 = 0 \text{ and } \hat{\delta}_{\underline{i}} \Omega = 0 \qquad (10.48)$$

and the values $\zeta_{\hat{i}} = \left(\zeta_{\underline{i}}, \zeta_5 = 0\right)$ are found as to be a unique solution of (10.23); for instance, if

$$\Omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_{\underline{i}}$ satisfy the equations

$$\partial_{\underline{i}}\Omega - (w_{\underline{i}} + \zeta_{\underline{i}})\Omega^* = 0. \qquad (10.49)$$

- One holds the same formulas (10.28)-(10.31) from the Theorem 3 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\underline{i} \to i$, and holonomic coordinates, $x^i \to x^{\underline{i}}$, i. e. in the 4D solutions there is not contained the variable x^1.

- The formulae (10.33) for the nontrivial coefficients of the Einstein tensor in 4D stated by the Corollary 1 are written

$$G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2. \qquad (10.50)$$

- For symmetries of the Einstein tensor (10.50) we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 2 by the conditions (10.34), which in 4D are transformed into

$$\Upsilon_2^2 = \Upsilon_3^3, \Upsilon_4^4 = \Upsilon_5^5. \qquad (10.51)$$

- In 4D Einstein gravity we are not having violations of the Newton law as it was state in Corollary 3 for 5D. Nevertheless, off–diagonal and anholonomic frames can induce an anholonomic particle and field dynamics, for instance, with deformations of horizons of black holes, which can be modelled by an effective anisotropic renormalization of constants if some conditions are satisfied [1, 2].

There were constructed and analyzed various classes of exact solutions of the Einstein equations (both in the vacuum, reducing to the system (10.13), (10.14), (10.15) and (10.16) and non–vacuum, reducing to (10.36), (10.37), (10.15) and (10.16), cases) in 3D, 4D and 5D gravity [1, 4]. The aim of the next Sections III – V is to prove that such solutions contain warped factors which in the vacuum case are induced by a second order anisotropy. We shall analyze some classes of such exact solutions with running constants and/or their anisotropic polarizations induced from extra dimension gravitational interactions.

10.3 5D Ellipsoidal Black Holes

Our goal is to apply the anholonomic frame method as to construct such exact solutions of vacuum 5D Einstein equations as they will be static ones but, for instance, with ellipsoidal horizon for a diagonal metric given with respect to some well defined anholonomic frames. If such metrics are redefined with respect to usual coordinate frames, they are described by some particular cases of off–diagonal ansatz of type (10.9), or (10.19) which results in a very sophysticate form of the Einstein equations. That why it was not possible to construct such solutions in the past, before elaboration of the anholonomic frame method with associated nonlinear connection structure which allows to find exact solutions of the Einstein equations for very general off–diagonal metric ansatz.

By using anholonomic transforms the Schwarzschild and Reissner-Nördstrom solutions were generalized in anisotropic forms with deformed horizons, anisoropic polarizations and running constants both in the Einstein and extra dimension gravity (see Refs. [1, 4]). It was shown that there are possible anisotropic solutions which preserve the local Lorentz symmetry. and that at large radial distances from the horizon the anisotropic configurations transform into the usual one with spherical symmetry. So, the solutions with anisotropic rotation ellipsoidal horizons do not contradict the well known Israel and Carter theorems [19] which were proved in the assumption of spherical symmetry at asymptotic. The vacuum metrics presented here differ from anisotropic black hole solutions investigated in Refs. [1, 4].

10.3.1 The Schwarzschild solution in ellipsoidal coordinates

Let us consider the system of *isotropic spherical coordinates* (ρ, θ, φ), where the isotropic radial coordinate ρ is related with the usual radial coordinate r via the relation $r = \rho\,(1 + r_g/4\rho)^2$ for $r_g = 2G_{[4]}m_0/c^2$ being the 4D gravitational radius of a point particle of mass m_0, $G_{[4]} = 1/M^2_{P[4]}$ is the 4D Newton constant expressed via Plank mass $M_{P[4]}$ (following modern string/brane theories, $M_{P[4]}$ can be considered as a value induced from extra dimensions). We put the light speed constant $c = 1$. This system of coordinates is considered for the so–called isotropic representation of the Schwarzschild solution [17]

$$dS^2 = \left(\frac{\widehat{\rho}-1}{\widehat{\rho}+1}\right)^2 dt^2 - \rho_g^2 \left(\frac{\widehat{\rho}+1}{\widehat{\rho}}\right)^4 \left(d\widehat{\rho}^2 + \widehat{\rho}^2 d\theta^2 + \widehat{\rho}^2 \sin^2\theta d\varphi^2\right), \tag{10.52}$$

where, for our further considerations, we re–scaled the isotropic radial coordinate as $\widehat{\rho} = \rho/\rho_g$, with $\rho_g = r_g/4$. The metric (10.52) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass m_0. It has a singularity for $r = 0$ and a spherical horizon for $r = r_g$, or, in re–scaled isotropic coordinates, for $\widehat{\rho} = 1$. We emphasize that this solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames.

We also introduce the *rotation ellipsoid coordinates* (in our case considered as alternatives to the isotropic radial coordinates) [16] (u, λ, φ) with $0 \le u < \infty, 0 \le \lambda \le \pi, 0 \le \varphi \le 2\pi$, where $\sigma = \cosh u \ge 1$ are related with the isotropic 3D Cartezian coordinates

$$(\tilde{x} = \widetilde{\rho}\sinh u \sin\lambda\cos\varphi, \tilde{y} = \widetilde{\rho}\sinh u \sin\lambda\sin\varphi, \tilde{z} = \widetilde{\rho}\cosh u\cos\lambda) \tag{10.53}$$

and define an elongated rotation ellipsoid hypersurface

$$\left(\tilde{x}^2 + \tilde{y}^2\right)/(\sigma^2 - 1) + \tilde{z}^2/\sigma^2 = \widetilde{\rho}^2. \tag{10.54}$$

with $\sigma = \cosh u$. The 3D metric on a such hypersurface is

$$dS^2_{(3D)} = g_{uu}du^2 + g_{\lambda\lambda}d\lambda^2 + g_{\varphi\varphi}d\varphi^2,$$

where

$$g_{uu} = g_{\lambda\lambda} = \widetilde{\rho}^2\left(\sinh^2 u + \sin^2\lambda\right), g_{\varphi\varphi} = \widetilde{\rho}^2 \sinh^2 u \sin^2\lambda.$$

We can relate the rotation ellipsoid coordinates (u, λ, φ) from (10.53) with the isotropic radial coordinates $(\widehat{\rho}, \theta, \varphi)$, scaled by the constant ρ_g, from (10.52) as

$$\widetilde{\rho} = 1, \sigma = \cosh u = \widehat{\rho}$$

and deform the Schwarzschild metric by introducing ellipsoidal coordinates and a new horizon defined by the condition that vanishing of the metric coefficient before dt^2 describe an elongated rotation ellipsoid hypersurface (10.54),

$$\begin{aligned} dS^2_{(S)} &= \left(\frac{\cosh u - 1}{\cosh u + 1}\right)^2 dt^2 - \rho_g^2 \left(\frac{\cosh u + 1}{\cosh u}\right)^4 (\sinh^2 u + \sin^2 \lambda) \\ &\times [du^2 + d\lambda^2 + \frac{\sinh^2 u \ \sin^2 \lambda}{\sinh^2 u + \sin^2 \lambda} d\varphi^2]. \end{aligned} \tag{10.55}$$

The ellipsoidally deformed metric (10.55) does not satisfy the vacuum Einstein equations, but at long distances from the horizon it transforms into the usual Schwarzchild solution (10.52).

For our further considerations we introduce two Classes (A and B) of 4D auxiliary pseudo–Riemannian metrics, also given in ellipsoid coordinates, being some conformal transforms of (10.55), like

$$dS^2_{(S)} = \Omega_{A,B}(u,\lambda)\, dS^2_{(A,B)}$$

but which are not supposed to be solutions of the Einstein equations:

- Metric of Class A:

$$dS^2_{(A)} = -du^2 - d\lambda^2 + a(u,\lambda)d\varphi^2 + b(u,\lambda)dt^2], \tag{10.56}$$

  where

$$a(u,\lambda) = -\frac{\sinh^2 u \ \sin^2 \lambda}{\sinh^2 u + \sin^2 \lambda} \text{ and } b(u,\lambda) = -\frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2 (\cosh u + 1)^6 (\sinh^2 u + \sin^2 \lambda)},$$

  which results in the metric (10.55) by multiplication on the conformal factor

$$\Omega_A(u,\lambda) = \rho_g^2 \frac{(\cosh u + 1)^4}{\cosh^4 u} (\sinh^2 u + \sin^2 \lambda). \tag{10.57}$$

- Metric of Class B:

$$dS^2 = g(u,\lambda)\left(du^2 + d\lambda^2\right) - d\varphi^2 + f(u,\lambda)dt^2, \tag{10.58}$$

  where

$$g(u,\lambda) = -\frac{\sinh^2 u + \sin^2 \lambda}{\sinh^2 u \ \sin^2 \lambda} \text{ and } f(u,\lambda) = \frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2 (\cosh u + 1)^6 \sinh^2 u \sin^2 \lambda},$$

  which results in the metric (10.55) by multiplication on the conformal factor

$$\Omega_B(u,\lambda) = \rho_g^2 \frac{(\cosh u + 1)^4}{\cosh^4 u} \sinh^2 u \sin^2 \lambda.$$

Now it is possible to generate exact solutions of the Einstein equations with rotation ellipsoid horizons and anisotropic polarizations and running of constants by performing corresponding anholonomic transforms as the solutions will have an horizon parametrized by a hypersurface like rotation ellipsoid and gravitational (extra dimensional or nonlinear 4D) renormalization of the constant ρ_g of the Schwarzschild solution, $\rho_g \to \overline{\rho}_g = \omega \rho_g$, where the dependence of the function ω on some holonomic or anholonomic coordinates depend on the type of anisotropy. For some solutions we can treat ω as a factor modelling running of the gravitational constant, induced, induced from extra dimension, in another cases we may consider ω as a nonlinear gravitational polarization which model some anisotropic distributions of masses and matter fields and/or anholonomic vacuum gravitational interactions.

10.3.2 Ellipsoidal 5D metrics of Class A

In this subsection we consider four classes of 5D vacuum solutions which are related to the metric of Class A (10.56) and to the Schwarzschild metric in ellipsoidal coordinates (10.55).

Let us parametrize the 5D coordinates as $(x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v, y^5 = p)$, where the solutions with the so–called φ–anisotropy will be constructed for $(v = \varphi, p = t)$ and the solutions with t–anisotropy will be stated for $(v = t, p = \varphi)$ (in brief, we shall write respective φ–solutions and t–solutions).

Class A solutions with ansatz (10.9):

We take an off–diagonal metric ansatz of type (10.9) (equivalently, (10.8)) by representing

$$g_1 = \pm 1, g_2 = -1, g_3 = -1, h_4 = \eta_4(x^i, v) h_{4(0)}(x^i) \text{ and } h_5 = \eta_5(x^i, v) h_{5(0)}(x^i),$$

where $\eta_{4,5}(x^i, v)$ are corresponding "gravitational renormalizations" of the metric coefficients $h_{4,5(0)}(x^i)$. For φ–solutions we state $h_{4(0)} = a(u, \lambda)$ and $h_{5(0)} = b(u, \lambda)$ (inversely, for t–solutions, $h_{4(0)} = b(u, \lambda)$ and $h_{5(0)} = a(u, \lambda)$).

Next we consider a renormalized gravitational 'constant' $\overline{\rho}_g = \omega \rho_g$, were for φ–solutions the receptivity $\omega = \omega(x^i, v)$ is included in the gravitational polarization η_5 as $\eta_5 = [\omega(x^i, \varphi)]^{-2}$, or for t–solutions is included in η_4, when $\eta_4 = [\omega(x^i, t)]^{-2}$. We can construct an exact solution of the 5D vacuum Einstein equations if, for explicit dependencies on anisotropic coordinate, the metric coefficients h_4 and h_5 are related by formula (10.29) with $h_{[0]}(x^i) = h_{(0)} = const$ (see the Theorem 3, with statements on formulas (10.29) and (8.10)), which in its turn imposes a corresponding relation between η_4 and η_5,

$$\eta_4 h_{4(0)}(x^i) = h_{(0)}^2 h_{5(0)}(x^i)\left[\left(\sqrt{|\eta_5|}\right)^*\right]^2.$$

In result, we express the polarizations η_4 and η_5 via the value of receptivity ω,

$$\eta_4(\chi,u,\lambda,\varphi) = h_{(0)}^2\frac{b(u,\lambda)}{a(u,\lambda)}\left\{\left[\omega^{-1}(\chi,u,\lambda,\varphi)\right]^*\right\}^2, \eta_5(\chi,u,\lambda,\varphi) = \omega^{-2}(\chi,u,\lambda,\varphi), \tag{10.59}$$

for φ–solutions , and

$$\eta_4(\chi,u,\lambda,t) = \omega^{-2}(\chi,u,\lambda,t), \eta_5(\chi,u,\lambda,t) = h_{(0)}^{-2}\frac{b(u,\lambda)}{a(u,\lambda)}\left[\int dt\omega^{-1}(\chi,u,\lambda,t)\right]^2, \tag{10.60}$$

for t–solutions, where $a(u,\lambda)$ and $b(u,\lambda)$ are those from (10.56).

For vacuum configurations, following the discussions of formula (8.10) in Theorem 3, we put $w_i = 0$. The next step is to find the values of n_i by introducing $h_4 = \eta_4 h_{4(0)}$ and $h_5 = \eta_5 h_{5(0)}$ into the formula (10.31), which, for convenience, is expressed via general coefficients η_4 and η_5, with the functions $n_{k[2]}(x^i)$ redefined as to contain the values $h_{(0)}^2$, $a(u,\lambda)$ and $b(u,\lambda)$

$$\begin{aligned} n_k &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int[\eta_4/(\sqrt{|\eta_5|})^3]dv, \ \eta_5^* \neq 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int \eta_4 dv, \qquad \eta_5^* = 0; \\ &= n_{k[1]}(x^i) + n_{k[2]}(x^i)\int[1/(\sqrt{|\eta_5|})^3]dv, \ \eta_4^* = 0. \end{aligned} \tag{10.61}$$

By introducing the formulas (10.59) for φ–solutions (or (10.60) for t–solutions) and fixing some boundary condition, in order to state the values of coefficients $n_{k[1,2]}(x^i)$ we can express the ansatz components $n_k(x^i,\varphi)$ as integrals of some functions of $\omega(x^i,\varphi)$ and $\partial_\varphi\omega(x^i,\varphi)$ (or, we can express the ansatz components $n_k(x^i,t)$ as integrals of some functions of $\omega(x^i,t)$ and $\partial_t\omega(x^i,t)$). We do not present an explicit form of such formulas because they depend on the type of receptivity $\omega = \omega(x^i,v)$, which must be defined experimentally, or from some quantum models of gravity in the quasi classical limit. We preserved a general dependence on coordinates x^i which reflect the fact that there is a freedom in fixing holonomic coordinates (for instance, on ellipsoidal hypersurface and its extensions to 4D and 5D spacetimes). For simplicity, we write that n_i are some functionals of $\{x^i, \omega\left(x^i, v\right), \omega^*\left(x^i, v\right)\}$

$$n_i\{x, \omega, \omega^*\} = n_i\{x^i, \omega\left(x^i, v\right), \omega^*\left(x^i, v\right)\}.$$

In conclusion, we constructed two exact solutions of the 5D vacuum Einstein equations, defined by the ansatz (10.9) with coordinates and coefficients stated by the data:

$$\begin{aligned}
\varphi\text{–solutions} &: (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= -1, g_3 = -1, h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \text{see (10.56)}; \\
h_4 &= \eta_4(x^i, \varphi) h_{4(0)}(x^i), h_5 = \eta_5(x^i, \varphi) h_{5(0)}(x^i), \\
\eta_4 &= h_{(0)}^2 \frac{b(u, \lambda)}{a(u, \lambda)} \left\{ \left[\omega^{-1}\left(\chi, u, \lambda, \varphi\right)\right]^* \right\}^2, \eta_5 = \omega^{-2}\left(\chi, u, \lambda, \varphi\right), \\
w_i &= 0, n_i\{x, \omega, \omega^*\} = n_i\{x^i, \omega\left(x^i, \varphi\right), \omega^*\left(x^i, \varphi\right)\}. \qquad (10.62)
\end{aligned}$$

and

$$\begin{aligned}
t\text{–solutions} &: (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \\
g_2 &= -1, g_3 = -1, h_{4(0)} = b(u, \lambda), h_{5(0)} = a(u, \lambda), \text{see (10.56)}; \\
h_4 &= \eta_4(x^i, t) h_{4(0)}(x^i), h_5 = \eta_5(x^i, t) h_{5(0)}(x^i), \\
\eta_4 &= \omega^{-2}\left(\chi, u, \lambda, t\right), \eta_5 = h_{(0)}^{-2} \frac{b(u, \lambda)}{a(u, \lambda)} \left[\int dt\ \omega^{-1}\left(\chi, u, \lambda, t\right)\right]^2, \\
w_i &= 0, n_i\{x, \omega, \omega^*\} = n_i\{x^i, \omega\left(x^i, t\right), \omega^*\left(x^i, t\right)\}. \qquad (10.63)
\end{aligned}$$

Both types of solutions have a horizon parametrized by a rotation ellipsoid hypersurface (as the condition of vanishing of the "time" metric coefficient states, i. e. when the function $b(u, \lambda) = 0$). These solutions are generically anholonomic (anisotropic) because in the locally isotropic limit, when $\eta_4, \eta_5, \omega \rightarrow 1$ and $n_i \rightarrow 0$, they reduce to the coefficients of the metric (10.56). The last one is not an exact solution of 4D vacuum Einstein equations, but it is a conformal transform of the 4D Schwarzschild solution with a further trivial extension to 5D. With respect to the anholonomic frames adapted to the coefficients n_i (see (10.11)), the obtained solutions have diagonal metric coefficients being very similar to the Schwarzschild metric (10.55) written in ellipsoidal coordinates. We can treat such solutions as black hole ones with a point particle mass put in one of the focuses of rotation ellipsoid hypersurface (for flattened ellipsoids the mass should be placed on the circle described by ellipse's focuses under rotation; we omit such details in this work which were presented for 4D gravity in Ref. [1]).

The initial data for anholonomic frames and the chosen configuration of gravitational interactions in the bulk lead to deformed "ellipsoidal" horizons even for static configurations. The solutions admit anisotropic polarizations on ellipsoidal and angular coordinates (u,λ) and running of constants on time t and/or on extra dimension coordinate χ. Such renormalizations of constants are defined by the nonlinear configuration of the 5D vacuum gravitational field and depend on introduced receptivity function $\omega\left(x^{i},v\right)$ which is to be considered an intrinsic characteristics of the 5D vacuum gravitational 'ether', emphasizing the possibility of nonlinear self–polarization of gravitational fields.

Finally, we note that the data (10.62) and (10.63) parametrize two very different classes of solutions. The first one is for static 5D vacuum black hole configurations with explicit dependence on anholonomic coordinate φ and possible renormalizations on the rest of 3D space coordinates u and λ and on the 5th coordinate χ. The second class of solutions are similar to the static solutions but with an emphasized anholonomic time running of constants and with possible anisotropic dependencies on coordinates (u,λ,χ).

Class A solutions with ansatz (10.19):

We construct here 5D vacuum φ– and t–solutions parametrized by an ansatz with conformal factor $\Omega(x^{i},v)$ (see (10.19) and (10.20)). Let us consider conformal factors parametrized as $\Omega=\Omega_{[0]}(x^{i})\Omega_{[1]}(x^{i},v)$. We can generate from the data (10.62) (or (10.63)) an exact solution of vacuum Einstein equations if there are satisfied the conditions (10.24) and (10.32), i. e.

$$\Omega_{[0]}^{q_1/q_2}\Omega_{[1]}^{q_1/q_2}=\eta_4 h_{4(0)},$$

for some integers q_1 and q_2, and there are defined the second anisotropy coefficients

$$\zeta_i=\left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^{*}+\left(\Omega_{[1]}^{*}\right)^{-1}\partial_i\Omega_{[1]}.$$

So, taking a φ– or t–solution with corresponding values of $h_4=\eta_4 h_{4(0)}$, for some q_1 and q_2, we obtain new exact solutions, called in brief, φ_c– or t_c–solutions (with the index "c" pointing to an ansatz with conformal factor), of the vacuum 5D Einstein equations given in explicit form by the data:

$$
\begin{aligned}
\varphi_c\text{–solutions} &: (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= -1, g_3 = -1, h_{4(0)} = a(u,\lambda), h_{5(0)} = b(u,\lambda), \text{see (10.56)}; \\
h_4 &= \eta_4(x^i,\varphi)h_{4(0)}(x^i), h_5 = \eta_5(x^i,\varphi)h_{5(0)}(x^i), \\
\eta_4 &= h^2_{(0)}\frac{b(u,\lambda)}{a(u,\lambda)}\left\{\left[\omega^{-1}(\chi,u,\lambda,\varphi)\right]^*\right\}^2, \eta_5 = \omega^{-2}(\chi,u,\lambda,\varphi), \qquad (10.64) \\
w_i &= 0, n_i\{x,\omega,\omega^*\} = n_i\{x^i,\omega\left(x^i,\varphi\right),\omega^*\left(x^i,\varphi\right)\}, \Omega = \Omega_{[0]}(x^i)\Omega_{[1]}(x^i,\varphi) \\
\zeta_i &= \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega^*_{[1]}\right)^{-1}\partial_i\Omega_{[1]}, \\
\eta_4 a &= \Omega^{q_1/q_2}_{[0]}(x^i)\Omega^{q_1/q_2}_{[1]}(x^i,\varphi).
\end{aligned}
$$

and

$$
\begin{aligned}
t_c\text{–solutions} &: (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \\
g_2 &= -1, g_3 = -1, h_{4(0)} = b(u,\lambda), h_{5(0)} = a(u,\lambda), \text{see (10.56)}; \\
h_4 &= \eta_4(x^i,t)h_{4(0)}(x^i), h_5 = \eta_5(x^i,t)h_{5(0)}(x^i), \\
\eta_4 &= \omega^{-2}(\chi,u,\lambda,t), \eta_5 = h^{-2}_{(0)}\frac{b(u,\lambda)}{a(u,\lambda)}\left[\int dt\ \omega^{-1}(\chi,u,\lambda,t)\right]^2, \qquad (10.65) \\
w_i &= 0, n_i\{x,\omega,\omega^*\} = n_i\{x^i,\omega\left(x^i,t\right),\omega^*\left(x^i,t\right)\}, \Omega = \Omega_{[0]}(x^i)\Omega_{[1]}(x^i,t) \\
\zeta_i &= \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega^*_{[1]}\right)^{-1}\partial_i\Omega_{[1]}, \eta_4 a = \Omega^{q_1/q_2}_{[0]}(x^i)\Omega^{q_1/q_2}_{[1]}(x^i,t).
\end{aligned}
$$

These solutions have two very interesting properties: 1) they admit a warped factor on the 5th coordinate, like $\Omega^{q_1/q_2}_{[1]} \sim \exp[-k|\chi|]$, which in our case is constructed for an anisotropic 5D vacuum gravitational configuration and not following a brane configuration like in Refs. [7]; 2) we can impose such conditions on the receptivity $\omega(x^i,v)$ as to obtain in the locally isotropic limit just the Schwarzschild metric (10.55) trivially embedded into the 5D spacetime.

Let us analyze the second property in details. We have to chose the conformal factor as to be satisfied three conditions:

$$
\Omega^{q_1/q_2}_{[0]} = \Omega_A, \Omega^{q_1/q_2}_{[1]}\eta_4 = 1, \Omega^{q_1/q_2}_{[1]}\eta_5 = 1, \qquad (10.66)
$$

were Ω_A is that from (10.24). The last two conditions are possible if

$$
\eta_4^{-q_1/q_2}\eta_5 = 1, \qquad (10.67)
$$

which selects a specific form of receptivity $\omega\left(x^{i},v\right).$ Putting into (10.67) the values η_4 and η_5 respectively from (10.64), or (10.65), we obtain some differential, or integral, relations of the unknown $\omega\left(x^{i},v\right)$, which results that

$$\begin{aligned}
\omega\left(x^{i},\varphi\right) &= \left(1-q_1/q_2\right)^{-1-q_1/q_2}\left[h_{(0)}^{-1}\sqrt{|a/b|}\varphi+\omega_{[0]}\left(x^{i}\right)\right], \text{ for } \varphi_c\text{–solutions;}\\
\omega\left(x^{i},t\right) &= \left[\left(q_1/q_2-1\right)h_{(0)}\sqrt{|a/b|}t+\omega_{[1]}\left(x^{i}\right)\right]^{1-q_1/q_2}, \text{ for } t_c\text{–solutions,} \quad (10.68)
\end{aligned}$$

for some arbitrary functions $\omega_{[0]}\left(x^{i}\right)$ and $\omega_{[1]}\left(x^{i}\right).$ So, receptivities of particular form like (10.68) allow us to obtain in the locally isotropic limit just the Schwarzschild metric.

We conclude this subsection by the remark: the vacuum 5D metrics solving the Einstein equations describe a nonlinear gravitational dynamics which under some particular boundary conditions and parametrizations of metric's coefficients can model anisotropic solutions transforming, in a corresponding locally isotropic limit, in some well known exact solutions like Schwarzschild, Reissner-Nördstrom, Taub NUT, various type of wormhole, solitonic and disk solutions (see details in Refs. [1, 2, 4]). Here we emphasize that, in general, an anisotropic solution (parametrized by an off–diagonal ansatz) could not have a locally isotropic limit to a diagonal metric with respect to some holonomic coordinate frames. By some boundary conditions and suggested type of horizons, singularities, symmetries and topological configuration such solutions model new classes of black hole/tori, wormholes and another type of solutions which defines a generic anholonomic gravitational field dynamics and has not locally isotropic limits.

10.3.3 Ellipsiodal 5D metrics of Class B

In this subsection we construct and analyze another two classes of 5D vacuum solutions which are related to the metric of Class B (10.58) and which can be reduced to the Schwarzschild metric in ellipsoidal coordinates (10.55) by corresponding parametrizations of receptivity $\omega\left(x^{i},v\right)$. We emphasize that because the function $g(u,\lambda)$ from (10.58) is not a solution of equation (10.13) we introduce an auxiliary factor $\varpi\ (u,\lambda)$ for which ϖg became a such solution, then we consider conformal factors parametrized as $\Omega=\varpi^{-1}$ $\Omega_{[2]}\left(x^{i},v\right)$ and find solutions parametrized by the ansatz (10.19) and anholonomic metric interval (10.20).

Because the method of definition of such solutions is similar to that from previous subsection, in our further considerations we shall omit intermediary computations and present directly the data which select the respective configurations for φ_c–solutions and t_c–solutions.

The Class B of 5D solutions with conformal factor are parametrized by the data:

$$
\begin{aligned}
\varphi_c\text{–solutions} &: & (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= & g_3 = \varpi(u,\lambda) g(u,\lambda), h_{4(0)} = -\varpi(u,\lambda), \\
h_{5(0)} &= & \varpi(u,\lambda) f(u,\lambda), \text{see (10.58)}; \\
\varpi &= & g^{-1}\varpi_0 \exp[a_2 u + a_3 \lambda],\ \varpi_0, a_2, a_3 = const;\ \text{see (10.26)} \\
h_4 &= & \eta_4(x^i,\varphi) h_{4(0)}(x^i), h_5 = \eta_5(x^i,\varphi) h_{5(0)}(x^i), \\
\eta_4 &= & -h^2_{(0)} f(u,\lambda) \left\{\left[\omega^{-1}(\chi,u,\lambda,\varphi)\right]^*\right\}^2, \eta_5 = \omega^{-2}(\chi,u,\lambda,\varphi), \qquad (10.69) \\
w_i &= & 0, n_i\{x,\omega,\omega^*\} = n_i\{x^i, \omega\left(x^i,\varphi\right), \omega^*\left(x^i,\varphi\right)\}, \Omega = \varpi^{-1}(u,\lambda)\Omega_{[2]}(x^i,\varphi) \\
\zeta_i &= & \partial_i \ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega^*_{[2]}\right)^{-1}\partial_i\Omega_{[2]}, \eta_4 = -\varpi^{-q_1/q_2}(x^i)\Omega^{q_1/q_2}_{[2]}(x^i,\varphi).
\end{aligned}
$$

and

$$
\begin{aligned}
t_c\text{–solutions} &: & (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \\
g_2 &= & g_3 = \varpi(u,\lambda) g(u,\lambda), h_{4(0)} = \varpi(u,\lambda) f(u,\lambda), \\
h_{5(0)} &= & -\varpi(u,\lambda), \text{see (10.58)}; \\
\varpi &= & g^{-1}\varpi_0 \exp[a_2 u + a_3 \lambda],\ \varpi_0, a_2, a_3 = const,\ \text{see (10.26)} \\
h_4 &= & \eta_4(x^i,t) h_{4(0)}(x^i), h_5 = \eta_5(x^i,t) h_{5(0)}(x^i), \\
\eta_4 &= & \omega^{-2}(\chi,u,\lambda,t), \eta_5 = -h^{-2}_{(0)} f(u,\lambda) \left[\int dt\ \omega^{-1}(\chi,u,\lambda,t)\right]^2, \qquad (10.70) \\
w_i &= & 0, n_i\{x,\omega,\omega^*\} = n_i\{x^i, \omega\left(x^i,t\right), \omega^*\left(x^i,t\right)\}, \Omega = \varpi^{-1}(u,\lambda)\Omega_{[2]}(x^i,t) \\
\zeta_i &= & \partial_i(\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega^*_{[2]}\right)^{-1}\partial_i\Omega_{[2]}, \eta_4 = -\varpi^{-q_1/q_2}(x^i)\Omega^{q_1/q_2}_{[2]}(x^i,t).
\end{aligned}
$$

where the coefficients n_i can be found explicitly by introducing the corresponding values η_4 and η_5 in formula (10.61).

By a procedure similar to the solutions of Class A (see previous subsection) we can find the conditions when the solutions (10.69) and (10.70) will have in the locally anisotropic limit the Schwarzschild solutions, which impose corresponding parametrizations and dependencies on $\Omega_{[2]}(x^i,v)$ and $\omega\left(x^i,v\right)$ like (10.66) and (10.68). We omit these formulas because, in general, for anholonomic configurations and nonlinear solutions there are not hard arguments to prefer any holonomic limits of such off–diagonal metrics.

Finally, in this Section, we remark that for the considered classes of ellipsoidal black hole solutions the so–called tt–components of metric contain modifications of the

Schwarzschild potential

$$\Phi = -\frac{M}{M_{P[4]}^2 r} \text{ into } \Phi = -\frac{M\omega\left(x^i, v\right)}{M_{P[4]}^2 r},$$

where $M_{P[4]}$ is the usual 4D Plank constant, and this is given with respect to the corresponding anholonomic frame of reference. The receptivity $\omega\left(x^i, v\right)$ could model corrections warped on extra dimension coordinate, χ, which for our solutions are induced by anholonomic vacuum gravitational interactions in the bulk and not from a brane configuration in AdS_5 spacetime. In the vacuum case k is a constant characterizing the receptivity for bulk vacuum gravitational polarizations.

10.4 4D Ellipsoidal Black Holes

For the ansatz (10.39), without conformal factor, some classes of ellipsoidal solutions of 4D Einstein equations were constructed in Ref. [1] with further generalizations and applications to brane physics [4] . The goal of this Section is to consider some alternative variants, both with and without conformal factors and for different coordinate parametrizations and types of anisotropies. The bulk of 5D solutions from the previous Section are reduced into corresponding 4D ones if one eliminates the 5th coordinate χ from the formulas and the off–diagonal ansatz (10.39) and (10.41) are considered.

10.4.1 Ellipsiodal 5D metrics of Class A

Let us parametrize the 4D coordinates as $(x^{\underline{i}}, y^a) = (x^2 = u, x^3 = \lambda, y^4 = v, y^5 = p)$; for the φ–solutions we shall take $(v = \varphi, p = t)$ and for the solutions t–solutions we shall consider $(v = t, p = \varphi)$. Following the prescription from subsection IIE we can write down the data for solutions without proofs and computations.

Class A solutions with ansat (10.39):

The off–diagonal metric ansatz of type (10.39) (equivalently, (10.8)) with the data

$$\begin{aligned}
\varphi\text{–solutions} &: \quad (x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t) \\
g_2 &= -1, g_3 = -1, h_{4(0)} = a(u,\lambda), h_{5(0)} = b(u,\lambda), \text{see } (10.56); \\
h_4 &= \eta_4(u,\lambda,\varphi) h_{4(0)}(u,\lambda), h_5 = \eta_5(u,\lambda,\varphi) h_{5(0)}(u,\lambda), \\
\eta_4 &= h_{(0)}^2 \frac{b(u,\lambda)}{a(u,\lambda)} \left\{\left[\omega^{-1}\left(u,\lambda,\varphi\right)\right]^*\right\}^2, \eta_5 = \omega^{-2}\left(u,\lambda,\varphi\right), \\
w_{\underline{i}} &= 0, n_i\{x,\omega,\omega^*\} = n_i\{u,\lambda,\omega\left(u,\lambda,\varphi\right),\omega^*\left(u,\lambda,\varphi\right)\}. \qquad (10.71)
\end{aligned}$$

and

$$
\begin{aligned}
t\text{–solutions} &: (x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi) \\
g_2 &= -1, g_3 = -1, h_{4(0)} = b(u,\lambda), h_{5(0)} = a(u,\lambda), \text{see (10.56)}; \\
h_4 &= \eta_4(u,\lambda,t) h_{4(0)}(u,\lambda), h_5 = \eta_5(u,\lambda,t) h_{5(0)}(u,\lambda), \\
\eta_4 &= \omega^{-2}(u,\lambda,t), \eta_5 = h_{(0)}^{-2} \frac{b(u,\lambda)}{a(u,\lambda)} \left[\int dt\ \omega^{-1}(u,\lambda,t) \right]^2, \\
w_{\underline{i}} &= 0, n_{\underline{i}}\{x,\omega,\omega^*\} = n_{\underline{i}}\{u,\lambda,\omega(u,\lambda,t), \omega^*(u,\lambda,t)\}. \qquad (10.72)
\end{aligned}
$$

where the $n_{\underline{i}}$ are computed

$$
\begin{aligned}
n_{\underline{k}} &= n_{\underline{k}[1]}(u,\lambda) + n_{\underline{k}[2]}(u,\lambda) \int [\eta_4/(\sqrt{|\eta_5|})^3] dv,\ \eta_5^* \neq 0; \qquad (10.73) \\
&= n_{\underline{k}[1]}(u,\lambda) + n_{\underline{k}[2]}(u,\lambda) \int \eta_4 dv, \qquad \eta_5^* = 0; \\
&= n_{\underline{k}[1]}(u,\lambda) + n_{\underline{k}[2]}(u,\lambda) \int [1/(\sqrt{|\eta_5|})^3] dv,\ \eta_4^* = 0.
\end{aligned}
$$

These solutions have the same ellipsoidal symmetries and properties stated for their 5D analogs (10.62) and for (10.63) with that difference that there are not any warped factors and extra dimension dependencies. We emphasize that the solutions defined by the formulas (10.71) and (10.72) do not result in a locally isotropic limit into an exact solution having diagonal coefficients with respect to some holonomic coordinate frames. The data introduced in this subsection are for generic 4D vacuum solutions of the Einstein equations parametrized by off–diagonal metrics. The renormalization of constants and metric coefficients have a 4D nonlinear vacuum gravitational origin and reflects a corresponding anholonomic dynamics.

Class A solutions with ansatz (10.41):

The 4D vacuum φ– and t–solutions parametrized by an ansatz with conformal factor $\Omega(u,\lambda,v)$ (see (10.41) and (10.45)). Let us consider conformal factors parametrized as $\Omega = \Omega_{[0]}(u,\lambda)\Omega_{[1]}(u,\lambda,v)$. The data are

$$
\begin{aligned}
\varphi_c\text{–solutions} &: (x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t)\\
g_2 &= -1, g_3 = -1, h_{4(0)} = a(u,\lambda), h_{5(0)} = b(u,\lambda), \text{see (10.56)};\\
h_4 &= \eta_4(u,\lambda,\varphi)h_{4(0)}(u,\lambda), h_5 = \eta_5(u,\lambda,\varphi)h_{5(0)}(u,\lambda),\\
\eta_4 &= h_{(0)}^2 \frac{b(u,\lambda)}{a(u,\lambda)} \left\{\left[\omega^{-1}(u,\lambda,\varphi)\right]^*\right\}^2, \eta_5 = \omega^{-2}(u,\lambda,\varphi), \qquad (10.74)\\
w_i &= 0, n_i\{x,\omega,\omega^*\} = n_i\{u,\lambda,\omega(u,\lambda,\varphi),\omega^*(u,\lambda,\varphi)\},\\
\Omega &= \Omega_{[0]}(u,\lambda)\Omega_{[1]}(u,\lambda,\varphi),\ \zeta_i = \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]},\\
\eta_4 a &= \Omega_{[0]}^{q_1/q_2}(u,\lambda)\Omega_{[1]}^{q_1/q_2}(u,\lambda,\varphi).
\end{aligned}
$$

and

$$
\begin{aligned}
t_c\text{–solutions} &: (x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi)\\
g_2 &= -1, g_3 = -1, h_{4(0)} = b(u,\lambda), h_{5(0)} = a(u,\lambda), \text{see (10.56)};\\
h_4 &= \eta_4(u,\lambda,t)h_{4(0)}(u,\lambda), h_5 = \eta_5(u,\lambda,t)h_{5(0)}(u,\lambda),\\
\eta_4 &= \omega^{-2}(u,\lambda,t), \eta_5 = h_{(0)}^{-2}\frac{b(u,\lambda)}{a(u,\lambda)}\left[\int dt\ \omega^{-1}(u,\lambda,t)\right]^2, \qquad (10.75)\\
w_i &= 0, n_i\{x,\omega,\omega^*\} = n_i\{u,\lambda,\omega(u,\lambda,t),\omega^*(u,\lambda,t)\},\\
\Omega &= \Omega_{[0]}(u,\lambda)\Omega_{[1]}(u,\lambda,t), \zeta_i = \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]},\\
\eta_4 a &= \Omega_{[0]}^{q_1/q_2}(u,\lambda)\Omega_{[1]}^{q_1/q_2}(u,\lambda,t),
\end{aligned}
$$

where the coefficients the $n_{\underline{i}}$ are given by the same formulas (10.73).

Contrary to the solutions (10.71) and for (10.72) theirs conformal anholonomic transforms, respectively, (10.74) and (10.75), can be subjected to such parametrizations of the conformal factor and conditions on the receptivity $\omega(u,\lambda,v)$ as to obtain in the locally isotropic limit just the Schwarzschild metric (10.55). These conditions are stated for $\Omega_{[0]}^{q_1/q_2} = \Omega_A$, $\Omega_{[1]}^{q_1/q_2}\eta_4 = 1$, $\Omega_{[1]}^{q_1/q_2}\eta_5 = 1$,were Ω_A is that from (10.24), which is possible if $\eta_4^{-q_1/q_2}\eta_5 = 1$,which selects a specific form of the receptivity ω. Putting the values η_4 and η_5, respectively, from (10.74), or (10.75), we obtain some differential, or integral, relations of the unknown $\omega(x^i,v)$, which results that

$$
\begin{aligned}
\omega(u,\lambda,\varphi) &= (1-q_1/q_2)^{-1-q_1/q_2}\left[h_{(0)}^{-1}\sqrt{|a/b|}\varphi + \omega_{[0]}(u,\lambda)\right], \text{for } \varphi_c\text{–solutions};\\
\omega(u,\lambda,t) &= \left[(q_1/q_2-1)\,h_{(0)}\sqrt{|a/b|}t + \omega_{[1]}(u,\lambda)\right]^{1-q_1/q_2}, \text{for } t_c\text{–solutions},
\end{aligned}
$$

for some arbitrary functions $\omega_{[0]}(u,\lambda)$ and $\omega_{[1]}(u,\lambda)$. The formulas for $\omega(u,\lambda,\varphi)$ and $\omega(u,\lambda,t)$ are 4D reductions of the formulas (10.66) and (10.68).

10.4.2 Ellipsiodal 4D metrics of Class B

We construct another two classes of 4D vacuum solutions which are related to the metric of Class B (10.58) and which can be reduced to the Schwarzschild metric in ellipsoidal coordinates (10.55) by corresponding parametrizations of receptivity $\omega(u,\lambda,v)$. The solutions contain a 2D conformal factor $\varpi\ (u,\lambda)$ for which ϖg becomes a solution of (10.13) and a 4D conformal factor parametrized as $\Omega = \varpi^{-1}\ \Omega_{[2]}(u,\lambda,v)$ in order to set the constructions into the ansatz (10.41) and anholonomic metric interval (10.45).

The data selecting the 4D configurations for φ_c–solutions and t_c–solutions:

$$
\begin{aligned}
\varphi_c\text{–solutions} &: (x^2=u, x^3=\lambda, y^4=v=\varphi, y^5=p=t)\\
g_2 &= g_3=\varpi(u,\lambda)g(u,\lambda),\\
h_{4(0)} &= -\varpi(u,\lambda), h_{5(0)}=\varpi(u,\lambda)f(u,\lambda), \text{see (10.58)};\\
\varpi &= g^{-1}\varpi_0\exp[a_2u+a_3\lambda],\ \varpi_0,a_2,a_3=const;\ \text{see (10.26)}\\
h_4 &= \eta_4(u,\lambda,\varphi)h_{4(0)}(u,\lambda), h_5=\eta_5(u,\lambda,\varphi)h_{5(0)}(u,\lambda),\\
\eta_4 &= -h^2_{(0)}f(u,\lambda)\left\{\left[\omega^{-1}(u,\lambda,\varphi)\right]^*\right\}^2, \eta_5=\omega^{-2}(u,\lambda,\varphi), \qquad (10.76)\\
w_i &= 0, n_i\{x,\omega,\omega^*\}=n_i\{u,\lambda,\omega(u,\lambda,\varphi),\omega^*(u,\lambda,\varphi)\},\\
\Omega &= \varpi^{-1}(u,\lambda)\Omega_{[2]}(u,\lambda,\varphi),\ \zeta_{\underline{i}}=\partial_{\underline{i}}\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^*+\left(\Omega^*_{[2]}\right)^{-1}\partial_{\underline{i}}\Omega_{[2]},\\
\eta_4 &= -\varpi^{-q_1/q_2}(u,\lambda)\Omega^{q_1/q_2}_{[2]}(u,\lambda,\varphi).
\end{aligned}
$$

and

$$
\begin{aligned}
t_c\text{–solutions} &: (x^2=u, x^3=\lambda, y^4=v=t, y^5=p=\varphi)\\
g_2 &= g_3=\varpi(u,\lambda)g(u,\lambda),\\
h_{4(0)} &= \varpi(u,\lambda)f(u,\lambda), h_{5(0)}=-\varpi(u,\lambda), \text{see (10.58)};\\
\varpi &= g^{-1}\varpi_0\exp[a_2u+a_3\lambda],\ \varpi_0,a_2,a_3=const,\ \text{see (10.26)}\\
h_4 &= \eta_4(u,\lambda,t)h_{4(0)}(x^i), h_5=\eta_5(u,\lambda,t)h_{5(0)}(x^i),
\end{aligned}
$$

$$
\begin{aligned}
\eta_4 &= \omega^{-2}(u,\lambda,t), \eta_5=-h^{-2}_{(0)}f(u,\lambda)\left[\int dt\ \omega^{-1}(u,\lambda,t)\right]^2, \qquad (10.77)\\
w_i &= 0, n_i\{x,\omega,\omega^*\}=n_i\{u,\lambda,\omega(u,\lambda,t),\omega^*(u,\lambda,t)\},
\end{aligned}
$$

$$\Omega = \varpi^{-1}(u,\lambda)\Omega_{[2]}(u,\lambda,t), \zeta_i = \partial_i(\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega^*_{[2]}\right)^{-1}\partial_i\Omega_{[2]},$$
$$\eta_4 = -\varpi^{-q_1/q_2}(u,\lambda)\Omega_{[2]}^{q_1/q_2}(u,\lambda,t).$$

where the coefficients n_i can be found explicitly by introducing the corresponding values η_4 and η_5 in formula (10.61).

For the 4D Class B solutions one can be imposed some conditions (see previous subsection) when the solutions (10.76) and (10.77) have in the locally anisotropic limit the Schwarzschild solution, which imposes some specific parametrizations and dependencies on $\Omega_{[2]}(u,\lambda,v)$ and $\omega(u,\lambda,v)$ like (10.66) and (10.68). We omit these considerations because for aholonomic configurations and nonlinear solutions there are not arguments to prefer any holonomic limits of such off–diagonal metrics.

We conclude this Section by noting that for the considered classes of ellipsoidal black hole 4D solutions the so–called t–component of metric contains modifications of the Schwarzschild potential

$$\Phi = -\frac{M}{M^2_{P[4]}r} \text{ into } \Phi = -\frac{M\omega(u,\lambda,v)}{M^2_{P[4]}r},$$

where $M_{P[4]}$ is the usual 4D Plank constant; the metric coefficients are given with respect to the corresponding anholonomic frame of reference. In 4D anholonomic gravity the receptivity $\omega(u,\lambda,v)$ is considered to renormalize the mass constant. Such gravitational self-polarizations are induced by anholonomic vacuum gravitational interactions. They should be defined experimentally or computed following a model of quantum gravity.

10.5 The Cosmological Constant and Anisotropy

In this Section we analyze the general properties of anholonomic Einstein equations in 5D and 4D gravity with cosmological constant and construct a 5D exact solution with cosmological constant.

10.5.1 4D and 5D Anholnomic Einstein spaces

There is a difference between locally anisotropic 4D and 5D gravity. The first theory admits an "isotropic" 4D cosmological constant $\Lambda_{[4]} = \Lambda$ even for anisotropic gravitational configurations. The second, 5D, theory admits extensions of vacuum anistoropic solutions to those with a cosmological constant only for anisotropic 5D sources parametrized like $\Lambda_{[5]\alpha\beta} = (2\Lambda g_{11}, \Lambda g_{\underline{\alpha\beta}})$ (see the Corollary 4 below). We emphasize that the conclusions from this subsection refer to the two classes of ansatz (10.9) and (10.19).

The simplest way to consider a source into the 4D Einstein equations, both with or not anistoropy, is to consider a gravitational constant Λ and to write the field equations

$$G_{\underline{\beta}}^{\underline{\alpha}} = \Lambda_{[4]}\delta_{\underline{\beta}}^{\underline{\alpha}} \tag{10.78}$$

which means that we introduced a "vacuum" energy–momentum tensor $\kappa\Upsilon_{\underline{\beta}}^{\underline{\alpha}} = \Lambda_{[4]}\delta_{\underline{\beta}}^{\underline{\alpha}}$ which is diagonal with respect to anholonomic frames and the conditions (10.51) transforms into $\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_4^4 = \Upsilon_5^5 = \kappa^{-1}\Lambda$. According to A. Z. Petrov [20] the spaces described by solutions of the Einstein equations

$$R_{\underline{\alpha\beta}} = \Lambda g_{\underline{\alpha\beta}}, \Lambda = const$$

are called the Einstein spaces. With respect to anisotropic frames we shall use the term anholonomic (equivalently, anisotropic) Einstein spaces.

In order to extend the equations (10.78) to 5D gravity we have to take into consideration the compatibility conditions for the energy–momentum tensors (10.34).

Corollary 10.5.4. *We are able to satisfy the conditions of the Corollary 2 if we consider a 5D diagonal source $\Upsilon_\beta^\alpha = \{2\Lambda,\ \Upsilon_{\underline{\beta}}^{\underline{\alpha}} = \Lambda\delta_{\underline{\beta}}^{\underline{\alpha}}\}$, for an anisotropic 5D cosmological constant source $(2\Lambda g_{11}, \Lambda g_{\underline{\alpha\beta}})$. The 5D Einstein equations with anisotropic cosmological "constants", for ansatz (10.9) are written in the form*

$$R_2^2 = S_4^4 = -\Lambda. \tag{10.79}$$

These equations without coordinate x^1 and g_{11} hold for the (10.39). We can extend the constructions for the ansatz with conformal factors, (10.19) and (10.41) by considering additional coefficients ζ_i satisfying the equations (10.25) and (10.49) for non vanishing values of w_i.

The proof follows from Corollaries 1 and 2 formulated respectively to 4D and 5D gravity (see formulas (10.50) and (10.51) and, correspondingly, (10.33) and (10.34)).

Theorem 10.5.4. *The nontrivial components of the 5D Einstein equations with anisotropic cosmological constant, $R_{11} = 2\Lambda g_{11}$ and $R_{\underline{\alpha\beta}} = \Lambda g_{\underline{\alpha\beta}}$, for the ansatz (10.19) and anholonomic metric (10.20) given with respect to anholonomic frames (10.21) and (10.22) are written in a form with separation of variables:*

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2} = 2\Lambda g_2 g_3, \tag{10.80}$$

$$h_5^{**} - h_5^{*}[\ln\sqrt{|h_4 h_5|}]^{*} = 2\Lambda h_4 h_5, \tag{10.81}$$

$$w_i\beta + \alpha_i = 0, \tag{10.82}$$

$$n_i^{**} + \gamma n_i^{*} = 0, \tag{10.83}$$

$$\partial_i\Omega - (w_i + \zeta_i)\Omega^{*} = 0. \tag{10.84}$$

where

$$\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \beta = 2\Lambda h_4 h_5, \gamma = 3h_5^*/2h_5 - h_4^*/h_4. \qquad (10.85)$$

The Theorem 4 is a generalization of the Theorem 2 for energy–momentum tensors induced by the an anisotropic 5D constant. The proof follows from (10.13)–(10.16) and (10.25), revised as to satisfy the formulas (10.36) and (10.37) with that substantial difference that $\beta \neq 0$ and in this case, in general, $w_i \neq 0$. We conclude that in the presence of a nonvanishing cosmological constant the equations (10.13) and (10.14) transform respectively into (10.80) and (10.81) which have a more general nonlinearity because of the $2\Lambda g_2 g_3$ and $2\Lambda h_4 h_5$ terms. For instance, the solutions with $g_2 = const$ and $g_3 = const$ (and $h_4 = const$ and $h_5 = const$) are not admitted. This makes more sophisticate the procedure of definition of g_2 for a given g_3 (or inversely, of definition of g_3 for a given g_2) from (10.80) [similarly of construction h_4 for a given h_5 from (10.81) and inversely], nevertheless, the separation of variables is not affected by introduction of cosmological constant and there is a number of possibilities to generate new exact solutions.

The general properties of solutions of the system (10.80)–(10.84) are stated by the

Theorem 10.5.5. *The system of second order nonlinear partial differential equations (10.80)-(10.83) and (10.84) can be solved in general form if there are given some values of functions $g_2(x^2, x^3)$ (or $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or $h_5(x^i, v)$) and $\Omega(x^i, v)$:*

- *The general solution of equation (10.80) is to be found from the equation*

$$\varpi\varpi^{\bullet\bullet} - (\varpi^{\bullet})^2 + \varpi\varpi'' - (\varpi')^2 = 2\Lambda\varpi^3. \qquad (10.86)$$

  for a coordinate transform coordinate transforms $x^{2,3} \to \widetilde{x}^{2,3}(u, \lambda)$ for which

$$g_2(u,\lambda)(du)^2 + g_3(u,\lambda)(d\lambda)^2 \to \varpi\left[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\right], \epsilon = \pm 1$$

  and $\varpi^{\bullet} = \partial\varpi/\partial\widetilde{x}^2$ and $\varpi' = \partial\varpi/\partial\widetilde{x}^3$.

- *The equation (10.81) relates two functions $h_4(x^i, v)$ and $h_5(x^i, v)$ with $h_5^* \neq 0$. If the function h_5 is given we can find h_4 as a solution of*

$$h_4^* + \frac{2\Lambda}{\tau}(h_4)^2 + 2\left(\frac{\tau^*}{\tau} - \tau\right) h_4 = 0, \qquad (10.87)$$

  where $\tau = h_5^*/2h_5$.

- *The exact solutions of (10.82) for* $\beta \neq 0$ *is*

$$\begin{aligned} w_k &= -\alpha_k/\beta, \\ &= \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \end{aligned} \quad (10.88)$$

for $\partial_v = \partial/\partial v$ *and* $h_5^* \neq 0$.

- *The exact solution of (10.83) is*

$$\begin{aligned} n_k &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [h_4/(\sqrt{|h_5|})^3] dv, \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [1/(\sqrt{|h_5|})^3] dv, \;\; h_4^* = 0, \end{aligned} \quad (10.89)$$

for some functions $n_{k[1,2]}\left(x^i\right)$ *stated by boundary conditions.*

- *The exact solution of (10.25) is given by*

$$\zeta_i = -w_i + (\Omega^*)^{-1}\partial_i\Omega, \quad \Omega^* \neq 0, \quad (10.90)$$

We note that by a corresponding re–parametrizations of the conformal factor $\Omega\left(x^i, v\right)$ we can reduce (10.86) to

$$\varpi\varpi^{\bullet\bullet} - (\varpi^{\bullet})^2 = 2\Lambda\varpi^3 \quad (10.91)$$

which has an exact solution $\varpi = \varpi\left(\widetilde{x}^2\right)$ to be found from

$$(\varpi^{\bullet})^2 = \varpi^3\left(C\varpi^{-1} + 4\Lambda\right), C = const,$$

(or, inversely, to reduce to

$$\varpi\varpi^{''} - (\varpi^{'})^2 = 2\Lambda\varpi^3$$

with exact solution $\varpi = \varpi\left(\widetilde{x}^3\right)$ found from

$$(\varpi^{\prime})^2 = \varpi^3\left(C\varpi^{-1} + 4\Lambda\right), C = const).$$

The inverse problem of definition of h_5 for a given h_4 can be solved in explicit form when $h_4^* = 0$, $h_4 = h_{4(0)}(x^i)$. In this case we have to solve

$$h_5^{**} + \frac{(h_5^*)^2}{2h_5} - 2\Lambda h_{4(0)} h_5 = 0, \quad (10.92)$$

which admits exact solutions by reduction to a Bernulli equation.

The proof of Theorem 5 is outlined in Appendix C.

The conditions of the Theorem 4 and 5 can be reduced to 4D anholonomic spacetimes with "isotropic" cosmological constant Λ. To do this we have to eliminate dependencies on the coordinate x^1 and to consider the 4D ansatz without g_{11} term as it was stated in the subsection II E.

10.5.2 5D anisotropic black holes with cosmological constant

We give an example of generalization of anisotropic black hole solutions of Class A , constructed in the Section III, as they will contain the cosmological constant Λ; we extend the solutions given by the data (10.64).

Our new 5D $\varphi-$ solution is parametrized by an ansatz with conformal factor $\Omega(x^i, v)$ (see (10.19) and (10.20)) as $\Omega = \varpi^{-1}(u)\Omega_{[0]}(x^i)\varpi^{-1}(u)\Omega_{[1]}(x^i, v)$. The factor $\varpi(u)$ is chosen to be a solution of (10.91). This conformal data must satisfy the conditions (10.24) and (10.32), i. e.

$$\varpi^{-q_1/q_2}\Omega_{[0]}^{q_1/q_2}\Omega_{[1]}^{q_1/q_2} = \eta_4 \varpi h_{4(0)}$$

for some integers q_1 and q_2, where η_4 is found as $h_4 = \eta_4 \varpi h_{4(0)}$ is a solution of equation (10.87). The factor $\Omega_{[0]}(x^i)$ could be chosen as to obtain in the locally isotropic limit and $\Lambda \to 0$ the Schwarzschild metric in ellipsoidal coordinates (10.55). Putting $h_5 = \eta_5 \varpi h_{5(0)}$, $\eta_5 h_{5(0)}$ in the ansatz for (10.64), for which we compute the value $\tau = h_5^*/2h_5$, we obtain from (10.87) an equation for η_4,

$$\eta_4^* + \frac{2\Lambda}{\tau}\varpi h_{4(0)}(\eta_4)^2 + 2\left(\frac{\tau^*}{\tau} - \tau\right)\eta_4 = 0$$

which is a Bernulli equation [18] and admit an exact solution, in general, in non explicit form, $\eta_4 = \eta_4^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b)$, were we emphasize the functional dependencies on functions ϖ, ω, a, b and cosmological constant Λ. Having defined $\eta_{4[bern]}$, η_5 and ϖ, we can compute the α_i-, $\beta-$, and $\gamma-$coefficients, expressed as

$$\alpha_i = \alpha_i^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b), \beta = \beta^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b)$$

and $\gamma = \gamma^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b)$, following the formulas (10.85).

The next step is to find

$$w_i = w_i^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b) \text{ and } n_i = n_i^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b)$$

as for the general solutions (10.88) and (10.89).

At the final step we are able to compute the the second anisotropy coefficients

$$\zeta_i = -w_i^{[bern]} + \left(\partial_i \ln |\varpi^{-1}\Omega_{[0]}) |\right) \left(\ln |\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1} \partial_i \Omega_{[1]},$$

which depends on an arbitrary function $\Omega_{[0]}(u, \lambda)$. If we state $\Omega_{[0]}(u, \lambda) = \Omega_A$, as for Ω_A from (10.58), see similar details with respect to formulas (10.66), (10.67) and (10.68).

The data for the exact solutions with cosmological constant for $v = \varphi$ can be stated in the form

$$
\begin{aligned}
\varphi_c\text{–solutions} &: (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= \varpi(u), g_3 = \varpi(u), \\
h_{4(0)} &= a(u,\lambda), h_{5(0)} = b(u,\lambda), \text{see (10.56) and (10.91)}; \\
h_4 &= \eta_4(x^i,\varphi)\varpi(u)h_{4(0)}(x^i), h_5 = \eta_5(x^i,\varphi)\varpi(u)h_{5(0)}(x^i), \\
\eta_4 &= \eta_4^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b), \eta_5 = \omega^{-2}(\chi, u, \lambda, \varphi), \qquad (10.93) \\
w_i &= w_i^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b), n_i\{x, \omega, \omega^*\} = n_i^{[bern]}(x^i, v, \Lambda, \varpi, \omega, a, b), \\
\Omega &= \varpi^{-1}(u)\Omega_{[0]}(x^i)\Omega_{[1]}(x^i,\varphi), \eta_4 a = \Omega_{[0]}^{q_1/q_2}(x^i)\Omega_{[1]}^{q_1/q_2}(x^i,\varphi). \\
\zeta_i &= -w_i^{[bern]} + \left(\partial_i \ln|\varpi^{-1}\Omega_{[0]}\right)|) \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]}.
\end{aligned}
$$

We note that a solution with $v = t$ can be constructed as to generalize (10.65) in order to contain Λ. We can not present such data in explicit form because in this case we have to define η_5 by integrating an equation like (10.81) for h_5, for a given h_4, with $h_4^* \neq 0$ which can not be integrated in explicit form.

The solution (10.93) has has the same the two very interesting properties as the solution (10.64): 1) it admits a warped factor on the 5th coordinate, like $\Omega_{[1]}^{q_1/q_2} \sim \exp[-k|\chi|]$, which in this case is constructed for an anisotropic 5D vacuum gravitational configuration with anisotropic cosmological constant and does not follow from a brane configuration like in Refs. [7]; 2) we can impose such conditions on the receptivity $\omega(x^i,\varphi)$ as to obtain in the locally isotropic limit just the Schwarzschild metric (10.55) trivially embedded into the 5D spacetime (the procedure is the same as in the subsection IIIB).

Finally, we note that in a similar manner like in the Sections III and IV we can construct another classes of anisotropic black holes solutions in 5D and 4D spacetimes with cosmological constants, being of Class A or Class B, with anisotropic φ–coordinate, or anisotropic t–coordinate. We omit the explicit data which are some nonlinear anholonomic generalizations of those solutions.

10.6 Conclusions

We formulated a new method of constructing exact solutions of Einstein equations with off–diagonal metrics in 4D and 5D gravity. We introduced ahnolonomic transforms which diagonalize metrics and simplify the system of gravitational field equations. The method works also for gravitational configurations with cosmological constants and for non–trivial matter sources. We constructed different classes of new exact solutions of the Einstein equations is 5D and 4D gravity which describe a generic anholonomic (anisotropic) dynamics modelled by off–diagonal metrics and anholonomic frames with mixed holonomic and anholonomic variables. They extend the class of exact solutions with linear extensions to the bulk 5D gravity [21].

We emphasized such exact solutions which can be associated to some black hole like configurations in 5D and 4D gravity. We consider that the constructed off–diagonal metrics define anisotropic black holes because they have a static horizon parametrized by a rotation ellipsoid hypersurface, they are singular in focuses of ellipsoid (or on the circle of focuses, for flattened ellipsoids) and they reduce in the locally anisotropic limit, with holonomic coordinates, to the Schwarzschild solution in ellipsoidal coordinates, or to some conformal transforms of the Schwarzschild metric.

The new classes of solutions admit variations of constants (in time and extra dimension coordinate) and anholonomic gravitational polarizations of masses which are induced by nonlinear gravitational interactions in the bulk of 5D gravity and by a constrained (anholonomic) dynamics of the fields in the 4D gravity. There are possible solutions with warped factors which are defined by some vacuum 5D gravitational interactions in the bulk and not by a specific brane configuration with energy–momentum tensor source. We emphasized anisotropies which in the effective 4D spacetime preserve the local Lorentz invariance but the method allows constructions with violation of local Lorentz symmetry like in Refs. [22]. In order to generate such solutions we should admit that the metric coefficients depends, for instance, anisotropically on extra dimension coordinate.

It should be noted that the anholonomic frame method deals with generic off–diagonal metrics and nonlinear systems of equations and allows to construct substantially nonlinear solutions. In general, such solutions could not have a locally isotropic limit with a holonomic analog. We can understand the physical properties of such solutions by analyzing both the metric coefficients stated with respect to an adapted anholonomic frame of reference and by a study of the coefficients defining such frames.

There is a subclass of static anisotropic black holes solutions, with static ellipsoidal horizons, which do not violate the well known Israel and Carter theorems [19] on spherical symmetry of solutions in asymptotically flat spacetimes. Those theorems were proved in the radial symmetry asymptotic limit and for holonomic coordinates. There is not a much difference between 3D static spherical and ellipsoidal horizons at long distances. In other turn, the statements of the mentioned theorems do not refers to generic off–diagonal gravitational metrics, anholonomic frames and anholonomic deformations of symmetries.

Finally, we note that the anholonomic frame method may have a number of applications in modern brane and string/M–theory gravity because it defines a general formalism of constructing exact solutions with off–diagonal metrics. It results in such prescriptions on anholonomic "mappings" of some known locally isotropic solutions from a gravity/string theory that new types of anisotropic solutions are generated:

A vacuum, or non-vacuum, solution, and metrics conformally equivalent to a such solution, parametrized by a diagonal matrix given with respect to a holonomic (coordinate) base, contained in a trivial form of ansatz (10.9), or (10.19), can be generalized to an anisotropic solution with similar but anisotropically renormalized physical constants and diagonal metric coefficients given with respect to adapted anholonomic frames; the new anholonomic metric defines an exact solution of a simplified form of the Einstein equations (10.13)–(10.16) and (10.25); such solutions are parametrized by off–diagonal metrics if they are re–defined with respect to coordinate frames .

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10.7 A: Anholonomic Frames and Nonlinear Connections

For convenience, we outline here the basic formulas for connections, curvatures and, induced by anholonomic frames, torsions on (pseudo) Riemannian spacetimes provided with N–coefficient bases (10.5) and (10.6) [1, 4]. The N–coefficients define an associated nonlinear connection (in brief, N–connection) structure. On (pseudo)–Riemannian spacetimes the N–connection structure can be treated as a "pure" anholonomic frame effect which is induced if we are dealing with mixed sets of holonomic–anholonomic basis vectors. When we are transferring our considerations only to coordinate frames (10.2) and (10.3) the N–connection coefficients are removed into both off–diagonal and diagonal components of the metric like in (10.9). In some cases the N–connection (anholonomic) structure is to be stated in a non–dynamical form by definition of some initial (boundary) conditions for the frame structure, following some prescribed symmetries of the gravitational–matter field interactions, or , in another cases, a subset of N–coefficients have to be treated as some dynamical variables defined as to satisfy the Einstein equations.

10.7.1 D–connections, d–torsions and d–curvatures

If a pseudo–Riemannian spacetime is enabled with a N–connection structure, the components of geometrical objects (for instance, linear connections and tensors) are distinguished into horizontal components (in brief h–components, labelled by indices like $i, j, k, ...$) and vertical components (in brief v–components, labelled by indices like $a, b, c, ..$). One call such objects, distinguished (d) by the N–connection structure, as d–tensors, d–connections, d–spinors and so on [12, 3, 1].

D–metrics and d-connections:

A metric of type (10.10), in general, with arbitrary coefficients $g_{ij}\left(x^k, y^a\right)$ and $h_{ab}\left(x^k, y^a\right)$ defined with respect to a N–elongated basis (10.6) is called a d–metric.

A linear connection $D_{\delta_\gamma}\delta_\beta = \Gamma^\alpha_{\ \beta\gamma}\left(x, y\right)\delta_\alpha$, associated to an operator of covariant derivation D, is compatible with a metric $g_{\alpha\beta}$ and N–connection structure on a 5D pseudo–Riemannian spacetime if $D_\alpha g_{\beta\gamma} = 0$. The linear d–connection is parametrized by irreducible h–v–components, $\Gamma^\alpha_{\ \beta\gamma} = \left(L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}\right)$, where

$$
\begin{aligned}
L^i_{\ jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \\
L^a_{\ bk} &= \partial_b N^a_k + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k\right), \\
C^i_{\ jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\ C^a_{\ bc} = \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right).
\end{aligned}
\tag{10.94}
$$

This defines a canonical linear connection (distinguished by a N–connection, in brief, the canonical d–connection) which is similar to the metric connection introduced by Christoffel symbols in the case of holonomic bases.

D–torsions and d–curvatures:

The anholonomic coefficients $W^\gamma_{\ \alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^\alpha_{\ \beta\gamma}\delta_\alpha$, where

$$
T^\alpha_{\ \beta\gamma} = \Gamma^\alpha_{\ \beta\gamma} - \Gamma^\alpha_{\ \gamma\beta} + w^\alpha_{\ \beta\gamma}, \tag{10.95}
$$

and for the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha$, where

$$
\begin{aligned}
R_{\beta\ \gamma\tau}^{\ \alpha} = {} & \delta_\tau\Gamma^\alpha_{\ \beta\gamma} - \delta_\gamma\Gamma^\alpha_{\ \beta\tau} \\
& + \Gamma^\varphi_{\ \beta\gamma}\Gamma^\alpha_{\ \varphi\tau} - \Gamma^\varphi_{\ \beta\tau}\Gamma^\alpha_{\ \varphi\gamma} + \Gamma^\alpha_{\ \beta\varphi}w^\varphi_{\ \gamma\tau}.
\end{aligned}
\tag{10.96}
$$

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components, $T^{\alpha}_{\ \beta\gamma} = \{T^{i}_{\ jk}, C^{i}_{\ ja}, S^{a}_{\ bc}, T^{a}_{\ ij}, T^{a}_{\ bi}\}$, where

$$\begin{array}{rcl} T^{i}_{.jk} & = & T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{ja} = C^{i}_{.ja}, \quad T^{i}_{aj} = -C^{i}_{ja}, \\ T^{i}_{.ja} & = & 0, \quad T^{a}_{.bc} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}, \\ T^{a}_{.ij} & = & -\Omega^{a}_{ij}, \quad T^{a}_{.bi} = \partial_b N^{a}_{i} - L^{a}_{.bi}, \quad T^{a}_{.ib} = -T^{a}_{.bi} \end{array} \tag{10.97}$$

(the d–torsion is computed by substituting the h–v–components of the canonical d–connection (10.94) and anholonomic coefficients (10.7) into the formula for the torsion coefficients (10.95)), where

$$\Omega^{a}_{ij} = \delta_j N^{a}_{i} - \delta_i N^{a}_{j}$$

is called the N–connection curvature (N–curvature).

The curvature d-tensor has the following irreducible, non-vanishing, h–v–components $R_{\beta\ \gamma\tau}^{\ \alpha} = \{R^{.i}_{h.jk}, R^{.a}_{b.jk}, P^{.i}_{j.ka}, P^{.c}_{b.ka}, S^{.i}_{j.bc}, S^{.a}_{b.cd}\}$, where

$$\begin{array}{rcl} R^{.i}_{h.jk} & = & \delta_k L^{i}_{.hj} - \delta_j L^{i}_{.hk} + L^{m}_{.hj} L^{i}_{mk} - L^{m}_{.hk} L^{i}_{mj} - C^{i}_{.ha} \Omega^{a}_{.jk}, \\ R^{.a}_{b.jk} & = & \delta_k L^{a}_{.bj} - \delta_j L^{a}_{.bk} + L^{c}_{.bj} L^{a}_{.ck} - L^{c}_{.bk} L^{a}_{.cj} - C^{a}_{.bc} \Omega^{c}_{.jk}, \\ P^{.i}_{j.ka} & = & \partial_a L^{i}_{.jk} + C^{i}_{.jb} T^{b}_{.ka} - (\delta_k C^{i}_{.ja} + L^{i}_{.lk} C^{l}_{.ja} - L^{l}_{.jk} C^{i}_{.la} - L^{c}_{.ak} C^{i}_{.jc}), \\ P^{.c}_{b.ka} & = & \partial_a L^{c}_{.bk} + C^{c}_{.bd} T^{d}_{.ka} - (\delta_k C^{c}_{.ba} + L^{c}_{.dk} C^{d}_{.ba} - L^{d}_{.bk} C^{c}_{.da} - L^{d}_{.ak} C^{c}_{.bd}), \\ S^{.i}_{j.bc} & = & \partial_c C^{i}_{.jb} - \partial_b C^{i}_{.jc} + C^{h}_{.jb} C^{i}_{.hc} - C^{h}_{.jc} C^{i}_{hb}, \\ S^{.a}_{b.cd} & = & \partial_d C^{a}_{.bc} - \partial_c C^{a}_{.bd} + C^{e}_{.bc} C^{a}_{.ed} - C^{e}_{.bd} C^{a}_{.ec} \end{array} \tag{10.98}$$

(the d–curvature components are computed in a similar fashion by using the formula for curvature coefficients (10.96)).

10.7.2 Einstein equations with holonomic–anholonomic variables

In this subsection we write and analyze the Einstein equations on 5D (pseudo) Riemannian spacetimes provided with anholonomic frame structures and associated N–connections.

Einstein equations with matter sources

The Ricci tensor $R_{\beta\gamma} = R_{\beta\ \gamma\alpha}^{\ \alpha}$ has the d–components

$$\begin{aligned} R_{ij} &= R_{i.jk}^{.k}, \quad R_{ia} = -{}^2P_{ia} = -P_{i.ka}^{.k}, \\ R_{ai} &= {}^1P_{ai} = P_{a.ib}^{.b}, \quad R_{ab} = S_{a.bc}^{.c}. \end{aligned} \tag{10.99}$$

In general, since ${}^1P_{ai} \neq {}^2P_{ia}$, the Ricci d-tensor is non-symmetric (this could be with respect to anholonomic frames of reference). The scalar curvature of the metric d–connection, $\overleftarrow{R} = g^{\beta\gamma}R_{\beta\gamma}$, is computed

$$\overleftarrow{R} = G^{\alpha\beta}R_{\alpha\beta} = \widehat{R} + S, \tag{10.100}$$

where $\widehat{R} = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$.

By substituting (10.99) and (10.100) into the 5D Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa\Upsilon_{\alpha\beta}, \tag{10.101}$$

where κ and $\Upsilon_{\alpha\beta}$ are respectively the coupling constant and the energy–momentum tensor we obtain the h-v-decomposition by N–connection of the Einstein equations

$$\begin{aligned} R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right)g_{ij} &= \kappa\Upsilon_{ij}, \\ S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right)h_{ab} &= \kappa\Upsilon_{ab}, \\ {}^1P_{ai} = \kappa\Upsilon_{ai},\ {}^2P_{ia} &= \kappa\Upsilon_{ia}. \end{aligned} \tag{10.102}$$

The definition of matter sources with respect to anholonomic frames is considered in Refs. [3, 1].

5D vacuum Einstein equations

The vacuum 5D, locally anisotropic gravitational field equations, in invariant h– v–components, are written

$$\begin{aligned} R_{ij} &= 0, S_{ab} = 0, \\ {}^1P_{ai} &= 0,\ {}^2P_{ia} = 0. \end{aligned} \tag{10.103}$$

The main 'trick' of the anholonomic frames method for integrating the Einstein equations in general relativity and various (super) string and higher / lower dimension gravitational theories is to find the coefficients N_j^a such that the block matrices g_{ij} and h_{ab} are diagonalized [3, 1, 4]. This greatly simplifies computations. With respect to such anholonomic frames the partial derivatives are N–elongated (locally anisotropic).

10.8 B: Proof of the Theorem 3

We prove step by step the items of the Theorem 3.

The first statement with respect to the solution of (10.13) is a connected with the well known result from 2D (pseudo) Riemannian gravity that every 2D metric can be redefined by using coordinate transforms into a conformally flat one.

The equation (10.14) can be treated as a second order differential equation on variable v, with parameters x^i, for the unknown function $h_5(x^i, v)$ if the value of $h_4(x^i, v)$ is given (or inversely as a first order differential equation on variable v, with parameters x^i, for the unknown function $h_4(x^i, v)$ if the value of $h_5(x^i, v)$ is given). The formulas (10.29) and (10.28) are consequences of integration on v of the equation (10.14) being considered also the degenerated cases when $h_5^* = 0$ or $h_4^* = 0$.

Having defined the values h_4 and h_5, we can compute the values the coefficients α_i, β and γ (10.17) and find the coefficients w_i and n_i The first set (8.10) for w_i is a solution of three independent first order algebraic equations (10.15) with known coefficients α_i and β. The second set of solutions (10.31) for n_i is found after two integrations on the anisotropic variable v of the independent equations (10.16) with known γ (the variables x^i being considered as parameters). In the formulas (10.31) we distinguish also the degenerated cases when $h_5^* = 0$ or $h_4^* = 0$.

Finally, we note that the formula (10.32) is a simple algebraic consequence from (10.25).

The Theorem 3 has been proven.

10.9 C: Proof of Theorem 5

We emphasize the first two items:

- The equation (10.80) imposes a constraint on coefficients of a diagonal 2D metric parametrized by coordinates $x^2 = u$ and $x^3 = \lambda$. By coordinate transforms $x^{2,3} \to \widetilde{x}^{2,3}(u, \lambda)$, see for instance, [20] we can reduce 2D every metric

$$ds^2_{[2]} = g_2(u, \lambda) du^2 + g_3(u, \lambda) d\lambda^2$$

to a conformally flat one

$$ds^2_{[2]} = \varpi(\widetilde{x}^2, \widetilde{x}^3) \left[d(\widetilde{x}^2)^2 + \epsilon d(\widetilde{x}^3)^2 \right], \epsilon = \pm 1.$$

with conformal factor $\varpi(\widetilde{x}^2, \widetilde{x}^3)$, for which (10.80) transforms into (10.86) with new 'dot' and 'prime' derivatives $\varpi^\bullet = \partial\varpi/\partial\widetilde{x}^2$ and $\varpi' = \partial\varpi/\partial\widetilde{x}^3$. It is not possible to find an explicit form of the general solution of (10.86). If we approximate, for instance, that $\varpi = \varpi\left(\widetilde{x}^2\right)$, the equation

$$\varpi\varpi^{\bullet\bullet} - (\varpi^{\bullet})^2 = 2\Lambda\varpi^3$$

has an exact solution (see 6.127 in [18]) which can be found from a Bernulli equation

$$(\varpi^{\bullet})^2 = \varpi^3\left(C\varpi^{-1} + 4\Lambda\right), C = const,$$

which allow us to find $\widetilde{x}^2(\varpi)$, or, in non explicit form $\varpi = \varpi\left(\widetilde{x}^2\right)$. We can chose a such solution as a background one and by using conformal factors $\Omega(\widetilde{x}^2, \widetilde{x}^3)$, transforming $\varpi(\widetilde{x}^2, \widetilde{x}^3)$ into $\varpi\left(\widetilde{x}^2\right)$ we can generate solutions of the 5D Einstein equations with anisotropic cosmological constant by inducing second order anisotropy ζ_i. The case when $\varpi = \varpi\left(\widetilde{x}^3\right)$ is to be obtained in a similar manner by changing the 'dot' derivative into 'prime' derivative.

- The equation (10.81) does not admit $h_5^* = 0$ because in this case we must have $h_5 = 0$. For a given value of h_5, introducing a new variable $\tau = h_5^*/2h_5$ we can transform (10.81) into a first order nonlinear equation for h_4 (10.104),which can be transformed [18] to a Ricatti, then to a Bernulli equation which admits exact solutions. We note that the holonomic coordinates are considered as parameters. The inverse problem, to find h_5 for a given h_4 is more complex because is connected with solution of a second order nonlinear differential equation

$$h_5^{**} + \frac{(h_5^*)^2}{2h_5} - \frac{h_4^*}{2h_4}h_5^* - 2\Lambda h_4 h_5 = 0, \tag{10.104}$$

  which can not integrated in general form. Nevertheless, a very general class of solutions can be found explicitly if $h_4^* = 0$, i. e. if h_4 depend only on holonomic coordinates. In this case the equation (10.104) can be reduced to a Bernulli equation [18] which admits exact solutions.

- The formulas (10.88), (10.89) and (10.90) solving respectively (10.82), (10.83) and (10.84) are proven similarly as for the Theorem 3 with that difference that in the presence of the cosmological term $h_5^* \neq 0, \beta \neq 0$ and, in general, $w_i \neq 0$.

The Theorem 5 has been proven.

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Chapter 11

Black Tori Solutions in Einstein and 5D Gravity

Abstract [1]

The 'anholonomic frame' method [1, 2, 3] is applied for constructing new classes of exact solutions of vacuum Einstein equations with off–diagonal metrics in 4D and 5D gravity. We examine several black tori solutions generated by anholonomic transforms with non–trivial topology of the Schwarzschild metric, which have a static toroidal horizon. We define ansatz and parametrizations which contain warping factors, running constants (in time and extra dimension coordinates) and effective nonlinear gravitational polarizations. Such anisotropic vacuum toroidal metrics, the first example was given in [1], differ substantially from the well known toroidal black holes [4] which were constructed as non–vacuum solutions of the Einstein–Maxwell gravity with cosmological constant. Finally, we analyze two anisotropic 5D and 4D black tori solutions with cosmological constant.

11.1 Introduction

Black hole - torus systems [5] and toroidal black holes [4, 1] became objects of astrophysical interest since it was shown that they are inevitable outcome of complete gravitational collapse of a massive star, cluster of stars, or can be present in the center of galactic systems.

Black hole and black tori solutions appear naturally as exact solutions in general relativity and extra dimension gravity theories. Such solutions can be constructed in both asymptotically flat spacetiems and in spacetimes with cosmological constant, posses a specific supersymmetry and could be with toroidal, cylindrical or planar topology [4].

String theory suggests that we may live in a fundamentally higher dimensional spacetime [6]. The recent approaches are based on the assumption that our Universe is realized as a three dimensional (in brief, 3D) brane, modelling a 4D pseudo–Riemannian spacetime, embedded in the 5D anti–de Sitter (AdS_5) bulk spacetime. It was proposed in the Rundall and Sundrum (RS) papers [7] that such models could be with relatively large extra dimension as a way to solve the hierarchy problem in high energy physics.

In the present paper we explore possible black tori solutions in 5D and 4D gravity. We obtain a new class of exact solutions to the 5D vacuum Einstein equations in the bulk, which have toroidal horizons and are related via anholonomic transforms with toroidal deformations of the Schwarzschild solutions. The solutions could be with warped factors, running constants and anisotropic gravitational polarizations. We than consider 4D black tori solutions and generalize both 5D and 4D constructions for spacetimes with cosmological constant.

We also discuss implications of existence of such anisotropic black tori solutions with non-trivial topology to the extra dimension gravity and general relativity theory. We prove that warped metrics can be obtained from vacuum 5D gravity and not only from a brane configurations with specific energy–momentum tensor.

We apply the Salam, Strathee and Peracci [8] idea on a gauge field like status of the coefficients of off–diagonal metrics in extra dimension gravity and develop it in a new fashion by applying the method of anholonomic frames with associated nonlinear connections on 5D and 4D (pseudo) Riemannian spaces [1, 2, 3].

We use the term 'locally anisotropic' spacetime (or 'anisotropic' spacetime) for a 5D (4D) pseudo-Riemannian spacetime provided with an anholonomic frame structure with mixed holonomic and anholonomic variables. The anisotropy of gravitational interactions is modelled by off–diagonal metrics, or, equivalently, by theirs diagonalized analogs given with respect to anholonomic frames.

The paper is organized as follow: In Sec. II we consider two off–diagonal metric ansatz, construct the corresponding exact solutions of 5D vacuum Einstein equations and illustrate the possibility of extension by introducing matter fields and the cosmological constant term. In Sec. III we construct two classes of 5D anisotropic black tori solutions and consider subclasses and reparemetizations of such solutions in order to generate new ones. Sec. IV is devoted to 4D black tori solutions. In Sec. V we extend the approach for anisotropic 5D and 4D spacetimes with cosmological constant and give two examples of 5D and 4D anisotropic black tori solution. Finally, in Sec. VI, we conclude and discuss the obtained results.

11.2 Off–Diagonal Metric Ansatz

We introduce the basic denotations and two ansatz for off–diagonal 5D metrics (see details in Refs. [1, 2, 3]) to be applied in definition of anisotropic black tori solutions.

Let us consider a 5D pseudo–Riemannian spacetime provided with local coordinates $u^\alpha = (x^i, y^4 = v, y^5)$, for indices like $i, j, k, .. = 1, 2, 3$ and $a, b, ... = 4, 5$. The x^i–coordinates are called holonomic and y^a–coordinates are called anholonomic (anisotropic); they are given respectively with respect to some holonomic and anholonomic subframes (see the formulae (10.12) and (11.8)). Every coordinate x^i or y^a could be a time like, 3D space, or the 5th (extra dimensional) coordinate; we shall fix on necessity different parametrizations.

We investigate two classes of 5D metrics:

The first type of metrics are given by a line element

$$ds^2 = g_{\alpha\beta}\left(x^i, v\right) du^\alpha du^\beta \tag{11.1}$$

with the metric coefficients $g_{\alpha\beta}$ parametrized with respect to the coordinate co–frame du^α, being dual to $\partial_\alpha = \partial/\partial u^\alpha$, by an off–diagonal matrix (ansatz)

$$\left[\begin{array}{ccccc} g_1 + {w_1}^2 h_4 + {n_1}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{array}\right], \tag{11.2}$$

where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g_1 &= \pm 1, g_{2,3} = g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^i, v), \\ w_i &= w_i(x^i, v), n_i = n_i(x^i, v). \end{aligned}$$

The second type of metrics are given by a line element (with a conformal factor $\Omega(x^i, v)$ and additional deformations of the metric via coefficients $\zeta_{\hat{i}}(x^i, v)$, indices with 'hat' take values like $\hat{i} = 1, 2, 3, 5)$) written as

$$ds^2 = \Omega^2(x^i, v)\hat{g}_{\alpha\beta}\left(x^i, v\right) du^\alpha du^\beta, \tag{11.3}$$

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz

$$\left[\begin{array}{ccccc} g_1 + ({w_1}^2 + {\zeta_1}^2) h_4 + {n_1}^2 h_5 & (w_1 w_2 + \zeta_1 \zeta_2) h_4 + n_1 n_2 h_5 & (w_1 w_3 + \zeta_1 \zeta_3) h_4 + n_1 n_3 h_5 & (w_1 + \zeta_1) h_4 & n_1 h_5 \\ (w_1 w_2 + \zeta_1 \zeta_2) h_4 + n_1 n_2 h_5 & g_2 + ({w_2}^2 + {\zeta_2}^2) h_4 + {n_2}^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\ (w_1 w_3 + \zeta_1 \zeta_3) h_4 + n_1 n_3 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & g_3 + ({w_3}^2 + {\zeta_3}^2) h_4 + {n_3}^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\ (w_1 + \zeta_1) h_4 & (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4 \end{array}\right] \tag{11.4}$$

For trivial values $\Omega = 1$ and $\zeta_{\hat{i}} = 0$, the line interval (11.3) transforms into (11.1).

The quadratic line element (11.1) with metric coefficients (11.2) can be diagonalized,

$$\delta s^2 = [g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2], \tag{11.5}$$

with respect to the anholonomic co–frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + w_i dx^i \text{ and } \delta y^5 = dy^5 + n_i dx^i \tag{11.6}$$

which is dual to the frame $(\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5. \tag{11.7}$$

The bases (11.6) and (11.7) are considered to satisfy some anholonomic relations of type

$$\delta_i \delta_j - \delta_j \delta_i = W^k_{ij} \delta_k \tag{11.8}$$

for some non–trivial values of anholonomy coefficients W^k_{ij}. We obtain a holonomic (coordinate) base if the coefficients W^k_{ij} vanish.

The quadratic line element (11.3) with metric coefficients (11.4) can be also diagonalized,

$$\delta s^2 = \Omega^2(x^i, v)[g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\delta y^5)^2], \tag{11.9}$$

but with respect to another anholonomic co–frame $\left(dx^i, \hat{\delta} v, \delta y^5\right)$, with

$$\delta v = dv + (w_i + \zeta_i) dx^i + \zeta_5 \delta y^5 \text{ and } \delta y^5 = dy^5 + n_i dx^i \tag{11.10}$$

which is dual to the frame $\left(\hat{\delta}_i, \partial_4, \hat{\partial}_5\right)$, where

$$\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i \partial_5, \hat{\partial}_5 = \partial_5 - \zeta_5 \partial_4. \tag{11.11}$$

The nontrivial components of the 5D Ricci tensor, $R^{\beta}{}_{\alpha}$, for the metric (11.5) given with respect to anholonomic frames (11.6) and (11.7) are

$$R^2_2 = R^3_3 = -\frac{1}{2g_2 g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'} g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}], \tag{11.12}$$

$$R^4_4 = R^5_5 = -\frac{\beta}{2h_4 h_5}, \tag{11.13}$$

$$R_{4i} = -w_i \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5}, \tag{11.14}$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right] \tag{11.15}$$

where

$$\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \beta = h_5^{**} - h_5^*[\ln \sqrt{|h_4 h_5|}]^*, \gamma = 3h_5^*/2h_5 - h_4^*/h_4. \quad (11.16)$$

For simplicity, the partial derivatives are denoted like $a^\times = \partial a/\partial x^1, a^\bullet = \partial a/\partial x^2, a^{'} = \partial a/\partial x^3, a^* = \partial a/\partial v.$

We obtain the same values of the Ricci tensor for the second ansatz (11.9) if there are satisfied the conditions

$$\hat{\delta}_i h_4 = 0 \text{ and } \hat{\delta}_i \Omega = 0 \quad (11.17)$$

and the values $\zeta_{\hat{i}} = (\zeta_i, \zeta_5 = 0)$ are found as to be a unique solution of (11.17); for instance, if

$$\Omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}), \quad (11.18)$$

the coefficients ζ_i must solve the equations

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0. \quad (11.19)$$

The system of 5D vacuum Einstein equations, $R^{\beta}_{\ \alpha} = 0$, reduces to a system of nonlinear equations with separation of variables,

$$R_2^2 = 0, \ R_4^4 = 0, \ R_{4i} = 0, R_{5i} = 0,$$

which together with (11.19) can be solved in general form [3]: For any given values of g_2 (or g_3), h_4 (or h_5) and Ω, and stated boundary conditions we can define consequently the set of metric coefficients g_3(or g_2), h_4 (or h_4)$, w_i, n_i$ and ζ_i.

The introduced ansatz can be used also for constructing solutions of 5D and 4D Einstein equations with nontrivial energy-momentum tensor

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}.$$

The non–trivial diagonal components of the Einstein tensor, $G^\alpha_\beta = R^\alpha_\beta - \frac{1}{2} R \delta^\alpha_\beta$, for the metric (11.5), given with respect to anholonomic frames, are

$$G_1^1 = -\left(R_2^2 + S_4^4\right), G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2. \quad (11.20)$$

So, we can extend the system of 5D vacuum Einstein equations by introducing matter fields for which the energy–momentum tensor $\Upsilon_{\alpha\beta}$ given with respect to anholonomic frames satisfy the conditions

$$\Upsilon_1^1 = \Upsilon_2^2 + \Upsilon_4^4, \Upsilon_2^2 = \Upsilon_3^3, \Upsilon_4^4 = \Upsilon_5^5. \quad (11.21)$$

We note that, in general, the tensor $\Upsilon_{\alpha\beta}$ may be not symmetric because with respect to anholonomic frames there are imposed constraints which makes non symmetric the Ricci and Einstein tensors [1, 2, 3].

In the simplest case we can consider a "vacuum" source induced by a non–vanishing 4D cosmological constant, Λ. In order to satisfy the conditions (11.21) the source induced by Λ should be in the form $\kappa\Upsilon_{\alpha\beta} = (2\Lambda g_{11}, \Lambda g_{\underline{\alpha\beta}})$, where underlined indices $\underline{\alpha}, \underline{\beta}, ...$ run 4D values $2, 3, 4, 5$. We note that in 4D anholonomic gravity the source $\kappa\Upsilon_{\underline{\alpha\beta}} = \Lambda g_{\underline{\alpha\beta}}$ satisfies the equalities $\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_4^4 = \Upsilon_5^5$.

By straightforward computations we obtain that the nontrivial components of the 5D Einstein equations with anisotropic cosmological constant, $R_{11} = 2\Lambda g_{11}$ and $R_{\underline{\alpha\beta}} = \Lambda g_{\underline{\alpha\beta}}$, for the ansatz (11.4) and anholonomic metric (11.9) given with respect to anholonomic frames (11.10) and (11.11), are written in a form with separated variables:

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'} g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2} = 2\Lambda g_2 g_3, \quad (11.22)$$

$$h_5^{**} - h_5^{*}[\ln\sqrt{|h_4 h_5|}]^{*} = 2\Lambda h_4 h_5, \quad (11.23)$$

$$w_i \beta + \alpha_i = 0, \quad (11.24)$$

$$n_i^{**} + \gamma n_i^{*} = 0, \quad (11.25)$$

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^{*} = 0. \quad (11.26)$$

where

$$\alpha_i = \partial_i h_5^{*} - h_5^{*} \partial_i \ln\sqrt{|h_4 h_5|}, \beta = 2\Lambda h_4 h_5, \gamma = 3h_5^{*}/2h_5 - h_4^{*}/h_4. \quad (11.27)$$

In the vacuum case (with $\Lambda = 0$) these equations are compatible if $\beta = \alpha_i = 0$ which results that $w_i(x^i, v)$ could be arbitrary functions; this reflects a freedom in definition of the holonomic coordinates. For simplicity, for vacuum solutions we shall put $w_i = 0$. Finally, we remark that we can "select" 4D Einstein solutions from an ansatz (11.2) or (11.4) by considering that the metric coefficients do not depend on variable x^1, which mean that in the system of equations (11.22)–(11.26) we have to deal with 4D values $w_{\underline{i}}(x^{\underline{k}}, v), n_{\underline{i}}(x^{\underline{k}}, v), \zeta_i(x^{\underline{k}}, v)$, and $h_4(x^{\underline{k}}, v), h_5(x^{\underline{k}}, v), \Omega(x^{\underline{k}}, v)$.

11.3 5D Black Tori

Our goal is to apply the anholonomic frame method as to construct such exact solutions of vacuum (and with cosmological constant) 5D Einstein equations as they have a static toroidal horizon for a metric ansatz (11.2) or (11.4) which can be diagonalized with respect to some well defined anholonomic frames. Such solutions are defined as some anholonomic transforms of the Schwarzschild solution to a toroidal configuration with non–trivial topology. In general form, they could be defined with warped factors, running constants (in time and extra dimension coordinate) and nonlinear polarizations.

11.3.1 Toroidal deformations of the Schwarzschild metric

Let us consider the system of *isotropic spherical coordinates* (ρ, θ, φ), where the isotropic radial coordinate ρ is related with the usual radial coordinate r via the relation $r = \rho\,(1 + r_g/4\rho)^2$ for $r_g = 2G_{[4]}m_0/c^2$ being the 4D gravitational radius of a point particle of mass m_0, $G_{[4]} = 1/M^2_{P[4]}$ is the 4D Newton constant expressed via Plank mass $M_{P[4]}$ (following modern string/brane theories, $M_{P[4]}$ can be considered as a value induced from extra dimensions). We put the light speed constant $c = 1$. This system of coordinates is considered for the so–called isotropic representation of the Schwarzschild solution [10]

$$ds^2 = \left(\frac{\widehat{\rho} - 1}{\widehat{\rho} + 1}\right)^2 dt^2 - \rho_g^2 \left(\frac{\widehat{\rho} + 1}{\widehat{\rho}}\right)^4 \left(d\widehat{\rho}^2 + \widehat{\rho}^2 d\theta^2 + \widehat{\rho}^2 \sin^2\theta d\varphi^2\right), \tag{11.28}$$

where, for our further considerations, we re–scaled the isotropic radial coordinate as $\widehat{\rho} = \rho/\rho_g$, with $\rho_g = r_g/4$. The metric (11.28) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass m_0. It has a singularity for $r = 0$ and a spherical horizon for $r = r_g$, or, in re–scaled isotropic coordinates, for $\widehat{\rho} = 1$. We emphasize that this solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames.

We also introduce the *toroidal coordinates* (in our case considered as alternatives to the isotropic radial coordinates) [9] (σ, τ, φ), running values $-\pi \leq \sigma < \pi, 0 \leq \tau \leq \infty, 0 \leq \varphi < 2\pi$, which are related with the isotropic 3D Cartezian coordinates via transforms

$$\tilde{x} = \frac{\widetilde{\rho}\sinh\tau}{\cosh\tau - \cos\sigma}\cos\varphi, \tilde{y} = \frac{\widetilde{\rho}\sinh\tau}{\cosh\tau - \cos\sigma}\sin\varphi, \tilde{z} = \frac{\widetilde{\rho}\sinh\sigma}{\cosh\tau - \cos\sigma} \tag{11.29}$$

and define a toroidal hypersurface

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \widetilde{\rho}\frac{\cosh\tau}{\sinh\tau}\right)^2 + \tilde{z}^2 = \frac{\widetilde{\rho}^2}{\sinh^2\tau}.$$

The 3D metric on a such toroidal hypersurface is

$$ds^2_{(3D)} = g_{\sigma\sigma}d\sigma^2 + g_{\tau\tau}d\tau^2 + g_{\varphi\varphi}d\varphi^2,$$

where

$$g_{\sigma\sigma} = g_{\tau\tau} = \frac{\widetilde{\rho}^2}{(\cosh\tau - \cos\sigma)^2}, g_{\varphi\varphi} = \frac{\widetilde{\rho}^2 \sinh^2\tau}{(\cosh\tau - \cos\sigma)^2}.$$

We can relate the toroidal coordinates (σ, τ, φ) from (10.53) with the isotropic radial coordinates $(\widehat{\rho}, \theta, \varphi)$, scaled by the constant ρ_g, from (11.28) as

$$\widetilde{\rho} = 1, \sinh^{-1}\tau = \widehat{\rho}$$

and transform the Schwarzschild solution into a new metric with toroidal coordinates by changing the 3D radial line element into the toroidal one and stating the tt–coefficient of the metric to have a toroidal horizon. The resulting metric is

$$ds^2_{(S)} = \left(\frac{\sinh\tau - 1}{\sinh\tau + 1}\right)^2 dt^2 - \rho_g^2 \frac{(\sinh\tau + 1)^4}{(\cosh\tau - \cos\sigma)^2}\left(d\sigma^2 + d\tau^2 + \sinh^2\tau d\varphi^2\right)], \quad (11.30)$$

Such deformed Schwarzschild like toroidal metric is not an exact solution of the vacuum Einstein equations, but at long radial distances it transform into usual Schwarzschild solution with the 3D line element parametrized by toroidal coordinates.

For our further considerations we introduce two Classes (A and B) of 4D auxiliary pseudo–Riemannian metrics, also given in toroidal coordinates, being some conformal transforms of (11.30), like

$$ds^2_{(S)} = \Omega_{A,B}(\sigma, \tau)\, ds^2_{(A,B)}$$

but which are not supposed to be solutions of the Einstein equations:

- Metric of Class A:

$$ds^2_{(A)} = -d\sigma^2 - d\tau^2 + a(\tau)d\varphi^2 + b(\sigma,\tau)dt^2], \quad (11.31)$$

  where

$$a(\tau) = -\sinh^2\tau \text{ and } b(\sigma,\tau) = -\frac{(\sinh\tau - 1)^2(\cosh\tau - \cos\sigma)^2}{\rho_g^2(\sinh\tau + 1)^6},$$

  which results in the metric (11.30) by multiplication on the conformal factor

$$\Omega_A(\sigma,\tau) = \rho_g^2 \frac{(\sinh\tau + 1)^4}{(\cosh\tau - \cos\sigma)^2}. \quad (11.32)$$

- Metric of Class B:

$$ds^2_{(B)} = g(\tau)\left(d\sigma^2 + d\tau^2\right) - d\varphi^2 + f(\sigma,\tau)dt^2, \tag{11.33}$$

where

$$g(\tau) = -\sinh^{-2}\tau \text{ and } f(\sigma,\tau) = \rho_g^2\left(\frac{\sinh^2\tau - 1}{\cosh\tau - \cos\sigma}\right)^2,$$

which results in the metric (11.30) by multiplication on the conformal factor

$$\Omega_B(\sigma,\tau) = \rho_g^{-2}\frac{(\cosh\tau - \cos\sigma)^2}{(\sinh\tau + 1)^2}.$$

We shall use the metrics (11.30), (11.31) and (11.33) in order to generate exact solutions of the Einstein equations with toroidal horizons and anisotropic polarizations and running of constants by performing corresponding anholonomic transforms as the solutions will have a horizon parametrized by a torus hypersurface and gravitational (extra dimensional, or nonlinear 4D) renormalizations of the constant ρ_g of the Schwarzschild solution, $\rho_g \to \overline{\rho}_g = \omega\rho_g$, where the dependence of the function ω on some holonomic or anholonomic coordinates will depend on the type of anisotropy. For some solutions we shall treat ω as a factor modelling running of the gravitational constant, induced, induced from extra dimension, in another cases we will consider ω as a nonlinear gravitational polarization which models some anisotropic distributions of masses and matter fields and/or anholonomic vacuum gravitational interactions.

11.3.2 Toroidal 5D metrics of Class A

In this subsection we consider four classes of 5D vacuum solutions which are related to the metric of Class A (11.31) and to the toroidally deformed Schwarzschild metric (11.30).

Let us parametrize the 5D coordinates as $\left(x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v, y^5 = p\right)$, where the solutions with the so–called φ–anisotropy will be constructed for $(v = \varphi, p = t)$ and the solutions with t–anisotropy will be stated for $(v = t, p = \varphi)$ (in brief, we write respectively, φ–solutions and t–solutions).

Class A of vacuum solutions with ansatz (11.2):

We take an off–diagonal metric ansatz of type (11.2) (equivalently, (11.1)) by representing

$$g_1 = \pm 1, g_2 = -1, g_3 = -1, h_4 = \eta_4(\sigma,\tau,v)h_{4(0)}(\sigma,\tau) \text{ and } h_5 = \eta_5(\sigma,\tau,v)h_{5(0)}(\sigma,\tau),$$

where $\eta_{4,5}(\sigma,\tau,v)$ are corresponding "gravitational renormalizations" of the metric coefficients $h_{4,5(0)}(\sigma,\tau)$. For φ–solutions we state $h_{4(0)}=a(\tau)$ and $h_{5(0)}=b(\sigma,\tau)$ (inversely, for t–solutions, $h_{4(0)}=b(\sigma,\tau)$ and $h_{5(0)}=a(\sigma,\tau)$).

Next we consider a renormalized gravitational 'constant' $\overline{\rho}_g=\omega\rho_g$, were for φ–solutions the receptivity $\omega=\omega(\sigma,\tau,v)$ is included in the gravitational polarization η_5 as $\eta_5=[\omega(\sigma,\tau,\varphi)]^{-2}$, or for t–solutions is included in η_4, when $\eta_4=[\omega(\sigma,\tau,t)]^{-2}$. We can construct an exact solution of the 5D vacuum Einstein equations if, for explicit dependencies on anisotropic coordinate, the metric coefficients h_4 and h_5 are related by the equation (11.23), which in its turn imposes a corresponding relation between η_4 and η_5,

$$\eta_4 h_{4(0)}=h_{(0)}^2 h_{5(0)}\left[\left(\sqrt{|\eta_5|}\right)^*\right]^2,\ h_{(0)}^2=const.$$

In result, we express the polarizations η_4 and η_5 via the value of receptivity ω,

$$\eta_4(\chi,\sigma,\tau,\varphi)=h_{(0)}^2\frac{b(\sigma,\tau)}{a(\tau)}\left\{\left[\omega^{-1}(\chi,\sigma,\tau,\varphi)\right]^*\right\}^2,\eta_5(\chi,\sigma,\tau,\varphi)=\omega^{-2}(\chi,\sigma,\tau,\varphi), \tag{11.34}$$

for φ–solutions , and

$$\eta_4(\chi,\sigma,\tau,t)=\omega^{-2}(\chi,\sigma,\tau,t),\eta_5(\chi,\sigma,\tau,t)=h_{(0)}^{-2}\frac{b(\sigma,\tau)}{a(\tau)}\left[\int dt\omega^{-1}(\chi,\sigma,\tau,t)\right]^2, \tag{11.35}$$

for t–solutions, where $a(\tau)$ and $b(\sigma,\tau)$ are those from (11.31).

For vacuum configurations, following (11.24), we put $w_i=0$. The next step is to find the values of n_i by introducing $h_4=\eta_4 h_{4(0)}$ and $h_5=\eta_5 h_{5(0)}$ into the formula (11.25), which, for convenience, is expressed via general coefficients η_4 and η_5. After two integrations on variable v, we obtain the exact solution

$$\begin{aligned} n_k(\sigma,\tau,v) &= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int[\eta_4/(\sqrt{|\eta_5|})^3]dv,\ \eta_5^*\neq 0; \\ &= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int\eta_4 dv, \qquad \eta_5^*=0; \\ &= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int[1/(\sqrt{|\eta_5|})^3]dv,\ \eta_4^*=0, \end{aligned} \tag{11.36}$$

with the functions $n_{k[2]}(\sigma,\tau)$ are defined as to contain the values $h_{(0)}^2$, $a(\tau)$ and $b(\sigma,\tau)$.

By introducing the formulas (11.34) for φ–solutions (or (11.35) for t–solutions) and fixing some boundary condition, in order to state the values of coefficients $n_{k[1,2]}(\sigma,\tau)$ we can express the ansatz components $n_k(\sigma,\tau,\varphi)$ as integrals of some functions of $\omega(\sigma,\tau,\varphi)$ and $\partial_\varphi\omega(\sigma,\tau,\varphi)$ (or, we can express the ansatz components $n_k(\sigma,\tau,t)$ as integrals of some functions of $\omega(\sigma,\tau,t)$ and $\partial_t\omega(\sigma,\tau,t)$). We do not present an explicit form of such formulas because they depend on the type of receptivity $\omega=\omega(\sigma,\tau,v)$, which must be defined experimentally, or from some quantum models of gravity in the quasi classical limit. We preserved a general dependence on coordinates (σ,τ) which reflect the fact that there is a freedom in fixing holonomic coordinates (for instance, on a toroidal hypersurface and its extensions to 4D and 5D spacetimes). For simplicity, we write that n_i are some functionals of $\{\sigma,\tau,\omega(\sigma,\tau,v),\omega^*(\sigma,\tau,v)\}$

$$n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,v),\omega^*(\sigma,\tau,v)\}.$$

In conclusion, we constructed two exact solutions of the 5D vacuum Einstein equations, defined by the ansatz (11.2) with coordinates and coefficients stated by the data:

$$\begin{aligned}
\varphi\text{–solutions} &: (x^1=\chi,x^2=\sigma,x^3=\tau,y^4=v=\varphi,y^5=p=t),g_1=\pm1,\\
g_2 &= -1,g_3=-1,h_{4(0)}=a(\tau),h_{5(0)}=b(\sigma,\tau),\text{see (11.31)};\\
h_4 &= \eta_4(\sigma,\tau,\varphi)h_{4(0)},h_5=\eta_5(\sigma,\tau,\varphi)h_{5(0)},\\
\eta_4 &= h_{(0)}^2\frac{b(\sigma,\tau)}{a(\tau)}\left\{\left[\omega^{-1}(\chi,\sigma,\tau,\varphi)\right]^*\right\}^2,\eta_5=\omega^{-2}(\chi,\sigma,\tau,\varphi),\\
w_i &= 0,n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,\varphi),\omega^*(\sigma,\tau,\varphi)\}. \qquad (11.37)
\end{aligned}$$

and

$$\begin{aligned}
t\text{–solutions} &: (x^1=\chi,x^2=\sigma,x^3=\tau,y^4=v=t,y^5=p=\varphi),g_1=\pm1,\\
g_2 &= -1,g_3=-1,h_{4(0)}=b(\sigma,\tau),h_{5(0)}=a(\tau),\text{see (11.31)};\\
h_4 &= \eta_4(\sigma,\tau,t)h_{4(0)},h_5=\eta_5(\sigma,\tau,t)h_{5(0)},\\
\eta_4 &= \omega^{-2}(\chi,\sigma,\tau,t),\eta_5=h_{(0)}^{-2}\frac{b(\sigma,\tau)}{a(\tau)}\left[\int dt\ \omega^{-1}(\chi,\sigma,\tau,t)\right]^2,\\
w_i &= 0,n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,t),\omega^*(\sigma,\tau,t)\}. \qquad (11.38)
\end{aligned}$$

Both types of solutions have a horizon parametrized by torus hypersurface (as the condition of vanishing of the "time" metric coefficient states, i. e. when the function $b(\sigma,\tau)=0$). These solutions are generically anholonomic (anisotropic) because in the locally isotropic limit, when $\eta_4,\eta_5,\ \omega\to1$ and $n_i\to0$, they reduce to the coefficients of the metric (11.31). The last one is not an exact solution of 4D vacuum Einstein equations, but it is a conformal transform of the 4D Schwarzschild metric deformed to a torus horizon, with a further trivial extension to 5D. With respect to the anholonomic frames adapted to the coefficients n_i (see (11.6)), the obtained solutions have diagonal metric coefficients being very similar to the metric (11.30) written in toroidal coordinates. We can treat such solutions as black tori with the mass distributed linearly on the circle which can not transformed in a point, in the center of torus.

The solutions are constructed as to have singularities on the mentioned circle. The initial data for anholonomic frames and the chosen configuration of gravitational interactions in the bulk lead to deformed toroidal horizons even for static configurations. The solutions admit anisotropic polarizations on toroidal coordinates (σ, τ) and running of constants on time t and/or on extra dimension coordinate χ. Such renormalizations of constants are defined by the nonlinear configuration of the 5D vacuum gravitational field and depend on introduced receptivity function $\omega(\sigma, \tau, v)$ which is to be considered an intrinsic characteristics of the 5D vacuum gravitational 'ether', emphasizing the possibility of nonlinear self–polarization of gravitational fields.

Finally, we point that the data (11.37) and (11.38) parametrize two very different classes of solutions. The first one is for static 5D vacuum black tori configurations with explicit dependence on anholonomic coordinate φ and possible renormalizations on the rest of 3D space coordinates σ and τ and on the 5th coordinate χ. The second class of solutions is similar to the static ones but with an emphasized anholonomic running on time of constants and with possible anisotropic dependencies on coordinates (σ, τ, χ).

Class A of vacuum solutions with ansatz (11.4):

We construct here 5D vacuum φ– and t–solutions parametrized by an ansatz with conformal factor $\Omega(\sigma, \tau, v)$ (see (11.4) and (11.9)). Let us consider conformal factors parametrized as $\Omega = \Omega_{[0]}(\sigma, \tau)\Omega_{[1]}(\sigma, \tau, v)$. We can generate from the data (11.37) (or (11.38)) an exact solution of vacuum Einstein equations if there are satisfied the conditions (11.17), (11.18) and (11.19), i. e.

$$\Omega_{[0]}^{q_1/q_2}\Omega_{[1]}^{q_1/q_2} = \eta_4 h_{4(0)},$$

for some integers q_1 and q_2, and there are defined the second anisotropy coefficients

$$\zeta_i = \left(\partial_i \ln|\Omega_{[0]}\right)|) \ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]}.$$

So, taking a φ– or t–solution with corresponding values of $h_4 = \eta_4 h_{4(0)}$, for some q_1 and q_2, we obtain new exact solutions, called in brief, φ_c– or t_c–solutions (with the index "c" pointing to an ansatz with conformal factor), of the vacuum 5D Einstein equations given in explicit form by the data:

$$\begin{aligned}
\varphi_c\text{–solutions} &: & (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= & -1, g_3 = -1, h_{4(0)} = a(\tau), h_{5(0)} = b(\sigma,\tau), \text{see (11.31)}; \\
h_4 &= & \eta_4(\sigma,\tau,\varphi) h_{4(0)}, h_5 = \eta_5(\sigma,\tau,\varphi) h_{5(0)}, \\
\eta_4 &= & h_{(0)}^2 \frac{b(\sigma,\tau)}{a(\tau)} \left\{ \left[\omega^{-1}(\chi,\sigma,\tau,\varphi) \right]^* \right\}^2, \eta_5 = \omega^{-2}(\chi,\sigma,\tau,\varphi), \\
w_i &= & 0, n_i\{\sigma,\tau,\omega,\omega^*\} = n_i\{\sigma,\tau,\omega(\sigma,\tau,\varphi), \omega^*(\sigma,\tau,\varphi)\}, \\
\zeta_i &= & \left(\partial_i \ln |\Omega_{[0]}) \,|\right) \left(\ln |\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1} \partial_i \Omega_{[1]}, \\
\eta_4 a &= & \Omega_{[0]}^{q_1/q_2}(\sigma,\tau) \Omega_{[1]}^{q_1/q_2}(\sigma,\tau,\varphi),\ \Omega = \Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,\varphi)
\end{aligned} \quad (11.39)$$

and

$$\begin{aligned}
t_c\text{–solutions} &: & (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1, \\
g_2 &= & -1, g_3 = -1, h_{4(0)} = b(\sigma,\tau), h_{5(0)} = a(\tau), \text{see (11.31)}; \\
h_4 &= & \eta_4(\sigma,\tau,t) h_{4(0)}, h_5 = \eta_5(\sigma,\tau,t) h_{5(0)}, \\
\eta_4 &= & \omega^{-2}(\chi,\sigma,\tau,t), \eta_5 = h_{(0)}^{-2} \frac{b(\sigma,\tau)}{a(\tau)} \left[\int dt\, \omega^{-1}(\chi,\sigma,\tau,t) \right]^2, \\
w_i &= & 0, n_i\{\sigma,\tau,\omega,\omega^*\} = n_i\{\sigma,\tau,\omega(\sigma,\tau,t), \omega^*(\sigma,\tau,t)\}, \\
\zeta_i &= & \left(\partial_i \ln |\Omega_{[0]}) \,|\right) \left(\ln |\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1} \partial_i \Omega_{[1]}, \\
\eta_4 a &= & \Omega_{[0]}^{q_1/q_2}(\sigma,\tau) \Omega_{[1]}^{q_1/q_2}(\sigma,\tau,t),\ \Omega = \Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,t)
\end{aligned} \quad (11.40)$$

These solutions have two very interesting properties: 1) they admit a warped factor on the 5th coordinate, like $\Omega_{[1]}^{q_1/q_2} \sim \exp[-k|\chi|]$, which in our case is constructed for an anisotropic 5D vacuum gravitational configuration and not following a brane configuration like in Refs. [7]; 2) we can impose such conditions on the receptivity $\omega(\sigma,\tau,v)$ as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric (11.30) trivially embedded into the 5D spacetime.

We analyze the second property in details. We have to chose the conformal factor as to be satisfied three conditions:

$$\Omega_{[0]}^{q_1/q_2} = \Omega_A, \Omega_{[1]}^{q_1/q_2} \eta_4 = 1, \Omega_{[1]}^{q_1/q_2} \eta_5 = 1, \quad (11.41)$$

were Ω_A is that from (11.32). The last two conditions are possible if

$$\eta_4^{-q_1/q_2} \eta_5 = 1, \quad (11.42)$$

which selects a specific form of receptivity $\omega(x^i, v)$. Putting into (11.42) the values η_4 and η_5 respectively from (11.39), or (11.40), we obtain some differential, or integral, relations of the unknown $\omega(\sigma, \tau, v)$, which results that

$$\begin{aligned} \omega(\sigma, \tau, \varphi) &= (1 - q_1/q_2)^{-1-q_1/q_2}\left[h_{(0)}^{-1}\sqrt{|a/b|}\varphi + \omega_{[0]}(\sigma, \tau)\right], \text{ for } \varphi_c\text{–solutions;} \\ \omega(\sigma, \tau, t) &= \left[(q_1/q_2 - 1)\, h_{(0)}\sqrt{|a/b|}t + \omega_{[1]}(\sigma, \tau)\right]^{1-q_1/q_2}, \text{ for } t_c\text{–solutions,} \end{aligned} \quad (11.43)$$

for some arbitrary functions $\omega_{[0]}(\sigma, \tau)$ and $\omega_{[1]}(\sigma, \tau)$. So, a receptivity of particular form like (11.43) allow us to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric.

We conclude this subsection: the vacuum 5D metrics solving the Einstein equations describe a nonlinear gravitational dynamics which under some particular boundary conditions and parametrizations of metric's coefficients can model anisotropic, topologically not–trivial, solutions transforming, in a corresponding locally isotropic limit, in some toroidal or ellipsoidal deformations of the well known exact solutions like Schwarzschild, Reissner-Nördstrom, Taub NUT, various type of wormhole, solitonic and disk solutions (see details in Refs. [1, 2, 3]). We emphasize that, in general, an anisotropic solution (parametrized by an off–diagonal ansatz) could not have a locally isotropic limit to a diagonal metric with respect to some holonomic coordinate frames. This was proved in explicit form by choosing a configuration with toroidal symmetry.

11.3.3 Toroidal 5D metrics of Class B

In this subsection we construct and analyze another two classes of 5D vacuum solutions which are related to the metric of Class B (11.33) and which can be reduced to the toroidally deformed Schwarzschild metric (11.30) by corresponding parametrizations of receptivity $\omega(\sigma, \tau, v)$. We emphasize that because the function $g(\sigma, \tau)$ from (11.33) is not a solution of equation(11.22) we introduce an auxiliary factor $\varpi(\sigma, \tau)$ for which ϖg became a such solution, then we consider conformal factors parametrized as $\Omega = \varpi^{-1}(\sigma, \tau)$ $\Omega_{[2]}(\sigma, \tau, v)$ and find solutions parametrized by the ansatz (11.4) and anholonomic metric interval (11.9).

Because the method of definition of such solutions is similar to that from previous subsection, in our further considerations we shall omit computations and present directly the data which select the respective configurations for φ_c–solutions and t_c–solutions.

The Class B of 5D solutions with conformal factor are parametrized by the data:

$$\begin{aligned}
\varphi_c\text{–solutions} &: (x^1=\chi, x^2=\sigma, x^3=\tau, y^4=v=\varphi, y^5=p=t), g_1=\pm 1, \\
g_2 &= g_3=\varpi(\sigma,\tau)g(\sigma,\tau), \\
h_{4(0)} &= -\varpi(\sigma,\tau), h_{5(0)}=\varpi(\sigma,\tau)f(\sigma,\tau), \text{see (11.33)}; \\
\varpi &= g^{-1}(\sigma,\tau)\varpi_0\exp[a_2\sigma+a_3\tau],\ \varpi_0,a_2,a_3=const; \\
h_4 &= \eta_4(\sigma,\tau,\varphi)h_{4(0)}, h_5=\eta_5(\sigma,\tau,\varphi)h_{5(0)}, \\
\eta_4 &= -h_{(0)}^2 f(\sigma,\tau)\left\{\left[\omega^{-1}(\chi,\sigma,\tau,\varphi)\right]^*\right\}^2, \eta_5=\omega^{-2}(\chi,\sigma,\tau,\varphi), \quad (11.44) \\
w_i &= 0, n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,\varphi),\omega^*(\sigma,\tau,\varphi)\}, \\
\zeta_i &= \partial_i \ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega^*_{[2]}\right)^{-1}\partial_i\Omega_{[2]}, \\
\eta_4 &= -\varpi^{-q_1/q_2}(\sigma,\tau)\Omega_{[2]}^{q_1/q_2}(\sigma,\tau,\varphi), \Omega=\varpi^{-1}(\sigma,\tau)\Omega_{[2]}(\sigma,\tau,\varphi)
\end{aligned}$$

and

$$\begin{aligned}
t_c\text{–solutions} &: (x^1=\chi, x^2=\sigma, x^3=\tau, y^4=v=t, y^5=p=\varphi), g_1=\pm 1, \\
g_2 &= g_3=\varpi(\sigma,\tau)g(\sigma,\tau), \\
h_{4(0)} &= \varpi(\sigma,\tau)f(\sigma,\tau), h_{5(0)}=-\varpi(\sigma,\tau), \text{see (11.33)}; \\
\varpi &= g^{-1}(\sigma,\tau)\varpi_0\exp[a_2\sigma+a_3\tau],\ \varpi_0,a_2,a_3=const, \\
h_4 &= \eta_4(\sigma,\tau,t)h_{4(0)}, h_5=\eta_5(\sigma,\tau,t)h_{5(0)}, \\
\eta_4 &= \omega^{-2}(\chi,\sigma,\tau,t), \eta_5=-h_{(0)}^{-2}f(\sigma,\tau)\left[\int dt\ \omega^{-1}(\chi,\sigma,\tau,t)\right]^2, (11.45) \\
w_i &= 0, n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,t),\omega^*(\sigma,\tau,t)\}, \\
\zeta_i &= \partial_i(\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega^*_{[2]}\right)^{-1}\partial_i\Omega_{[2]}, \\
\eta_4 &= -\varpi^{-q_1/q_2}(\sigma,\tau)\Omega_{[2]}^{q_1/q_2}(\sigma,\tau,t),\ \Omega=\varpi^{-1}(\sigma,\tau)\Omega_{[2]}(\sigma,\tau,t),
\end{aligned}$$

where the coefficients n_i can be found explicitly by introducing the corresponding values η_4 and η_5 in formula (11.36).

By a procedure similar to the solutions of Class A (see previous subsection) we can find the conditions when the solutions (11.44) and (11.45) will have in the locally anisotropic limit the toroidally deformed Schwarzschild solutions, which impose corresponding parametrizations and dependencies on $\Omega_{[2]}(\sigma,\tau,v)$ and $\omega(\sigma,\tau,v)$ like (11.41) and (11.43). We omit these formulas because, in general, for anholonomic configurations and nonlinear solutions there are not hard arguments to prefer any holonomic limits of such off–diagonal metrics.

Finally, in this Section, we remark that for the considered classes of black tori solutions the so–called t–components of metric contain modifications of the Schwarzschild potential

$$\Phi = -\frac{M}{M_{P[4]}^2 r} \text{ into } \Phi = -\frac{M\omega(\sigma,\tau,v)}{M_{P[4]}^2 r},$$

where $M_{P[4]}$ is the usual 4D Plank constant, and this is given with respect to the corresponding anholonomic frame of reference. The receptivity $\omega(\sigma,\tau,v)$ could model corrections warped on extra dimension coordinate, χ, which for our solutions are induced by anholonomic vacuum gravitational interactions in the bulk and not from a brane configuration in AdS_5 spacetime. In the vacuum case k is a constant which characterizes the receptivity for bulk vacuum gravitational polarizations.

11.4 4D Black Tori

For the ansatz (11.2), with trivial conformal factor, a black torus solution of 4D vacuum Einstein equations was constructed in Ref. [1]. The goal of this Section is to consider some alternative variants, with trivial or nontrivial conformal factors and for different coordinate parametrizations and types of anisotropies. The bulk of 5D solutions from the previous Section are reduced into corresponding 4D ones if we eliminate the 5th coordinate χ from the the off–diagonal ansatz (11.2) and (11.4) and corresponding formulas and solutions.

11.4.1 Toroidal 4D vacuum metrics of Class A

Let us parametrize the 4D coordinates as $(x^{\underline{i}}, y^a) = (x^2 = \sigma, x^3 = \tau, y^4 = v, y^5 = p)$; for the φ–solutions we shall take $(v = \varphi, p = t)$ and for the solutions t–solutions will consider $(v = t, p = \varphi)$. For simplicity, we write down the data for solutions without proofs and computations.

Class A of vacuum solutions with ansatz (11.2):

The off–diagonal metric ansatz of type (11.2) (equivalently, (11.5)) with the data

$$
\begin{aligned}
\varphi\text{–solutions} &: \left(x^2=\sigma, x^3=\tau, y^4=v=\varphi, y^5=p=t\right) \\
g_2 &= -1, g_3=-1, h_{4(0)}=a(\tau), h_{5(0)}=b(\sigma,\tau), \text{see (11.31)}; \\
h_4 &= \eta_4(\sigma,\tau,\varphi) h_{4(0)}, h_5=\eta_5(\sigma,\tau,\varphi) h_{5(0)}, \\
\eta_4 &= h_{(0)}^2 \frac{b(\sigma,\tau)}{a(\tau)} \left\{\left[\omega^{-1}(\sigma,\tau,\varphi)\right]^*\right\}^2, \eta_5=\omega^{-2}(\sigma,\tau,\varphi), \\
w_{\underline{i}} &= 0, n_{\underline{i}}\{\sigma,\tau,\omega,\omega^*\}=n_{\underline{i}}\{\sigma,\tau,\omega(\sigma,\tau,\varphi), \omega^*(\sigma,\tau,\varphi)\}. \qquad (11.46)
\end{aligned}
$$

and

$$
\begin{aligned}
t\text{–solutions} &: \left(x^2=\sigma, x^3=\tau, y^4=v=t, y^5=p=\varphi\right) \\
g_2 &= -1, g_3=-1, h_{4(0)}=b(\sigma,\tau), h_{5(0)}=a(\tau), \text{see (11.31)}; \\
h_4 &= \eta_4(\sigma,\tau,t) h_{4(0)}, h_5=\eta_5(\sigma,\tau,t) h_{5(0)}, \\
\eta_4 &= \omega^{-2}(\sigma,\tau,t), \eta_5=h_{(0)}^{-2} \frac{b(\sigma,\tau)}{a(\tau)} \left[\int dt\, \omega^{-1}(\sigma,\tau,t)\right]^2, \\
w_{\underline{i}} &= 0, n_{\underline{i}}\{\sigma,\tau,\omega,\omega^*\}=n_{\underline{i}}\{\sigma,\tau,\omega(\sigma,\tau,t), \omega^*(\sigma,\tau,t)\}. \qquad (11.47)
\end{aligned}
$$

where the $n_{\underline{i}}$ are computed

$$
\begin{aligned}
n_k(\sigma,\tau,v) &= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int[\eta_4/(\sqrt{|\eta_5|})^3]dv, \ \eta_5^* \neq 0; \qquad (11.48) \\
&= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int \eta_4 dv, \qquad \eta_5^*=0; \\
&= n_{k[1]}(\sigma,\tau)+n_{k[2]}(\sigma,\tau)\int[1/(\sqrt{|\eta_5|})^3]dv, \ \eta_4^*=0.
\end{aligned}
$$

when the integration variable is taken $v=\varphi$, for (11.46), or $v=t$, for (11.47). These solutions have the same toroidal symmetries and properties stated for their 5D analogs (11.37) and for (11.38) with that difference that there are not any warped factors and extra dimension dependencies. Such solutions defined by the formulas (11.46) and (11.47) do not result in a locally isotropic limit into an exact solution having diagonal coefficients with respect to some holonomic coordinate frames. The data introduced in this subsection are for generic 4D vacuum solutions of the Einstein equations parametrized by off–diagonal metrics. The renormalization of constants and metric coefficients have a 4D nonlinear vacuum gravitational nature and reflects a corresponding anholonomic dynamics.

Class A of vacuum solutions with ansatz (11.4):

The 4D vacuum φ– and t–solutions parametrized by an ansatz with conformal factor $\Omega(\sigma,\tau,v)$ (see (11.4) and (11.9)). Let us consider conformal factors parametrized as $\Omega=\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,v)$. The data are

$$\begin{aligned}
\varphi_c\text{–solutions} &: (x^2=\sigma, x^3=\tau, y^4=v=\varphi, y^5=p=t)\\
g_2 &= -1, g_3=-1, h_{4(0)}=a(\tau), h_{5(0)}=b(\sigma,\tau), \text{see (11.31)};\\
h_4 &= \eta_4(\sigma,\tau,\varphi)h_{4(0)}, h_5=\eta_5(\sigma,\tau,\varphi)h_{5(0)}, \Omega=\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,\varphi),\\
\eta_4 &= h^2_{(0)}\frac{b(\sigma,\tau)}{a(\tau)}\left\{\left[\omega^{-1}(\sigma,\tau,\varphi)\right]^*\right\}^2, \eta_5=\omega^{-2}(\sigma,\tau,\varphi), \qquad (11.49)\\
w_i &= 0, n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\lambda,\omega(\sigma,\tau,\varphi),\omega^*(\sigma,\tau,\varphi)\},\\
\zeta_i &= \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega^*_{[1]}\right)^{-1}\partial_i\Omega_{[1]},\\
\eta_4 a &= \Omega^{q_1/q_2}_{[0]}(\sigma,\tau)\Omega^{q_1/q_2}_{[1]}(\sigma,\tau,\varphi).
\end{aligned}$$

and

$$\begin{aligned}
t_c\text{–solutions} &: (x^2=\sigma, x^3=\tau, y^4=v=t, y^5=p=\varphi)\\
g_2 &= -1, g_3=-1, h_{4(0)}=b(\sigma,\tau), h_{5(0)}=a(\tau), \text{see (11.31)};\\
h_4 &= \eta_4(\sigma,\tau,t)h_{4(0)}, h_5=\eta_5(\sigma,\tau,t)h_{5(0)}, \Omega=\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,t),\\
\eta_4 &= \omega^{-2}(\sigma,\tau,t), \eta_5=h^{-2}_{(0)}\frac{b(\sigma,\tau)}{a(\tau)}\left[\int dt\ \omega^{-1}(\sigma,\tau,t)\right]^2, \qquad (11.50)\\
w_i &= 0, n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,t),\omega^*(\sigma,\tau,t)\},\\
\zeta_i &= \left(\partial_i \ln|\Omega_{[0]}\right)|)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega^*_{[1]}\right)^{-1}\partial_i\Omega_{[1]},\\
\eta_4 a &= \Omega^{q_1/q_2}_{[0]}(\sigma,\tau)\Omega^{q_1/q_2}_{[1]}(\sigma,\tau,t),
\end{aligned}$$

where the coefficients the $n_{\underline{i}}$ are given by the same formulas (11.48).

Contrary to the solutions (11.46) and for (11.47) theirs conformal anholonomic transforms, respectively, (11.49) and (11.50), can be subjected to such parametrizations of the conformal factor and conditions on the receptivity $\omega(\sigma,\tau,v)$ as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric (11.30). These conditions are stated for $\Omega^{q_1/q_2}_{[0]}=\Omega_A$, $\Omega^{q_1/q_2}_{[1]}\eta_4=1$, $\Omega^{q_1/q_2}_{[1]}\eta_5=1$,were Ω_A is that from (11.32), which is possible if $\eta_4^{-q_1/q_2}\eta_5=1$,which selects a specific form of the receptivity ω. Putting the values η_4 and η_5, respectively, from (11.49), or (11.50), we obtain some differential, or integral, relations of the unknown $\omega(\sigma,\tau,v)$, which results that

$$\begin{aligned}
\omega(\sigma,\tau,\varphi) &= (1-q_1/q_2)^{-1-q_1/q_2}\left[h_{(0)}^{-1}\sqrt{|a/b|}\varphi+\omega_{[0]}(\sigma,\tau)\right], \text{ for } \varphi_c\text{–solutions;}\\
\omega(\sigma,\tau,t) &= \left[(q_1/q_2-1)\,h_{(0)}\sqrt{|a/b|}t+\omega_{[1]}(\sigma,\tau)\right]^{1-q_1/q_2}, \text{ for } t_c\text{–solutions,}
\end{aligned}$$

for some arbitrary functions $\omega_{[0]}(\sigma,\tau)$ and $\omega_{[1]}(\sigma,\tau)$. The formulas for $\omega(\sigma,\tau,\varphi)$ and $\omega(\sigma,\tau,t)$ are 4D reductions of the formulas (11.41) and (11.43).

11.4.2 Toroidal 4D vacuum metrics of Class B

We construct another two classes of 4D vacuum solutions which are related to the metric of Class B (11.33) and can be reduced to the toroidally deformed Schwarzshild metric (11.30) by corresponding parametrizations of receptivity $\omega(\sigma,\tau,v)$. The solutions contain a 2D conformal factor $\varpi(\sigma,\tau)$ for which ϖg becomes a solution of (11.22) and a 4D conformal factor parametrized as $\Omega=\varpi^{-1}\,\Omega_{[2]}(\sigma,\tau,v)$ in order to set the constructions into the ansatz (11.4) and anholonomic metric interval (11.9).

The data selecting the 4D configurations for φ_c–solutions and t_c–solutions:

$$\begin{aligned}
\varphi_c\text{–solutions} &: (x^2=\sigma,x^3=\tau,y^4=v=\varphi,y^5=p=t)\\
g_2 &= g_3=\varpi(\sigma,\tau)g(\sigma,\tau),\\
h_{4(0)} &= -\varpi(\sigma,\tau),h_{5(0)}=\varpi(\sigma,\tau)f(\sigma,\tau),\text{see (11.33);}\\
\varpi &= g^{-1}\varpi_0\exp[a_2\sigma+a_3\tau],\ \varpi_0,a_2,a_3=const;\\
h_4 &= \eta_4(\sigma,\tau,\varphi)h_{4(0)},h_5=\eta_5(\sigma,\tau,\varphi)h_{5(0)},\Omega=\varpi^{-1}(\sigma,\tau)\Omega_{[2]}(\sigma,\tau,\varphi)\\
\eta_4 &= -h_{(0)}^2 f(\sigma,\tau)\left\{\left[\omega^{-1}(\sigma,\tau,\varphi)\right]^*\right\}^2,\eta_5=\omega^{-2}(\sigma,\tau,\varphi), \qquad (11.51)\\
w_i &= 0,n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,\varphi),\omega^*(\sigma,\tau,\varphi)\},\\
\zeta_{\underline{i}} &= \partial_{\underline{i}}\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^*+\left(\Omega_{[2]}^*\right)^{-1}\partial_{\underline{i}}\Omega_{[2]},\\
\eta_4 &= -\varpi^{-q_1/q_2}(\sigma,\tau)\Omega_{[2]}^{q_1/q_2}(\sigma,\tau,\varphi)
\end{aligned}$$

and

$$
\begin{array}{rcl}
t_c\text{–solutions} & : & (x^2=\sigma, x^3=\tau, y^4=v=t, y^5=p=\varphi) \\
g_2 & = & g_3=\varpi(\sigma,\tau)g(\sigma,\tau), \\
h_{4(0)} & = & \varpi(\sigma,\tau)f(\sigma,\tau), h_{5(0)}=-\varpi(\sigma,\tau), \text{see } (11.33); \\
\varpi & = & g^{-1}\varpi_0\exp[a_2\sigma+a_3\tau],\ \varpi_0,a_2,a_3=const, \\
h_4 & = & \eta_4(\sigma,\tau,t)h_{4(0)}, h_5=\eta_5(\sigma,\tau,t)h_{5(0)}, \Omega=\varpi^{-1}(\sigma,\tau)\Omega_{[2]}(\sigma,\tau,t) \\
\eta_4 & = & \omega^{-2}(\sigma,\tau,t), \eta_5=-h_{(0)}^{-2}f(\sigma,\tau)\left[\int dt\ \omega^{-1}(\sigma,\tau,t)\right]^2, \\
w_i & = & 0, n_i\{\sigma,\tau,\omega,\omega^*\}=n_i\{\sigma,\tau,\omega(\sigma,\tau,t),\omega^*(\sigma,\tau,t)\}, \\
\zeta_i & = & \partial_i(\ln|\varpi|)\ \left(\ln|\Omega_{[2]}|\right)^*+\left(\Omega_{[2]}^*\right)^{-1}\partial_i\Omega_{[2]}, \\
\eta_4 & = & -\varpi^{-q_1/q_2}(\sigma,\tau)\Omega_{[2]}^{q_1/q_2}(\sigma,\tau,t).
\end{array} \quad (11.52)
$$

where the coefficients n_i can be found explicitly by introducing the corresponding values η_4 and η_5 in formula (11.36).

For the 4D Class B solutions, some conditions can be imposed (see previous subsection) when the solutions (11.51) and (11.52) have in the locally anisotropic limit the toroidally deformed Schwarzschild solution, which imposes some specific parametrizations and dependencies on $\Omega_{[2]}(\sigma,\tau,v)$ and $\omega(\sigma,\tau,v)$ like (11.41) and (11.43). We omit these considerations because for anholonomic configurations and nonlinear solutions there are not arguments to prefer any holonomic limits of such off–diagonal metrics.

We conclude this Section by noting that for the constructed classes of 4D black tori solutions the so–called t–component of metric contains modifications of the Schwarzschild potential

$$
\Phi=-\frac{M}{M_{P[4]}^2 r} \text{ into } \Phi=-\frac{M\omega(\sigma,\tau,v)}{M_{P[4]}^2 r},
$$

where $M_{P[4]}$ is the usual 4D Plank constant; the metric coefficients are given with respect to the corresponding anholonomic frame of reference. In 4D anholonomic gravity the receptivity $\omega(\sigma,\tau,v)$ is considered to renormalize the mass constant. Such gravitational self-polarizations are induced by anholonomic vacuum gravitational interactions. They should be defined experimentally or computed following a model of quantum gravity.

11.5 The Cosmological Constant and Anisotropy

In this Section we analyze the general properties of anholonomic Einstein equations in 5D and 4D gravity with cosmological constant and consider two examples of 5D and

4D exact solutions.

A non–vanishing Λ term in the system of Einstein's equations results in substantial differences because t $\beta \neq 0$ and, in this case, one could be $w_i \neq 0$; The equations (11.22) and (11.23) are of more general nonlinearity because of presence of the $2\Lambda g_2 g_3$ and $2\Lambda h_4 h_5$ terms. In this case, the solutions with $g_2 = const$ and $g_3 = const$ (and $h_4 = const$ and $h_5 = const$) are not admitted. This makes more sophisticate the procedure of definition of g_2 for a stated g_3 (or inversely, of definition of g_3 for a stated g_2) from (11.22) [similarly of constructing h_4 for a given h_5 from (11.23) and inversely], nevertheless, the separation of variables is not affected by introduction of cosmological constant and there is a number of possibilities to generate exact solutions.

The general properties of solutions of the system (11.22)–(11.26), with cosmological constant Λ, are stated in the form:

- The general solution of equation (11.22) is to be found from the equation

$$\varpi\varpi^{\bullet\bullet} - (\varpi^{\bullet})^2 + \varpi\varpi^{''} - (\varpi^{'})^2 = 2\Lambda\varpi^3. \quad (11.53)$$

  for a coordinate transform coordinate transforms $x^{2,3} \to \widetilde{x}^{2,3}(u,\lambda)$ for which

$$g_2(\sigma,\tau)(d\sigma)^2 + g_3(\sigma,\tau)(d\tau)^2 \to \varpi\left[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\right], \epsilon = \pm 1$$

  and $\varpi^{\bullet} = \partial\varpi/\partial\widetilde{x}^2$ and $\varpi^{'} = \partial\varpi/\partial\widetilde{x}^3$.

- The equation (11.23) relates two functions $h_4(\sigma,\tau,v)$ and $h_5(\sigma,\tau,v)$ with $h_5^* \neq 0$. If the function h_5 is given we can find h_4 as a solution of

$$h_4^* + \frac{2\Lambda}{\pi}(h_4)^2 + 2\left(\frac{\pi^*}{\pi} - \pi\right)h_4 = 0, \quad (11.54)$$

  where $\pi = h_5^*/2h_5$.

- The exact solutions of (11.24) for $\beta \neq 0$ is

$$\begin{aligned} w_k &= -\alpha_k/\beta, \quad (11.55)\\ &= \partial_k \ln[\sqrt{|h_4h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4h_5|}/|h_5^*|], \end{aligned}$$

  for $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$.

- The exact solution of (11.25) is

$$\begin{aligned} n_k &= n_{k[1]}(\sigma,\tau) + n_{k[2]}(\sigma,\tau)\int [h_4/(\sqrt{|h_5|})^3]dv, \\ &= n_{k[1]}(\sigma,\tau) + n_{k[2]}(\sigma,\tau)\int [1/(\sqrt{|h_5|})^3]dv, \ h_4^* = 0, \end{aligned} \quad (11.56)$$

  for some functions $n_{k[1,2]}(\sigma,\tau)$ stated by boundary conditions.

- The exact solution of (11.26) is given by

$$\zeta_i = -w_i + (\Omega^*)^{-1}\partial_i\Omega, \quad \Omega^* \neq 0, \quad (11.57)$$

We note that by a corresponding re–parametrizations of the conformal factor $\Omega(\sigma,\tau,v)$ we can reduce (11.53) to

$$\varpi\varpi^{\bullet\bullet} - (\varpi^{\bullet})^2 = 2\Lambda\varpi^3 \quad (11.58)$$

which gives and exact solution $\varpi = \varpi(\widetilde{x}^2)$ found from

$$(\varpi^{\bullet})^2 = \varpi^3\left(C\varpi^{-1} + 4\Lambda\right), C = const,$$

(or, inversely, to reduce to

$$\varpi\varpi'' - (\varpi')^2 = 2\Lambda\varpi^3$$

with exact solution $\varpi = \varpi(\widetilde{x}^3)$ found from

$$(\varpi')^2 = \varpi^3\left(C\varpi^{-1} + 4\Lambda\right), C = const).$$

The inverse problem of definition of h_5 for a given h_4 can be solved in explicit form when $h_4^* = 0$, $h_4 = h_{4(0)}(\sigma,\tau)$. In this case we have to solve

$$h_5^{**} + \frac{(h_5^*)^2}{2h_5} - 2\Lambda h_{4(0)}h_5 = 0, \quad (11.59)$$

which admits exact solutions by reduction to a Bernulli equation.

The outlined properties of solutions with cosmological constant hold also for 4D anholonomic spacetimes with "isotropic" cosmological constant Λ. To transfer general solutions from 5D to 4D we have to eliminate dependencies on the coordinate x^1 and to consider the 4D ansatz without g_{11} term.

11.5.1 A 5D anisotropic black torus solution with cosmological constant

We give an example of generalization of ansiotropic black hole solutions of Class A , constructed in the Section III as they will contain the cosmological constant Λ; we extend the solutions given by the data (11.39).

Our new 5D $\varphi-$ solution is parametrized by an ansatz with conformal factor $\Omega(x^i,v)$ (see (11.4) and (11.9)) as $\Omega = \varpi^{-1}(\sigma)\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,v)$. The factor $\varpi(\sigma,\tau)$ is chosen as to be a solution of (11.58). This conformal data must satisfy the condition (11.18), i. e.

$$\varpi^{-q_1/q_2}\Omega_{[0]}^{q_1/q_2}\Omega_{[1]}^{q_1/q_2} = \eta_4 \varpi h_{4(0)}$$

for some integers q_1 and q_2, where η_4 is found as $h_4 = \eta_4 \varpi h_{4(0)}$ satisfies the equation (11.54) and $\Omega_{[0]}(\sigma,\tau)$ could be chosen as to obtain in the locally isotropic limit and $\Lambda \to 0$ the toroidally deformed Schwarzschild metric (11.30). Choosing $h_5 = \eta_5 \varpi h_{5(0)}$, $\eta_5 h_{5(0)}$ is for the ansatz for (11.39), for which we compute the value $\pi = h_5^*/2h_5$, we obtain from (11.54) an equation for η_4,

$$\eta_4^* + \frac{2\Lambda}{\pi}\varpi h_{4(0)}(\eta_4)^2 + 2\left(\frac{\pi^*}{\pi} - \pi\right)\eta_4 = 0$$

which is a Bernulli equation [11] and admit an exact solution, in general, in non explicit form, $\eta_4 = \eta_4^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b)$, were we emphasize the functional dependencies on functions ϖ,ω,a,b and cosmological constant Λ. Having defined $\eta_{4[bern]}$, η_5 and ϖ, we can compute the α_i-, $\beta-$, and $\gamma-$coefficients, expressed as

$$\alpha_i = \alpha_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \beta = \beta^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b)$$

and $\gamma = \gamma^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b)$, following the formulas (11.16).

The next step is to find

$$w_i = w_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b) \text{ and } n_i = n_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b)$$

as for the general solutions (11.55) and (11.56).

At the final step we are able to compute the the second anisotropy coefficients

$$\zeta_i = -w_i^{[bern]} + \left(\partial_i \ln|\varpi^{-1}\Omega_{[0]}|\right)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]},$$

which depends on an arbitrary function $\Omega_{[0]}(\sigma,\tau)$. If we state $\Omega_{[0]}(\sigma,\tau) = \Omega_A$, as for Ω_A from (11.33), see similar details with respect to formulas (11.41), (11.42) and (11.43).

The data for the exact solutions with cosmological constant for $v = \varphi$ can be stated in the form

$$\begin{aligned}
\varphi_c\text{–solutions} &: (x^1 = \chi, x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1, \\
g_2 &= \varpi(\sigma), g_3 = \varpi(\sigma), \\
h_{4(0)} &= a(\tau), h_{5(0)} = b(\sigma,\tau), \text{see (11.31) and (11.58)}; \\
h_4 &= \eta_4(\sigma,\tau,\varphi)\varpi(\sigma)h_{4(0)}, h_5 = \eta_5(\sigma,\tau,\varphi)\varpi(\sigma)h_{5(0)}, \\
\eta_4 &= \eta_4^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \eta_5 = \omega^{-2}(\chi,\sigma,\tau,\varphi), \qquad (11.60) \\
w_i &= w_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \\
n_i &= n_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \\
\Omega &= \varpi^{-1}(\sigma)\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,\varphi), \eta_4 a = \Omega_{[0]}^{q_1/q_2}(\sigma,\tau)\Omega_{[1]}^{q_1/q_2}(\sigma,\tau,\varphi). \\
\zeta_i &= -w_i^{[bern]} + \left(\partial_i \ln|\varpi^{-1}\Omega_{[0]}\right)|) \ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]}.
\end{aligned}$$

We note that a solution with $v = t$ can be constructed as to generalize (11.40) to the presence of Λ. We can not present such data in explicit form because in this case we have to define η_5 by solving a solution like (11.23) for h_5, for a given h_4, which can not be integrated in explicit form.

The solution (11.60) preserves the two interesting properties of (11.39): 1) it admits a warped factor on the 5th coordinate, like $\Omega_{[1]}^{q_1/q_2} \sim \exp[-k|\chi|]$, which in this case is constructed for an anisotropic 5D vacuum gravitational configuration with anisotropic cosmological constant but not following a brane configuration like in Refs. [7]; 2) we can impose such conditions on the receptivity $\omega(\sigma,\tau,\varphi)$ as to obtain in the locally isotropic limit just the toroidally deformed Schwarzschild metric (11.30) trivially embedded into the 5D spacetime.

11.5.2 A 4D anisotropic black torus solution with cosmological constant

The simplest way to construct a such solution is to take the data (11.60), for $v = \varphi$, to eliminate the variable χ and to reduce the 5D indices to 4D ones. We obtain the 4D data:

$$
\begin{array}{rcl}
\varphi_c\text{–solutions} & : & (x^2 = \sigma, x^3 = \tau, y^4 = v = \varphi, y^5 = p = t), \\
g_2 & = & \varpi(\sigma), g_3 = \varpi(\sigma), \\
h_{4(0)} & = & a(\tau), h_{5(0)} = b(\sigma,\tau), \text{see (11.31) and (11.58);} \\
h_4 & = & \eta_4(\sigma,\tau,\varphi)\varpi(\sigma)h_{4(0)}, h_5 = \eta_5(\sigma,\tau,\varphi)\varpi(\sigma)h_{5(0)}, \\
\eta_4 & = & \eta_4^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \eta_5 = \omega^{-2}(\sigma,\tau,\varphi), \\
w_i & = & w_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \\
n_i & = & n_i^{[bern]}(\sigma,\tau,v,\Lambda,\varpi,\omega,a,b), \\
\Omega & = & \varpi^{-1}(\sigma)\Omega_{[0]}(\sigma,\tau)\Omega_{[1]}(\sigma,\tau,\varphi), \eta_4 a = \Omega_{[0]}^{q_1/q_2}(\sigma,\tau)\Omega_{[1]}^{q_1/q_2}(\sigma,\tau,\varphi), \\
\zeta_i & = & -w_i^{[bern]} + \left(\partial_i \ln|\varpi^{-1}\Omega_{[0]})\,|\right)\ \left(\ln|\Omega_{[1]}|\right)^* + \left(\Omega_{[1]}^*\right)^{-1}\partial_i\Omega_{[1]}.
\end{array}
\tag{11.61}
$$

The solution (11.61) describes a static black torus solution in 4D gravity with cosmological constant, Λ. The parameters of solutions depends on the Λ as well are renormalized by nonlinear anholonomic gravitational interactions. We can consider that the mass associated to such toroidal configuration can be anisotropically distributed in the interior of the torus and gravitationally polarized.

Finally, we note that in a similar manner like in the Sections III and IV we can construct another classes of anisotropic black holes solutions in 5D and 4D spacetimes with cosmological constants, being of Class A or Class B, with anisotropic φ–coordinate, or anisotropic t–coordinate. We omit the explicit data which are some nonlinear anholonomic generalizations of those solutions.

11.6 Conclusions and Discussion

We have shown that static black tori solutions can be constructed both in vacuum Einstein and five dimensional (5D) gravity. The solutions are parametrized by off–diagonal metric ansatz which can diagonalized with respect to corresponding anholonomic frames with mixtures of holonomic and anholonomic variables. Such metrics contain a toroidal horizon being some deformations with non-trivial topology of the Schwarzschild black hole solution.

The solutions were constructed by using the anholonomic frame method [1, 2, 3] which results in a very substantial simplification of the Einstein equations which admit general integrals for solutions.

The constructed black tori metrics depend on classes of two dimensional and three dimensional functions which reflect the freedom in definition of toroidal coordinates as well the possibility to state by boundary conditions various configurations with running constants, anisotropic gravitational polarizations and (in presence of extra dimensions) with warping geometries. The new toroidal solutions can be extended for spacetimes with cosmological constant.

In view of existence of such solutions, the old problem of the status of frames in gravity theories rises once again, now in the context of "effective" diagonalization of off–diagonal metrics by using anholonomic transforms. The bulk of solutions with spherical, cylindrical and plane symmetries were constructed in gravitational theories of diverse dimensions by using diagonal metrics (sometimes with off–diagonal terms) given with respect to "pure" coordinate frames. Such solutions can be equivalently re–defined with respect to arbitrary frames of reference and usually the problem of fixing some reference bases in order to state the boundary conditions is an important physical problem but not a dynamical one. This problem becomes more sophisticate when we deal with generic off–diagonal metrics and anholonomic frames. In this case some 'dynamical, components of metrics can be transformed into "non–dynamical" components of frame bases, which, following a more rigorous mathematical approach, reflects a constrained nonlinear dynamics for gravitational and matter fields with both holonomic (unconstrained) and anholonomic (constrained) variables. In result there are more possibilities in definition of classes of exact solutions with non–trivial topology, anisotropies and nonlinear interactions.

The solutions obtained in this paper contain as particular cases (for corresponding parametrizations of considered ansatz) the 'black ring' metrics with event horizon of topology $S^1 \times S^2$ analyzed in Refs. [12]. In our case we emphasized the presence of off–diagonal terms which results in warping, anisotropy and running of constants. Here it should be noted that the generic nonlinear character of the Einstein equations written with respect to anholonomic frames connected with diagonalization of off–diagonal metrics allow us to construct different classes of exact 5D and 4D solutions with the same or different topology; such solutions can define very different vacuum gravitational and gravitational–matter field configurations.

The method and results presented in this paper provide a prescription on anholonomic transforming of some known locally isotropic solutions from a gravity/string theory into corresponding classes of anisoropic solutions of the same, or of an extended theory:

A vacuum, or non-vacuum, solution, and metrics conformally equivalent to a known solution, parametrized by a diagonal matrix given with respect to a holonomic (coordinate) base, contained as a trivial form of ansatz (11.2), or (11.4), can be transformed into a metric with non-trivial topological horizons and then generalized to be an anisotropic solution with similar but anisotropically renormalized physical constants and diagonal metric coefficients, given with respect to adapted anholonomic frames; the new anholo-

nomic metric defines an exact solution of a simplified form of the Einstein equations (11.22)–(11.26) and (11.19); such types of solutions are parametrized by off–diagonal metrics if they are re–defined with respect to usual coordinate frames.

We emphasize that the anholonomic frame method and constructed black tori solutions conclude in a general formalism of generating exact solutions with off–diagonal metrics in gravity theories and may have a number of applications in modern astrophysics and string/M–theory gravity.

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Chapter 12

Ellipsoidal Black Hole – Black Tori Systems in 4D Gravity

Abstract [1]

We construct new classes of exact solutions of the 4D vacuum Einstein equations which describe ellipsoidal black holes, black tori and combined black hole – black tori configurations. The solutions can be static or with anisotropic polarizations and running constants. They are defined by off–diagonal metric ansatz which may be diagonalized with respect to anholonomic moving frames. We examine physical properties of such anholonomic gravitational configurations and discuss why the anholonomy may remove the restriction that horizons must be with spherical topology.

12.1 Introduction

Torus configurations of matter around black hole – neutron star objects are intensively investigated in modern astrophysics [1]. One considers that such tori may radiate gravitational radiation powered by the spin energy of the black hole in the presence of non–axisymmetries; long gamma–ray bursts from rapidly spinning black hole–torus systems may represent hypernovae or black hole–neutron star coalescence. Thus the topic of constructing of exact vacuum and non–vacuum solutions with non–trivial topology in the framework of general relativity and extra dimension gravitational theories becomes of special importance and interest.

In the early 1990s, new solutions with non–spherical black hole horizons (black tori) were found [2] for different states of matter and for locally anti-de Sitter space times; for a recent review, see [3]. Static ellipsoidal black hole, black tori, anisotropic wormhole and Taub NUT metrics and solitonic solutions of the vacuum and non–vacuum Einstein equations were constructed in Refs. [4, 5]. Non–trivial topology configurations (for instance, black rings) are intensively studied in extra dimension gravity [6, 7, 8].

For four dimensional gravity (4D), it is considered that a number of classical theorems [9] impose that a stationary, asymptotically flat, vacuum black hole solution is completely characterized by its mass and spin and event horizons of non–spherical topology are forbidden [10]; see [11] for further discussion of this issue.

Nevertheless, there were constructed various classes of exact solutions in 4D and 5D gravity with non–trivial topology, anisotropies, solitonic configurations, running constants and warped factors, under certain conditions defining static configurations in 4D vacuum gravity. Such metrics were parametrized by off–diagonal ansazt (for coordinate frames) which can be effectively diagonalized with respect to certain anholonomic frames with mixtures of holonomic and anholonomic variables. The system of vacuum Einstein equations for such ansatz becomes exactly integrable and describe a new "anholonomic nonlinear dynamics" of vacuum gravitational fields, which posses generic local anisotropy. The new classes of solutions may have locally isotropic limits, or can be associated to metric coefficients of some well known, for instance, black hole, cylindrical, or wormhole soutions.

There is one important question if such anholonomic (anisotropic) solutions can exist only in extra dimension gravity, with some specific effective reductions to lower dimensions, or the anholonomic transforms generate a new class of solutions even in general relativity theory which might be not restricted by the conditions of Israel–Carter–Robinson uniqueness and Hawking cosmic cenzorship theorems [9, 10]?

In the present paper, we explore possible 4D ellipsoidal black hole – black torus systems which are defined by generic off–diagonal matrices and describe anholonomic vacuum gravitational configurations. We present a new class of exact solutions of 4D vacuum Einstein equations which can be associated to some exact solutions with ellipsoidal/toroidal horizons and signularities, and theirs superpositions, being of static configuration, or, in general, with nonlinear gravitational polarization and running constants. We also discuss implications of these anisotropic solutions to gravity theories and ponder possible ways to solve the problem with topologically non–trivial and deformed horizons.

The organization of this paper is as follows: In Sec. II, we consider ellipsoidal and torus deformations and anistoropic conformal transforms of the Schwarzschild metric. We introduce an off–diagonal ansatz which can be diagonalized by anholonomic transforms and compute the non–trivial components of the vacuum Einstein equations in Sec. III. In Sec. IV, we construct and analyze three types of exact static solutions with ellisoidal–torus horizons. Sec. V is devoted to generalization of such solutions for configurations with running constants and anisotropic polarizations. The conclusion and discussion are presented in Sec. VI.

12.2 Ellipsoidal/Torus Deformations of Metrics

In this Section we analyze anholonomic transforms with ellipsoidal/torus deformations of the Schwarzschild solution to some off–diagonal metrics. We define the conditions when the new 'deformed' metrics are exact solutions of vacuum Einstein equations.

The Schwarzschild solution may be written in *isotropic spherical coordinates* (ρ, θ, φ) [12]

$$dS^2 = -\rho_g^2 \left(\frac{\widehat{\rho}+1}{\widehat{\rho}}\right)^4 \left(d\widehat{\rho}^2 + \widehat{\rho}^2 d\theta^2 + \widehat{\rho}^2 \sin^2\theta d\varphi^2\right) + \left(\frac{\widehat{\rho}-1}{\widehat{\rho}+1}\right)^2 dt^2, \quad (12.1)$$

where the isotropic radial coordinate ρ is related with the usual radial coordinate r via the relation $r = \rho\,(1 + r_g/4\rho)^2$ for $r_g = 2G_{[4]}m_0/c^2$ being the 4D gravitational radius of a point particle of mass m_0, $G_{[4]} = 1/M^2_{P[4]}$ is the 4D Newton constant expressed via Plank mass $M_{P[4]}$. In our further considerations, we put the light speed constant $c = 1$ and re–scale the isotropic radial coordinate as $\widehat{\rho} = \rho/\rho_g$, with $\rho_g = r_g/4$. The metric (12.1) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass m_0. It has a singularity for $r = 0$ and a spherical horizon for $r = r_g$, or, in re–scaled isotropic coordinates, for $\widehat{\rho} = 1$. We emphasize that this solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames.

We may introduce a new 'exponential' radial coordinate $\varsigma = \ln\widehat{\rho}$ and write the Schwarzschild metric as

$$ds^2 = -\rho_g^2 b(\varsigma)\left(d\varsigma^2 + d\theta^2 + \sin^2\theta d\varphi^2\right) + a(\varsigma)\, dt^2, \quad (12.2)$$

$$a(\varsigma) = \left(\frac{\exp\varsigma - 1}{\exp\varsigma + 1}\right)^2, b(\varsigma) = \frac{(\exp\varsigma + 1)^4}{(\exp\varsigma)^2}. \quad (12.3)$$

The condition of vanishing of coefficient $a(\varsigma)$, $\exp\varsigma = 1$, defines the horizon 3D spherical hypersurface

$$\varsigma = \varsigma\left[\widehat{\rho}\left(\sqrt{x^2+y^2+z^2}\right)\right],$$

where x, y and z are usual Cartezian coordinates.

The 3D spherical line element

$$ds^2_{(3)} = d\varsigma^2 + d\theta^2 + \sin^2\theta d\varphi^2,$$

may be written in arbitrary ellipsoidal, or toroidal, coordinates which transforms the spherical horizon equation into very sophisticate relations (with respect to new coordinates).

Our idea is to deform (renormalize) the coefficients (12.3), $a(\varsigma) \rightarrow A(\varsigma, \theta)$ and $b(\varsigma) \rightarrow B(\varsigma, \theta)$, as they would define a rotation ellipsoid and/or a toroidal horizon and symmetry (for simplicity, we shall consider the elongated ellipsoid configuration; the flattened ellipsoids may be analyzed in a similar manner). But such a diagonal metric with respect to ellipsoidal, or toroidal, local coordinate frame does not solve the vacuum Einstein equations. In order to generate a new vacuum solution we have to "elongate" the differentials $d\varphi$ and dt, i. e. to introduce some "anholonomic transforms" (see details in [7]), like

$$\begin{aligned} d\varphi &\rightarrow \delta\varphi + w_\varsigma(\varsigma, \theta, v)\, d\varsigma + w_\theta(\varsigma, \theta, v)\, d\theta, \\ dt &\rightarrow \delta t + n_\varsigma(\varsigma, \theta, v)\, d\varsigma + n_\theta(\varsigma, \theta, v)\, d\theta, \end{aligned}$$

for $v = \varphi$ (static configuration), or $v = t$ (running in time configuration) and find the conditions when w- and n–coefficients and the renormalized metric coefficients define off–diagonal metrics solving the Einstein equations and possessing some ellipsoidal and/or toroidal horizons and symmetries.

We shall define the 3D space ellipsoid – toroidal configuration in this manner: in the center of Cartezian coordinates we put an rotation ellipsoid elongated along axis z (its intersection by the xy–coordinate plane describes a circle of radius $\rho_g^{[e]} = \sqrt{x^2 + y^2} \sim \rho_g$); the ellipsoid is surrounded by a torus with the same z axis of symmetry, when $-z_0 \leq z \leq z_0$, and the intersections of the torus with the xy–coordinate plane describe two circles of radia $\rho_g^{[t]} - z_0 = \sqrt{x^2 + y^2}$ and $\rho_g^{[t]} + z_0 = \sqrt{x^2 + y^2}$; the parameters $\rho_g^{[e]}, \rho_g^{[t]}$ and z_0 are chosen as to define not intersecting toroidal and ellipsoidal horizons, i. e. the conditions

$$\rho_g^{[t]} - z_0 > \rho_g^{[e]} > 0 \qquad (12.4)$$

are imposed.

12.2.1 Ellipsoidal Configurations

We shall consider the *rotation ellipsoid coordinates* [13] (u, λ, φ) with $0 \leq u < \infty, 0 \leq \lambda \leq \pi, 0 \leq \varphi \leq 2\pi$, where $\sigma = \cosh u \geq 1$, are related with the isotropic 3D Cartezian coordinates (x, y, z) as

$$\begin{aligned}
(x &= \widetilde{\rho}\sinh u \sin\lambda\cos\varphi, \\
y &= \widetilde{\rho}\sinh u \sin\lambda\sin\varphi, z = \widetilde{\rho}\cosh u\cos\lambda)
\end{aligned} \tag{12.5}$$

and define an elongated rotation ellipsoid hypersurface

$$\left(x^2 + y^2\right)/(\sigma^2 - 1) + \tilde{z}^2/\sigma^2 = \widetilde{\rho}^2. \tag{12.6}$$

with $\sigma = \cosh u$. The 3D metric on a such hypersurface is

$$dS^2_{(3D)} = g_{uu}du^2 + g_{\lambda\lambda}d\lambda^2 + g_{\varphi\varphi}d\varphi^2,$$

where

$$\begin{aligned}
g_{uu} &= g_{\lambda\lambda} = \widetilde{\rho}^2\left(\sinh^2 u + \sin^2\lambda\right), \\
g_{\varphi\varphi} &= \widetilde{\rho}^2\sinh^2 u \sin^2\lambda.
\end{aligned}$$

We can relate the rotation ellipsoid coordinates (u, λ, φ) from (12.5) with the isotropic radial coordinates $(\widehat{\rho}, \theta, \varphi)$, scaled by the constant ρ_g, from (12.1), equivalently with coordinates $(\varsigma, \theta, \varphi)$ from (12.2), as

$$\widetilde{\rho} = 1, \cosh u = \widehat{\rho} = \exp\varsigma$$

and deform the Schwarzschild metric by introducing ellipsoidal coordinates and a new horizon defined by the condition that vanishing of the metric coefficient before dt^2 describe an elongated rotation ellipsoid hypersurface (12.6),

$$\begin{aligned}
ds^2_E &= -\rho_g^2\left(\frac{\cosh u + 1}{\cosh u}\right)^4(\sinh^2 u + \sin^2\lambda) \\
&\times[du^2 + d\lambda^2 + \frac{\sinh^2 u\ \sin^2\lambda}{\sinh^2 u + \sin^2\lambda}d\varphi^2] \\
&+\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^2 dt^2.
\end{aligned} \tag{12.7}$$

The ellipsoidaly deformed metric (12.7) does not satisfy the vacuum Einstein equations, but at long distances from the horizon it transforms into the usual Schwarzschild solution (12.1).

We introduce two Classes (A and B) of 4D auxiliary pseudo–Riemannian metrics, also given in ellipsoid coordinates, being some conformal transforms of (12.7), like

$$ds^2_E = \Omega_{A(B)E}\,(u,\lambda)\, ds^2_{A(B)E}$$

which also are not supposed to be solutions of the Einstein equations:

Metric of Class A:

$$ds^2_{(AE)} = -du^2 - d\lambda^2 + a_E(u,\lambda)d\varphi^2 + b_E(u,\lambda)dt^2], \tag{12.8}$$

where

$$\begin{aligned} a_E(u,\lambda) &= -\frac{\sinh^2 u\ \sin^2\lambda}{\sinh^2 u + \sin^2\lambda}, \\ b_E(u,\lambda) &= \frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2(\cosh u + 1)^6(\sinh^2 u + \sin^2\lambda)}, \end{aligned} \tag{12.9}$$

which results in the metric (12.7) by multiplication on the conformal factor

$$\Omega_{AE}\,(u,\lambda) = \rho_g^2 \frac{(\cosh u + 1)^4}{\cosh^4 u}(\sinh^2 u + \sin^2\lambda). \tag{12.10}$$

Metric of Class B:

$$ds^2_{BE} = g_E(u,\lambda)\left(du^2 + d\lambda^2\right) - d\varphi^2 + f_E(u,\lambda)dt^2, \tag{12.11}$$

where

$$\begin{aligned} g_E(u,\lambda) &= -\frac{\sinh^2 u + \sin^2\lambda}{\sinh^2 u\ \sin^2\lambda}, \\ f_E(u,\lambda) &= \frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2(\cosh u + 1)^6 \sinh^2 u \sin^2\lambda}, \end{aligned} \tag{12.12}$$

which results in the metric (12.7) by multiplication on the conformal factor

$$\Omega_{BE}\,(u,\lambda) = \rho_g^2 \frac{(\cosh u + 1)^4}{\cosh^4 u}\sinh^2 u \sin^2\lambda.$$

In Ref. [7] we proved that there are anholonomic transforms of the metrics (12.7), (12.8) and (12.11) which results in exact ellipsoidal black hole solutions of the vacuum Einstein equations.

12.2.2 Toroidal Configurations

Fixing a scale parameter $\rho_g^{[t]}$ which satisfies the conditions (12.4) we define the *toroidal coordinates* (σ, τ, φ) (we emphasize that in in this paper we use different letters for ellipsoidal and toroidal coordinates introduced in Ref. [13]). These coordinates run the values $-\pi \leq \sigma < \pi, 0 \leq \tau \leq \infty, 0 \leq \varphi < 2\pi$. They are related with the isotropic 3D Cartezian coordinates via transforms

$$\begin{aligned} \tilde{x} &= \frac{\widetilde{\rho}\sinh\tau}{\cosh\tau - \cos\sigma}\cos\varphi, \qquad (12.13) \\ \tilde{y} &= \frac{\widetilde{\rho}\sinh\tau}{\cosh\tau - \cos\sigma}\sin\varphi, \tilde{z} = \frac{\widetilde{\rho}\sinh\sigma}{\cosh\tau - \cos\sigma} \end{aligned}$$

and define a toroidal hypersurface

$$\left(\sqrt{\tilde{x}^2 + \tilde{y}^2} - \widetilde{\rho}\frac{\cosh\tau}{\sinh\tau}\right)^2 + \tilde{z}^2 = \frac{\widetilde{\rho}^2}{\sinh^2\tau}.$$

The 3D metric on a such toroidal hypersurface is

$$ds^2_{(3D)} = g_{\sigma\sigma}d\sigma^2 + g_{\tau\tau}d\tau^2 + g_{\varphi\varphi}d\varphi^2,$$

where

$$\begin{aligned} g_{\sigma\sigma} &= g_{\tau\tau} = \frac{\widetilde{\rho}^2}{(\cosh\tau - \cos\sigma)^2}, \\ g_{\varphi\varphi} &= \frac{\widetilde{\rho}^2\sinh^2\tau}{(\cosh\tau - \cos\sigma)^2}. \end{aligned}$$

We can relate the toroidal coordinates (σ, τ, φ) from (12.13) with the isotropic radial coordinates $\left(\widehat{\rho}^{[t]}, \theta, \varphi\right)$, scaled by the constant $\rho_g^{[t]}$, as

$$\widetilde{\rho} = 1, \sinh^{-1}\tau = \widehat{\rho}^{[t]}$$

and transform the Schwarzschild solution into a new metric with toroidal coordinates by changing the 3D radial line element into the toroidal one and stating the tt–coefficient of the metric to have a toroidal horizon. The resulting metric is

$$\begin{aligned} ds^2_T &= -\left(\rho_g^{[t]}\right)^2 \frac{(\sinh\tau + 1)^4}{(\cosh\tau - \cos\sigma)^2} \times \qquad (12.14) \\ & \left(d\sigma^2 + d\tau^2 + \sinh^2\tau d\varphi^2\right) + \left(\frac{\sinh\tau - 1}{\sinh\tau + 1}\right)^2 dt^2, \end{aligned}$$

Such a deformed Schwarzschild like toroidal metric is not an exact solution of the vacuum Einstein equations, but at long radial distances it transforms into the usual Schwarzschild solution with effective horizon $\rho_g^{[t]}$ with the 3D line element parametrized by toroidal coordinates.

We introduce two Classes (A and B) of 4D auxiliary pseudo–Riemannian metrics, also given in toroidal coordinates, being some conformal transforms of (12.14), like

$$ds_T^2 = \Omega_{A(B)T}\,(\sigma,\tau)\,ds^2_{A(B)T}$$

but which are not supposed to be solutions of the Einstein equations:

Metric of Class A:

$$ds^2_{AT} = -d\sigma^2 - d\tau^2 + a_T(\tau)d\varphi^2 + b_T(\sigma,\tau)dt^2, \tag{12.15}$$

where

$$\begin{aligned} a_T(\tau) &= -\sinh^2\tau, \\ b_T(\sigma,\tau) &= \frac{(\sinh\tau - 1)^2\,(\cosh\tau - \cos\sigma)^2}{\rho_g^{[t]2}\,(\sinh\tau + 1)^6}, \end{aligned} \tag{12.16}$$

which results in the metric (12.14) by multiplication on the conformal factor

$$\Omega_{AT}\,(\sigma,\tau) = \rho_g^{[t]2}\frac{(\sinh\tau + 1)^4}{(\cosh\tau - \cos\sigma)^2}. \tag{12.17}$$

Metric of Class B:

$$ds^2_{BT} = g_T(\tau)\left(d\sigma^2 + d\tau^2\right) - d\varphi^2 + f_T(\sigma,\tau)dt^2, \tag{12.18}$$

where

$$\begin{aligned} g_T(\tau) &= -\sinh^{-2}\tau, \\ f_T(\sigma,\tau) &= \rho_g^{[t]2}\left(\frac{\sinh^2\tau - 1}{\cosh\tau - \cos\sigma}\right)^2, \end{aligned}$$

which results in the metric (12.14) by multiplication on the conformal factor

$$\Omega_{BT}\,(\sigma,\tau) = \left(\rho_g^{[t]}\right)^{-2}\frac{(\cosh\tau - \cos\sigma)^2}{(\sinh\tau + 1)^2}. \tag{12.19}$$

In Ref. [8] we used the metrics (12.14), (12.15) and (12.18) in order to generate exact solutions of the Einstein equations with toroidal horizons and anisotropic polarizations and running constants by performing corresponding anholonomic transforms.

12.3 The Metric Ansatz and Vacuum Einstein Equations

Let us denote the local system of coordinates as $u^\alpha = (x^i, y^a)$, where $x^1 = u$ and $x^2 = \lambda$ for ellipsoidal coordinates ($x^1 = \sigma$ and $x^2 = \tau$ for toroidal coordinates) and $y^3 = v = \varphi$ and $y^4 = t$ for the so–called φ–anisotropic configurations ($y^4 = v = t$ and $y^5 = \varphi$ for the so–called t–anisotropic configurations). Our spacetime is modelled as a 4D pseudo–Riemannian space of signature $(-,-,-,+)$ (or $(-,-,+,-)$), which in general may be enabled with an anholonomic frame structure (tetrads, or vierbiend) $e_\alpha = A_\alpha^\beta(u^\gamma)\partial/\partial u^\beta$ subjected to some anholonomy relations

$$e_\alpha e_\beta - e_\beta e_\alpha = W_{\alpha\beta}^\gamma(u^\varepsilon)\, e_\gamma, \tag{12.20}$$

where $W_{\alpha\beta}^\gamma(u^\varepsilon)$ are called the coefficients of anholonomy.

The anholonomically and conformally transformed 4D line element is

$$ds^2 = \Omega^2(x^i,v)\hat{g}_{\alpha\beta}\left(x^i,v\right) du^\alpha du^\beta, \tag{12.21}$$

were the coefficients $\hat{g}_{\alpha\beta}$ are parametrized by the ansatz

$$\left[\begin{array}{cccc} g_1+{\zeta_1}^2h_3+{n_3}^2h_4 & \zeta_1\zeta_2h_3+n_1n_2h_4 & \zeta_1h_3 & n_1h_4 \\ \zeta_1\zeta_2h_3+n_1n_2h_4 & g_2+{\zeta_2}^2h_3+{n_3}^2h_4 & +\zeta_2h_3 & n_2h_4 \\ \zeta_1h_3 & \zeta_2h_3 & h_3 & 0 \\ n_1h_4 & n_2h_4 & 0 & h_4 \end{array}\right], \tag{12.22}$$

with $g_i = g_i(x^i), h_a = h_{ai}\left(x^k,v\right), n_i = n_i\left(x^k,v\right),\ \zeta_i = \zeta_i\left(x^k,v\right)$ and $\Omega = \Omega\left(x^k,v\right)$ being some functions of necessary smoothly class or even singular in some points and finite regions. So, the g_i–components of our ansatz depend only on "holonomic" variables x^i and the rest of coefficients may also depend on "anisotropic" (anholonomic) variable $y^3 = v$; our ansatz does not depend on the second anisotropic variable y^4.

We may diagonalize the line element

$$\delta s^2 = \Omega^2[g_1(dx^1)^2 + g_2(dx^2)^2 + h_3(\delta v)^2 + h_4(\delta y^4)^2], \tag{12.23}$$

with respect to the anholonomic co–frame
$\delta^\alpha = (dx^i, \delta v, \delta y^4)$, where

$$\delta v = dv + \zeta_i dx^i \text{ and } \delta y^4 = dy^4 + n_i dx^i, \tag{12.24}$$

which is dual to the frame $\delta_\alpha = (\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i + \zeta_i\partial_3 + n_i\partial_4. \tag{12.25}$$

The tetrads δ_α and δ^α are anholonomic because, in general, they satisfy some non–trivial anholonomy relations (12.20). The anholonomy is induced by the coefficients ζ_i and n_i which "elongate" partial derivatives and differentials if we are working with respect to anholonomic frames. This result in a more sophisticate differential and integral calculus (a usual situation in 'tetradic' and 'spinor' gravity), but simplifies substantially tensor computations, because we are dealing with diagonalized metrics.

The vacuum Einstein equations for the (12.22) (equivalently, for (12.23)), $R_\alpha^\beta = 0$, computed with respect to anholonomic frames (12.24) and (12.25), transforms into a system of partial differential equations [4, 7, 8]:

$$
\begin{aligned}
R_1^1 = R_2^2 = -\frac{1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} & \\
+ g_1^{''} - \frac{g_1^{'} g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}] &= 0, \quad (12.26)\\
R_3^3 = R_4^4 = \frac{-1}{2h_3 h_4}\left[h_4^{**} - h_4^*\left(\ln\sqrt{|h_3 h_4|}\right)^*\right] &= 0, \quad (12.27)\\
R_{4i} = -\frac{h_4}{2h_3}\left[n_i^{**} + \gamma n_i^*\right] &= 0, \quad (12.28)
\end{aligned}
$$

where

$$\gamma = 3h_4^*/2h_4 - h_3^*/h_3, \quad (12.29)$$

and the partial derivatives are written in brief like $g_1^\bullet = \partial g_1/\partial x^1, g_1^{'} = \partial g_1/\partial x^2$ and $h_3^* = \partial h_3/\partial v$. The coefficients ζ_i are found as to consider non–trivial conformal factors $\Omega:$ we compensate by ζ_i possible conformal deformations of the Ricci tensors, computed with respect to anholonomic frames. The conformal invariance of such anholonomic transforms holds if

$$\Omega^{q_1/q_2} = h_3 \ (q_1 \text{ and } q_2 \text{ are integers}), \quad (12.30)$$

and there are satisfied the equations

$$\partial_i \Omega - \zeta_i \Omega^* = 0. \quad (12.31)$$

The system of equations (12.26)–(12.28) and (12.31) can be integrated in general form [7]. Physical solutions are defined from some additional boundary conditions, imposed types of symmetries, nonlinearities and singular behavior and compatibility in locally anisotropic limits with some well known exact solutions.

In this paper we give some examples of ellipsoidal and toroidal solutions and investigate some classes of metrics for combined ellipsoidal black hole – black tori configurations.

12.4 Static Black Hole – Black Torus Metrics

We analyzed in detail the method of anholonomic frames and constructed 4D and 5D ellipsoidal black hole and black tori solutions in Refs. [4, 7, 8]. In this Section we give same new examples of metrics describing one static 4D black hole or one static 4D black torus configurations. Then we extend the constructions for metrics describing combined variants of black hole – black torus solutions. We shall analyze solutions with trivial and non–trivial conformal factors.

In this section the 4D local coordinates are written as $(x^1, x^2, y^3 = v = \varphi, y^4 = t)$, where we take $x^i = (u, \lambda)$ for ellipsoidal configurations and $x^i = (\sigma, \tau)$ for toroidal configurations. Here we note that, we can introduce a "general" 2D space ellipsoidal coordinate system, $u = u(\sigma, \tau)$ and $\lambda = \tau$, for both ellipsoidal and toroidal configurations if, for instance, we identify the ellipsoidal coordinate λ with the toroidal τ, and relate u with σ and τ as

$$\sinh u = \frac{1}{\cosh\tau - \cos\sigma}.$$

In the vicinity of $\tau = 0$ we can approximate $\cosh\tau \approx 1$ and to write $u = u(\sigma)$and $\lambda = \tau$. For $\tau \gg 1$ we have

$$\sinh u \approx \frac{1}{\cosh\tau}\left(1 + \frac{1}{\cos\sigma}\right).$$

In general, we consider that the "holonomic" coordinates are some functions $x^i = x^i(\sigma, \tau) = \widetilde{x}^i(u, \lambda)$ for which the 2D line element can be written in conformal metric form,

$$ds^2_{[2]} = -\mu^2\left(x^i\right)\left[\left(dx^1\right)^2 + \left(dx^2\right)^2\right].$$

For simplicity, we consider 4D coordinate parametrizations when the angular coordinate φ and the time like coordinate t are not affected by any transforms of x–coordinates.

12.4.1 Static anisotropic black hole/torus solutions

An example of ellipsoidal black hole configuration

The simplest way to generate a static but anisotropic ellipsoidal black hole solution with an anholonomically diagonalized metric (12.23) is to take a metric of type (12.8), to "elongate" its differentials,

$$\begin{aligned} d\varphi &\rightarrow \delta\varphi = d\varphi + \zeta_i\left(x^k, \varphi\right) dx^i, \\ dt &\rightarrow \delta t = dt + n_i\left(x^k, \varphi\right) dx^i, \end{aligned}$$

than to multiply on a conformal factor

$$\Omega^2\left(x^k,\varphi\right) = \omega^2\left(x^k,\varphi\right)\Omega^2_{AE}(x^k),$$

the factor $\omega^2\left(x^k,\varphi\right)$ is obtained by re–scaling the constant ρ_g from (12.10),

$$\rho_g \rightarrow \overline{\rho}_g = \omega\left(x^k,\varphi\right)\rho_g, \tag{12.32}$$

in the simplest case we can consider only "angular" on φ anisotropies. Then we 'renormalize' (by introducing x^i coordinates) the g_1, g_2 and h_3 coefficients,

$$\begin{aligned} g_{1,2} &= -1 \rightarrow -\mu^2\left(x^i\right), & (12.33)\\ h_3 &= h_{3[0]} = a_E(u,\lambda) \rightarrow h_3 = -\Omega^{-2}\left(x^k,\varphi\right), & (12.34)\end{aligned}$$

we fix a relation of type (12.30), and take $h_4 = h_{4[0]} = b_E(x^i)$. The anholonomically transformed metric is parametrized in the form

$$\begin{aligned} \delta s^2 &= \Omega^2\{-\mu^2\left(x^i\right)\left[(dx^1)^2+(dx^2)^2\right] & (12.35)\\ &\quad -\Omega^{-2}\left(x^k,\varphi\right)(\delta v)^2 + b_E(x^i)(\delta y^4)^2\}, \end{aligned}$$

where μ, ζ_i and n_i are to be defined respectively from the equations (12.26), (12.31) and (12.28). We note that the equation (12.27) is already solved because in our case $h_4^* = 0$.

The equation (12.26), with partial derivations on coordinates x^i and parametrizations (12.33) has the general solution

$$\mu^2 = \mu^2_{[0]}\exp\left[c_{[1]}x^1\left(u,\lambda\right) + c_{[2]}x^2\left(u,\lambda\right)\right], \tag{12.36}$$

where $\mu_{[0]}, c_{[1]}$ and $c_{[2]}$ are some constants which should be defined from boundary conditions and by fixing a corresponding 2D system of coordinates; we pointed that we may redefine the factor (12.36) in 'pure' ellipsoidal coordinates (u,λ).

The general solution of (12.31) for renormalization (12.32) and parametrization (12.34) is

$$\begin{aligned} \zeta_i\left(x^k,\varphi\right) &= \left(\omega^*\right)^{-1}\partial_i\omega + \partial_i\ln|\Omega_{AE}|/\left(\ln|\omega|\right)^*, & (12.37)\\ &= \partial_i\ln|\Omega_{AE}|/\left(\ln|\omega|\right)^* \text{ for } \omega=\omega\left(\varphi\right). \end{aligned}$$

For a given h_3 with $h_4^* = 0$, we can compute the coefficient γ from (12.29). After two integrations on φ in (12.28) we find

$$n_i\left(x^k,\varphi\right) = n_{i[0]}\left(x^k\right) + n_{i[1]}\left(x^k\right)\int\omega^{-2}d\varphi. \tag{12.38}$$

The set of functions (12.36), (12.37) and (12.38) for any given $\Omega_{AE}(x^i)$ and $\omega\left(x^k,\varphi\right)$ defines an exact static solution of the vacuum Einstein equations parametrized by an off–diagonal metric of type (12.35). This solution have an ellipsoidal horizon defined by the condition of vanishing of the coefficient $h_{4[0]}=b_E(x^i)$, see the coefficients for the auxiliary metric (12.8) and an anisotropic effective constant (12.32). This is a general solution depending on arbitrary functions $\omega\left(x^k,\varphi\right)$ and $n_{i[0,1]}\left(x^k\right)$ and constants $\mu_{[0]},c_{[1]}$ and $c_{[2]}$ which have to be stated from some additional physical arguments.

For instance, if we wont to impose the condition that our solution, far away from the ellipsoidal horizon, transform into the Schwarzschild solution with an effective anisotropic "mass", or a renormalized gravitational Newton constant, we may put $\mu_{[0]}=1$ and fix the x^i–coordinates and constants $c_{[1,2]}$ as to obtain the linear interval

$$ds^2_{[2]}=-\left[du^2+d\lambda^2\right].$$

The coefficients $n_{i[0,1]}\left(x^k\right)$ and $\omega\left(x^k,\varphi\right)$ may be taken as at long distances from the horizon one holds the limits $n_{i[0,1]}\left(x^k\right)\rightarrow 0$ and $\zeta_i\left(x^k,\varphi\right)\rightarrow 0$ for $\omega\left(x^k,\varphi\right)\rightarrow 0$. In this case, at asymptotic, our solution will transform into a Schwarzschild like solution with "renormalized" parameter $\overline{\rho}_g\rightarrow const.$

Nevertheless, we consider that it is not obligatory to select only such type of ellipsoidal solutions (with imposed asymptotic spherical symmetry) parametrized by metrics of class (12.35). The system of vacuum gravitational equations (12.26)–(12.31) for the ansatz (12.35) defines a nonlinear static configuration (an alternative vacuum Einstein configuration with ellipsoidal horizon) which, in general, is not equivalent to the Schwarzschild vacuum. This points to some specific properties of the gravitational vacuum which follow from the nonlinear character of the Einstein equations. In quantum field theory the nonlinear effects may result in unitary non–equivalent vacua; in classical gravitational theories we could obtain a similar behavior if we are dealing with off–diagonal metrics and anholonomic frames.

The constructed new static vacuum solution (12.35) for a 4D ellipsoidal black hole is stated by the coefficients

$$\begin{aligned}
g_{1,2} &= -1,\mu=1,\overline{\rho}_g=\omega\left(x^k,\varphi\right)\rho_g,\Omega^2=\omega^2\Omega^2_{AE},\\
h_3 &= -\Omega^{-2}\left(x^k,\varphi\right),h_4=b_E(x^i),(\text{ see (12.8),(12.10)}),\\
\zeta_i &= \left(\omega^*\right)^{-1}\partial_i\omega+\partial_i\ln|\Omega_{AE}|/\left(\ln|\omega|\right)^*,\\
n_i &= n_{i[0]}\left(x^k\right)+n_{i[1]}\left(x^k\right)\int\omega^{-2}d\varphi.
\end{aligned}\qquad(12.39)$$

These data define an ellipsoidal configuration, see Fig. 12.1.

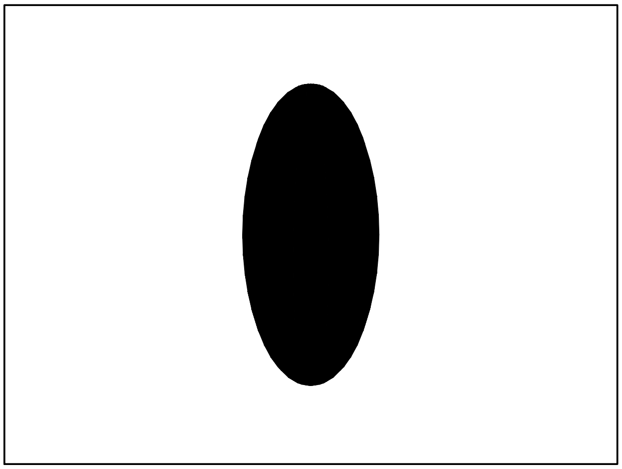

Figure 12.1: Ellipsoidal Configuration

Finally, we remark that we have generated a vacuum ellipsoidal gravitational configuration starting from the metric (12.8), i. e. we constructed an ellipsoidal φ–solution of Class A (see details on classification in [7]). In a similar manner we can define anholonomic deformations of the metric (12.11) and renormalization of conformal factor $\Omega_{BE}\left(u,\lambda\right)$ in order to construct an ellipsoidal φ–solution of Class B. We omit such considerations in this paper but present, in the next subsection, an example of toroidal φ–solution of Class B.

An example of toroidal black hole configuration

We start with the metric (12.18), "elongate" its differentials $d\varphi\rightarrow\delta\varphi$ and $dt\rightarrow\delta t$ and than multiply on a conformal factor

$$\Omega^{2}\left(x^{k},\varphi\right)=\varpi^{2}\left(x^{k},\varphi\right)\Omega_{BT}^{2}(x^{k})g_{T}\left(\tau\right),$$

see (12.19) which is connected with the renormalization of constant $\rho_{g}^{[t]}$,

$$\rho_{g}^{[t]}\rightarrow\overline{\rho}_{g}^{[t]}=\varpi\left(x^{k},\varphi\right)\rho_{g}^{[t]}. \qquad (12.40)$$

For toroidal configurations it is naturally to use 2D toroidal holonomic coordinates $x^{i}=(\sigma,\tau)$.

The anholonomically transformed metric is parametrized in the form

$$\begin{aligned}\delta s^{2} &= \Omega^{2}\{-\left[d\sigma^{2}+d\tau^{2}\right]-\eta_{3}\left(\sigma,\tau,\varphi\right)g_{T}^{-1}\left(\tau\right)\delta\varphi^{2}\\ &\quad +f_{T}(\sigma,\tau)g_{T}^{-1}\left(\tau\right)\delta t^{2}\}. \end{aligned} \qquad (12.41)$$

We state the coefficients

$$h_3 = -\eta_3(\sigma,\tau,\varphi)\, g_T^{-1}(\tau) \text{ and } h_4 = f_T(\sigma,\tau) g_T^{-1}(\tau),$$

where the polarization

$$\eta_3(\sigma,\tau,\varphi) = \varpi^{-2}(\sigma,\tau,\varphi)\, \Omega_{BT}^{-2}(\sigma,\tau)$$

is found from the condition (12.30) as $h_3 = -\Omega^{-2}$. The equation (12.27) is solved by arbitrary couples $h_3(\sigma,\tau,\varphi)$ and $h_4(\sigma,\tau)$ when $h_4^* = 0$. The procedure of definition of $\zeta_i(\sigma,\tau,\varphi)$ and $n_i(\sigma,\tau,\varphi)$ is similar to that from the previous subsection. We present the final results as the data

$$\begin{aligned}
g_{1,2} &= -1, \overline{\rho}_g = \varpi(\sigma,\tau,\varphi)\,\rho_g, \Omega^2 = \varpi^2 \Omega_{BT}^2 g_T(\tau), \\
h_3 &= -\eta_3(\sigma,\tau,\varphi)\, g_T^{-1}(\tau), h_4 = f_T(\sigma,\tau) g_T^{-1}(\tau), \\
\eta_3 &= \varpi^{-2}(\sigma,\tau,\varphi)\,\Omega_{BT}^{-2}(\sigma,\tau), (\text{ see } (12.18), (12.19)), \\
\zeta_i &= (\varpi^*)^{-1}\partial_i \varpi + \partial_i \ln|\Omega_{BT}\sqrt{g_T}|/(\ln|\varpi|)^*, \\
n_i &= n_{i[0]}(\sigma,\tau) + n_{i[1]}(\sigma,\tau)\int \varpi^{-2} d\varphi
\end{aligned} \tag{12.42}$$

for the ansatz (12.41) which defines an exact static solution of the vacuum Einstein equations with toroidal symmetry, of Class B, with anisotropic dependence on coordinate φ, see the torus configuration from Fig. 12.2. The off–diagonal solution is non–trivial for anisotropic linear distributions of mass on the circle contained in the torus ring, or alternatively, if there is a renormalized gravitational constant with anisotropic dependence on angle φ. This class of solutions have a toroidal horizon defined by the condition of vanishing of the coefficient h_4 which holds if $f_T(\sigma,\tau) = 0$. The functions $\varpi(\sigma,\tau,\varphi)$ and $n_{i[0,1]}(\sigma,\tau)$ may be stated in a form that at long distance from the toroidal horizon the (12.41) with data (12.42) will have asymptotic like the Schwarzschild metric. We can also consider alternative toroidal vacuum configurations. We note that instead of relations like $h_3 = -\Omega^{-2}$ we can use every type $h_3 \sim \Omega^{p/q}$, like is stated by (12.30); it depends on what type of nonlinear configuration and asymptotic limits we wont to obtain.

We remark also that in a similar manner we can generate toroidal configurations of Class A, starting from the auxiliary metric (12.15). In the next subsection we elucidate this possibility by interfering it with a Class B ellipsoidal configuration.

12.4.2 Static Ellipsoidal Black Hole – Black Torus solutions

There are different possibilities to combine static ellipsoidal black hole and black torus solutions as they will give configurations with two horizons. In this subsection

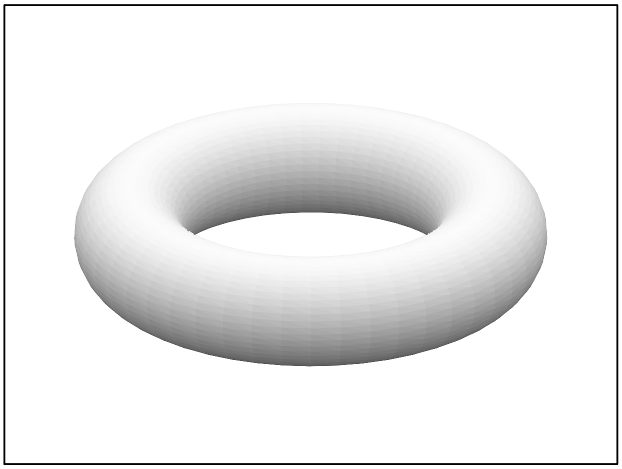

Figure 12.2: Toroidal Configuration

we analyze two such variants. We consider a 2D system of holonomic coordinates x^i, which may be used both on the 'ellipsoidal' and 'toroidal' sectors via transforms like $u = u(x^i), \lambda = \tau\left(x^i\right)$ and $\sigma = \sigma\left(x^i\right)$.

Ellipsoidal–torus black configurations of Class BA

We construct a 4D vacuum metric with posses two type of horizons, ellipsoidal and toroidal one, having both type characteristics like a metric of Class B for ellipsoidal configurations and a metric of Class A for toroidal configurations (we conventionally call this ellipsoidal torus metric to be of Class BA). The ansatz is taken

$$\begin{aligned} \delta s^2 &= \Omega^2\{-\mu^2\left(x^i\right)\left[(dx^1)^2+(dx^2)^2\right] \\ &\quad -\eta_3\left(x^k,\varphi\right)a_T\left(x^i\right)\delta\varphi^2+\frac{b_T(x^i)f_E(x^i)}{g_E(x^i)}\delta t^2\}, \end{aligned} \tag{12.43}$$

with

$$\begin{aligned} \Omega^2 &= \omega^2\left(x^k,\varphi\right)\varpi^2\left(x^k,\varphi\right)\Omega^2_{AT}\left(x^i\right)\Omega^2_{BE}\left(x^i\right), \\ \eta_3 &= -a_T^{-1}\left(x^i\right)\Omega^{-2}, h_3=-\eta_3\left(x^k,\varphi\right)a_T\left(x^i\right), \\ h_4 &= b_T(x^i)f_E(x^i)/g_E(x^i), \\ \mu^2 &= \mu^2_{[0]}\exp\left[c_{[1]}x^1+c_{[2]}x^2\right]. \end{aligned}$$

So, in general we may having both type of anisotropic renormalizations of constants ρ_g and $\rho_g^{[t]}$ as in (12.32) and (12.40). The prolongations of differentials $\delta\varphi$ and δt are defined

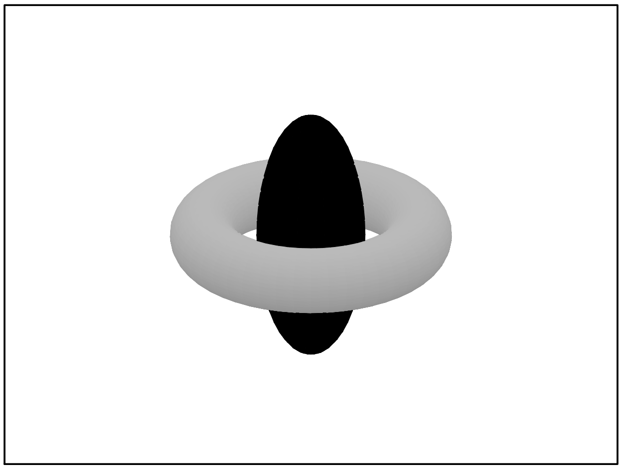

Figure 12.3: Ellipsoidal–Torus Configuration

by the coefficients

$$\begin{aligned}\zeta_i\left(x^k,\varphi\right) &= \left(\Omega^*\right)^{-1}\partial_i\Omega,\\ n_i\left(x^k,\varphi\right) &= n_{i[0]}\left(x^k\right)+n_{i[1]}\left(x^k\right)\int\omega^{-2}\varpi^{-2}d\varphi.\end{aligned}$$

The constants $\mu^2_{[0]}, c_{[1,2]}$, functions $\omega^2\left(x^k,\varphi\right), \varpi^2\left(x^k,\varphi\right)$ and $n_{i[0,1]}\left(x^k\right)$ and relation $h_3\sim\Omega^{p/q}$ may be selected as to obtain at asymptotic a Schwarzschild like behavior. The metric (12.43) has two horizons, a toroidal one, defined by the condition $b_T(x^i)=0$, and an ellipsoidal one, defined by the condition $f_E(x^i)=0$ (see respectively these functions in (12.16) and (12.12)).

The ellipsoidal–torus configuration is illustrated in Fig. 12.3.

We can consider different combinations of ellipsoidal black hole an black torus metrics in order to construct solutions of Class AA, AB and BB (we omit such similar constructions).

A second example of ellipsoidal black hole – black torus system

In the simplest case we can construct a solution with an ellipsoidal and toroidal horizon which have a trivial conformal factor Ω and vanishing coefficients $\zeta_i=0$ (see (12.31)). Establishing a global 3D toroidal space coordinate system, we consider the ansatz

$$\begin{aligned} \delta s^2 &= \{-\left[d\sigma^2 + d\tau^2\right] - \eta_3(\sigma,\tau,\varphi)\, h_{3[0]}(\sigma,\tau)\, \delta\varphi^2 \\ &\quad + \eta_4(\sigma,\tau,\varphi)\, h_{4[0]}(\sigma,\tau)\, \delta t^2\}, \end{aligned} \tag{12.44}$$

where (in order to construct a Class AA solution) we put

$$\begin{aligned} h_{3[0]} &= a_E(\sigma,\tau)\, a_T(\sigma,\tau), h_{4[0]} = b_E(\sigma,\tau)\, b_T(\sigma,\tau), \\ \eta_4 &= \omega^{-2}(\sigma,\tau,\varphi)\, \varpi^{-2}(\sigma,\tau,\varphi), \end{aligned}$$

considering anisotropic renormalizations of constants as in (12.32) and (12.40). The polarization η_3 is to be found from the relation

$$h_3 = h_{[0]}^2[(\sqrt{|h_4|})^*]^2, h_{[0]}^2 = const, \tag{12.45}$$

which defines a solution of equation (12.27) for $h_4^* \neq 0$, when $h_3 = -\eta_3 h_{3[0]}$ and $h_4 = \eta_4 h_{4[0]}$. Substituting the last values in (12.45) we get

$$|\eta_3| = h_{[0]}^2 \frac{b_E b_T}{a_E a_T}\left(\frac{\omega^* + \varpi^*}{\omega\varpi}\right)^2.$$

Then, computing the coefficient γ, see (12.29), after two integrations on φ we find

$$\begin{aligned} n_i(\sigma,\tau,\varphi) &= n_{i[0]}(\sigma,\tau) + n_{i[1]}(\sigma,\tau)\int [\eta_3/\left(\sqrt{|\eta_3|}\right)^3] d\varphi \\ &= n_{i[0]}(\sigma,\tau) + \tilde{n}_{i[1]}(\sigma,\tau)\int \omega\varpi\left(\omega^* + \varpi^*\right)^2 d\varphi, \end{aligned}$$

where we re-defined the function $n_{i[1]}(\sigma,\tau)$ into a new $\tilde{n}_{i[1]}(\sigma,\tau)$ by including all factors and constants like $h_{[0]}^2$, b_E, b_T, a_E and a_T.

The constructed solution (12.44) does not has as locally isotropic limit the Schwarzschild metric. It has also a toroidal and ellipsoidal horizons defined by the conditions of vanishing of b_E and b_T, but this solution is different from the metric (12.43): it has a trivial conformal factor and vanishing coefficients ζ_i which means that in this case we are having a splitting of dynamics into three holonomic and one anholonomic coordinate. We can select such functions $n_{i[0,1]}(\sigma,\tau)$, $\omega(\sigma,\tau,\varphi)$ and $\varpi(\sigma,\tau,\varphi)$, when at asymptotic one obtains the Minkowski metric.

12.5 Anisotropic Polarizations and Running Constants

In this Section we consider non–static vacuum anholonomic ellipsoidal and/or toroidal configurations depending explicitly on time variable t and on holonomic coordinates x^i, but not on angular coordinate φ. Such solutions are generated by dynamical anholonomic deformations and conformal transforms of the Schwarzschild metric. For simplicity, we analyze only Class A and AA solutions.

The coordinates are parametrized: x^i are holonomic ones, in particular, $x^i = (u, \lambda)$, for ellipsoidal configurations, and $x^i = (\sigma, \tau)$, for toroidal configurations; $y^3 = v = t$ and $y^4 = \varphi$. The metric ansatz is stated in the form

$$\begin{aligned} \delta s^2 &= \Omega^2 \left(x^i, t\right) [-(dx^1)^2 - (dx^2)^2 \\ &+ h_3 \left(x^i, t\right) \delta t^2 + h_4 \left(x^i, t\right) \delta\varphi^2], \end{aligned} \quad (12.46)$$

where the differentials are elongated

$$\begin{aligned} d\varphi &\rightarrow \delta\varphi = d\varphi + \zeta_i \left(x^k, t\right) dx^i, \\ dt &\rightarrow \delta t = dt + n_i \left(x^k, t\right) dx^i. \end{aligned}$$

The ansatz (12.46) is related with some ellipsoidal and/ or toroidal anholonomic deformations of the Schwarzschild metric (see respectively, (12.7), (12.8), (12.11) and (12.14), (12.15), (12.18)) via time running renormalizations of ellipsoidal and toroidal constants (instead of the static ones, (12.32) and (12.40)),

$$\rho_g \rightarrow \widehat{\rho}_g = \omega \left(x^k, t\right) \rho_g, \quad (12.47)$$

and

$$\rho_g^{[t]} \rightarrow \widehat{\rho}_g^{[t]} = \varpi \left(x^k, t\right) \rho_g^{[t]}. \quad (12.48)$$

As particular cases we shall consider trivial values $\Omega^2 = 1$. The horizons of such classes of solutions are defined by the condition of vanishing of the coefficient $h_3 (x^i, t)$.

12.5.1 Ellipsoidal/toroidal solutions with running constants

Trivial conformal factors, $\Omega^2 = 1$

The simplest way to generate a t–depending ellipsoidal (or toroidal) configuration is to take the metric (12.8) (or (12.15)) and to renormalize the constant as (12.47) (or

(12.48)). In result we obtain a metric (12.46) with $\Omega^2 = 1$, $h_3 = \eta_3(x^i,t)\, h_{3[0]}(x^i)$ and $h_4 = h_{4[0]}(x^i)$, where

$$\begin{aligned} \eta_3 &= \omega^{-2}(u,\lambda,t), h_{3[0]} = b_E(u,\lambda), h_{4[0]} = a_E(u,\lambda), \\ (\eta_3 &= \varpi^{-2}(\sigma,\tau,t), h_{3[0]} = b_T(\sigma,\tau), h_{4[0]} = a_T(\tau)). \end{aligned}$$

The equation (12.27) is satisfied by these data because $h_4^* = 0$ and the condition (12.31) holds for $\zeta_i = 0$. The coefficient γ from (12.29) is defined only by polarization η_3, which allow us to write the integral of (12.28) as

$$n_i = n_{i[0]}\left(x^i\right) + n_{i[1]}\left(x^i\right) \int \eta_3\left(x^i,t\right) dt.$$

The corresponding ellipsoidal (or toroidal) configuration may be transformed into asymptotically Minkowschi metric if the functions $\omega^{-2}(u,\lambda,t)$ (or $\varpi^{-2}(\sigma,\tau,t)$) and $n_{i[0,1]}(x^i)$ are such way determined by boundary conditions that $\eta_3 \to const$ and $n_{i[0,1]}(x^i) \to 0$, far away from the horizons, which are defined by the conditions $b_E(u,\lambda) = 0$ (or $b_T(\sigma,\tau) = 0$).

Such vacuum gravitational configurations may be considered as to posses running of gravitational constants in a local spacetime region. For instance, in Ref [4] we suggested the idea that a vacuum gravitational soliton may renormalize effectively the constants, but at asymptotic we have static configurations.

Non–trivial conformal factors

The previous configuration can not be related directly with the Schwarzschild metric (we used its conformal transforms). A more direct relation is possible if we consider non–trivial conformal factors. For ellipsoidal (or toroidal) configurations we renromalize (as in (12.47), or (12.48)) the conformal factor (12.10) (or (12.17)),

$$\begin{aligned} \Omega^2\left(x^k,t\right) &= \omega^2\left(x^k,t\right)\Omega^2_{AE}(x^k) b_E^{-1}\left(x^k\right), \\ (\Omega^2\left(x^k,t\right) &= \varpi^2\left(x^k,t\right)\Omega^2_{AT}(x^k) b_T^{-1}\left(x^k\right)). \end{aligned}$$

In order to satisfy the condition (12.30) we choose $h_3 = \Omega^{-2}$ but $h_4 = h_{4[0]}$ as in previous subsection: this solves the equation (12.27). The non–trivial values of ζ_i and n_i are defined from (12.31) and (12.28),

$$\begin{aligned} \zeta_i\left(x^k,t\right) &= \left(\Omega^*\right)^{-1}\partial_i\Omega, \\ n_i\left(x^k,t\right) &= n_{i[0]}\left(x^k\right) + n_{i[1]}\left(x^k\right)\int h_3\left(x^i,t\right) dt. \end{aligned}$$

We note that the conformal factor Ω^2 is singular on horizon, which is defined by the condition of vanishing of the coefficient h_3, i. e. of b_E (or b_T). By a corresponding parametrization of functions $\omega^2\left(x^k,t\right)$ (or $\varpi^2\left(x^k,t\right)$) and $n_{i[0,1]}\left(x^k\right)$ we may generate asymptotically flat solutions, very similar to the Schwarzschild solution, which have anholonomic running constants in a local region of spacetime.

12.5.2 Black Ellipsoid – Torus Metrics with Running Constants

Now we consider nonlinear superpositions of the previous metrics as to construct solutions with running constants and two horizons (one ellipsoidal and another toroidal).

Trivial conformal factor, $\Omega^2 = 1$

The simplest way to generate such metrics with two horizons is to establish, for instance, a common toroidal system of coordinate, to take the ellipsoidal and toroidal metrics constructed in subsection V.A.1 and to multiply correspondingly their coefficients. The corresponding data, defining a new solution for the ansazt (12.46), are

$$\begin{aligned}
g_{1,2} &= -1, \widehat{\rho}_g = \omega\left(x^k,t\right)\rho_g, \widehat{\rho}_g^{[t]} = \varpi\left(x^k,t\right)\rho_g^{[t]}, \Omega = 1, \\
h_3 &= \eta_3\left(x^i,t\right)h_{3[0]}\left(x^i\right), \eta_3 = \omega^{-2}\left(x^k,t\right)\varpi^{-2}\left(x^k,t\right), \\
h_{3[0]} &= b_E\left(x^k\right)b_T\left(x^k\right), h_4 = h_{4[0]} = a_E\left(x^i\right)a_T(x^i), (\\
\zeta_i &= 0, n_i = n_{i[0]}\left(x^k\right) + n_{i[1]}\left(x^k\right)\int \omega^{-2}\varpi^{-2}dt, \qquad (12.49)
\end{aligned}$$

where the functions a_E, a_T and b_E, b_T are given by formulas (12.9) and (12.16). Analyzing the data (12.49) we conclude that we have two horizons, when $b_E\left(x^k\right) = 0$ and $b_T\left(x^k\right) = 0$, parametrized respectively as ellipsoidal and torus hypersurfaces. The boundary conditions on running constants and on off–diagonal components of the metric may be imposed as the solution would result in an asymptotic flat metric. In a finite region of spacetime we may consider various dependencies in time.

Non–trivial conformal factor

In a similar manner, we can multiply the conformal factors and coefficients of the metrics from subsection V.A.2 in order to generate a solution parametrized by the (12.46) with nontrivial conformal factor Ω and non-vanishing coefficients ζ_i. The data are

$$
\begin{aligned}
g_{1,2} &= -1, \widehat{\rho}_g = \omega\left(x^k,t\right)\rho_g, \widehat{\rho}_g^{[t]} = \varpi\left(x^k,t\right)\rho_g^{[t]}, \\
\Omega^2 &= \omega^2\left(x^k,t\right)\varpi^2\left(x^k,t\right)\Omega^2_{AE}(x^k)\times \\
& \quad \Omega^2_{AT}(x^k)b_E^{-1}\left(x^k\right)b_T^{-1}\left(x^k\right), (\text{ see (12.9),(12.16)}), \\
h_3 &= \Omega^{-2}, h_{3[0]} = b_E\left(x^k\right)b_T\left(x^k\right), \\
h_4 &= h_{4[0]} = a_E\left(x^i\right)a_T(x^i), \zeta_i\left(x^k,t\right) = \left(\Omega^*\right)^{-1}\partial_i\Omega, \\
n_i &= n_{i[0]}\left(x^k\right) + n_{i[1]}\left(x^k\right)\int \omega^{-2}\varpi^{-2}dt.
\end{aligned} \tag{12.50}
$$

The data (12.50) define a new type of solution than that given by (12.49). It this case there is a singular on horizons conformal factor. The behavior nearly horizons is very complicated. By corresponding parametrizations of functions $\omega\left(x^k,t\right)$, $\varpi\left(x^k,t\right)$ and $n_{i[0,1]}\left(x^k\right)$, which approximate $\omega, \varpi \rightarrow const$ and $\zeta_i, n_i \rightarrow 0$ we may obtain a stationary flat asymptotic.

Finally, we note that instead of Class AA solutions with anisotropic and running constants we may generate solutions with two horizons (ellipsoidal and toroidal) by considering nonlinear superpositions, anholonomic deformations, conformal transforms and combinations of solutions of Classes A, B. The method of construction is similar to that considered in this Section.

12.6 Conclusions and Discussion

We constructed new classes of exact solutions of vacuum Einstein equations by considering anholonomic deformations and conformal transforms of the Schwarzschild black hole metric. The solutions posses ellipsoidal and/ or toroidal horizons and symmetries and could be with anisotropic renormalizations and running constants. Some of such solutions define static configurations and have Schwarzschild like (in general, multiplied to a conformal factor) asymptotically flat limits. The new metrics are parametrized by off–diagonal metrics which can be diagonalized with respect to certain anholonomic frames. The coefficients of diagonalized metrics are similar to the Schwarzschild metric coefficients but describe deformed horizons and contain additional dependencies on one 'anholonomic' coordinate.

We consider that such vacuum gravitational configurations with non–trivial topology and deformed horizons define a new class of ellipsoidal black hole and black torus objects and/or their combinations.

Toroidal and ellipsoidal black hole solutions were constructed for different models of extra dimension gravity and in the four dimensional (4D) gravity with cosmological constant and specific configurations of matter [2, 3, 6]. There were defined also vacuum configurations for such objects [4, 5, 7, 8]. However, we must solve the very important problems of physical interpretation of solutions with anholonomy and to state their compatibility with the black hole uniqueness theorems [9] and the principle of topological censorship [10, 11].

It is well known that the Schwarzschild metric is no longer the unique asymptotically flat static solution if the 4D gravity is derived as an effective theory from extra dimension like in recent Randall and Sundrum theories (see basic results and references in [14]). The Newton law may be modified at sub-millimeter scales and there are possible configurations with violation of local Lorentz symmetry [15]. Guided by modern conjectures with extra dimension gravity and string/M–theory, we have to answer the question: it is possible to give a physical meaning to the solutions constructed in this paper only from a viewpoint of a generalized effective 4D Einstein theory, or they also can be embedded into the framework of general relativity theory?

It should be noted that the Schwarzschild solution was constructed as the unique static solution with spherical symmetry which was connected to the Newton spherical gravitational potential $\sim 1/r$ and defined as to result in the Minkowski flat spacetime, at long distances. This potential describes the static gravitational field of a point particle with "isotropic" mass m_0. The spherical symmetry is imposed at the very beginning and it is not a surprising fact that the spherical topology and spherical symmetry of horizons are obtained for well defined states of matter with specific energy conditions and in the vacuum limits. Here we note that the spherical coordinates and systems of reference are holonomic ones and the considered ansatz for the Schwarzschild metric is diagonal in the more "natural" spherical coordinate frame.

We can approach in a different manner the question of constructing 4D static vacuum metrics. We might introduce into consideration off–diagonal ansatz, prescribe instead of the spherical symmetry a deformed one (ellipsoidal, toroidal, or their superposition) and try to check if a such configurations may be defined by a metric as to satisfy the 4D vacuum Einstein equations. Such metrics were difficult to be found because of cumbersome calculus if dealing with off–diagonal ansatz. But the problem was substantially simplified by an equivalent transferring of calculations with respect to anholonomic frames [4, 7, 8]. Alternative exact static solutions, with ellipsoidal and toroidal horizons (with possible extensions for nonlinear polarizations and running constants), were constructed and related to some anholonomic and conformal transforms of the Schwarzschild metric.

It is not difficult to suit such solutions with the asymptotic limit to the locally isotropic Minkowschi spacetime: "an egg and/or a ring look like spheres far away from their non–trivial horizons". The unsolved question is that what type of modified Newton potentials should be considered in this case as they would be compatible with non–spherical symmetries of solutions? The answer may be that at short distances the masses and constants are renormalized by specific nonlinear vacuum gravitational interactions which can induce anisotropic effective masses, ellipsoidal or toroidal polarizations and running constants. For instance, the Laplace equation for the Newton potential can be solved in ellipsoidal coordinates [12]: this solution could be a background for constructing ellipsoidal Schwarzschild like metrics. Such nonlinear effects should be treated, in some approaches, as certain quasi–classical approximations for some 4D quantum gravity models, or related to another type of theories of extra dimension classical or quantum gravity.

Independently of the type of little, or more, internal structure of black holes with non–spherical horizons we search for physical justification, it is a fact that exact vacuum solutions with prescribed non–spherical symmetry of horizons can be constructed even in the framework of general relativity theory. Such solutions are parametrized by off–diagonal metrics, described equivalently, in a more simplified form, with respect to associated anholonomic frames; they define some anholonomic vacuum gravitational configurations of corresponding symmetry and topology. Considering certain characteristic initial value problems we can select solutions which at asymptotic result in the Minkowschi flat spacetime, or into an anti–de Sitter (AdS) spacetime, and have a causal behavior of geodesics with the equations solved with respect to anholonomic frames.

It is known that the topological censorship principle was reconsidered for AdS black holes [11]. But such principles and uniqueness black hole theorems have not yet been proven for spacetimes defined by generic off–diagonal metrics with prescribed non–spherical symmetries and horizons and with associated anholonomic frames with mixtures of holonomic and anholonomic variables. It is clear that we do not violate the conditions of such theorems for those solutions which are locally anisotropic and with nontrivial topology in a finite region of spacetime and have locally isotropic flat and trivial topology limits. We can select for physical considerations only the solutions which satisfy the conditions of the mentioned restrictive theorems and principles but with respect to well defined anholonomic frames with holonomic limits. As to more sophisticate nonlinear vacuum gravitational configurations with global non–trivial topology we conclude that there are required a more deep analysis and new physical interpretations.

The off–diagonal metrics and associated anholonomic frames extend the class of vacuum gravitational configurations as to be described by a nonlinear, anholonomic and anisotropic dynamics which, in general, may not have any well known locally isotropic and holonomic limits. The formulation and proof of some uniqueness theorems and principles of topological censorship as well analysis of physical consequences of such anholonomic vacuum solutions is very difficult. We expect that it is possible to reconsider the statements of the Israel–Carter–Robinson and Hawking theorems with respect to anholonomic frames and spacetimes with non–spherical topology and anholonomically deformed spherical symmetries. These subjects are currently under our investigation.

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Part III

Noncommutative Riemann–Lagrange Geometry

Chapter 13

Noncommutative Finsler Geometry, Gauge Fields and Gravity

Abstract [1]

The work extends the A. Connes' noncommutative geometry to spaces with generic local anisotropy. We apply the E. Cartan's anholonomic frame approach to geometry models and physical theories and develop the nonlinear connection formalism for projective module spaces. Examples of noncommutative generation of anholonomic Riemann, Finsler and Lagrange spaces are analyzed. We also present a research on noncommutative Finsler–gauge theories, generalized Finsler gravity and anholonomic (pseudo) Riemann geometry which appear naturally if anholonomic frames (vierbeins) are defined in the context of string/M–theory and extra dimension Riemann gravity.

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13.1 Introduction

In the last twenty years, there has been an increasing interest in noncommutative and/or quantum geometry with applications both in mathematical and particle physics. It is now generally considered that at very high energies, the spacetime can not be described by a usual manifold structure. Because of quantum fluctuations, it is difficult to define localized points and the quantum spacetime structure is supposed to posses generic noncommutative, nonlocal and locally anisotropic properties. Such ideas originate from the suggestion that the spacetime coordinates do not commute at a quantum level [44], they are present in the modern string theory [12, 38] and background the noncommutative physics and geometry [8] and quantum geometry [33].

Many approaches can be taken to introducing noncommutative geometry and developing noncommutative physical theories (Refs. [8, 14, 15, 20, 27, 28, 31, 63] emphasize some basic monographs and reviews). This paper has three aims: First of all we would like to give an exposition of some basic facts on anholonomic frames and associated nonlinear connection structures both on commutative and noncommutative spaces (respectively modelled in vector bundles and in projective modules of finite type). Our second goal is to state the conditions when different variants of Finsler, Lagrange and generalized Lagrange geometries, in commutative and noncommutative forms, can be defined by corresponding frame, metric and connection structures. The third aim is to construct and analyze properties of gauge and gravitational noncommutative theories with generic local anisotropy and to prove that such models can be elaborated in the framework of noncommutative approaches to Riemannian gravity theories.

This paper does not concern the topic of Finsler like commutative and noncommutative structures in string/M-theories (see the Ref. [59], which can be considered as a string partner of this work).

We are inspired by the geometrical ideas from a series of monographs and works by E. Cartan [6] where a unified moving frame approach to the Riemannian and Finsler geometry, Einstein gravity and Pffaf systems, bundle spaces and spinors, as well the preliminary ideas on nonlinear connections and various generalizations of gravity theories were developed. By considering anholonomic frames on (pseudo) Riemannian manifolds and in tangent and vector bundles, we can model very sophisticate geometries with local anisotropy. We shall apply the concepts and methods developed by the Romanian school on Finsler geometry and generalizations [35, 36, 3, 54] from which we leaned that the Finsler and Cartan like geometries may be modelled on vector (tangent) and covector (cotangent) bundles if the constructions are adapted to the corresponding nonlinear connection structure via anholonomic frames. In this case the geometric "picture" and physical models have a number of common points with those from the usual Einstein–Cartan theory and/or extra dimension (pseudo) Riemannian geometry. As general references on Finsler geometry and applications we cite the monographs [41, 35, 36, 3, 54, 62]) and point the fact that the bulk of works on Finsler geometry and generalizations emphasize differences with the usual Riemannian geometry rather than try to approach them from a unified viewpoint (as we propose in this paper).

By applying the formalism of nonlinear connections (in brief, N–connection) and adapted anholonomic frames in vector bundles and superbundles we extended the geometry of Clifford structures and spinors for generalized Finsler spaces and their higher order extensions in vector–covector bundles [49, 62], constructed and analyzed different models of gauge theories and gauge gravity with generic anisotropy [61], defined an anisotropic stochastic calculus in bundle and superbundle spaces provided with nonlinear connection structure [50, 54], with a number of applications in the theory of anisotropic kinetic and thermodynamic processes [55], developed supersymmetric theories with local anisotropy [51, 54, 52] and proved that Finsler like (super) geometries are contained alternatively in modern string theory [52, 54]. One should be emphasized here that in our approach we have not proposed any "exotic" locally anisotropic string theories modifications but demonstrated that anisotropic structures, Finsler like or another ones, may appear alternatively to the Riemannian geometry, or even can be modelled in the framework of a such geometry, in the low energy limit of the string theory, because we are dealing with frame, vierbein, constructions.

The most surprising fact was that the Finsler like structures arise in the usual (pseudo) Riemannian geometry of lower and higher dimensions and even in the Einstein gravity. References [56] contain investigations of a number of exact solutions in modern gravity theories (Einstein, Kaluza–Klein and string/brane gravity) which describe locally anisotropic wormholes, Taub NUT spaces, black ellipsoid/torus solutions, solitonic and another type configurations. It was proposed a new consequent method of constructing exact solutions of the Einstein equations for off–diagonal metrics, in spaces of dimension $d > 2$, depending on three and more isotropic and anisotropic variables which are effectively diagonalized by anholonomic frame transforms. The vacuum and matter field equations are reduced to very simplified systems of partial differential equations which can be integrated in general form [57].

A subsequent research in Riemann–Finsler and noncommutative geometry and physics requires the investigation of the fact if the A. Connes functional analytic approach to noncommutative geometry and gravity may be such way generalized as to include the Finsler, and of another type anisotropy, spaces. The first attempt was made in Refs. [58] where some models of noncommutative gauge gravity (in the commutative limit being equivalent to the Einstein gravity, or to different generalizations to de Sitter, affine, or Poincare gauge gravity with, or not, nonlinear realization of the gauge groups) were analyzed. Further developments in formulation of noncommutative geometries with anholonomic and anisotropic structures and their applications in modern particle physics lead to a rigorous study of the geometry of anholonomic noncommutative frames with associated N–connection structure, to which are devoted our present researches.

The paper has the following structure: in section 2 we present the necessary definitions and results on the functional approach to commutative and noncommutative geometry. Section 3 is devoted to the geometry of vector bundles and theirs noncommutative generalizations as finite projective modules. We define the nonlinear connection in commutative and noncommutative spaces, introduce locally anisotropic Clifford/spinor structures and consider the gravity and gauge theories from the viewpoint of anholonomic frames with associated nonlinear connection structures. In section 4 we prove that various type of gravity theories with generic anisotropy, constructed on anholonomic Riemannian spaces and their Kaluza–Klein and Finsler like generalizations can be derived from the A. Connes' functional approach to noncommutative geometry by applying the canonical triple formalism but extended to vector bundles provided with nonlinear connection structure. In section 5, we elaborate and investigate noncommutative gauge like gravity models (which in different limits contain the standard Einstein's general relativity and various its anisotropic and gauge generalizations). The approach holds true also for (pseudo) Riemannian metrics, but is based on noncommutative extensions of the frame and connection formalism. This variant is preferred instead of the usual metric models which seem to be more difficult to be tackled in the framework of noncommutative geometry if we are dealing with pseudo–Euclidean signatures and with complex and/or nonsymmetic metrics. Finally, we present a discussion and conclusion of the results in section 6.

13.2 Commutative and Noncommutative Spaces

The A. Connes' functional analytic approach [8] to the noncommutative topology and geometry is based on the theory of noncommutative C^*–algebras. Any commutative C^*–algebra can be realized as the C^*–algebra of complex valued functions over locally compact Hausdorff space. A noncommutative C^*–algebra can be thought of as the algebra of continuous functions on some 'noncommutative space' (see details in Refs. [8, 15, 20, 28, 31, 63]).

Commutative gauge and gravity theories stem from the notions of connections (linear and nonlinear), metrics and frames of references on manifolds and vector bundle spaces. The possibility of extending such theories to some noncommutative models is based on the Serre–Swan theorem [46] stating that there is a complete equivalence between the category of (smooth) vector bundles over a smooth compact space (with bundle maps) and the category of projective modules of finite type over commutative algebras and module morphisms. Following that theorem, the space $\Gamma(E)$ of smooth sections of a vector bundle E over a compact space is a projective module of finite type over the algebra $C(M)$ of smooth functions over M and any finite projective $C(M)$–module can be realized as the module of sections of some vector bundle over M. This construction may be extended if a noncommutative algebra $\mathcal{A}$ is taken as the starting ingredient: the noncommutative analogue of vector bundles are projective modules of finite type over $\mathcal{A}$. This way one developed a theory of linear connections which culminates in the definition of Yang–Mills type actions or, by some much more general settings, one reproduced Lagrangians for the Standard model with its Higgs sector or different type of gravity and Kaluza–Klein models (see, for instance, Refs [11, 7, 29, 31]).

This section is devoted to the theory of nonlinear connections in projective modules of finite type over a noncommutative algebra $\mathcal{A}$. We shall introduce the basic definitions and present the main results connected with anhlolonomic frames and metric structures in such noncommutative spaces.

13.2.1 Algebras of functions and (non) commutative spaces

The general idea of noncommutative geometry is to shift from spaces to the algebras of functions defined on them. In this subsection, we give some general facts about algebras of continuous functions on topological spaces, analyze the concept of modules as bundles and define the nonlinear connections. We present mainly the objects we shall need later on while referring to [8, 14, 15, 20, 27, 28, 31, 63] for details.

We start with some necessary definitions on C^*–algebras and compact operators

In this work any algebra $\mathcal{A}$ is an algebra over the field of complex numbers $\mathbb{C}$, i. e. $\mathcal{A}$ is a vector space over $\mathbb{C}$ when the objects like $\alpha a \pm \beta b$, with $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, make sense. Also, there is defined (in general) a noncommutative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ when for every elements (a, b) and a, b, $\mathcal{A} \times \mathcal{A} \ni (a, b) \to ab \in \mathcal{A}$ the conditions of distributivity,

$$a(b + c) = ab + ac, \ (a + b)c = ac + bc,$$

for any $a, b, c \in \mathcal{A}$, in general, $ab \neq ba$. It is assumed that there is a unity $I \in \mathcal{A}$.

The algebra $\mathcal{A}$ is considered to be a so–called "$*$–algebra", for which an (antilinear) involution $* : \mathcal{A} \to \mathcal{A}$ is defined by the properties

$$a^{**} = a, \ \ (ab)^* = b^* a^*, \ \ (\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*,$$

where the bar operation denotes the usual complex conjugation.

One also considers $\mathcal{A}$ to be a normed algebra with a norm $||\cdot|| : \mathcal{A} \to \mathbb{R}$, where $\mathbb{R}$ the real number field, satisfying the properties

$$\begin{aligned} ||\alpha a|| &= |\alpha| \, ||a|| \, ; \ \ ||a|| \geq 0, \ \ ||a|| = 0 \Leftrightarrow a = 0; \\ ||a + b|| &\leq ||a|| + ||b|| \, ; \ \ ||ab|| \leq ||a|| \, ||b|| \, . \end{aligned}$$

This allows to define the 'norm' or 'uniform' topology when an ε–neighborhood of any $a \in \mathcal{A}$ is given by

$$U(a, \varepsilon) = \{ b \in \mathcal{A}, ||a - b|| < \varepsilon \}, \varepsilon > 0.$$

A Banach algebra is a normed algebra which is complete in the uniform topology and a Banach $*$–algebra is a normed $*$–algebra which is complete and such that $||a^*|| = ||a||$ for every $a \in \mathcal{A}$. We can define now a C^*–algebra $\mathcal{A}$ as a Banach $*$–algebra with the norm satisfying the additional identity $||a^*a|| = ||a||^2$ for every $a \in \mathcal{A}$.

We shall use different commutative and noncommutative algebras:

By $\mathcal{C}(M)$ one denotes the algebra of continuous functions on a compact Hausdorf topological space M, with $*$ treated as the complex conjugation and the norm given by the supremum norm, $||f||_\infty = \sup_{x\in M} |f(x)|$. If the space M is only locally compact, one writes $\mathcal{C}_0(M)$ for the algebra of continuous functions vanishing at infinity (this algebra has no unit).

The $\mathcal{B}(\mathcal{H})$ is used for the noncommutative algebra of bounded operators on an infinite dimensional Hilbert space $\mathcal{H}$ with the involution $*$ given by the adjoint and the norm defined as the operator norm

$$||A|| = \sup\{||A\zeta||; \zeta \in \mathcal{H},\ A \in \mathcal{B}(\mathcal{H}),\ ||\zeta|| \leq 1\}.$$

One considers the noncommutative algebra $M_n(\mathbb{C})$ of $n \times n$ matrices T with complex entries, when T^* is considered as the Hermitian conjugate of T. We may define a norm as

$$||T|| = \{\text{the positive square root of the largest eigenvalue of } T^*T\}$$

or as

$$||T||' = \sup[T_{ij}],\ T = \{T_{ij}\}.$$

The last definition does not define a C^*–norm, but both norms are equivalent as Banach norm because they define the same topology on $M_n(\mathbb{C})$.

A left (right) ideal $\mathcal{T}$ is a subalgebra $\mathcal{A} \in \mathcal{T}$ if $a \in \mathcal{A}$ and $b \in \mathcal{T}$ imply that $ab \in \mathcal{T}$ $(ba \in \mathcal{T})$. A two sided ideal is a subalgebra (subspace) which is both a left and right ideal. An ideal $\mathcal{T}$ is called maximal if there is not other ideal of the same type which contain it. For a Banach $*$–algebra $\mathcal{A}$ and two–sided $*$–ideal $\mathcal{T}$ (which is closed in the norm topology) we can make $\mathcal{A}/\mathcal{T}$ a Banach $*$–algebra. This allows to define the quotient $\mathcal{A}/\mathcal{T}$ to be a C^*–algebra if $\mathcal{A}$ is a C^*–algebra. A C^*–algebra is called simple if it has no nontrivial two–sided ideals. A two–sided ideal is called essential in a C^*–algebra if any other non–zero ideal in this algebra has a non–zero intersection with it.

One defines the resolvent set $r(a)$ of an element $a \in \mathcal{A}$ as a the subset of complex numbers given by $r(a) = \{\lambda \in \mathbb{C} | a - \lambda I \text{ is invertible}\}$. The resolvent of a at any $\lambda \in r(a)$ is given by the inverse $(a - \lambda I)^{-1}$. The spectrum $\sigma(a)$ of an element a is introduced as the complement of $r(a)$ in $\mathbb{C}$. For C^*–algebras the spectrum of any element is a nonempty compact subset of $\mathbb{C}$. The spectral radius $\rho(a)$ of $a \in \mathcal{A}$ is defined $\rho(a) = \sup\{|\lambda|, \lambda \in r(a)\}$; for $\mathcal{A}$ being a C^*–algebra, one obtains $\rho(a) = ||a||$ for every $a \in \mathcal{A}$. This distinguishes the C^*–algebras as those for which the norm may be uniquely determined by the algebraic structure. One considers self–adjoint elements for which $a = a^*$, such elements have real spectra and satisfy the conditions $\sigma(a) \subseteq [-||a||, ||a||]$ and $\sigma(a^2) \subseteq [0|, ||a||]$. An element a is positive, i. e. $a > 0$, if its spectrum belongs to the positive half–line. This is possible if and only if $a = bb^*$ for some $b \in \mathcal{A}$.

One may consider $*$–morphisms between two C^*–algebras $\mathcal{A}$ and $\mathcal{B}$ as some $\mathbb{C}$–linear maps $\pi : \mathcal{A} \to \mathcal{B}$ which are subjected to the additional conditions

$$\pi(ab) = \pi(a)\pi(b), \ \pi(a^*) = \pi(a)^*$$

which imply that π are positive and continuous and that $\pi(\mathcal{A})$ is a C^*–subalgebra of $\mathcal{B}$ (see, for instance, [28]). We note that a $*$–morphism which is bijective as a map defines a $*$–isomorphism for which the inverse map π^{-1} is automatically a $*$–morphism.

In order to construct models of noncommutative geometry one uses representations of a C^*–algebra $\mathcal{A}$ as pairs $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a $*$–morphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ with $\mathcal{B}(\mathcal{H})$ being the C^*–algebra of bounded operators on $\mathcal{H}$. There are different particular cases of representations: A representation $(\mathcal{H}, \pi)$ is faithful if $\ker \pi = \{0\}$, i. e. π is a $*$–isomorphism between $\mathcal{A}$ and $\pi(\mathcal{A})$ which holds if and only if $||\pi(a)|| = ||a||$ for any $a \in \mathcal{A}$ or $\pi(a) > 0$ for all $a > 0$. A representation is irreducible if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\pi(\mathcal{A})$ are the trivial subspaces $\{0\}$ and $\mathcal{H}$. It can be proven that if the set of the elements in $\mathcal{B}(\mathcal{H})$ commute with each element in $\pi(\mathcal{A})$, i. e. the set consists of multiples of the identity operator, the representation is irreducible. Here we note that two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said to be (unitary) equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_1(a) = U^*\pi_2(a)U$ for every $a \in \mathcal{A}$.

A subspace (subalgebra) $\mathcal{T}$ of the C^*–algebra $\mathcal{A}$ is a primitive ideal if $\mathcal{T} = \ker \pi$ for some irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$. In this case $\mathcal{T}$ is automatically a closed two–sided ideal. One say that $\mathcal{A}$ is a primitive C^*–algebra if $\mathcal{A}$ has a faithful irreducible representation on some Hilbert space for which the set $\{0\}$ is a primitive ideal. One denotes by $\Pr im\mathcal{A}$ the set of all primitive ideals of a C^*–algebra $\mathcal{A}$.

Now we recall some basic definitions and properties of compact operators on Hilbert spaces [40]:

Let us first consider the class of operators which may be thought as some infinite dimensional matrices acting on an infinite dimensional Hilbert space $\mathcal{H}$. More exactly, an operator on the Hilbert space $\mathcal{H}$ is said to be of finite rank if the orthogonal component of its null space is finite dimensional. An operator T on $\mathcal{H}$ which can be approximated in norm by finite rank operators is called compact. It can be characterized by the property that for every $\varepsilon > 0$ there is a finite dimensional subspace $E \subset \mathcal{H} : ||T_{|E^{\perp}}|| < \varepsilon$, where the orthogonal subspace $E^{\perp}$ is of finite codimension in $\mathcal{H}$. This way we may define the set $\mathcal{K}(\mathcal{H})$ of all compact operators on the Hilbert spaces which is the largest two–sided ideal in the C^{*}–algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators. This set is also a C^{*}–algebra with no unit, since the operator I on an infinite dimensional Hilbert space is not compact, it is the only norm closed and two–sided when $\mathcal{H}$ is separable. We note that the defining representation of $\mathcal{K}(\mathcal{H})$ by itself is irreducible and it is the only irreducible representation up to equivalence.

For an arbitrary C^{*}–algebra $\mathcal{A}$ acting irreducibly on a Hilbert space $\mathcal{H}$ and having non–zero intersection with $\mathcal{K}(\mathcal{H})$ one holds $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$. In the particular case of finite dimensional Hilbert spaces, for instance, for $\mathcal{H} = \mathbb{C}^{n}$, we may write $\mathcal{B}(\mathbb{C}^{n}) = \mathcal{K}(\mathbb{C}^{n}) = M_{n}(\mathbb{C})$, which is the algebra of $n \times n$ matrices with complex entries. Such algebra has only one irreducible representation (the defining one).

13.2.2 Commutative spaces

Let us denote by $\mathcal{C}$ a fixed commutative C^{*}–algebra with unit and by $\widehat{\mathcal{C}}$ the corresponding structure space defined as the space of equivalence classes of irreducible representations of $\mathcal{C}$ ($\widehat{\mathcal{C}}$ does not contains the trivial representation $\mathcal{C} \to \{0\}$). One can define a non–trivial $*$–linear multiplicative functional $\phi : \mathcal{C} \to \mathbb{C}$ with the property that $\phi(ab) = \phi(a)\phi(b)$ for any a and b from $\mathcal{C}$ and $\phi(I) = 1$ for every $\phi \in \widehat{\mathcal{C}}$. Every such multiplicative functional defines a character of $\mathcal{C}$, i. e. $\widehat{\mathcal{C}}$ is also the space of all characters of $\mathcal{C}$.

The Gel'fand topology is the one with point wise convergence on $\mathcal{C}$. A sequence $\{\phi_{\varpi}\}_{\varpi \in \Xi}$ of elements of $\widehat{\mathcal{C}}$, where Ξ is any directed set, converges to $\phi(c) \in \widehat{\mathcal{C}}$ if and only if for any $c \in \mathcal{C}$, the sequence $\{\phi_{\varpi}(c)\}_{\varpi \in \Xi}$ converges to $\phi(c)$ in the topology of $\mathbb{C}$. If the algebra $\mathcal{C}$ has a unite, $\widehat{\mathcal{C}}$ is a compact Hausdorff space (a topological space is Hausdorff if for any two points of the space there are two open disjoint neighborhoods each containing one of the point, see Ref. [25]). The space $\widehat{\mathcal{C}}$ is only compact if $\mathcal{C}$ is without unit. This way the space $\widehat{\mathcal{C}}$ (called the Gel'fand space) is made a topological space. We may also consider $\widehat{\mathcal{C}}$ as a space of maximal ideals, two sided, of $\mathcal{C}$ instead of the space of irreducible representations. If there is no unit, the ideals to be considered should be regular (modular), see details in Ref. [13]. Considering $\phi \in \mathbb{C}$, we can decompose $\mathcal{C} = Ker(\phi) \oplus \mathbb{C}$, where $Ker(\phi)$ is an ideal of codimension one and so is a maximal ideal of $\mathcal{C}$. Considered in terms of maximal ideals, the space $\widehat{\mathcal{C}}$ is given the Jacobson topology, equivalently, hull kernel topology (see next subsection for general definitions for both commutative and noncommutative spaces), producing a space which is homeomorphic to the one constructed by means of the Gel'fand topology.

Let us consider an example when the algebra $\mathcal{C}$ generated by s commuting self–adjoint elements $x_1, ...x_s$. The structure space $\widehat{\mathcal{C}}$ can be identified with a compact subset of $\mathbb{R}^s$ by the map $\phi(c) \in \widehat{\mathcal{C}} \to [\phi(x_1), ..., \phi(x_s)] \in \mathbb{R}^s$. This map has a joint spectrum of $x_1, ...x_s$ as the set of all s–tuples of eigenvalues corresponding to common eigenvectors.

In general, we get an interpretation of elements $\mathcal{C}$ as $\mathbb{C}$–valued continuous functions on $\widehat{\mathcal{C}}$. The Gel'fand–Naimark theorem (see, for instance, [13]) states that all continuous functions on $\widehat{\mathcal{C}}$ are of the form $\widehat{c}(\phi) = \phi\,(c)$, which defines the so–called Gel'fand transform for every $\phi(c) \in \widehat{\mathcal{C}}$ and the map $\widehat{c} : \widehat{\mathcal{C}} \to \mathbb{C}$ being continuous for each c. A transform $c \to \widehat{c}$ is isometric for every $c \in \mathcal{C}$ if $||\widehat{c}||_\infty = ||c||$, with $||...||_\infty$ defined at the supremum norm on $\mathcal{C}\left(\widehat{\mathcal{C}}\right)$.

The Gel'fand transform can be extended for an arbitrary locally compact topological space M for which there exists a natural C^*–algebra $\mathcal{C}\ (M)$. On can be identified both set wise and topologically the Gel'fand space $\widehat{\mathcal{C}}(M)$ and the space M itself through the evaluation map

$$\phi_x : \mathcal{C}(M) \to \mathbb{C},\ \phi_x(f) = f(x)$$

for each $x \in M$, where $\phi_x \in \widehat{\mathcal{C}}(M)$ gives a complex homomorphism. Denoting by $\mathcal{I}_x = \ker \phi_x$, which is the maximal ideal of $\mathcal{C}\ (M)$ consisting of all functions vanishing at x, one proves [13] that the map ϕ_x is a homomorphism of M onto $\widehat{\mathcal{C}}(M)$, and, equivalently, every maximal ideal of $\mathcal{C}\ (M)$ is of the form $\mathcal{I}_x$ for some $x \in M$.

We conclude this subsection: There is a one–to–one correspondence between the $*$–isomorphism classes of commutative C^*–algebras and the homomorphism classes of locally compact Hausdorff spaces (such commutative C^*–algebras with unit correspond to compact Hausdorff spaces). This correspondence defines a complete duality between the category of (locally) compact Hausdorff spaces and (proper, when a map f relating two locally compact Hausdorff spaces $f : X \to Y$ has the property that $f^{-1}(K)$ is a compact subset of X when K is a compact subset of Y, and) continuous maps and the category of commutative (non necessarily) unital C^*–algebras and $*$–homomorphisms. In result, any commutative C^*–algebra can be realized as the C^*–algebra of complex valued functions over a (locally) compact Hausdorff space. It should be mentioned that the space M is a metrizable topological space, i. e. its topology comes from a metric, if and only if the C^*–algebra is norm separable (it admits a dense in norm countable subset). This space is connected topologically if the corresponding algebra has no projectors which are self–adjoint, $p^* = p$ and satisfy the idempotentity condition $p^2 = p$.

We emphasize that the constructions considered for commutative algebras cannot be directly generalized for noncommutative C^*–algebras.

13.2.3 Noncommutative spaces

For a given noncommutative C^*–algebra, there is more than one candidate for the analogue of the topological space M. Following Ref. [28] (see there the proofs of results and Appendices), we consider two possibilities:

- To use the space $\widehat{\mathcal{A}}$, called the structure space of the noncommutative C^*–algebra $\mathcal{A}$, which is the space of all unitary equivalence classes of irreducible $*$–representations.
- To use the space $\Pr im\mathcal{A}$, called the primitive spectrum of $\mathcal{A}$, which is the space of kernels of irreducible $*$–representations (any element of $\Pr im\mathcal{A}$ is automatically a two–sided $*$–ideal of $\mathcal{A}$).

The spaces $\widehat{\mathcal{A}}$ and $\Pr im\mathcal{A}$ agree for a commutative C^*–algebra, for instance, $\widehat{\mathcal{A}}$ may be very complicate while $\Pr im\mathcal{A}$ consisting of a single point.

Let us examine a simple example of generalization to noncommutative C^*–algebra given by the 2×2 complex matrix algebra

$$M_2(\mathbb{C}) = \{ \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right],\ a_{ij} \in \mathbb{C}\}.$$

The commutative subalgebra of diagonal matrices $\mathcal{C} = \{diag[\lambda_1, \lambda_2],\ \lambda_{1,2} \in \mathbb{C}\}$ has a structure space consisting of two points given by the characters $\phi_{1,2}(\left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right]) = \lambda_{1,2}$. These two characters extend as pure states to the full algebra $M_2(\mathbb{C})$ by the maps $\widetilde{\phi}_{1,2} : M_2(\mathbb{C}) \to \mathbb{C}$,

$$\widetilde{\phi}_1 \left(\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \right) = a_{11},\ \widetilde{\phi}_2 \left(\left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \right) = a_{22}.$$

Further details are given in Appendix B to Ref. [28].

It is possible to define natural topologies on $\widehat{\mathcal{A}}$ and $\Pr im\mathcal{A}$, for instance, by means of a closure operation. For a subset $Q \subset \Pr im\mathcal{A}$, the closure $\overline{Q}$ is by definition the subset of all elements in $\Pr im\mathcal{A}$ containing the intersection $\cap Q$ of the elements of Q, $\overline{Q} \doteqdot \{\mathcal{I} \in \Pr im\mathcal{A} : \cap Q \subseteq \mathcal{I}\}$. It is possible to check that such subsets satisfy the Kuratowski topology axioms and this way defined topology on $\Pr im\mathcal{A}$ is called the Jacobson topology or hull–kernel topology, for which $\cap Q$ is the kernel of Q and $\overline{Q}$ is the hull of $\cap Q$ (see [28, 13] on the properties of this type topological spaces).

13.3 Nonlinear Connections in Noncommutative Spaces

In this subsection we define the nonlinear connections in module spaces, i. e. in noncommutative spaces. The concept on nonlinear connection came from Finsler geometry (as a set of coefficients it is present in the works of E. Cartan [6], then the concept was elaborated in a more explicit fashion by A. Kawaguchi [24]). The global formulation in commutative spaces is due to W. Barthel [2] and it was developed in details for vector, covector and higher order bundles [36, 35, 3], spinor bundles [49, 62], superspaces and superstrings [51, 54, 52] and in the theory of exact off–diagonal solutions of the Einstein equations [56, 57]. The concept of nonlinear connection can be extended in a similar manner from commutative to noncommutative spaces if a differential calculus is fixed on a noncommutative vector (or covector) bundle.

13.3.1 Modules as bundles

A vector bundle $E \to M$ over a manifold M is completely characterized by the space $\mathcal{E} = \Gamma(E, M)$ over its smooth sections defined as a (right) module over the algebra of $C^\infty(M)$ of smooth functions over M. It is known the Serre–Swan theorem [46] which states that locally trivial, finite–dimensional complex vector bundles over a compact Hausdorff space M correspond canonically to finite projective modules over the algebra $\mathcal{A} = C^\infty(M)$. Inversely, for $\mathcal{E}$ being a finite projective modules over $C^\infty(M)$, the fiber E_m of the associated bundle E over the point $x \in M$ is the space $E_x = \mathcal{E}/\mathcal{E}\mathcal{I}_x$ where the ideal is given by

$$\mathcal{I}_x = \ker\{\xi_x : C^\infty(M) \to \mathbb{C}; \xi_x(x) = f(x)\} = \{f \in C^\infty(M) \,|\, f(x) = 0\} \in \mathcal{C}(M).$$

If the algebra $\mathcal{A}$ is taken to play the role of smooth functions on a noncomutative, instead of the commutative algebra smooth functions $C^\infty(M)$, the analogue of a vector bundle is provided by a projective module of finite type (equivalently, finite projective module) over $\mathcal{A}$. On considers the proper construction of projective modules of finite type generalizing the Hermitian bundles as well the notion of Hilbert module when $\mathcal{A}$ is a C^*–algebra in the Appendix C of Ref. [28].

A vector space $\mathcal{E}$ over the complex number field $\mathbb{C}$ can be defined also as a right module of an algebra $\mathcal{A}$ over $\mathbb{C}$ which carries a right representation of $\mathcal{A}$, when for every map of elements $\mathcal{E} \times \mathcal{A} \ni (\eta, a) \to \eta a \in \mathcal{E}$ one hold the properties

$$\lambda(ab) = (\lambda a)b,\ \lambda(a+b) = \lambda a + \lambda b,\ (\lambda + \mu)a = \lambda a + \mu a$$

fro every $\lambda, \mu \in \mathcal{E}$ and $a, b \in \mathcal{A}$.

Having two $\mathcal{A}$–modules $\mathcal{E}$ and $\mathcal{F}$, a morphism of $\mathcal{E}$ into $\mathcal{F}$ is any linear map $\rho : \mathcal{E} \to \mathcal{F}$ which is also $\mathcal{A}$–linear, i. e. $\rho(\eta a) = \rho(\eta) a$ for every $\eta \in \mathcal{E}$ and $a \in \mathcal{A}$.

We can define in a similar (dual) manner the left modules and theirs morphisms which are distinct from the right ones for noncommutative algebras $\mathcal{A}$. A bimodule over an algebra $\mathcal{A}$ is a vector space $\mathcal{E}$ which carries both a left and right module structures. We may define the opposite algebra $\mathcal{A}^o$ with elements a^o being in bijective correspondence with the elements $a \in \mathcal{A}$ while the multiplication is given by $a^o b^o = (ba)^o$.A right (respectively, left) $\mathcal{A}$–module $\mathcal{E}$ is connected to a left (respectively right) $\mathcal{A}^o$–module via relations $a^o \eta = \eta a^o$ (respectively, $a\eta = \eta a$).

One introduces the enveloping algebra $\mathcal{A}^\varepsilon = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^o$; any $\mathcal{A}$–bimodule $\mathcal{E}$ can be regarded as a right [left] $\mathcal{A}^\varepsilon$–module by setting $\eta \left(a \otimes b^o \right) = b \eta a \quad \left[\left(a \otimes b^o \right) \eta = a \eta b \right].$

For a (for instance, right) module $\mathcal{E}$, we may introduce a family of elements $\left(e_t \right)_{t \in T}$ parametrized by any (finite or infinite) directed set T for which any element $\eta \in \mathcal{E}$ is expressed as a combination (in general, in more than one manner) $\eta = \sum_{t \in T} e_t a_t$ with $a_t \in \mathcal{A}$ and only a finite number of non vanishing terms in the sum. A family $\left(e_t \right)_{t \in T}$ is free if it consists from linearly independent elements and defines a basis if any element $\eta \in \mathcal{E}$ can be written as a unique combination (sum). One says a module to be free if it admits a basis. The module $\mathcal{E}$ is said to be of finite type if it is finitely generated, i. e. it admits a generating family of finite cardinality.

Let us consider the module $\mathcal{A}^K \doteqdot \mathbb{C}^K \otimes_{\mathbb{C}} \mathcal{A}$. The elements of this module can be thought as K–dimensional vectors with entries in $\mathcal{A}$ and written uniquely as a linear combination $\eta = \sum_{t=1}^K e_t a_t$ were the basis e_t identified with the canonical basis of $\mathbb{C}^K$. This is a free and finite type module. In general, we can have bases of different cardinality. However, if a module $\mathcal{E}$ is of finite type there is always an integer K and a module surjection $\rho : \mathcal{A}^K \to \mathcal{E}$ with a base being a image of a free basis, $\epsilon_j = \rho(e_j); j = 1, 2, ..., K$.

In general, it is not possible to solve the constraints among the basis elements as to get a free basis. The simplest example is to take a sphere S^2 and the Lie algebra of smooth vector fields on it, $\mathcal{G} = \mathcal{G}(S^2)$ which is a module of finite type over $C^\infty \left(S^2 \right)$, with the basis defined by $X_i = \sum_{j,k=1}^3 \varepsilon_{ijk} x_k \partial / \partial x^k ; i, j, k = 1, 2, 3$, and coordinates x_i such that $\sum_{j=1}^3 x_j^2 = 1$. The introduced basis is not free because $\sum_{j=1}^3 x_j X_j = 0$; there are not global vector field on S^2 which could form a basis of $\mathcal{G}(S^2)$. This means that the tangent bundle TS^2 is not trivial.

We say that a right $\mathcal{A}$–module $\mathcal{E}$ is projective if for every surjective module morphism $\rho : \mathcal{M} \to \mathcal{N}$ splits, i. e. there exists a module morphism $s : \mathcal{E} \to \mathcal{M}$ such that $\rho \circ s = id_{\mathcal{E}}$. There are different definitions of porjective modules (see Ref. [28] on properties of such modules). Here we note the property that if a $\mathcal{A}$–module $\mathcal{E}$ is projective, there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ (being a priory projective) such that $\mathcal{F}=\mathcal{E}\oplus\mathcal{E}'$.

For the right $\mathcal{A}$–module $\mathcal{E}$ being projective and of finite type with surjection $\rho:\mathcal{A}^K\rightarrow\mathcal{E}$ and following the projective property we can find a lift $\widetilde{\lambda}:\mathcal{E}\rightarrow\mathcal{A}^K$ such that $\rho\circ\widetilde{\lambda}=id_{\mathcal{E}}$. There is a proof of the property that the module $\mathcal{E}$ is projective of finite type over $\mathcal{A}$ if and only if there exists an idempotent $p\in End_{\mathcal{A}}\mathcal{A}^K=M_K(\mathcal{A})$, $p^2=p$, the $M_K(\mathcal{A})$ denoting the algebra of $K\times K$ matrices with entry in $\mathcal{A}$, such that $\mathcal{E}=p\mathcal{A}^K$. We may associate the elements of $\mathcal{E}$ to K–dimensional column vectors whose elements are in $\mathcal{A}$, the collection of which are invariant under the map p, $\mathcal{E}=\{\xi=(\xi_1,...,\xi_K);\xi_j\in\mathcal{A},\ p\xi=\xi\}$. For simplicity, we shall use the term finite projective to mean projective of finite type.

The noncommutative variant of the theory of vector bundles may be constructed by using the Serre and Swan theorem [46, 28] which states that for a compact finite dimensional manifold M, a $C^\infty(M)$–module $\mathcal{E}$ is isomorphic to a module $\Gamma(E,M)$ of smooth sections of a bundle $E\rightarrow M$, if and only if it is finite projective. If E is a complex vector bundle over a compact manifold M of dimension n, there exists a finite cover $\{U_i,i=1,...,n\}$ of M such that $E_{|U_i}$ is trivial. Thus, the integer K which determines the rank of the free bundle from which to project onto sections of the bundle is determined by the equality $N=mn$ where m is the rank of the bundle (i. e. of the fiber) and n is the dimension of M.

13.3.2 The commutative nonlinear connection geometry

Let us remember the definition and the main results on nonlinear connections in commutative vector bundles as in Ref. [36].

Vector bundles, Riemannian spaces and nonlinear connections

We consider a vector bundle $\xi=(E,\mu,M)$ whose fibre is $\mathbb{R}^m$ and $\mu^T:TE\rightarrow TM$ denotes the differential of the map $\mu:E\rightarrow M$. The map μ^T is a fibre–preserving morphism of the tangent bundle (TE,τ_E,E) to E and of tangent bundle (TM,τ,M) to M. The kernel of the morphism μ^T is a vector subbundle of the vector bundle (TE,τ_E,E). This kernel is denoted (VE,τ_V,E) and called the vertical subbundle over E. We denote by $i:VE\rightarrow TE$ the inclusion mapping and the local coordinates of a point $u\in E$ by $u^\alpha=(x^i,y^a)$, where indices $i,j,k,...=1,2,...,n$ and $a,b,c,...=1,2,...,m$.

A vector $X_u\in TE$, tangent in the point $u\in E$, is locally represented $\left(x,y,X,\widetilde{X}\right)=(x^i,y^a,X^i,X^a)$, where $(X^i)\in\mathbb{R}^n$ and $(X^a)\in\mathbb{R}^m$ are defined by the equality $X_u=X^i\partial_i+X^a\partial_a$ [$\partial_\alpha=(\partial_i,\partial_a)$ are usual partial derivatives on respective coordinates x^i and y^a]. For instance, $\mu^T\left(x,y,X,\widetilde{X}\right)=(x,X)$ and the submanifold VE contains elements of type $\left(x, y, 0, \widetilde{X}\right)$ and the local fibers of the vertical subbundle are isomorphic to $\mathbb{R}^m$. Having $\mu^T(\partial_a) = 0$, one comes out that ∂_a is a local basis of the vertical distribution $u \rightarrow V_u E$ on E, which is an integrable distribution.

A nonlinear connection (in brief, N–connection) in the vector bundle $\xi = (E, \mu, M)$ is the splitting on the left of the exact sequence

$$0 \rightarrow VE \rightarrow TE/VE \rightarrow 0,$$

i. e. a morphism of vector bundles $N : TE \rightarrow VE$ such that $C \circ i$ is the identity on VE.

The kernel of the morphism N is a vector subbundle of (TE, τ_E, E), it is called the horizontal subbundle and denoted by (HE, τ_H, E). Every vector bundle (TE, τ_E, E) provided with a N–connection structure is Whitney sum of the vertical and horizontal subbundles, i. e.

$$TE = HE \oplus VE. \tag{13.1}$$

It is proven that for every vector bundle $\xi = (E, \mu, M)$ over a compact manifold M there exists a nonlinear connection [36].

Locally a N–connection N is parametrized by a set of coefficients $N_i^a(u^\alpha) = N_i^a(x^j, y^b)$ which transforms as

$$N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a$$

under coordinate transforms on the vector bundle $\xi = (E, \mu, M)$,

$$x^{i'} = x^{i'}\left(x^i\right) \text{ and } y^{a'} = M_a^{a'}(x) y^a.$$

If a N–connection structure is defined on ξ, the operators of local partial derivatives $\partial_\alpha = (\partial_i, \partial_a)$ and differentials $d^\alpha = du^\alpha = (d^i = dx^i, d^a = dy^a)$ should be elongated as to adapt the local basis (and dual basis) structure to the Whitney decomposition of the vector bundle into vertical and horizontal subbundles, (13.1):

$$\partial_\alpha = (\partial_i, \partial_a) \rightarrow \delta_\alpha = \left(\delta_i = \partial_i - N_i^b \partial_b, \partial_a\right), \tag{13.2}$$
$$d^\alpha = \left(d^i, d^a\right) \rightarrow \delta^\alpha = \left(d^i, \delta^a = d^a + N_i^b d^i\right). \tag{13.3}$$

The transforms can be considered as some particular case of frame (vielbein) transforms of type

$$\partial_\alpha \rightarrow \delta_\alpha = e_\alpha^\beta \partial_\beta \text{ and } d^\alpha \rightarrow \delta^\alpha = (e^{-1})_\beta^\alpha \delta^\beta,$$

$e_\alpha^\beta (e^{-1})_\beta^\gamma = \delta_\alpha^\gamma$, when the "tetradic" coefficients δ_α^β are induced by using the Kronecker symbols δ_a^b, δ_j^i and N_i^b.

The bases δ_α and δ^α satisfy in general some anholonomy conditions, for instance,

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W^\gamma_{\alpha\beta} \delta_\gamma, \tag{13.4}$$

where $W^\gamma_{\alpha\beta}$ are called the anholonomy coefficients.

Tensor fields on a vector bundle $\xi = (E, \mu, M)$ provided with N–connection structure N, we shall write ξ_N, may be decomposed with in N–adapted form with respect to the bases δ_α and δ^α, and their tensor products. For instance, for a tensor of rang (1,1) $T = \{T_\alpha^{\ \beta} = \left(T_i^{\ j}, T_i^{\ a}, T_b^{\ j}, T_a^{\ b}\right)\}$ we have

$$T = T_\alpha^{\ \beta} \delta^\alpha \otimes \delta_\beta = T_i^{\ j} d^i \otimes \delta_i + T_i^{\ a} d^i \otimes \partial_a + T_b^{\ j} \delta^b \otimes \delta_j + T_a^{\ b} \delta^a \otimes \partial_b. \tag{13.5}$$

Every N–connection with coefficients N_i^b automatically generates a linear connection on ξ as $\Gamma^{(N)\gamma}_{\alpha\beta} = \{N^a_{bi} = \partial N^a_i(x,y)/\partial y^b\}$ which defines a covariant derivative $D^{(N)}_\alpha A^\beta = \delta_\alpha A^\beta + \Gamma^{(N)\beta}_{\alpha\gamma} A^\gamma$.

Another important characteristic of a N–connection is its curvature $\Omega = \{\Omega^a_{ij}\}$ with the coefficients

$$\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j = \partial_j N^a_i - \partial_i N^a_j + N^b_i N^a_{bj} - N^b_j N^a_{bi}.$$

In general, on a vector bundle we consider arbitrary linear connection and, for instance, metric structure adapted to the N–connection decomposition into vertical and horizontal subbundles (one says that such objects are distinguished by the N–connection, in brief, d–objects, like the d-tensor (13.5), d–connection, d–metric:

- the coefficients of linear d–connections $\Gamma = \{\Gamma^\beta_{\alpha\gamma} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^b_{ac}\right)\}$ are defined for an arbitrary covariant derivative D on ξ being adapted to the N–connection structure as $D_{\delta_\alpha}(\delta_\beta) = \Gamma^\gamma_{\beta\alpha}\delta_\gamma$ with the coefficients being invariant under horizontal and vertical decomposition

$$D_{\delta_i}(\delta_j) = L^k_{ji}\delta_k,\ D_{\delta_i}(\partial_a) = L^b_{ai}\partial_b,\ D_{\partial_c}(\delta_j) = C^k_{jc}\delta_k,\quad D_{\partial_c}(\partial_a) = C^b_{ac}\partial_b.$$

- the d–metric structure $G = g_{\alpha\beta}\delta^a \otimes \delta^b$ which has the invariant decomposition as $g_{\alpha\beta} = (g_{ij}, g_{ab})$ following from

$$G = g_{ij}(x,y) d^i \otimes d^j + g_{ab}(x,y)\delta^a \otimes \delta^b. \tag{13.6}$$

We may impose the condition that a d–metric and a d–connection are compatible, i. e. there are satisfied the conditions

$$D_\gamma g_{\alpha\beta} = 0. \tag{13.7}$$

With respect to the anholonomic frames (13.2) and (13.3), there is a linear connection, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i. e. being constructed only from the metric components and satisfying the metricity conditions (13.7). It is parametrized by the coefficients, ${\Gamma^{\alpha}}_{\beta\gamma} = \left({L^i}_{jk}, {L^a}_{bk}, {C^i}_{jc}, {C^a}_{bc} \right)$ with the coefficients

$$
\begin{aligned}
{L^i}_{jk} &= \frac{1}{2} g^{in} \left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk} \right), \\
{L^a}_{bk} &= \partial_b N_k^a + \frac{1}{2} h^{ac} \left(\delta_k h_{bc} - h_{dc} \partial_b N_k^d - h_{db} \partial_c N_k^d \right), \\
{C^i}_{jc} &= \frac{1}{2} g^{ik} \partial_c g_{jk},\ {C^a}_{bc} = \frac{1}{2} h^{ad} \left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc} \right).
\end{aligned} \tag{13.8}
$$

We note that on Riemannian spaces the N–connection is an object completely defined by anholonomic frames, when the coefficients of frame transforms, $e_\alpha^\beta \left(u^\gamma \right)$, are parametrized explicitly via certain values $\left(N_i^a, \delta_i^j, \delta_b^a \right)$, where δ_i^j and δ_b^a are the Kronecker symbols. By straightforward calculations we can compute that the coefficients of the Levi–Civita metric connection

$$
\Gamma_{\alpha\beta\gamma}^{\bigtriangledown} = g\left(\delta_\alpha, \bigtriangledown_\gamma \delta_\beta \right) = g_{\alpha\tau} \Gamma_{\beta\gamma}^{\bigtriangledown\tau},
$$

associated to a covariant derivative operator $\bigtriangledown$, satisfying the metricity condition $\bigtriangledown_\gamma g_{\alpha\beta} = 0$ for $g_{\alpha\beta} = \left(g_{ij}, h_{ab} \right)$,

$$
\Gamma_{\alpha\beta\gamma}^{\bigtriangledown} = \frac{1}{2} \left[\delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\beta\alpha} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau} W_{\gamma\beta}^\tau + g_{\beta\tau} W_{\alpha\gamma}^\tau - g_{\gamma\tau} W_{\beta\alpha}^\tau \right], \tag{13.9}
$$

are given with respect to the anholonomic basis (13.3) by the coefficients

$$
\Gamma_{\beta\gamma}^{\bigtriangledown\tau} = \left({L^i}_{jk}, {L^a}_{bk}, {C^i}_{jc} + \frac{1}{2} g^{ik} \Omega_{jk}^a h_{ca}, {C^a}_{bc} \right). \tag{13.10}
$$

A specific property of off–diagonal metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [24] (Details can be found in Ref. [36], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N–connection structure).

With respect to anholonomic frames, we can not distinguish the Levi–Civita connection as the unique both metric and torsionless one. For instance, both linear connections

(13.8) and (13.10) contain anholonomically induced torsion coefficients, are compatible with the same metric and transform into the usual Levi–Civita coefficients for vanishing N–connection and "pure" holonomic coordinates. This means that to an off–diagonal metric in general relativity one may be associated different covariant differential calculi, all being compatible with the same metric structure (like in the non–commutative geometry, which is not a surprising fact because the anolonomic frames satisfy by definition some non–commutative relations (13.4)). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N–coefficients. We note that if $\Omega^a_{jk} = 0$ the connections (13.8) and (13.10) coincide, i. e. $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\bigtriangledown\alpha}_{\beta\gamma}$.

D–torsions and d–curvatures:

The anholonomic coefficients $W^{\gamma}{}_{\alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^{\alpha}{}_{\beta\gamma}\delta_\alpha$, where

$$T^{\alpha}_{\ \beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} + w^{\alpha}{}_{\beta\gamma}, \tag{13.11}$$

and for the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha$, where

$$\begin{aligned} R_{\beta\ \gamma\tau}^{\ \alpha} &= \delta_\tau \Gamma^{\alpha}{}_{\beta\gamma} - \delta_\gamma \Gamma^{\alpha}{}_{\beta\tau} \\ &\quad + \Gamma^{\varphi}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\varphi\tau} - \Gamma^{\varphi}{}_{\beta\tau}\Gamma^{\alpha}{}_{\varphi\gamma} + \Gamma^{\alpha}{}_{\beta\varphi} w^{\varphi}{}_{\gamma\tau}. \end{aligned} \tag{13.12}$$

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible (horizontal–vertical) h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components,
$T^{\alpha}{}_{\beta\gamma} = \{T^{i}{}_{jk}, C^{i}{}_{ja}, S^{a}{}_{bc}, T^{a}{}_{ij}, T^{a}{}_{bi}\}$, where

$$\begin{aligned} T^{i}_{.jk} &= T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{ja} = C^{i}_{.ja}, \quad T^{i}_{aj} = -C^{i}_{ja}, \\ T^{i}_{.ja} &= 0, \quad T^{a}_{.bc} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}, \\ T^{a}_{.ij} &= -\Omega^{a}_{ij}, \quad T^{a}_{.bi} = \partial_b N^{a}_{i} - L^{a}_{.bi}, \quad T^{a}_{.ib} = -T^{a}_{.bi} \end{aligned} \tag{13.13}$$

(the d–torsion is computed by substituting the h–v–components of the canonical d–connection (13.8) and anholonomy coefficients(13.4) into the formula for the torsion coefficients (13.11)).

The curvature d-tensor has the following irreducible, non-vanishing, h–v–components $R_{\beta\ \gamma\tau}^{\ \alpha} = \{R_{h.jk}^{.i}, R_{b.jk}^{.a}, P_{j.ka}^{.i}, P_{b.ka}^{.c}, S_{j.bc}^{.i}, S_{b.cd}^{.a}\}$, where

$$
\begin{aligned}
R_{h.jk}^{.i} &= \delta_k L_{.hj}^{i} - \delta_j L_{.hk}^{i} + L_{.hj}^{m} L_{mk}^{i} - L_{.hk}^{m} L_{mj}^{i} - C_{.ha}^{i} \Omega_{.jk}^{a}, \qquad (13.14)\\
R_{b.jk}^{.a} &= \delta_k L_{.bj}^{a} - \delta_j L_{.bk}^{a} + L_{.bj}^{c} L_{.ck}^{a} - L_{.bk}^{c} L_{.cj}^{a} - C_{.bc}^{a} \Omega_{.jk}^{c},\\
P_{j.ka}^{.i} &= \partial_a L_{.jk}^{i} + C_{.jb}^{i} T_{.ka}^{b} - (\delta_k C_{.ja}^{i} + L_{.lk}^{i} C_{.ja}^{l} - L_{.jk}^{l} C_{.la}^{i} - L_{.ak}^{c} C_{.jc}^{i}),\\
P_{b.ka}^{.c} &= \partial_a L_{.bk}^{c} + C_{.bd}^{c} T_{.ka}^{d} - (\delta_k C_{.ba}^{c} + L_{.dk}^{c} C_{.ba}^{d} - L_{.bk}^{d} C_{.da}^{c} - L_{.ak}^{d} C_{.bd}^{c}),\\
S_{j.bc}^{.i} &= \partial_c C_{.jb}^{i} - \partial_b C_{.jc}^{i} + C_{.jb}^{h} C_{.hc}^{i} - C_{.jc}^{h} C_{hb}^{i},\\
S_{b.cd}^{.a} &= \partial_d C_{.bc}^{a} - \partial_c C_{.bd}^{a} + C_{.bc}^{e} C_{.ed}^{a} - C_{.bd}^{e} C_{.ec}^{a}
\end{aligned}
$$

(the d–curvature components are computed in a similar fashion by using the formula for curvature coefficients (13.12)).

Einstein equations in d–variables

In this subsection we write and analyze the Einstein equations on spaces provided with anholonomic frame structures and associated N–connections.

The Ricci tensor $R_{\beta\gamma} = R_{\beta\ \gamma\alpha}^{\ \alpha}$ has the d–components

$$
\begin{aligned}
R_{ij} &= R_{i.jk}^{.k}, \quad R_{ia} = -{}^{2}P_{ia} = -P_{i.ka}^{.k}, \qquad (13.15)\\
R_{ai} &= {}^{1}P_{ai} = P_{a.ib}^{.b}, \quad R_{ab} = S_{a.bc}^{.c}.
\end{aligned}
$$

In general, since ${}^{1}P_{ai} \neq {}^{2}P_{ia}$, the Ricci d-tensor is non-symmetric (this could be with respect to anholonomic frames of reference). The scalar curvature of the metric d–connection, $\overleftarrow{R} = g^{\beta\gamma} R_{\beta\gamma}$, is computed

$$
\overleftarrow{R} = G^{\alpha\beta} R_{\alpha\beta} = \widehat{R} + S, \qquad (13.16)
$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

By substituting (13.15) and (13.16) into the Einstein equations

$$
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \qquad (13.17)
$$

where κ and $\Upsilon_{\alpha\beta}$ are respectively the coupling constant and the energy–momentum tensor we obtain the h-v-decomposition by N–connection of the Einstein equations

$$
\begin{aligned}
R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right) g_{ij} &= \kappa \Upsilon_{ij}, \qquad (13.18)\\
S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right) h_{ab} &= \kappa \Upsilon_{ab},\\
{}^{1}P_{ai} = \kappa \Upsilon_{ai}, \ {}^{2}P_{ia} &= \kappa \Upsilon_{ia}.
\end{aligned}
$$

The definition of matter sources with respect to anholonomic frames is considered in Refs. [49, 54, 36].

The vacuum 5D, locally anisotropic gravitational field equations, in invariant h– v–components, are written

$$\begin{aligned} R_{ij} &= 0, S_{ab} = 0, \\ {}^1P_{ai} &= 0,\ {}^2P_{ia} = 0. \end{aligned} \tag{13.19}$$

We emphasize that vector bundles and even the (pseudo) Riemannian space-times admit non–trivial torsion components, if off–diagonal metrics and anholomomic frames are introduced into consideration. This is a "pure" anholonomic frame effect: the torsion vanishes for the Levi–Civita connection stated with respect to a coordinate frame, but even this metric connection contains some torsion coefficients if it is defined with respect to anholonomic frames (this follows from the W–terms in (3.10)). For (pseudo) Riemannian spaces we conclude that the Einstein theory transforms into an effective Einstein–Cartan theory with anholonomically induced torsion if the general relativity is formulated with respect to general frame bases (both holonomic and anholonomic).

The N–connection geometry can be similarly formulated for a tangent bundle TM of a manifold M (which is used in Finsler and Lagrange geometry [36]), on cotangent bundle T^*M and higher order bundles (higher order Lagrange and Hamilton geometry [35]) as well in the geometry of locally anisotropic superspaces [51], superstrings [53], anisotropic spinor [49] and gauge [61] theories or even on (pseudo) Riemannian spaces provided with anholonomic frame structures [62].

13.3.3 Nonlinear connections in projective modules

The nonlinear connection (N–connection) for noncommutative spaces can be defined similarly to commutative spaces by considering instead of usual vector bundles theirs noncommutative analogs defined as finite projective modules over noncommutative algebras. The explicit constructions depend on the type of differential calculus we use for definition of tangent structures and theirs maps.

In general, there can be several differential calculi over a given algebra $\mathcal{A}$ (for a more detailed discussion within the context of noncommutative geometry see Refs. [8, 31, 15]; a recent approach is connected with Lie superalgebra structures on the space of multiderivations [18]). For simplicity, in this work we fix a differential calculus on $\mathcal{A}$, which means that we choose a (graded) algebra $\Omega^*(\mathcal{A}) = \cup_p \Omega^p(\mathcal{A})$ which gives a differential structure to $\mathcal{A}$. The elements of $\Omega^p(\mathcal{A})$ are called p–forms. There is a linear map d which takes p–forms into $(p+1)$–forms and which satisfies a graded Leibniz rule as well the condition $d^2 = 0$. By definition $\Omega^0(\mathcal{A}) = \mathcal{A}$.

The differential df of a real or complex variable on a vector bundle ξ_N

$$
\begin{aligned}
df &= \delta_i f \ dx^i + \partial_a f \ \delta y^a, \\
\delta_i f &= \partial_i f - N_i^a \partial_a f \ , \ \delta y^a = dy^a + N_i^a dx^i
\end{aligned}
$$

in the noncommutative case is replaced by a distinguished commutator (d–commutator)

$$\overline{d}f = [F, f] = \left[F^{[h]}, f\right] + \left[F^{[v]}, f\right]$$

where the operator $F^{[h]}$ $(F^{[v]})$ is acting on the horizontal (vertical) projective submodule being defined by some fixed differential calculus $\Omega^*(\mathcal{A}^{[h]})$ $(\Omega^*(\mathcal{A}^{[v]}))$ on the so–called horizontal (vertical) $\mathcal{A}^{[h]}$ $(\mathcal{A}^{[v]})$ algebras.

Let us consider instead of a vector bundle ξ an $\mathcal{A}$–module $\mathcal{E}$ being projective and of finite type. For a fixed differential calculus on $\mathcal{E}$ we define the tangent structures $T\mathcal{E}$ and TM. A nonlinear connection N in an $\mathcal{A}$–module $\mathcal{E}$ is defined by an exact sequence of finite projective $\mathcal{A}$–moduli

$$0 \to V\mathcal{E} \to T\mathcal{E}/V\mathcal{E} \to 0,$$

where all subspaces are constructed as in the commutative case with that difference that the vector bundle objects are substituted by theirs projective modules equivalents. A projective module provided with N–connection structures will be denoted as $\mathcal{E}_N$. All objects on a $\mathcal{E}_N$ have a distinguished invariant character with respect to the horizontal and vertical subspaces.

To understand how the N–connection structure may be taken into account on noncommutative spaces we analyze in the next subsection an example.

13.3.4 Commutative and noncommutative gauge d–fields

Let us consider a vector bundle ξ_N and a another vector bundle $\beta = (B, \pi, \xi_N)$ with $\pi : B \to \xi_N$ with a typical k-dimensional vector fiber. In local coordinates a linear connection (a gauge field) in β is given by a collection of differential operators

$$\bigtriangledown_\alpha = D_\alpha + B_\alpha(u),$$

acting on $T\xi_N$ where

$$D_\alpha = \delta_\alpha \pm \Gamma^{\cdot}_{\cdot\alpha} \text{ with } D_i = \delta_i \pm \Gamma^{\cdot}_{\cdot i} \text{ and } D_a = \partial_a \pm \Gamma^{\cdot}_{\cdot a}$$

is a d–connection in ξ_N $(\alpha = 1, 2, ..., n+m)$, with δ_α N–elongated as in (13.2), $u = (x, y) \in \xi_N$ and B_α are $k \times k$–matrix valued functions. For every vector field

$$X = X^\alpha(u)\delta_\alpha = X^i(u)\delta_i + X^a(u)\partial_a \in T\xi_N$$

we can consider the operator

$$X^\alpha(u) \bigtriangledown_\alpha (f \cdot s) = f \cdot \bigtriangledown_X s + \delta_X f \cdot s \qquad (13.20)$$

for any section $s \in \mathcal{B}$ and function $f \in C^\infty(\xi_N)$, where

$$\delta_X f = X^\alpha \delta_\alpha \text{ and } \bigtriangledown_{fX} = f \bigtriangledown_X .$$

In the simplest definition we assume that there is a Lie algebra $\mathcal{GL}B$ that acts on associative algebra B by means of infinitesimal automorphisms (derivations). This means that we have linear operators $\delta_X : B \to B$ which linearly depend on X and satisfy

$$\delta_X(a \cdot b) = (\delta_X a) \cdot b + a \cdot (\delta_X b)$$

for any $a, b \in B$. The mapping $X \to \delta_X$ is a Lie algebra homomorphism, i. e. $\delta_{[X,Y]} = [\delta_X, \delta_Y]$.

Now we consider respectively instead of vector bundles ξ and β the finite projective $\mathcal{A}$–module $\mathcal{E}_N$, provided with N–connection structure, and the finite projective $\mathcal{B}$–module $\mathcal{E}_\beta$.

A d–connection $\bigtriangledown_X$ on $\mathcal{E}_\beta$ is by definition a set of linear d–operators, adapted to the N–connection structure, depending linearly on X and satisfying the Leibniz rule

$$\bigtriangledown_X(b \cdot e) = b \cdot \bigtriangledown_X(e) + \delta_X b \cdot e \qquad (13.21)$$

for any $e \in \mathcal{E}_\beta$ and $b \in \mathcal{B}$. The rule (13.21) is a noncommutative generalization of (13.20). We emphasize that both operators $\bigtriangledown_X$ and δ_X are distinguished by the N–connection structure and that the difference of two such linear d–operators, $\bigtriangledown_X - \bigtriangledown'_X$ commutes with action of B on $\mathcal{E}_\beta$, which is an endomorphism of $\mathcal{E}_\beta$. Hence, if we fix some fiducial connection $\bigtriangledown'_X$ (for instance, $\bigtriangledown'_X = D_X$) on $\mathcal{E}_\beta$ an arbitrary connection has the form

$$\bigtriangledown_X = D_X + B_X,$$

where $B_X \in End_B \mathcal{E}_\beta$ depend linearly on X.

The curvature of connection $\bigtriangledown_X$ is a two–form F_{XY} which values linear operator in $\mathcal{B}$ and measures a deviation of mapping $X \to \bigtriangledown_X$ from being a Lie algebra homomorphism,

$$F_{XY} = [\bigtriangledown_X, \bigtriangledown_Y] - \bigtriangledown_{[X,Y]}.$$

The usual curvature d–tensor is defined as

$$F_{\alpha\beta} = [\bigtriangledown_\alpha, \bigtriangledown_\beta] - \bigtriangledown_{[\alpha,\beta]}.$$

The simplest connection on a finite projective $\mathcal{B}$–module $\mathcal{E}_\beta$ is to be specified by a projector $P : \mathcal{B}^k \otimes \mathcal{B}^k$ when the d–operator δ_X acts naturally on the free module $\mathcal{B}^k$. The operator $\bigtriangledown_X^{LC} = P \cdot \delta_X \cdot P$ is called the Levi–Civita operator and satisfy the condition $Tr[\bigtriangledown_X^{LC}, \phi] = 0$ for any endomorphism $\phi \in End_B\mathcal{E}_\beta$. From this identity, and from the fact that any two connections differ by an endomorphism that

$$Tr[\bigtriangledown_X, \phi] = 0$$

for an arbitrary connection $\bigtriangledown_X$ and an arbitrary endomorphism ϕ, that instead of $\bigtriangledown_X^{LC}$ we may consider equivalently the canonical d–connection, constructed only from d-metric and N–connection coefficients.

13.4 Distinguished Spectral Triples

In this section we develop the basic ingredients introduced by A. Connes [8] to define the analogue of differential calculus for noncommutative distinguished algebras. The N–connection structures distinguish a commutative or a noncommutative spaces into horizontal and vertical subspaces. The geometric objects possess a distinguished invariant character with respect to a such splitting. The basic idea in definition of spectral triples generating locally anisotropic spaces (Rimannian spaces with anholonomic structure, or, for more general constructions, Finsler and Lagrange spaces) is to consider pairs of noncommutative algebras $\mathcal{A}_{[d]} = (\mathcal{A}_{[h]}, \mathcal{A}_{[v]})$, given by respective pairs of elements $a = \left(a_{[h]}, a_{[v]}\right) \in \mathcal{A}_{[d]}$, called also distinguished algebras (in brief, d–algebras), together with d-operators $D_{[d]} = (D_{[h]}, D_{[v]})$ on a Hilbert space $\mathcal{H}$ (for simplicity we shall consider one Hilbert space, but a more general construction can be provided for Hilbert d-spaces, $\mathcal{H}_{[d]} = \left(\mathcal{H}_{[h]}, \mathcal{H}_{[v]}\right)$.

The formula of Wodzicki–Adler–Manin–Guillemin residue (see, for instance, [28]) may be written for vector bundles provided with N–connection structure. It is necessary to introduce the N–elongated differentials (13.2) in definition of the measure: Let Q be a pseudo–differential operator of order $-n$ acting on sections of a complex vector bundle $E \to M$ over an n–dimensional compact Riemannian manifold M. The residue $ResQ$ of Q is defined by the formula

$$ResQ =: \frac{1}{n\,(2\pi)^n} \int\limits_{S^*M} tr_E \sigma_{-n}(Q) \delta\mu,$$

where $\sigma_{-n}(Q)$ is the principal symbol (a matrix–valued function on T^*M which is homogeneous of degree $-n$ in the fiber coordinates), the integral is taken over the unit co–sphere $S^*M = \{(x,y) \in T^*M : ||y|| = 1\} \subset T^*M$, the tr_E is the matrix trace over "internal indices" and the measures $\delta\mu = dx^i \delta y^a$.

A spectral d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ is given by an involutive d–algebra of d–operators $D^{[d]}$ consisting from pairs of bounded operators $D_{[h]}$ and $D_{[v]}$ on the Hilbert space $\mathcal{H}$, together with the self–adjoint operation $D_{[d]} = D^*_{[d]}$ for respective h- and v–components on $\mathcal{H}$ being satisfied the properties:

1. The resolvents $(D_{[h]} - \lambda_{[h]})^{-1}$ and $(D_{[v]} - \lambda_{[v]})^{-1}$, $\lambda_{[h]}, \lambda_{[v]} \in \mathbb{R}$, are compact operators on $\mathcal{H}$;

2. The commutators $\left[D_{[h,]}, a_{[h]}\right] \doteqdot D_{[h]}a_{[h]} - a_{[h]}D_{[h]} \in \mathcal{B}(\mathcal{H})$ and $\left[D_{[v,]}, a_{[v]}\right] \doteqdot D_{[v]}a_{[v]} - a_{[v]}D_{[v]} \in \mathcal{B}(\mathcal{H})$ for any $a \in \mathcal{A}_{[d]}$, where by $\mathcal{B}(\mathcal{H})$ we denote the algebra of bounded operators on $\mathcal{H}$.

The $h(v)$–component of a d–triple is said to be even if there is a $\mathbb{Z}_2$–grading for $\mathcal{H}$, i. e. an operator Υ on $\mathcal{H}$ such that

$$\Upsilon = \Upsilon^*, \Upsilon^2 = 1, \ \Upsilon D_{[h(v)]} - D_{[h(v)]}\Upsilon = 0, \ \Upsilon a - a\Upsilon = 0$$

for every $a \in \mathcal{A}_{[d]}$. If such a grading does not exist, the $h(v)$–component of a d–triple is said to be odd.

13.4.1 Canonical triples over vector bundles

The basic examples of spectral triples in connections with noncommutative field theory and geometry models were constructed by means of the Dirac operator on a closed n–dimensional Riemannian spin manifold (M, g) [8, 11]. In order to generate by using functional methods some anisotropic geometries, it is necessary to generalize the approach to vector and covector bundles provided with compatible N–connection, d–connection and metric structures. The theory of spinors on locally anisotropic spaces was developed in Refs. [49, 62]. This section is devoted to the spectral d–triples defined by the Dirac operators on closed regions of $(n+m)$–dimensional spin–vector manifolds. We note that if we deal with off–diagonal metrics and/or anholonomic frames there is an infinite number of d–connections which are compatible with d–metric and N–connection structures, see discussion and details in Ref. [56]. For simplicity, we restrict our consideration only to the Euclidean signature of metrics of type (13.6) (on attempts to define triples with Minkowskian signatures see, for instance, Refs. [16]).

For a spectral d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ associated to a vector bundle ξ_N one takes the components:

1. $\mathcal{A}_{[d]} = \mathcal{F}(\xi_N)$ is the algebra of complex valued functions on ξ_N.
2. $\mathcal{H} = L^2(\xi_N, S)$ is the Hilbert space of square integrable sections of the irreducible d–spinor bundle (of rank $2^{(n+m)/2}$ over ξ_N [49, 62]. The scalar product in $L^2(\xi_N, S)$ is the defined by the measure associated to the d–metric (13.6),

$$(\psi, \phi) = \int \delta\mu(g)\overline{\psi}(u)\phi(u)$$

were the bar indicates to the complex conjugation and the scalar product in d–spinor space is the natural one in $\mathbb{C}^{2[n/2]} \oplus \mathbb{C}^{2[m/2]}$.

3. D is a Dirac d–operator associated to one of the d–metric compatible d–connection, for instance, with the Levi–Civita connection, canonical d–connection or another one, denoted with a general symbol $\Gamma = \Gamma_\mu \delta u^\mu$.

We note that the elements of the algebra $\mathcal{A}_{[d]}$ acts as multiplicative operators on $\mathcal{H}$,

$$(a\psi)(u) =: f(u)\psi(u),$$

for every $a \in \mathcal{A}_{[d]}, \psi \in \mathcal{H}$.

Distinguished spinor structures

Let us analyze the connection between d–spinor structures and spectral d–triples over a vector bundle ξ_N. One consider a $(n+m)$–bein (frame) decomposition of the d–metric $g_{\alpha\beta}$ (13.6) (and its inverse $g^{\alpha\beta}$),

$$g^{\alpha\beta}(u) = e^{\alpha}_{\underline{\alpha}}(u)e^{\beta}_{\underline{\beta}}(u)\eta^{\underline{\alpha\beta}},\ \eta_{\underline{\alpha\beta}} = e^{\alpha}_{\underline{\alpha}}(u)e^{\beta}_{\underline{\beta}}(u)g_{\alpha\beta},$$

$\eta_{\underline{\alpha\beta}}$ it the diagonal Euclidean $(n+m)$–metric, which is adapted to the N–connection structure because the coefficients $g_{\alpha\beta}$ are defined with respect to the dual N–distinguished basis (13.3). We can define compatible with this decomposition d–connections $\Gamma^{\underline{\alpha}}_{\underline{\beta}\mu}$ (for instance, the Levi–Civita connection, which with respect to anholonomic frames contains torsions components, or the canonical d–connection), defined by

$$D_\mu e_{\underline{\beta}} = \Gamma^{\underline{\alpha}}_{\underline{\beta}\mu} e_{\underline{\alpha}},$$

as the solution of the equations

$$\delta_\mu e^{\underline{\nu}}_{v} - \delta_\nu e^{\underline{\nu}}_{\mu} = \Gamma^{\underline{\nu}}_{\underline{\beta}\mu} e^{\underline{\beta}}_{v} - \Gamma^{\underline{\nu}}_{\underline{\beta}\nu} e^{\underline{\beta}}_{\mu}.$$

We define by $C(\xi_N)$ the Clifford bundle over ξ_N with the fiber at $u \in \xi_N$ being just the complexified Clifford d–algebra $Cliff_{\mathbb{C}}(T^*_u\xi_N)$, $T^*_u\xi_N$ being dual to $T_u\xi_N$, and $\Gamma[\xi_N, C(\xi_N)]$ is the module of corresponding sections. By defining the maps

$$\gamma(\delta^\alpha) = \left(\gamma\left(d^i\right), \gamma(\delta^a)\right) \doteqdot \gamma^\alpha(u) = \gamma^{\underline{\alpha}} e^\alpha_{\underline{\alpha}}(u) = \left(\gamma^{\underline{\alpha}} e^i_{\underline{\alpha}}(u), \gamma^{\underline{\alpha}} e^a_{\underline{\alpha}}(u)\right),$$

extended as an algebra map by $\mathcal{A}_{[d]}$–linearity, we construct an algebra morphism

$$\gamma : \Gamma(\xi_N, C(\xi_N)) \to \mathcal{B}(\mathcal{H}). \tag{13.22}$$

The indices of the "curved" $\gamma^\alpha(u)$ and "flat" $\gamma^{\underline{\alpha}}$ gamma matrices can be lowered by using respectively the d-metric components $g_{\alpha\beta}(u)$ and $\eta_{\underline{\alpha}\underline{\beta}}$, i. e. $\gamma_\beta(u) = \gamma^\alpha(u)\, g_{\alpha\beta}(u)$ and $\gamma_{\underline{\beta}} = \gamma^{\underline{\alpha}}\eta_{\underline{\alpha}\underline{\beta}}$. We take the gamma matrices to be Hermitian and to obey the relations,

$$\begin{aligned} \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha &= -2g_{\alpha\beta}\ \left(\gamma_i\gamma_j + \gamma_j\gamma_i = -2g_{ij},\ \gamma_a\gamma_b + \gamma_b\gamma_a = -2g_{ab}\right), \\ \gamma_{\underline{\alpha}}\gamma_{\underline{\beta}} + \gamma_{\underline{\beta}}\gamma_{\underline{\alpha}} &= -2\eta_{\underline{\alpha}\underline{\beta}}. \end{aligned}$$

Every d–connection $\Gamma^{\underline{\nu}}_{\underline{\beta}\mu}$ can be shifted as a d–covariant operator $\bigtriangledown^{[S]}_\mu = \left(\bigtriangledown^{[S]}_i, \bigtriangledown^{[S]}_a\right)$ on the bundle of d–spinors,

$$\bigtriangledown^{[S]}_\mu = \delta_\mu + \frac{1}{2}\Gamma_{\underline{\alpha}\underline{\beta}\mu}\gamma^{\underline{\alpha}}\gamma^{\underline{\beta}},\ \Gamma^{[S]}_\mu = \frac{1}{2}\Gamma_{\underline{\alpha}\underline{\beta}\mu}\gamma^{\underline{\alpha}}\gamma^{\underline{\beta}},$$

which defines the Dirac d–operator

$${}_{[d]}D =: \gamma \circ \bigtriangledown^{[S]}_\mu = \gamma^\alpha(u)\left(\delta_\mu + \Gamma^{[S]}_\mu\right) = \gamma^{\underline{\alpha}} e^\mu_{\underline{\alpha}}\left(\delta_\mu + \Gamma^{[S]}_\mu\right). \tag{13.23}$$

Such formulas were introduced in Refs. [49] for distinguished spinor bundles (of first and higher order). In this paper we revise them in connection to spectral d–triples and noncommutative geometry. On such spaces one also holds a variant of Lichnerowicz formula [4] for the square of the Dirac d–operator

$${}_{[d]}D^2 = \bigtriangledown^{[S]} + \frac{1}{4}\overleftarrow{R}, \tag{13.24}$$

where the formulas for the scalar curvature $\overleftarrow{R}$ is given in (13.16) and

$$\bigtriangledown^{[S]} = -g^{\mu\nu}\left(\bigtriangledown^{[S]}_\mu \bigtriangledown^{[S]}_v - \Gamma^\rho_{\mu\nu}\bigtriangledown^{[S]}_\rho\right).$$

In a similar manner as in Ref. [28] but reconsidering all computations on a vector bundle ξ_N we can prove that for every d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ one holds the properties:

1. The vector bundle ξ_N is the structure space of the algebra $\overline{\mathcal{A}}_{[d]}$ of continuous functions on ξ_N (the bar here points to the norm closure of $\mathcal{A}_{[d]}$).

2. The geodesic distance ρ between two points $p_1, p_2 \in \xi_N$ is defined by using the Dirac d–operator,
$$\rho(p_1, p_2) = \sup_{f\in} \left\{ |f(p) - f(q)| : \ ||[D_{[d]}, f]|| \leq 1 \right\}.$$

3. The Dirac d–operator also defines the Riemannian measure on ξ_N,
$$\int_{\xi_N} f = c\,(n+m)\, tr_\Gamma \left(f |D_{[d]}|^{-(n+m)} \right)$$
for every $f \in \mathcal{A}_{[d]}$ and $c\,(n+m) = 2^{[n+m-(n+m)/2-1]} \pi^{(n+m)/2} (n+m) \Gamma\left(\frac{n+m}{2}\right), \Gamma$ being the gamma function.

The spectral d–triple formalism has the same properties as the usual one with that difference that we are working on spaces provided with N–connection structures and the bulk of constructions and objects are distinguished by this structure.

Noncommutative differential forms

To construct a differential algebra of forms out a spectral d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ one follows universal graded differential d–algebras defined as couples of universal ones, respectively associated to the h– and v–components of some splitting to subspaces defined by N–connection structures. Let $\mathcal{A}_{[d]}$ be an associative d–algebra (for simplicity, with unit) over the field of complex numbers $\mathbb{C}$. The universal d–algebra of differential forms $\Omega\mathcal{A}_{[d]} = \oplus_p \Omega^p \mathcal{A}_{[d]}$ is introduced as a graded d–algebra when $\Omega^0\mathcal{A}_{[d]} = \mathcal{A}_{[d]}$ and the space $\Omega^1\mathcal{A}_{[d]}$ of one–forms is generated as a left $\mathcal{A}_{[d]}$–module by symbols of degree $\delta a, a \in \mathcal{A}_{[d]}$ satisfying the properties
$$\delta(ab) = (\delta a)b + a\delta b \text{ and } \delta(\alpha a + \beta b) = \alpha(\delta a) + \beta\delta b$$
from which follows $\delta 1 = 0$ which in turn implies $\delta\mathbb{C} = 0$. These relations state the Leibniz rule for the map
$$\delta : \mathcal{A}_{[d]} \rightarrow \Omega^1\mathcal{A}_{[d]}$$
An element $\varpi \in \Omega^1\mathcal{A}_{[d]}$ is expressed as a finite sum of the form
$$\varpi = \sum_{\underline{i}} a_{\underline{i}} b_{\underline{i}}$$

for $a_{\underline{i}}, b_{\underline{i}} \in \mathcal{A}_{[d]}$. The left $\mathcal{A}_{[d]}$–module $\Omega^1\mathcal{A}_{[d]}$ can be also endowed with a structure of right $\mathcal{A}_{[d]}$–module if the elements are imposed to satisfy the conditions

$$\left(\sum_{\underline{i}} a_{\underline{i}}\delta b_{\underline{i}}\right) c =: \sum_{\underline{i}} a_{\underline{i}}(\delta b_{\underline{i}})c = \sum_{\underline{i}} a_{\underline{i}}\delta(b_{\underline{i}}c) - \sum_{\underline{i}} a_{\underline{i}}b_{\underline{i}}\delta c.$$

Given a spectral d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$, one constructs and exterior d–algebra of forms by means of a suitable representation of the universal algebra $\Omega\mathcal{A}_{[d]}$ in the d–algebra of bounded operators on $\mathcal{H}$ by considering the map

$$\begin{aligned} \pi &: \Omega\mathcal{A}_{[d]} \rightarrow \mathcal{B}(\mathcal{H}), \\ \pi\left(a_0\delta a_1...\delta a_p\right) &= : a_0\left[D, a_1\right]...\left[D, a_p\right] \end{aligned}$$

which is a homomorphism since both δ and $[D, .]$ are distinguished derivations on $\mathcal{A}_{[d]}$. More than that, since $[D, a]^* = -\left[D, a^*\right]$, we have $\pi\left(\varpi\right)^* = \pi\left(\varpi^*\right)$ for any d–form $\varpi \in \Omega\mathcal{A}_{[d]}$ and π being a $*$–homomorphism.

Let $J_0 =: \oplus_p J_0^p$ be the graded two–sided ideal of $\Omega\mathcal{A}_{[d]}$ given by $J_0^p =: \{\pi\left(\varpi\right) = 0\}$ when $J = J_0 + \delta J_0$ is a graded differential two–sided ideal of $\Omega\mathcal{A}_{[d]}$. At the next step we can define the graded differential algebra of Connes' forms over the d–algebra $\mathcal{A}_{[d]}$ as

$$\Omega_D\mathcal{A}_{[d]} =: \Omega\mathcal{A}_{[d]}/J \simeq \pi\left(\Omega\mathcal{A}_{[d]}\right)/\pi\left(\delta J_0\right).$$

It is naturally graded by the degrees of $\Omega\mathcal{A}_{[d]}$ and J with the space of p–forms being given by $\Omega_D^p\mathcal{A}_{[d]} = \Omega^p\mathcal{A}_{[d]}/J^p$. Being J a differential ideal, the exterior differential δ defines a differential on $\Omega_D\mathcal{A}_{[d]}$,

$$\delta : \Omega_D^p\mathcal{A}_{[d]} \rightarrow \Omega_D^{p+1}\mathcal{A}_{[d]},\ \delta[\varpi] =: [\delta\varpi]$$

with $\varpi \in \Omega_D^p\mathcal{A}_{[d]}$ and $[\varpi]$ being the corresponding class in $\Omega_D^p\mathcal{A}_{[d]}$.

We conclude that the theory of distinguished d–forms generated by d–algebras, as well of the graded differential d–algebra of Connes' forms, is constructed in a usual form (see Refs. [8, 28]) but for two subspaces (the horizontal and vertical ones) defined by a N–connection structure.

The exterior d–algebra

The differential d–form formalism when applied to the canonical d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ over an ordinary vector bundle ξ_N provided with N–connection structure reproduce the usual exterior d–algebra over this vector bundle. Consider our d–triple on a closed $(n+m)$–dimensional Riemannian *spin*^{c} manifold as described in subsection 13.4.1 when $\mathcal{A}_{[d]} = \mathcal{F}(\xi_N)$ is the algebra of smooth complex valued functions on ξ_N and $\mathcal{H} = L^2(\xi_N, S)$ is the Hilbert space of square integrable sections of the irreducible d–spinor bundle (of rank $2^{(n+m)/2}$ over ξ_N. We can identify

$$\pi\left(\delta f\right) =: [{}_{[d]}D, f] = \gamma^{\mu}(u)\delta_{\mu} f = \gamma\left(\delta_{\xi_N} f\right) \tag{13.25}$$

for every $f \in \mathcal{A}_{[d]}$, see the formula for the Dirac d–operator (13.23), where γ is the d–algebra morphism (13.22) and δ_{ξ_N} denotes the usual exterior derivative on ξ_N. In a more general case, with $f_{[i]} \in \mathcal{A}_{[d]}, [i] = [1], ..., [p]$, we can write

$$\pi\left(f_{[0]}\delta f_{[1]}...\delta f_{[p]}\right) =: f_{[0]}[{}_{[d]}D, f_{[1]}]...[{}_{[d]}D, f_{[p]}] = \gamma\left(f_{[0]}\delta_{\xi_N} f_{[1]} \cdot ... \cdot \delta_{\xi_N} f_{[p]}\right), \tag{13.26}$$

where the d–differentials $\delta_{\xi_N} f_{[1]}$ are regarded as sections of the Clifford d–bundle $C_1(\xi_N)$, while $f_{[i]}$ can be thought of as sections of $C_0(\xi_N)$ and the *dot* $\cdot$ the Clifford product in the fibers of $C(\xi_N) = \oplus_k C_k(\xi_N)$, see details in Refs. [49, 62].

A generic differential 1–form on ξ_N can be written as $\sum_{[i]} f_0^{[i]} \delta_{\xi_N} f_1^{[i]}$ with $f_0^{[i]}, f_1^{[i]} \in \mathcal{A}_{[d]}$. Following the definitions (13.25) and (13.26), we can identify the distinguished Connes' 1–forms $\Omega_D^p \mathcal{A}_{[d]}$ with the usual distinguished differential 1–forms, i. e.

$$\Omega_D^p \mathcal{A}_{[d]} \simeq \Lambda^p\left(\xi_N\right).$$

For each $u \in \xi_N$, we can introduce a natural filtration for the Clifford d–algebra, $C_u(\xi_N) = \cup C_u^{(p)}$, where $C_u^{(p)}$ is spanned by products of type $\chi_{[1]} \cdot \chi_{[2]} \cdot ... \cdot \chi_{[p']}, p' \leq p,\ \chi_{[i]} \in T_u^* \xi_N$. One defines a natural graded d–algebra,

$$grC_u =: \sum_p gr_p C_u,\ gr_p C_u = C_u^{(p)}/C_u^{(p-1)}, \tag{13.27}$$

for which the natural projection is called the symbol map,

$$\sigma_p : C_u^{(p)} \rightarrow gr_p C_u.$$

The natural graded d–algebra is canonical isomorphic to the complexified exterior d–algebra $\Lambda_{\mathbb{C}}\left(T_u^* \xi_N\right)$, the isomorphism being defined as

$$\Lambda_{\mathbb{C}}\left(T_u^* \xi_N\right) \ni \chi_{[1]} \wedge \chi_{[2]} \wedge ... \wedge \chi_{[p]} \rightarrow \sigma_p\left(\chi_{[1]} \cdot \chi_{[2]} \cdot ... \cdot \chi_{[p]}\right) \in gr_p C_u. \tag{13.28}$$

As a consequence of formulas (13.27) and (13.28), for a canonical d–triple $\left[\mathcal{A}_{[d]}, \mathcal{H}, D_{[d]}\right]$ over the vector bundle ξ_N, one follows the property: a pair of operators

Q_1 and Q_2 on $\mathcal{H}$ is of the form $Q_1 = \pi(\varpi)$ and $Q_2 = \pi(\delta\varpi)$ for some universal form $\varpi \in \Omega^p\mathcal{A}_{[d]}$, if and only if there are sections ρ_1 of $C^{(p)}$ and ρ_2 of $C^{(p+1)}$ such that

$$Q_1 = \gamma\left(\rho_1\right) \text{ and } Q_2 = \gamma\left(\rho_2\right)$$

for which

$$\delta_{\xi_N}\sigma_p\left(\rho_1\right) = \sigma_{p+1}\left(\rho_2\right).$$

The introduced symbol map defines the canonical isomorphism

$$\sigma_p : \Omega_D^p\mathcal{A}_{[d]} \simeq \Gamma\left(\Lambda_{\mathbb{C}}^p T^*\xi_N\right) \tag{13.29}$$

which commutes with the differential. With this isomorphism the inner product on $\Omega_D^p\mathcal{A}_{[d]}$ (the scalar product of forms) is proportional to the Riemannian inner product of distinguished p–forms on ξ_N,

$$< \varpi_1, \varpi_2 >_p = (-1)^p \frac{2^{(n+m)/2+1-(n+m)}\pi^{-(n+m)/2}}{(n+m)\Gamma\left((n+m)/2\right)} \int_{\xi_N} \varpi_1 \wedge *\varpi_2 \tag{13.30}$$

for every $\varpi_1, \varpi_2 \in \Omega_D^p\mathcal{A}_{[d]} \simeq \Gamma\left(\Lambda_{\mathbb{C}}^p T^*\xi_N\right)$.

The proofs of formulas (13.29) and (13.30) are similar to those given in [28] for $\xi_N = M$.

13.4.2 Noncommutative Geometry and Anholonomic Gravity

We introduce the concepts of generalized Lagrange and Finsler geometry and outline the conditions when such structures can be modelled on a Riemannian space by using anholnomic frames.

Anisotropic spacetimes

Different classes of commutative anisotropic spacetimes are modelled by corresponding parametriztions of some compatible (or even non–compatible) N–connection, d–connection and d–metric structures on (pseudo) Riemannian spaces, tangent (or cotangent) bundles, vector (or covector) bundles and their higher order generalizations in their usual manifold, supersymmetric, spinor, gauge like or another type approaches (see Refs. [56, 35, 36, 3, 49, 61, 54, 62]). Here we revise the basic definitions and formulas which will be used in further noncommutative embedding and generalizations.

Anholonomic structures on Riemannian spaces: We can generate an anholonomic (equivalently, anisotropic) structure on a Rieman space of dimension $(n+m)$ space (let us denote this space $V^{(n+m)}$ and call it as a anholonomic Riemannian space) by fixing an anholonomic frame basis and co-basis with associated N–connection $N_i^a(x,y)$, respectively, as (13.2) and (13.3) which splits the local coordinates $u^\alpha = (x^i, y^a)$ into two classes: n holonomic coorinates, x^i, and m anholonomic coordinates, y^a. The d–metric (13.6) on $V^{(n+m)}$,

$$G^{[R]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)\delta y^a \otimes \delta y^b \tag{13.31}$$

written with respect to a usual coordinate basis $du^\alpha = (dx^i, dy^a)$,

$$ds^2 = \underline{g}_{\alpha\beta}(x,y)\, du^\alpha du^\beta$$

is a generic off–diagonal Riemannian metric parametrized as

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b g_{ab} & h_{ab}N_i^a \\ h_{ab}N_j^b & g_{ab} \end{array} \right]. \tag{13.32}$$

Such type of metrics were largely investigated in the Kaluza–Klein gravity [42], but also in the Einstein gravity [56]. An off–diagonal metric (13.32) can be reduced to a block $(n \times n) \oplus (m \times m)$ form (g_{ij}, g_{ab}), and even effectively diagonalized in result of a superposition of ahnolonomic N–transforms. It can be defined as an exact solution of the Einstein equations. With respect to anholonomic frames, in general, the Levi–Civita connection obtains a torsion component (3.10). Every class of off–diagonal metrics can be anholonomically equivalent to another ones for which it is not possible to a select the Levi–Civita metric defied as the unique torsionless and metric compatible linear connection. The conclusion is that if anholonomic frames of reference, which authomatically induce the torsion via anholonomy coefficients, are considered on a Riemannian space we have to postulate explicitly what type of linear connection (adapted both to the anholonomic frame and metric structure) is chosen in order to construct a Riemannian geometry and corresponding physical models. For instance, we may postulate the connection (13.10) or the d–connection (13.8). Both these connections are metric compatible and transform into the usual Christoffel symbols if the N–connection vanishes, i. e. the local frames became holonomic. But, in general, anholonomic frames and off–diagonal Riemannian metrics are connected with anisotropic configurations which allow, in principle, to model even Finsler like structures in (pseudo) Riemannian spaces [55, 56].

Finsler geometry and its almost Kahlerian model: The modern approaches to Finsler geometry are outlined in Refs. [41, 36, 35, 3, 54, 62]. Here we emphasize that a Finsler metric can be defined on a tangent bundle TM with local coordinates $u^\alpha = (x^i, y^a \to y^i)$ of dimension $2n$, with a d–metric (13.6) for which the Finsler metric, i. e. the quadratic form

$$g_{ij}^{[F]} = g_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive definite, is defined in this way: 1) A Finsler metric on a real manifold M is a function $F : TM \to \mathbb{R}$ which on $\widetilde{TM} = TM\backslash\{0\}$ is of class C^∞ and F is only continuous on the image of the null cross–sections in the tangent bundle to M. 2) $F(x, \lambda y) = \lambda F(x, \lambda y)$ for every $\mathbb{R}_+^*$. 3) The restriction of F to $\widetilde{TM}$ is a positive function. 4) $rank\left[g_{ij}^{[F]}(x,y)\right] = n.$

The Finsler metric $F(x, y)$ and the quadratic form $g_{ij}^{[F]}$ can be used to define the Christoffel symbols (not those from the usual Riemannian geometry)

$$c_{jk}^{\iota}(x,y) = \frac{1}{2}g^{ih}\left(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk}\right)$$

which allows to define the Cartan nonlinear connection as

$$N_j^i(x,y) = \frac{1}{4}\frac{\partial}{\partial y^j}\left[c_{lk}^{\iota}(x,y)y^l y^k\right] \tag{13.33}$$

where we may not distinguish the v- and h- indices taking on TM the same values.

In Finsler geometry there were investigated different classes of remarkable Finsler linear connections introduced by Cartan, Berwald, Matsumoto and other ones (see details in Refs. [41, 36, 3]). Here we note that we can introduce $g_{ij}^{[F]} = g_{ab}$ and $N_j^i(x,y)$ in (13.6) and construct a d–connection via formulas (13.8).

A usual Finsler space $F^n = (M, F(x,y))$ is completely defined by its fundamental tensor $g_{ij}^{[F]}(x,y)$ and Cartan nonlinear connection $N_j^i(x,y)$ and its chosen d–connection structure. But the N–connection allows us to define an almost complex structure I on TM as follows

$$I(\delta_i) = -\partial/\partial y^i \text{ and } I\left(\partial/\partial y^i\right) = \delta_i$$

for which $I^2 = -1$.

The pair $\left(G^{[F]}, I\right)$ consisting from a Riemannian metric on TM,

$$G^{[F]} = g_{ij}^{[F]}(x,y)dx^i \otimes dx^j + g_{ij}^{[F]}(x,y)\delta y^i \otimes \delta y^j \tag{13.34}$$

and the almost complex structure I defines an almost Hermitian structure on $\widetilde{TM}$ associated to a 2–form

$$\theta = g_{ij}^{[F]}(x,y)\delta y^i \wedge dx^j.$$

This model of Finsler geometry is called almost Hermitian and denoted H^{2n} and it is proven [36] that is almost Kahlerian, i. e. the form θ is closed. The almost Kahlerian space $K^{2n} = \left(\widetilde{TM}, G^{[F]}, I\right)$ is also called the almost Kahlerian model of the Finsler space F^n.

On Finsler (and their almost Kahlerian models) spaces one distinguishes the almost Kahler linear connection of Finsler type, $D^{[I]}$ on $\widetilde{TM}$ with the property that this covariant derivation preserves by parallelism the vertical distribution and is compatible with the almost Kahler structure $\left(G^{[F]}, I\right)$, i.e.

$$D_X^{[I]} G^{[F]} = 0 \text{ and } D_X^{[I]} I = 0$$

for every d–vector field on $\widetilde{TM}$. This d–connection is defined by the data

$$\Gamma = \left(L^i_{jk}, L^a_{bk} = 0, C^i_{ja} = 0, C^a_{bc} \rightarrow C^i_{jk}\right)$$

with L^i_{jk} and C^i_{jk} computed as in the formulas (13.8) by using $g_{ij}^{[F]}$ and N^i_j from (13.33).

We emphasize that a Finsler space F^n with a d–metric (13.34) and Cartan's N–connection structure (13.33), or the corresponding almost Hermitian (Kahler) model H^{2n}, can be equivalently modelled on a Riemannian space of dimension $2n$ provided with an off–diagonal Riemannian metric (13.32). From this viewpoint a Finsler geometry is a corresponding Riemannian geometry with a respective off–diagonal metric (or, equivalently, with an anholonomic frame structure with associated N–connection) and a corresponding prescription for the type of linear connection chosen to be compatible with the metric and N–connection structures.

Lagrange and generalized Lagrange geometry: The Lagrange spaces were introduced in order to generalize the fundamental concepts in mechanics [26] and investigated in Refs. [36] (see [49, 61, 51, 53, 54, 62] for their spinor, gauge and supersymmetric generalizations).

A Lagrange space $L^n = (M, L(x,y))$ is defined as a pair which consists of a real, smooth n–dimensional manifold M and regular Lagrangian $L : TM \rightarrow \mathbb{R}$. Similarly as for Finsler spaces one introduces the symmetric d–tensor field

$$g_{ij}^{[L]} = g_{ab} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}. \qquad (13.35)$$

So, the Lagrangian $L(x, y)$ is like the square of the fundamental Finsler metric, $F^2(x, y)$, but not subjected to any homogeneity conditions.

In the rest me can introduce similar concepts of almost Hermitian (Kahlerian) models of Lagrange spaces as for the Finsler spaces, by using the similar definitions and formulas as in the previous subsection, but changing $g_{ij}^{[F]} \rightarrow g_{ij}^{[L]}$.

R. Miron introduced the concept of generalized Lagrange space, GL–space (see details in [36]) and a corresponding N–connection geometry on TM when the fundamental metric function $g_{ij} = g_{ij}(x, y)$ is a general one, not obligatory defined as a second derivative from a Lagrangian as in (13.35). The corresponding almost Hermitian (Kahlerian) models of GL–spaces were investigated and applied in order to elaborate generalizations of gravity and gauge theories [36, 61].

Finally, a few remarks on definition of gravity models with generic local anisotropy on anholonomic Riemannian, Finsler or (generalized) Lagrange spaces and vector bundles. So, by choosing a d-metric (13.6) (in particular cases (13.31), or (13.34) with $g_{ij}^{[F]}$, or $g_{ij}^{[L]}$) we may compute the coefficients of, for instance, d–connection (13.8), d–torsion (13.13) and (13.14) and even to write down the explicit form of Einstein equations (13.18) which define such geometries. For instance, in a series of works [55, 56, 62] we found explicit solutions when Finsler like and another type anisotropic configurations are modelled in anisotropic kinetic theory and irreversible thermodynamics and even in Einstein or low/extra–dimension gravity as exact solutions of the vacuum (13.18) and nonvacuum (13.19) Einstein equations. From the viewpoint of the geometry of anholonomic frames is not much difference between the usual Riemannian geometry and its Finsler like generalizations. The explicit form and parametrizations of coefficients of metric, linear connections, torsions, curvatures and Einstein equations in all types of mentioned geometric models depends on the type of anholomic frame relations and compatibility metric conditions between the associated N–connection structure and linear connections we fixed. Such structures can be correspondingly picked up from a noncommutative functional model, for instance from some almost Hermitian structures over projective modules and/or generalized to some noncommutative configurations.

13.4.3 Noncommutative Finsler like gravity models

We shall briefly describe two possible approaches to the construction of gravity models with generic anisotropy following from noncommutative geometry which while agreeing for the canonical d–triples associated with vector bundles provided with N–connection structure. Because in the previous section we proved that the Finsler geometry and its extensions are effectively modelled by anholonomic structures on Riemannian manifolds (bundles) we shall only emphasize the basic ideas how from the beautiful result by Connes [8, 9] we may select an anisotropic gravity (possible alternative approaches to noncommutative gravity are examined in Refs. [7, 31, 16, 29, 30]; by introducing anholonomic frames with associated N–connections those models also can be transformed into certain anisotropic ones; we omit such considerations in the present work).

Anisotropic gravity a la Connes–Deximier–Wodzicki

The first scheme to construct gravity models in noncommutative geometry (see details in [8, 28]) may be extend for vector bundles provided with N–connection structure (i. e. to projective finite distinguished moduli) and in fact to reconstruct the full anisotorpic (for instance, Finsler) geometry from corresponding distinguishing of the Diximier trace and the Wodzicki residue.

Let us consider a smooth compact vector bundle ξ_N without boundary and of dimension $n+m$ and $D_{[t]}$ as a "symbol" for a time being operator and denote $\mathcal{A}_{[d]} = C^{\infty}(\xi_N)$. For a unitary representation $[\mathcal{A}_\pi, D_\pi]$ of the couple $\left(\mathcal{A}_{[d]}, D_{[t]}\right)$ as operators on an Hilbert space $\mathcal{H}_\pi$ provided with a real structure operator J_π, such that $[\mathcal{A}_\pi, D_\pi, \mathcal{H}, J_\pi]$ satisfy all axioms of a real spectral d–triple. Then, one holds the properties:

1. There is a unique Riemannian d–metric g_π on ξ_N such the geodesic distance in the total space of the vector bundle between every two points $u_{[1]}$ and $u_{[2]}$ is given by

$$d\left(u_{[1]}, u_{[2]}\right) = \sup_{a \in \mathcal{A}_{[d]}} \left\{ \left| a(u_{[1]}) - a(u_{[2]}) \right| : \| D_\pi, \pi(a) \|_{\mathcal{B}(\mathcal{H}_\pi)} \leq 1 \right\}.$$

2. The d–metric g_π depends only on the unitary equivalence class of the representations π. The fibers of the map $\pi \to g_\pi$ form unitary equivalence classes of representations to metrics define a finite collection of affine spaces $\mathcal{A}_\sigma$ parametrized by the spin structures σ on ξ_N. These spin structures depends on the type of d–metrics we are using in ξ_N.

3. The action functional given by the Diximier trace

$$G\left(D_{[t]}\right) = tr_{\varpi}\left(D_{[t]}^{n+m-2}\right)$$

is a positive quadratic d–form with a unique minimum π_σ for each $\mathcal{A}_\sigma$. At the minimum, the values of $G\left(D_{[t]}\right)$ coincides with the Wodzicki residue of D_σ^{n+m-2}

and is proportional to the Hilbert–Einstein action for a fixed d–connection,

$$G(D_\sigma) = Res_W\left(D_\sigma^{n+m-2}\right) = :\frac{1}{(n+m)(2\pi)^{n+m}} \int\limits_{S^*\xi_N} tr\left[\sigma_{-(n+m)}(u,u')\,\delta u \delta u'\right]$$
$$= c_{n+m}\int\limits_{\xi_N} \overleftarrow{R}\,\delta u,$$

where

$$c_{n+m} = \frac{n+m-2}{12}\frac{2^{[(n+m)/2]}}{(4\pi)^{(n+m)/2}\,\Gamma\left(\frac{n+m}{2}+1\right)},$$

$\sigma_{-(n+m)}(u,u')$ is the part of order $-(n+m)$ of the total symbol of D_σ^{n+m-2}, $\overleftarrow{R}$ is the scalar curvature (13.16) on ξ_N and tr is a normalized Clifford trace.

4. It is defined a representation of $\left(\mathcal{A}_{[d]}, D_{[t]}\right)$ for every minimum π_σ on the Hilbert space of square integrable d–spinors $\mathcal{H} = L^2(\xi_N, S_\sigma)$ where $\mathcal{A}_{[d]}$ acts by multiplicative operators and D_σ is the Dirac operator of chosen d–connection. If there is no real structure J, one has to replace *spin* by *spin*^{c} (for d–spinors investigated in Refs. [49, 54, 62]). In this case there is not a uniqueness and the minimum of the functional $G(D)$ is reached on a linear subspace of $\mathcal{A}_\sigma$ with σ a fixed *spin*^{c} structure. This subspace is parametrized by the $U(1)$ gauge potentials entering in the *spin*^{c} Dirac operator (the rest properties hold).

The properties 1-4 are proved in a similar form as in [23, 16, 28], but all computations are distinguished by the N–connection structure and a fixed type of d–connection (we omit such details). We can generate an anholonomic Riemannian, Finsler or Lagrange gravity depending on the class of d–metrics ((13.31), (13.34), (13.35), or a general one for vector bundles (13.6)) we choose.

Spectral anisotropic Gravity

Consider a canonical d–triple $\left[\mathcal{A}_{[d]} = C^\infty(\xi_N), \mathcal{H} = L^2(\xi_N), {}_{[d]}D\right]$ defined in subsection 13.4.1 for a vector bundle ξ_N, where ${}_{[d]}D$ is the Dirac d–operator (13.23) defined for a d–connection on ξ_N. We are going to compute the action

$$S_G\left({}_{[d]}D, \Lambda\right) = tr_{\mathcal{H}}\left[\chi\left(\frac{{}_{[d]}D^2}{\Lambda^2}\right)\right], \tag{13.36}$$

depending on the spectrum of ${}_{[d]}D$, were $tr_{\mathcal{H}}$ is the usual trace in the Hilbert space, Λ is the cutoff parameter and χ will be closed as a suitable cutoff function which cut off all eigenvalues of ${}_{[d]}D^2$ larger than Λ^2. By using the Lichnerowicz formula, in our case with operators for a vector bundle, and the heat kernel expansion (similarly as for the proof summarized in Ref. [28])

$$S_G\left({}_{[d]}D,\Lambda\right)=\sum_{k\geq 0} f_k a_k\left({}_{[d]}D^2/\Lambda^2\right),$$

were the coefficients f_k are computed

$$f_0=\int_0^\infty \chi(z)\, z dz,\quad f_2=\int_0^\infty \chi(z)\, dz,\qquad f_{2(k'+2)}=(-1)^{k'}\chi^{(k')}(0)\,, k'\geq 0,$$

$\chi^{(k')}$ denotes the k'th derivative on its argument, the so–called non–vanishing Seeley–de Witt coefficients $a_k\left({}_{[d]}D^2/\Lambda^2\right)$ are defined for even values of k as integrals

$$a_k\left({}_{[d]}D^2/\Lambda^2\right)=\int_{\xi_N} a_k\left(u;_{[d]} D^2/\Lambda^2\right)\sqrt{g}\delta u$$

with the first three subintegral functions given by

$$\begin{aligned}
a_0\left(u;_{[d]} D^2/\Lambda^2\right) &= \Lambda^4 (4\pi)^{-(n+m)/2}\, trI_{2^{[(n+m)/2]}},\\
a_2\left(u;_{[d]} D^2/\Lambda^2\right) &= \Lambda^2 (4\pi)^{-(n+m)/2}\left(-\overleftarrow{R}/6+E\right) trI_{2^{[(n+m)/2]}},\\
a_4\left(u;_{[d]} D^2/\Lambda^2\right) &= (4\pi)^{-(n+m)/2}\frac{1}{360}(-12D_\mu D^\mu\overleftarrow{R}+5\overleftarrow{R}^2-2R_{\mu\nu}R^{\mu\nu}\\
&\quad -\frac{7}{4}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-60RE+180E^2+60D_\mu D^\mu\overleftarrow{E})trI_{2^{[(n+m)/2]}},
\end{aligned}$$

and $\overleftarrow{E}=:_{[d]} D^2-\bigtriangledown^{[S]}=\overleftarrow{R}/4$, see (13.24). We can use for the function χ the characteristic value of the interval $[0,1]$, namely $\chi(z)=1$ for $z\leq 1$ and $\chi(z)=0$ for $z\geq 1$, possibly 'smoothed out' at $z=1$, we get

$$f_0=1/2, f_2=1, f_{2(k'+2)}=0, k'\geq 0.$$

We compute (a similar calculus is given in [28]; we only distinguish the curvature scalar, the Ricci and curvature d–tensor) the action (13.36),

$$S_G\left({}_{[d]}D,\Lambda\right)=\Lambda^4\frac{2^{(n+m)/2-1}}{(4\pi)^{(n+m)/2}}\int_{\xi_N}\sqrt{g}\delta u+\frac{\Lambda^2}{6}\frac{2^{(n+m)/2-1}}{(4\pi)^{(n+m)/2}}\int_{\xi_N}\sqrt{g}\overleftarrow{R}\delta u.$$

This action is dominated by the first term with a huge cosmological constant. But this constant can be eliminated [30] if the function $\chi(z)$ is replaced by $\widetilde{\chi}(z) = \chi(z) - \alpha\chi(\beta z)$ with any two numbers α and β such that $\alpha = \beta^2$ and $\beta \geq 0, \beta \neq 1$. The final form of the action becomes

$$S_G\left({}_{[d]}D, \Lambda\right) = \left(1 - \frac{\alpha}{\beta^2}\right) f_2 \frac{\Lambda^2}{6} \frac{2^{(n+m)/2-1}}{(4\pi)^{(n+m)/2}} \int_{\xi_N} \sqrt{g}\overleftarrow{R}\delta u + O\left((\Lambda^2)^0\right). \quad (13.37)$$

From the action (13.37) we can generate different models of anholonomic Riemannian, Finsler or Lagrange gravity depending on the class of d–metrics ((13.31), (13.34), (13.35), or a general one for vector bundles (13.6)) we parametrize for computations. But this construction has a problem connected with "spectral invariance versus diffeomorphysm invariance on manifolds or vector bundles. Let us denote by $spec\left(\xi_N, {}_{[d]}D\right)$ the spectrum of the Dirac d–operator with each eigenvalue repeated according to its multiplicity. Two vector bundles ξ_N and ξ_N' are called isospectral if $spec\left(\xi_N, {}_{[d]}D\right) = spec\left(\xi_N', {}_{[d]}D\right)$, which defines an invariant transform of the action (13.36). There are manifolds (and in consequence vector bundles) which are isospectral without being isometric (the converse is obviously true). This is known as a fact that one cannot 'hear the shape of a drum [19] because the spectral invariance is stronger that usual diffeomorphysm invariance.

In spirit of spectral gravity, the eigenvalues of the Dirac operator are diffeomorphic invariant functions of the geometry and therefore true observable in general relativity. As we have shown in this section they can be taken as a set of variables for invariant descriptions to the anholonomic dynamics of the gravitational field with (or not) local anisotropy in different approaches of anholonomic Riemannian gravity and Finsler like generalizations. But in another turn there exist isospectral vector bundles which fail to be isometric. Thus, the eigenvalues of the Dirac operator cannot be used to distinguish among such vector bundles (or manifolds). A rigorous analysis is also connected with the type of d–metric and d–connection structures we prescribe for our geometric and physical models.

Finally, we remark that there are different models of gravity with noncommutative setting (see, for instance, Refs. [7, 31, 16, 29, 30, 11, 23]). By introducing nonlinear connections in a respective commutative or noncommutative variant we can transform such theories to be anholonomic, i. e. locally anisotropic, in different approaches with (pseudo) Riemannian geometry and Finsler/Lagrange or Hamilton extensions.

13.5 Noncommutative Finsler–Gauge Theories

The bulk of noncommutative models extending both locally isotropic and anisotropic gravity theories are confrunted with the problem of definition of noncommutative variants of pseudo–Eucliedean and pseudo–Riemannian metrics. The problem is connected with the fact of generation of noncommutative metric structures via the Moyal results in complex and noncommutative metrics. In order to avoid this difficulty we elaborated a model of noncommutative gauge gravity (containing as particular case the Einstein general relativity theory) starting from a variant of gauge gravity being equivalent to the Einstein gravity and emphasizing in a such approach the tetradic (frame) and connection structures, but not the metric configuration (see Refs. [58]). The metric for such theories is induced from the frame structure which can be holonomic or anholonomic. The aim of this section is to generalize our results on noncommutative gauge gravity as to include also possible anisotropies in different variants of gauge realization of anholonomic Einstein and Finsler like generalizations formally developed in Refs. [61, 54, 62].

A still presented drawback of noncommutative geometry and physics is that there is not yet formulated a generally accepted approach to interactions of elementary particles coupled to gravity. There are improved Connes–Lott and Chamsedine–Connes models of nocommutative geometry [9, 11] which yielded action functionals typing together the gravitational and Yang–Mills interactions and gauge bosons the Higgs sector (see also the approaches [16] and, for an outline of recent results, [34]).

In the last years much work has been made in noncommutative extensions of physical theories (see reviews and original results in Refs. [14, 45]). It was not possible to formulate gauge theories on noncommutative spaces [10, 43, 21, 32] with Lie algebra valued infinitesimal transformations and with Lie algebra valued gauge fields. In order to avoid the problem it was suggested to use enveloping algebras of the Lie algebras for setting this type of gauge theories and showed that in spite of the fact that such enveloping algebras are infinite–dimensional one can restrict them in a way that it would be a dependence on the Lie algebra valued parameters, the Lie algebra valued gauge fields and their spacetime derivatives only.

We follow the method of restricted enveloping algebras [21] and construct gauge gravitational theories by stating corresponding structures with semisimple or nonsemisimple Lie algebras and their extensions. We consider power series of generators for the affine and non linear realized de Sitter gauge groups and compute the coefficient functions of all the higher powers of the generators of the gauge group which are functions of the coefficients of the first power. Such constructions are based on the Seiberg–Witten map [43] and on the formalism of $*$–product formulation of the algebra [65] when for functional objects, being functions of commuting variables, there are associated some algebraic noncommutative properties encoded in the ∗–product.

The concept of gauge theory on noncommutative spaces was introduced in a geometric manner [32] by defining the covariant coordinates without speaking about derivatives and this formalism was developed for quantum planes [64]. In this section we shall prove the existence for noncommutative spaces of gauge models of gravity which agrees with usual gauge gravity theories being equivalent, or extending, the general relativity theory (see works [39, 47] for locally isotropic and anisotropic spaces and corresponding reformulations and generalizations respectively for anholonomic frames [60] and locally anisotropic (super) spaces [61, 51, 52, 54]) in the limit of commuting spaces.

13.5.1 Star–products and enveloping algebras in noncommutative spaces

For a noncommutative space the coordinates $\hat{u}^i$, $(i = 1, ..., N)$ satisfy some noncommutative relations

$$[\hat{u}^i, \hat{u}^j] = \left\{ \begin{array}{cl} i\theta^{ij}, & \theta^{ij} \in \mathbb{C},\ \text{canonical structure;} \\ if_k^{ij}\hat{u}^k, & f_k^{ij} \in \mathbb{C},\ \text{Lie structure;} \\ iC_{kl}^{ij}\hat{u}^k\hat{u}^l, & C_{kl}^{ij} \in \mathbb{C},\ \text{quantum plane} \end{array} \right. \tag{13.38}$$

where $\mathbb{C}$ denotes the complex number field.

The noncommutative space is modelled as the associative algebra of $\mathbb{C}$; this algebra is freely generated by the coordinates modulo ideal $\mathcal{R}$ generated by the relations (one accepts formal power series) $\mathcal{A}_u = \mathbb{C}[[\hat{u}^1, ..., \hat{u}^N]]/\mathcal{R}$. One restricts attention [22] to algebras having the (so–called, Poincare–Birkhoff–Witt) property that any element of $\mathcal{A}_u$ is defined by its coefficient function and vice versa,

$$\widehat{f} = \sum_{L=0}^{\infty} f_{i_1,...,i_L} : \hat{u}^{i_1} \ldots \hat{u}^{i_L} : \qquad \text{when } \widehat{f} \sim \{f_i\},$$

where $: \hat{u}^{i_1} \ldots \hat{u}^{i_L} :$ denotes that the basis elements satisfy some prescribed order (for instance, the normal order $i_1 \leq i_2 \leq \ldots \leq i_L$, or, another example, are totally symmetric). The algebraic properties are all encoded in the so–called diamond ($\diamond$) product which is defined by

$$\widehat{f}\widehat{g} = \widehat{h} \ \sim \ \{f_i\} \diamond \{g_i\} = \{h_i\}.$$

In the mentioned approach to every function $f(u) = f(u^1, \ldots, u^N)$ of commuting variables $u^1, \ldots, u^N$ one associates an element of algebra $\widehat{f}$ when the commuting variables are substituted by anticommuting ones,

$$f(u) = \sum f_{i_1 \ldots i_L} u^1 \cdots u^N \rightarrow \widehat{f} = \sum_{L=0}^{\infty} f_{i_1,\ldots,i_L} : \hat{u}^{i_1} \ldots \hat{u}^{i_L} :$$

when the $\diamond$–product leads to a bilinear $*$–product of functions (see details in [32])

$$\{f_i\} \diamond \{g_i\} = \{h_i\} \sim (f * g)(u) = h(u).$$

The $*$–product is defined respectively for the cases (13.38)

$$f * g = \begin{cases} \exp[\frac{i}{2}\frac{\partial}{\partial u^i}\theta^{ij}\frac{\partial}{\partial u'^j}] f(u) g(u')|_{u' \rightarrow u}, \\ \exp[\frac{i}{2} u^k g_k(i\frac{\partial}{\partial u'}, i\frac{\partial}{\partial u''})] f(u') g(u'')|_{u'' \rightarrow u}^{u' \rightarrow u}, \\ q^{\frac{1}{2}(-u'\frac{\partial}{\partial u'}v\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}v'\frac{\partial}{\partial v'})} f(u,v) g(u',v')|_{v' \rightarrow v}^{u' \rightarrow u}, \end{cases}$$

where there are considered values of type

$$\begin{aligned} e^{ik_n \widehat{u}^n} \quad e^{ip_{nl}\widehat{u}^n} &= e^{i\{k_n + p_n + \frac{1}{2} g_n(k,p)\}\widehat{u}^n}, \\ g_n(k,p) &= -k_i p_j f^{ij}_{\ \ n} + \frac{1}{6} k_i p_j (p_k - k_k) f^{ij}_{\ \ m} f^{mk}_{\ \ \ n} + \ldots, \\ e^A e^B &= e^{A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]])} + \ldots \end{aligned} \quad (13.39)$$

and for the coordinates on quantum (Manin) planes one holds the relation $uv = qvu$.

A non–abelian gauge theory on a noncommutative space is given by two algebraic structures, the algebra $\mathcal{A}_u$ and a non–abelian Lie algebra $\mathcal{A}_I$ of the gauge group with generators $I^1, \ldots, I^S$ and the relations

$$[I^{\underline{s}}, I^{\underline{p}}] = i f^{\underline{sp}}_{\ \ \underline{t}} I^{\underline{t}}. \quad (13.40)$$

In this case both algebras are treated on the same footing and one denotes the generating elements of the big algebra by $\widehat{u}^i$,

$$\widehat{z}^{\underline{i}} = \{\widehat{u}^1, \ldots, \widehat{u}^N, I^1, \ldots, I^S\}, \mathcal{A}_z = \mathbb{C}[[\widehat{u}^1, \ldots, \widehat{u}^{N+S}]]/\mathcal{R}$$

and the $*$–product formalism is to be applied for the whole algebra $\mathcal{A}_z$ when there are considered functions of the commuting variables u^i $(i,j,k,\ldots = 1,\ldots,N)$ and $I^{\underline{s}}$ $(s,p,\ldots = 1,\ldots,S)$.

For instance, in the case of a canonical structure for the space variables u^i we have

$$(F * G)(u) = e^{\frac{i}{2}\left(\theta^{ij}\frac{\partial}{\partial u'^i}\frac{\partial}{\partial u''^j} + t^s g_s\left(i\frac{\partial}{\partial t'}, i\frac{\partial}{\partial t''}\right)\right) \times F(u',t') G(u'',t'')|_{t' \rightarrow t, t'' \rightarrow t}^{u' \rightarrow u, u'' \rightarrow u}}. \quad (13.41)$$

This formalism was developed in [22] for general Lie algebras. In this section we consider those cases when in the commuting limit one obtains the gauge gravity and general relativity theories or some theirs anisotropic generalizations..

13.5.2 Enveloping algebras for gauge gravity connections

In order to construct gauge gravity theories on noncommutative space we define the gauge fields as elements the algebra $\mathcal{A}_u$ that form representation of the generator I–algebra for the de Sitter gauge group. For commutative spaces it is known [39, 47, 61] that an equivalent re–expression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic spacetimes, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames (to this class one belong the gauge Poincare theories; on metric–affine and gauge gravity models see original results and reviews in [48]). By using auxiliary biliniear forms, instead of degenerated Killing form for the affine structural group, on fiber spaces, the gauge models of gravity can be formulated to be variational. After projection on the base spacetime, for the so–called Cartan connection form, the Yang–Mills equations transforms equivalently into the Einstein equations for general relativity [39]. A variational gauge gravitational theory can be also formulated by using a minimal extension of the affine structural group $\mathcal{A}f_{3+1}(\mathbb{R})$ to the de Sitter gauge group $S_{10} = SO(4+1)$ acting on $\mathbb{R}^{4+1}$ space.

Nonlinear gauge theories of de Sitter group in commutative spaces

Let us consider the de Sitter space Σ^4 as a hypersurface given by the equations $\eta_{AB}u^A u^B = -l^2$ in the four dimensional flat space enabled with diagonal metric η_{AB}, $\eta_{AA} = \pm 1$ (in this section $A, B, C, ... = 1, 2, ..., 5$), where $\{u^A\}$ are global Cartesian coordinates in $\mathbb{R}^5$; $l > 0$ is the curvature of de Sitter space. The de Sitter group $S_{(\eta)} = SO_{(\eta)}(5)$ is defined as the isometry group of Σ^5–space with 6 generators of Lie algebra $so_{(\eta)}(5)$ satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}. \quad (13.42)$$

Decomposing indices $A, B, ...$ as $A = (\underline{\alpha}, 5), B = (\underline{\beta}, 5), ...$, the metric η_{AB} as $\eta_{AB} = \left(\eta_{\underline{\alpha}\underline{\beta}}, \eta_{55}\right)$, and operators M_{AB} as $M_{\underline{\alpha}\underline{\beta}} = \mathcal{F}_{\underline{\alpha}\underline{\beta}}$ and $P_{\underline{\alpha}} = l^{-1}M_{5\underline{\alpha}}$, we can write (13.42) as

$$\begin{aligned} \left[\mathcal{F}_{\underline{\alpha}\underline{\beta}}, \mathcal{F}_{\underline{\gamma}\underline{\delta}}\right] &= \eta_{\underline{\alpha}\underline{\gamma}}\mathcal{F}_{\underline{\beta}\underline{\delta}} - \eta_{\underline{\beta}\underline{\gamma}}\mathcal{F}_{\underline{\alpha}\underline{\delta}} + \eta_{\underline{\beta}\underline{\delta}}\mathcal{F}_{\underline{\alpha}\underline{\gamma}} - \eta_{\underline{\alpha}\underline{\delta}}\mathcal{F}_{\underline{\beta}\underline{\gamma}}, \\ \left[P_{\underline{\alpha}}, P_{\underline{\beta}}\right] &= -l^{-2}\mathcal{F}_{\underline{\alpha}\underline{\beta}}, \left[P_{\underline{\alpha}}, \mathcal{F}_{\underline{\beta}\underline{\gamma}}\right] = \eta_{\underline{\alpha}\underline{\beta}}P_{\underline{\gamma}} - \eta_{\underline{\alpha}\underline{\gamma}}P_{\underline{\beta}}, \end{aligned} \quad (13.43)$$

where we decompose the Lie algebra $so_{(\eta)}(5)$ into a direct sum, $so_{(\eta)}(5) = so_{(\eta)}(4) \oplus V_4$, where V_4 is the vector space stretched on vectors $P_{\underline{\alpha}}$. We remark that $\Sigma^4 = S_{(\eta)}/L_{(\eta)}$, where $L_{(\eta)} = SO_{(\eta)}(4)$. For $\eta_{AB} = diag(1,-1,-1,-1)$ and $S_{10} = SO(1,4), L_6 = SO(1,3)$ is the group of Lorentz rotations.

In this paper the generators $I^{\underline{a}}$ and structure constants $f^{sp}_{\underline{t}}$ from (13.40) are parametrized just to obtain de Sitter generators and commutations (13.43).

The action of the group $S_{(\eta)}$ can be realized by using 4×4 matrices with a parametrization distinguishing the subgroup $L_{(\eta)}$:

$$B = bB_L, \tag{13.44}$$

where

$$B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix},$$

$L \in L_{(\eta)}$ is the de Sitter bust matrix transforming the vector $(0,0,...,\rho) \in \mathbb{R}^5$ into the arbitrary point $(V^1, V^2, ..., V^5) \in \Sigma^5_\rho \subset \mathcal{R}^5$ with curvature ρ, $(V_A V^A = -\rho^2, V^A = t^A \rho)$. Matrix b can be expressed as

$$b = \begin{pmatrix} \delta^{\underline{\alpha}}{}_{\underline{\beta}} + \frac{t^{\underline{\alpha}} t_{\underline{\beta}}}{(1+t^5)} & t^{\underline{\alpha}} \\ t_{\underline{\beta}} & t^5 \end{pmatrix}.$$

The de Sitter gauge field is associated with a $so_{(\eta)}(5)$–valued connection 1–form

$$\widetilde{\Omega} = \begin{pmatrix} \omega^{\underline{\alpha}}{}_{\underline{\beta}} & \widetilde{\theta}^{\underline{\alpha}} \\ \widetilde{\theta}_{\underline{\beta}} & 0 \end{pmatrix}, \tag{13.45}$$

where $\omega^{\underline{\alpha}}{}_{\underline{\beta}} \in so(4)_{(\eta)}, \widetilde{\theta}^{\underline{\alpha}} \in \mathcal{R}^4, \widetilde{\theta}_{\underline{\beta}} \in \eta_{\underline{\beta}\underline{\alpha}} \widetilde{\theta}^{\underline{\alpha}}$.

Because $S_{(\eta)}$–transforms mix the components of the matrix $\omega^{\underline{\alpha}}{}_{\underline{\beta}}$ and $\widetilde{\theta}^{\underline{\alpha}}$ fields in (13.45) (the introduced parametrization is invariant on action on $SO_{(\eta)}(4)$ group we cannot identify $\omega^{\underline{\alpha}}{}_{\underline{\beta}}$ and $\widetilde{\theta}^{\underline{\alpha}}$, respectively, with the connection $\Gamma^\alpha{}_{\beta\gamma}$ and the fundamental form χ^α in a metric–affine spacetime). To avoid this difficulty we consider [47] a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$, namely, we introduce into consideration the nonlinear gauge field

$$\Gamma = b^{-1}\widetilde{\Omega} b + b^{-1} db = \begin{pmatrix} \Gamma^{\underline{\alpha}}{}_{\underline{\beta}} & \theta^{\underline{\alpha}} \\ \theta_{\underline{\beta}} & 0 \end{pmatrix}, \tag{13.46}$$

where

$$\begin{aligned}
\Gamma^{\underline{\alpha}}{}_{\underline{\beta}} &= \omega^{\underline{\alpha}}{}_{\underline{\beta}} - \left(t^{\underline{\alpha}} D t_{\underline{\beta}} - t_{\underline{\beta}} D t^{\underline{\alpha}}\right) / \left(1+t^5\right), \\
\theta^{\underline{\alpha}} &= t^5 \widetilde{\theta}^{\underline{\alpha}} + D t^{\underline{\alpha}} - t^{\underline{\alpha}} \left(dt^5 + \widetilde{\theta}_{\underline{\gamma}} t^{\underline{\gamma}}\right) / \left(1+t^5\right), \\
D t^{\underline{\alpha}} &= dt^{\underline{\alpha}} + \omega^{\underline{\alpha}}{}_{\underline{\beta}} t^{\underline{\beta}}.
\end{aligned}$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms

$$\Gamma' = L'\Gamma \left(L'\right)^{-1} + L' d \left(L'\right)^{-1}, \theta' = L\theta,$$

where the nonlinear matrix–valued function

$$L' = L'\left(t^{\alpha}, b, B_T\right)$$

is defined from $B_b = b' B_{L'}$ (see the parametrization (13.44)). The de Sitter algebra with generators (13.43) and nonlinear gauge transforms of type (13.46) is denoted $\mathcal{A}_I^{(dS)}$.

De Sitter nonlinear gauge gravity and Einstein and Finsler like gravity

Let us consider the de Sitter nonlinear gauge gravitational connection (13.46) rewritten in the form

$$\Gamma = \begin{pmatrix} \Gamma^{\underline{\alpha}}{}_{\underline{\beta}} & l_0^{-1}\chi^{\underline{\alpha}} \\ l_0^{-1}\chi_{\underline{\beta}} & 0 \end{pmatrix} \tag{13.47}$$

where

$$\begin{aligned}
\Gamma^{\underline{\alpha}}{}_{\underline{\beta}} &= \Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu} \delta u^{\mu}, \\
\Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu} &= \chi^{\underline{\alpha}}{}_{\alpha} \chi^{\beta}{}_{\underline{\beta}} \Gamma^{\alpha}{}_{\beta\gamma} + \chi^{\underline{\alpha}}{}_{\alpha} \delta_{\mu} \chi^{\alpha}{}_{\underline{\beta}}, \chi^{\underline{\alpha}} = \chi^{\underline{\alpha}}{}_{\mu} \delta u^{\mu},
\end{aligned}$$

and

$$G_{\alpha\beta} = \chi^{\underline{\alpha}}{}_{\alpha} \chi^{\underline{\beta}}{}_{\beta} \eta_{\underline{\alpha}\underline{\beta}},$$

$\eta_{\underline{\alpha}\underline{\beta}} = (1, -1, ..., -1)$ and l_0 is a dimensional constant. As $\Gamma^{\alpha}{}_{\beta\gamma}$ we take the Christoffel symbols for the Einstein theory, or every type of d–connection (13.8) for an anisotropic spacetime. Correspondingly, $G_{\alpha\beta}$ can be the pseudo–Rieamannian metric in general relativity or any d–metric (13.6), which can be particularized for the anholonomic Einstein gravity (13.31) or for a Finsler type gravity (13.34).

The curvature of (13.47),

$$\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \bigwedge \Gamma,$$

can be written

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}} + l_0^{-1}\pi^{\underline{\alpha}}_{\underline{\beta}} & l_0^{-1}T^{\underline{\alpha}} \\ l_0^{-1}T^{\underline{\beta}} & 0 \end{pmatrix}, \tag{13.48}$$

where

$$\pi^{\underline{\alpha}}_{\underline{\beta}} = \chi^{\underline{\alpha}} \bigwedge \chi_{\underline{\beta}}, \mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}} = \frac{1}{2}\mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu}\delta u^{\mu} \bigwedge \delta u^{\nu},$$

and

$$\mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu} = \chi_{\underline{\beta}}{}^{\beta}\chi_{\alpha}{}^{\underline{\alpha}}R^{\alpha}{}_{\beta_{\mu\nu}}.$$

with the $R^{\alpha}{}_{\beta\mu\nu}$ being the metric–affine (for Einstein-Cartan–Weyl spaces), or the (pseudo) Riemannian curvature, or for anisotropic spaces the d–curvature (13.14). The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational theory with the Lagrangian

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined

$$L_{(G)} = \frac{1}{4\pi}Tr\left(\mathcal{R}^{(\Gamma)} \bigwedge *_G\mathcal{R}^{(\Gamma)}\right) = \mathcal{L}_{(G)}\left|G\right|^{1/2}\delta^{n+m}u,$$

with

$$\mathcal{L}_{(G)} = \frac{1}{2l^2}T^{\underline{\alpha}}{}_{\mu\nu}T_{\underline{\alpha}}{}^{\mu\nu} + \frac{1}{8\lambda}\mathcal{R}^{\underline{\alpha}}{}_{\underline{\beta}\mu\nu}\mathcal{R}^{\underline{\beta}}{}_{\underline{\alpha}}{}^{\mu\nu} - \frac{1}{l^2}\left(\overleftarrow{R}\left(\Gamma\right) - 2\lambda_1\right),$$

$\delta^4 u$ being the volume element, $T^{\underline{\alpha}}{}_{\mu\nu} = \chi^{\underline{\alpha}}{}_{\alpha}T^{\alpha}{}_{\mu\nu}$ (the gravitational constant l^2 satisfies the relations $l^2 = 2l_0^2\lambda, \lambda_1 = -3/l_0$), Tr denotes the trace on $\underline{\alpha}, \underline{\beta}$ indices, and the matter field Lagrangian is defined

$$L_{(m)} = -1\frac{1}{2}Tr\left(\Gamma \bigwedge *_G\mathcal{I}\right) = \mathcal{L}_{(m)}\left|G\right|^{1/2}\delta^{n+m}u,$$

where

$$\mathcal{L}_{(m)} = \frac{1}{2}\Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu}S^{\underline{\beta}}{}_{\alpha}{}^{\mu} - t^{\mu}{}_{\underline{\alpha}}l^{\underline{\alpha}}{}_{\mu}.$$

The matter field source $\mathcal{J}$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on Γ and is parametrized as

$$\mathcal{J} = \begin{pmatrix} S^{\underline{\alpha}}{}_{\underline{\beta}} & -l_0 t^{\underline{\alpha}} \\ -l_0 t_{\underline{\beta}} & 0 \end{pmatrix}$$

with $t^{\underline{\alpha}} = t^{\underline{\alpha}}{}_{\mu}\delta u^{\mu}$ and $S^{\underline{\alpha}}{}_{\underline{\beta}} = S^{\underline{\alpha}}{}_{\underline{\beta}\mu}\delta u^{\mu}$ being respectively the canonical tensors of energy–momentum and spin density.

Varying the action

$$S=\int \delta^4 u\left(\mathcal{L}_{(G)}+\mathcal{L}_{(m)}\right)$$

on the Γ–variables (1a), we obtain the gauge–gravitational field equations, in general, with local anisotropy,

$$d\left(*\mathcal{R}^{(\Gamma)}\right)+\Gamma\bigwedge\left(*\mathcal{R}^{(\Gamma)}\right)-\left(*\mathcal{R}^{(\Gamma)}\right)\bigwedge\Gamma=-\lambda\left(*\mathcal{J}\right), \qquad (13.49)$$

were the Hodge operator $*$ is used.

Specifying the variations on $\Gamma^{\underline{\alpha}}{}_{\underline{\beta}}$ and χ–variables, we rewrite (13.49)

$$\begin{aligned}
\widehat{\mathcal{D}}\left(*\mathcal{R}^{(\Gamma)}\right) & +\frac{2\lambda}{l^2}(\widehat{\mathcal{D}}\left(*\pi\right)+\chi\bigwedge\left(*T^T\right)-\left(*T\right)\bigwedge\chi^T)=-\lambda\left(*S\right), \\
\widehat{\mathcal{D}}\left(*T\right) & - \left(*\mathcal{R}^{(\Gamma)}\right)\bigwedge\chi-\frac{2\lambda}{l^2}\left(*\pi\right)\bigwedge\chi=\frac{l^2}{2}\left(*t+\frac{1}{\lambda}*\tau\right),
\end{aligned}$$

where

$$\begin{aligned}
T^t &= \{T_{\underline{\alpha}}=\eta_{\underline{\alpha}\underline{\beta}}T^{\underline{\beta}},\ T^{\underline{\beta}}=\frac{1}{2}T^{\underline{\beta}}{}_{\mu\nu}\delta u^\mu\bigwedge\delta u^\nu\}, \\
\chi^T &= \{\chi_{\underline{\alpha}}=\eta_{\underline{\alpha}\underline{\beta}}\chi^{\underline{\beta}},\ \chi^{\underline{\beta}}=\chi^{\widehat{\beta}}{}_\mu\delta u^\mu\}, \qquad \widehat{\mathcal{D}}=d+\widehat{\Gamma},
\end{aligned}$$

($\widehat{\Gamma}$ acts as $\Gamma^{\underline{\alpha}}{}_{\underline{\beta}\mu}$ on indices $\underline{\gamma},\underline{\delta},...$ and as $\Gamma^{\alpha}{}_{\beta\mu}$ on indices $\gamma,\delta,...$). The value τ defines the energy–momentum tensor of the gauge gravitational field $\widehat{\Gamma}$:

$$\tau_{\mu\nu}\left(\widehat{\Gamma}\right)=\frac{1}{2}Tr\left(\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu}-\frac{1}{4}\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}G_{\mu\nu}\right).$$

Equations (13.49) make up the complete system of variational field equations for nonlinear de Sitter gauge anisotropic gravity.

We note that we can obtain a nonvariational Poincare gauge gravitational theory if we consider the contraction of the gauge potential (13.47) to a potential with values in the Poincare Lie algebra

$$\Gamma=\left(\begin{array}{cc}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0\end{array}\right)\rightarrow\Gamma=\left(\begin{array}{cc}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ 0 & 0\end{array}\right). \qquad (13.50)$$

A similar gauge potential was considered in the formalism of linear and affine frame bundles on curved spacetimes by Popov and Daikhin [39]. They treated (13.50) as the

Cartan connection form for affine gauge like gravity and by using 'pure' geometric methods proved that the Yang–Mills equations of their theory are equivalent, after projection on the base, to the Einstein equations. The main conclusion for a such approach to Einstein gravity is that this theory admits an equivalent formulation as a gauge model but with a nonsemisimple structural gauge group. In order to have a variational theory on the total bundle space it is necessary to introduce an auxiliary bilinear form on the typical fiber, instead of degenerated Killing form; the coefficients of auxiliary form disappear after projection on the base. An alternative variant is to consider a gauge gravitational theory when the gauge group was minimally extended to the de Sitter one with nondegenerated Killing form. The nonlinear realizations have to be introduced if we consider in a common fashion both the frame (tetradic) and connection components included as the coefficients of the potential (13.47). Finally, we note that the models of de Sitter gauge gravity were generalized for Finsler and Lagrange theories in Refs. [61, 54].

Enveloping nonlinear de Sitter algebra valued connection

Let now us consider a noncommutative space. In this case the gauge fields are elements of the algebra $\widehat{\psi} \in \mathcal{A}_I^{(dS)}$ that form the nonlinear representation of the de Sitter algebra $so_{(\eta)}(5)$ when the whole algebra is denoted $\mathcal{A}_z^{(dS)}$. Under a nonlinear de Sitter transformation the elements transform as follows

$$\delta\widehat{\psi} = i\widehat{\gamma}\widehat{\psi}, \widehat{\psi} \in \mathcal{A}_u, \widehat{\gamma} \in \mathcal{A}_z^{(dS)}.$$

So, the action of the generators (13.43) on $\widehat{\psi}$ is defined as this element is supposed to form a nonlinear representation of $\mathcal{A}_I^{(dS)}$ and, in consequence, $\delta\widehat{\psi} \in \mathcal{A}_u$ despite $\widehat{\gamma} \in \mathcal{A}_z^{(dS)}$. It should be emphasized that independent of a representation the object $\widehat{\gamma}$ takes values in enveloping de Sitter algebra and not in a Lie algebra as would be for commuting spaces. The same holds for the connections that we introduce [32], in order to define covariant coordinates,

$$\widehat{U}^{\nu} = \widehat{u}^{v} + \widehat{\Gamma}^{\nu}, \widehat{\Gamma}^{\nu} \in \mathcal{A}_z^{(dS)}.$$

The values $\widehat{U}^{\nu}\widehat{\psi}$ transform covariantly,

$$\delta\widehat{U}^{\nu}\widehat{\psi} = i\widehat{\gamma}\widehat{U}^{\nu}\widehat{\psi},$$

if and only if the connection $\widehat{\Gamma}^{\nu}$ satisfies the transformation law of the enveloping nonlinear realized de Sitter algebra,

$$\delta\widehat{\Gamma}^{\nu}\widehat{\psi} = -i[\widehat{u}^{v}, \widehat{\gamma}] + i[\widehat{\gamma}, \widehat{\Gamma}^{\nu}],$$

where $\delta\widehat{\Gamma}^{\nu} \in \mathcal{A}_{z}^{(dS)}$. The enveloping algebra–valued connection has infinitely many component fields. Nevertheless, it was shown that all the component fields can be induced from a Lie algebra–valued connection by a Seiberg–Witten map ([43, 21] and [5] for $SO(n)$ and $Sp(n)$). In this subsection we show that similar constructions could be proposed for nonlinear realizations of de Sitter algebra when the transformation of the connection is considered

$$\delta\widehat{\Gamma}^{\nu} = -i[u^{\nu},^{*}\ \widehat{\gamma}] + i[\widehat{\gamma},^{*}\ \widehat{\Gamma}^{\nu}].$$

For simplicity, we treat in more detail the canonical case with the star product (13.41). The first term in the variation $\delta\widehat{\Gamma}^{\nu}$ gives

$$-i[u^{\nu},^{*}\ \widehat{\gamma}] = \theta^{\nu\mu}\frac{\partial}{\partial u^{\mu}}\gamma.$$

Assuming that the variation of $\widehat{\Gamma}^{\nu} = \theta^{\nu\mu}Q_{\mu}$ starts with a linear term in θ, we have

$$\delta\widehat{\Gamma}^{\nu} = \theta^{\nu\mu}\delta Q_{\mu},\ \delta Q_{\mu} = \frac{\partial}{\partial u^{\mu}}\gamma + i[\widehat{\gamma},^{*}\ Q_{\mu}].$$

We follow the method of calculation from the papers [32, 22] and expand the star product (13.41) in θ but not in g_a and find to first order in θ,

$$\gamma = \gamma_{\underline{a}}^{1} I^{\underline{a}} + \gamma_{\underline{ab}}^{1} I^{\underline{a}} I^{\underline{b}} + ..., Q_{\mu} = q_{\mu,\underline{a}}^{1} I^{\underline{a}} + q_{\mu,\underline{ab}}^{2} I^{\underline{a}} I^{\underline{b}} + ... \qquad (13.51)$$

where $\gamma_{\underline{a}}^{1}$ and $q_{\mu,\underline{a}}^{1}$ are of order zero in θ and $\gamma_{\underline{ab}}^{1}$ and $q_{\mu,\underline{ab}}^{2}$ are of second order in θ. The expansion in $I^{\underline{b}}$ leads to an expansion in g_a of the $*$–product because the higher order $I^{\underline{b}}$–derivatives vanish. For de Sitter case as $I^{\underline{b}}$ we take the generators (13.43), see commutators (13.40), with the corresponding de Sitter structure constants $f_{\ \underline{d}}^{\underline{bc}} \simeq f_{\ \underline{\beta}}^{\underline{\alpha\beta}}$ (in our further identifications with spacetime objects like frames and connections we shall use Greek indices).

The result of calculation of variations of (13.51), by using g_a to the order given in (13.39), is

$$\begin{aligned}
\delta q_{\mu,\underline{a}}^{1} &= \frac{\partial \gamma_{\underline{a}}^{1}}{\partial u^{\mu}} - f_{\ \underline{a}}^{\underline{bc}} \gamma_{\underline{b}}^{1} q_{\mu,\underline{c}}^{1},\\
\delta Q_{\tau} &= \theta^{\mu\nu}\partial_{\mu}\gamma_{\underline{a}}^{1}\partial_{\nu} q_{\tau,\underline{b}}^{1} I^{\underline{a}} I^{\underline{b}} + ...,\\
\delta q_{\mu,\underline{ab}}^{2} &= \partial_{\mu}\gamma_{\underline{ab}}^{2} - \theta^{\nu\tau}\partial_{\nu}\gamma_{\underline{a}}^{1}\partial_{\tau} q_{\mu,\underline{b}}^{1} - 2 f_{\ \underline{a}}^{\underline{bc}}\{\gamma_{\underline{b}}^{1} q_{\mu,\underline{cd}}^{2} + \gamma_{\underline{bd}}^{2} q_{\mu,\underline{c}}^{1}\}.
\end{aligned}$$

Next, we introduce the objects ε, taking the values in de Sitter Lie algebra and W_{μ}, being enveloping de Sitter algebra valued,

$$\varepsilon = \gamma_{\underline{a}}^{1} I^{\underline{a}} \text{ and } W_{\mu} = q_{\mu,\underline{ab}}^{2} I^{\underline{a}} I^{\underline{b}},$$

with the variation δW_μ satisfying the equation [32, 22]

$$\delta W_\mu = \partial_\mu(\gamma^2_{\underline{ab}} I^{\underline{a}} I^{\underline{b}}) - \frac{1}{2}\theta^{\tau\lambda}\{\partial_\tau \varepsilon, \partial_\lambda q_\mu\} + i[\varepsilon, W_\mu] + i[(\gamma^2_{\underline{ab}} I^{\underline{a}} I^{\underline{b}}), q_\nu].$$

This equation has the solution (found in [32, 43])

$$\gamma^2_{\underline{ab}} = \frac{1}{2}\theta^{\nu\mu}(\partial_\nu \gamma^1_{\underline{a}}) q^1_{\mu,\underline{b}},\ q^2_{\mu,\underline{ab}} = -\frac{1}{2}\theta^{\nu\tau} q^1_{\nu,\underline{a}}\left(\partial_\tau q^1_{\mu,\underline{b}} + R^1_{\tau\mu,\underline{b}}\right)$$

where

$$R^1_{\tau\mu,\underline{b}} = \partial_\tau q^1_{\mu,\underline{b}} - \partial_\mu q^1_{\tau,\underline{b}} + f^{\underline{ec}}_{\ \ \underline{d}} q^1_{\tau,\underline{e}} q^1_{\mu,\underline{e}}$$

could be identified with the coefficients $\mathcal{R}^{\underline{\alpha}}_{\ \ \underline{\beta}\mu\nu}$ of de Sitter nonlinear gauge gravity curvature (see formula (13.48)) if in the commutative limit $q^1_{\mu,\underline{b}} \simeq \begin{pmatrix} \Gamma^{\underline{\alpha}}_{\ \ \underline{\beta}} & l_0^{-1}\chi^{\underline{\alpha}} \\ l_0^{-1}\chi_{\underline{\beta}} & 0 \end{pmatrix}$ (see (13.47)).

The below presented procedure can be generalized to all the higher powers of θ.

13.5.3 Noncommutative Gravity Covariant Gauge Dynamics

First order corrections to gravitational curvature

The constructions from the previous section are summarized by the conclusion that the de Sitter algebra valued object $\varepsilon = \gamma^1_{\underline{a}}(u) I^{\underline{a}}$ determines all the terms in the enveloping algebra

$$\gamma = \gamma^1_{\underline{a}} I^{\underline{a}} + \frac{1}{4}\theta^{\nu\mu}\partial_\nu \gamma^1_{\underline{a}}\ q^1_{\mu,\underline{b}}\left(I^{\underline{a}} I^{\underline{b}} + I^{\underline{b}} I^{\underline{a}}\right) + ...$$

and the gauge transformations are defined by $\gamma^1_{\underline{a}}(u)$ and $q^1_{\mu,\underline{b}}(u)$, when

$$\delta_{\gamma^1}\psi = i\gamma\left(\gamma^1, q^1_\mu\right) * \psi.$$

For de Sitter enveloping algebras one holds the general formula for compositions of two transformations

$$\delta_\gamma \delta_\varsigma - \delta_\varsigma \delta_\gamma = \delta_{i(\varsigma * \gamma - \gamma * \varsigma)}$$

which is also true for the restricted transformations defined by γ^1,

$$\delta_{\gamma^1}\delta_{\varsigma^1} - \delta_{\varsigma^1}\delta_{\gamma^1} = \delta_{i(\varsigma^1 * \gamma^1 - \gamma^1 * \varsigma^1)}.$$

Applying the formula (13.41) we calculate

$$\begin{aligned}[\gamma,^{*}\zeta] &= i\gamma_{\underline{a}}^{1}\zeta_{\underline{b}}^{1}f_{\ \underline{c}}^{\underline{ab}}I^{\underline{c}}+\frac{i}{2}\theta^{\nu\mu}\{\partial_{v}\left(\gamma_{\underline{a}}^{1}\zeta_{\underline{b}}^{1}f_{\ \underline{c}}^{\underline{ab}}\right)q_{\mu,\underline{c}} \\ &+\left(\gamma_{\underline{a}}^{1}\partial_{v}\zeta_{\underline{b}}^{1}-\zeta_{\underline{a}}^{1}\partial_{v}\gamma_{\underline{b}}^{1}\right)q_{\mu,\underline{b}}f_{\ \underline{c}}^{\underline{ab}}+2\partial_{v}\gamma_{\underline{a}}^{1}\partial_{\mu}\zeta_{\underline{b}}^{1}\}I^{\underline{d}}I^{\underline{c}}.\end{aligned}$$

Such commutators could be used for definition of tensors [32]

$$\widehat{S}^{\mu\nu}=[\widehat{U}^{\mu},\widehat{U}^{\nu}]-i\widehat{\theta}^{\mu\nu}, \tag{13.52}$$

where $\widehat{\theta}^{\mu\nu}$ is respectively stated for the canonical, Lie and quantum plane structures. Under the general enveloping algebra one holds the transform

$$\delta\widehat{S}^{\mu\nu}=i[\widehat{\gamma},\widehat{S}^{\mu\nu}].$$

For instance, the canonical case is characterized by

$$\begin{aligned}S^{\mu\nu} &= i\theta^{\mu\tau}\partial_{\tau}\Gamma^{\nu}-i\theta^{\nu\tau}\partial_{\tau}\Gamma^{\mu}+\Gamma^{\mu}*\Gamma^{\nu}-\Gamma^{\nu}*\Gamma^{\mu} \\ &= \theta^{\mu\tau}\theta^{\nu\lambda}\{\partial_{\tau}Q_{\lambda}-\partial_{\lambda}Q_{\tau}+Q_{\tau}*Q_{\lambda}-Q_{\lambda}*Q_{\tau}\}.\end{aligned}$$

By introducing the gravitational gauge strength (curvature)

$$R_{\tau\lambda}=\partial_{\tau}Q_{\lambda}-\partial_{\lambda}Q_{\tau}+Q_{\tau}*Q_{\lambda}-Q_{\lambda}*Q_{\tau}, \tag{13.53}$$

which could be treated as a noncommutative extension of de Sitter nonlinear gauge gravitational curvature (2a), we calculate

$$R_{\tau\lambda,\underline{a}}=R_{\tau\lambda,\underline{a}}^{1}+\theta^{\mu\nu}\{R_{\tau\mu,\underline{a}}^{1}R_{\lambda\nu,\underline{b}}^{1}-\frac{1}{2}q_{\mu,\underline{a}}^{1}\left[(D_{\nu}R_{\tau\lambda,\underline{b}}^{1})+\partial_{\nu}R_{\tau\lambda,\underline{b}}^{1}\right]\}I^{\underline{b}},$$

where the gauge gravitation covariant derivative is introduced,

$$(D_{\nu}R_{\tau\lambda,\underline{b}}^{1})=\partial_{\nu}R_{\tau\lambda,\underline{b}}^{1}+q_{\nu,\underline{c}}R_{\tau\lambda,\underline{d}}^{1}f_{\ \underline{b}}^{\underline{cd}}.$$

Following the gauge transformation laws for γ and q^{1} we find

$$\delta_{\gamma^{1}}R_{\tau\lambda}^{1}=i\left[\gamma,^{*}R_{\tau\lambda}^{1}\right]$$

with the restricted form of γ.

Such formulas were proved in references [43] for usual gauge (nongravitational) fields. Here we reconsidered them for gravitational gauge fields.

Gauge covariant gravitational dynamics

Following the nonlinear realization of de Sitter algebra and the $*$–formalism we can formulate a dynamics of noncommutative spaces. Derivatives can be introduced in such a way that one does not obtain new relations for the coordinates. In this case a Leibniz rule can be defined that

$$\widehat{\partial}_\mu \widehat{u}^\nu = \delta^\nu_\mu + d^{\nu\tau}_{\mu\sigma}\ \widehat{u}^\sigma\ \widehat{\partial}_\tau$$

where the coefficients $d^{\nu\tau}_{\mu\sigma} = \delta^\nu_\sigma \delta^\tau_\mu$ are chosen to have not new relations when $\widehat{\partial}_\mu$ acts again to the right hand side. In consequence one holds the $*$–derivative formulas

$$\partial_\tau * f = \frac{\partial}{\partial u^\tau} f + f * \partial_\tau,\ [\partial_l, {}^*(f * g)] = ([\partial_l, {}^* f]) * g + f * ([\partial_l, {}^* g])$$

and the Stokes theorem

$$\int [\partial_l, f] = \int d^N u [\partial_l, {}^* f] = \int d^N u \frac{\partial}{\partial u^l} f = 0,$$

where, for the canonical structure, the integral is defined,

$$\int \widehat{f} = \int d^N u f\left(u^1, ..., u^N\right).$$

An action can be introduced by using such integrals. For instance, for a tensor of type (13.52), when

$$\delta \widehat{L} = i\left[\widehat{\gamma}, \widehat{L}\right],$$

we can define a gauge invariant action

$$W = \int d^N u\ Tr \widehat{L},\ \delta W = 0,$$

were the trace has to be taken for the group generators.

For the nonlinear de Sitter gauge gravity a proper action is

$$L = \frac{1}{4} R_{\tau\lambda} R^{\tau\lambda},$$

where $R_{\tau\lambda}$ is defined by (13.53) (in the commutative limit we shall obtain the connection (13.47)). In this case the dynamic of noncommutative space is entirely formulated in the framework of quantum field theory of gauge fields. In general, we are dealing with anisotropic gauge gravitational interactions. The method works for matter fields as well to restrictions to the general relativity theory.

13.6 Outlook and Conclusions

In this work we have extended the A. Connes' approach to noncommutative geometry by introducing into consideration anholonomic frames and locally anisotropic structures. We defined nonlinear connections for finite projective module spaces (noncommutative generalization of vector bundles) and related this geometry with the E. Cartan's moving frame method.

We have explicitly shown that the functional analytic approach and noncommutative C^*–algebras may be transformed into arena of modelling geometries and physical theories with generic local anisotropy, for instance, the anholonomic Riemannian gravity and generalized Finsler like geometries. The formalism of spectral triples elaborated for vector bundles provided with nonlinear connection structure allows a functional and algebraic generation of new types of anholonomic/ anisotropic interactions.

A novel future in our work is that by applying anholonomic transforms associated to some nonlinear connections we may generate various type of spinor, gauge and gravity models, subjected to some anholonomic constraints and/or with generic anisotropic interactions, which can be included in noncommutative field theory.

We can address a number of questions which were put or solved partially in this paper and may have further generalizations:

One of the question is how to combine the noncommutative geometry contained in string theory with locally anisotropic configurations arising in the low energy limits. It is known that the nonsymmetric background field results in effective noncommutative coordinates. In other turn, a (super) frame set consisting from mixed subsets of holonomic and anholonomic vectors may result in an anholonomic geometry with associated nonlinear connection structure. A further work is to investigate the conditions when from a string theory one appears explicit variants of commutative–anisotropic, commutative–isotropic, noncommutative–isotropic and, finally, nocommutative–anisotropic geometries.

A second question is connected with the problem of definition of noncommutative (pseudo) Riemannian metric structures which is connected with nonsymmetric and/or complex metrics. We have elaborated variants of noncommutative gauge gravity with noncommutative representations of the affine and de Sitter algebras which contains in the commutative limit an Yang–Mills theory (with nonsemisimple structure group) being equivalent to the Einstein theory. The gauge connection in such theories is constructed from the frame and linear connection coefficients. Metrics, in this case, arise as some effective configurations which avoid problems with their noncommutative definition. The approach can be generated as to include anholonomic frames and, in consequence, to define anisotropic variants of commutative and noncommutative gauge gravity with the

Einstein type or Finsler generalizations.

Another interesting open question is to establish a relation between quantum groups and geometries with anisotropic models of gravity and field theories. Different variants of quantum generalizations for anholonomic frames with associated nonlinear connection structures are possible.

Finally, we give some historical remarks. An approach to Finsler and spinor like spaces of infinite dimensions (in Banach and/or Hilbert spaces) and to nonsymmetric locally anisotropic metrics was proposed by some authors belonging to the Romanian school on Finsler geometry and generalizations (see, Refs. [36, 17, 1, 37]). It could not be finalized before elaboration of the A. Connes' models of noncommutative geometry and gravity and before definition of Clifford and spinor distinguished structures [49, 62], formulation of supersymmetric variants of Finsler spaces [51] and establishing theirs relation to string theory [52, 53, 54]. This paper concludes a noncommutative interference and a development of the mentioned results.

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Chapter 14

(Non) Commutative Finsler Geometry from String/ M–Theory

Abstract [1]

We synthesize and extend the previous ideas about appearance of both noncommutative and Finsler geometry in string theory with nonvanishing B–field and/or anholonomic (super) frame structures [42, 43, 48, 50]. There are investigated the limits to the Einstein gravity and string generalizations containing locally anisotropic structures modelled by moving frames. The relation of anholonomic frames and nonlinear connection geometry to M–theory and possible noncommutative versions of locally anisotropic supergravity and D–brane physics is discussed. We construct and analyze new classes of exact solutions with noncommutative local anisotropy describing anholonomically deformed black holes (black ellipsoids) in string gravity, embedded Finsler–string two dimensional structures, solitonically moving black holes in extra dimensions and wormholes with noncommutativity and anisotropy induced from string theory.

14.1 Introduction

The idea that string/M–theory results in a noncommutative limit of field theory and spacetime geometry is widely investigated by many authors both from mathematical and physical perspectives [57, 38, 9] (see, for instance, the reviews [13]). It is now generally accepted that noncommutative geometry and quantum groups [8, 16, 19] play a fundamental role in further developments of high energy particle physics and gravity theory.

First of all we would like to give an exposition of some basic facts about the geometry of anholonomic frames (vielbeins) and associated nonlinear connection (N–connection) structures which emphasize surprisingly some new results: We will consider N–connections in commutative geometry and we will show that locally anisotropic spacetimes (anholonomic Riemannian, Finsler like and their generalizations) can be obtained from the string/M–theory. We shall discuss the related low energy limits to Einstein and gauge gravity. Our second goal is to extend A. Connes' differential noncommutative geometry as to include geometries with anholonomic frames and N–connections and to prove that such 'noncommutative anisotropies' also arise very naturally in the framework of strings and extra dimension gravity. We will show that the anholonomic frame method is very useful in investigating of new symmetries and nonperturbative states and for constructing new exact solutions in string gravity with anholonomic and/or noncommutative variables. We remember here that some variables are considered anholonomic (equivalently, nonholonomic) if they are subjected to some constraints (equivalently, anholonomy conditions).

Almost all of the physics paper dealing with the notion of (super) frame in string theory do not use the well developed apparatus of E. Cartan's 'moving frame' method [6] which gave an unified approach to the Riemannian and Finsler geometry, to bundle spaces and spinors, to the geometric theory of systems of partial equations and to Einstein (and the so–called Einstein–Cartan–Weyl) gravity. It is considered that very "sophisticate" geometries like the Finsler and Cartan ones, and theirs generalizations, are less related to real physical theories. In particular, the bulk of frame constructions in string and gravity theories are given by coefficients defined with respect to coordinate frames or in abstract form with respect to some general vielbein bases. It is completely disregarded the fact that via anholonomic frames on (pseudo) Riemannian manifolds and on (co) tangent and (co) vector bundles we can model different geometries and interactions with local anisotropy even in the framework of generally accepted classical and quantum theories. For instance, there were constructed a number of exact solutions in general relativity and its lower/higher dimension extensions with generic local anisotropy, which under certain conditions define Finsler like geometries [45, 44, 46, 53, 54, 52]. It was demonstrated that anholonomic geometric constructions are inevitable in the theory of anisotropic stochastic, kinetic and thermodynamic processes in curved spacetimes [44] and proved that Finsler like (super) geometries are contained alternatively in modern string theory [42, 43, 47].

We emphasize that we have not proposed any "exotic" locally anisotropic modifications of string theory and general relativity but demonstrated that such anisotropic structures, Finsler like or another type ones, may appear alternatively to the Riemannian geometry, or even can be modelled in the framework of a such geometry, in the low energy limit of the string theory, if we are dealing with frame (vielbein) constructions. One of our main goals is to give an accessible exposition of some important notions and results of N–connection geometry and to show how they can be applied to concrete problems in string theory, noncommutative geometry and gravity. We hope to convince a reader–physicist, who knows that 'the B–field' in string theory may result in noncommutative geometry, that the anholonomic (super) frames could define nonlinear connections and Finsler like commutative and/ or noncommutative geometries in string theory and (super) gravity and this holds true in certain limits to general relativity.

We address the present work to physicists who would like to learn about some new geometrical methods and to apply them to mathematical problems arising at the forefront of modern theoretical physics. We do not assume that such readers have very deep knowledge in differential geometry and nonlinear connection formalism (for convenience, we give an Appendix outlining the basic results on the geometry of commutative spaces provided with N–connection structures [29, 47, 55]) but consider that they are familiar with some more geometric approaches to gravity [14, 30] and string theories [15].

Finally, we note that the first attempts to relate Riemann–Finsler spaces (and spaces with anisotropy of another type) to noncommutative geometry and physics were made in Refs. [48] where some models of noncommutative gauge gravity (in the commutative limit being equivalent to the Einstein gravity, or to different generalizations to de Sitter, affine, or Poincare gauge gravity with, or not, nonlinear realization of the gauge groups) were analyzed. Further developments of noncommutative geometries with anholonomic/ anisotropic structures and their applications in modern particle physics lead to a rigorous study of the geometry of noncommutative anholonomic frames with associated N–connection structure [50] (that work should be considered as the non–string partner of the present paper).

The paper has the following structure:

In Section 2 we consider stings in general manifolds and bundles provided with anholonomic frames and associated nonlinear connection structures and analyze the low energy string anholonomic field equations. The conditions when anholonomic Einstein or Finsler like gravity models can be derived from string theory are stated.

Section 3 outlines the geometry of locally anisotropic supergravity models contained in superstring theory. Superstring effective actions and anisotropic toroidal compactifications are analyzed. The corresponding anholonomic field equations with distinguishing of anholonomic Riemannian–Finesler (super) gravities are derived.

In Section 4 we formulate the theory of noncommutative anisotropic scalar and gauge fields interactions and examine their anholonomic symmetries.

In Section 5 we emphasize how noncommutative anisotropic structures are embedded in string/M–theory and discuss their connection to anholonomic geometry.

Section 6 is devoted to locally anisotropic gravity models generated on noncommutative D–branes.

In Section 7 we construct four classes of exact solutions with noncommutative and locally anisotropic structures. We analyze solutions describing locally anisotropic black holes in string theory, define a class of Finsler–string structures containing two dimensional Finsler metrics, consider moving solitonic string–black hole configurations and give an examples of anholonomic noncommutative wormhole solution induced from string theory.

Finally, in Section 8, some additional comments and questions for further developments are presented. The Appendix outlines the necessary results from the geometry of nonlinear connections and generalized Finsler–Riemannian spaces.

14.2 String Theory and Commutative Riemann–Finsler Gravity

The string gravitational effects are computed from corresponding low–energy effective actions and moving equations of stings in curved spacetimes (on string theory, see monographs [15]). The basic idea is to consider propagation of a string not only of a flat 26–dimensional space with Minkowski metric $\eta_{\mu\nu}$ but also its propagation in a background more general manifold with metric tensor $g_{\mu\nu}$ from where one derived string–theoretic corrections to general relativity when the vacuum Einstein equations $R_{\mu\nu} = 0$ correspond to vanishing of the one–loop beta function in corresponding sigma model. More rigorous theories were formulated by adding an antisymmetric tensor field $B_{\mu\nu}$, the dilaton field Φ and possible other background fields, by introducing supersymmetry, higher loop corrections and another generalizations. It should be noted here that propagation of (super) strings may be considered on arbitrary (super) manifolds. For instance, in Refs. [42, 43, 47], the corresponding background (super) spaces were treated as (super) bundles provided with nonlinear connection (N–connection) structure and, in result, there were constructed some types of generalized (super) Finsler corrections to the usual Einstein and to locally anisotropic (Finsler type, or theirs generalizations) gravity theories.

The aim of this section is to demonstrate that anisotropic corrections and extensions may be computed both in Einstein and string gravity [derived for string propagation in usual (pseudo) Riemannian backgrounds] if the approach is developed following a more rigorous geometrical formalism with off–diagonal metrics and anholonomic frames. We note that (super) frames [vielbeins] were used in general form, for example, in order to introduce spinors and supersymmetry in sting theory but the anholonomic transforms with mixed holonomic–anholonomic variables, resulting in diagonalization of off–diagonal (super) metrics and effective anisotropic structures, were not investigated in the previous literature on string/M–theory.

14.2.1 Strings in general manifolds and bundles

Generalized nonlinear sigma models (some basics)

The first quantized string theory was constructed in flat Minkowski spacetime of dimension $k \geq 4$. Then the analysis was extended to more general manifolds with (pseudo) Riemannian metric $\underline{g}_{\mu\nu}$, antisymmetric $B_{\mu\nu}$ and dilaton field Φ and possible other background fields, including tachyonic matter associated to a field U in a tachyon state. The starting point in investigating the string dynamics in the background of these fields is the generalized nonlinear sigma model action for the maps $u : \Sigma \rightarrow M$ from a two dimensional surface Σ to a spacetime manifold M of dimension k,

$$S = S_{\underline{g},B} + S_{\Phi} + S_U, \tag{14.1}$$

with

$$\begin{aligned} S_{\underline{g},B}[u,g] &= \frac{1}{8\pi l^2} \int\limits_{\Sigma} d\mu_g \partial_A u^\mu \partial_B u^\nu \left[g_{[2]}^{AB} \underline{g}_{\mu\nu}(u) + \varepsilon^{AB} B_{\mu\nu}(u) \right], \\ S_{\Phi}[u,g] &= \frac{1}{2\pi} \int\limits_{\Sigma} d\mu_g R_g \Phi(u), \; S_U[u,g] = \frac{1}{4\pi} \int\limits_{\Sigma} d\mu_g U(u), \end{aligned}$$

where $B_{\mu\nu}$ is the pullback of a two–form $B = B_{\mu\nu} du^\mu \wedge du^\nu$ under the map u, written out in local coordinates u^μ; $g_{[2]AB}$ is the metric on the two dimensional surface Σ (indices $A, B = 0, 1$); $\varepsilon^{AB} = \overline{\varepsilon}^{AB}/\sqrt{\det |g_{AB}|}, \overline{\varepsilon}^{01} = -\overline{\varepsilon}^{10} = 1$; the integration measure $d\mu_g$ is defined by the coefficients of the metric g_{AB}, R_g is the Gauss curvature of Σ. The constants in the action are related as

$$k = \frac{1}{4\pi\alpha'} = \frac{1}{8\pi\ell^2}, \alpha' = 2\ell^2$$

where α' is the Regge slope parameter α' and $\ell \sim 10^{-33} cm$ is the Planck length scale. The metric coefficients $\underline{g}_{\mu\nu}(u)$ are defined by the quadratic metric element given with respect to the coordinate co–basis $d^\mu = du^\mu$ (being dual to the local coordinate basis $\partial_\mu = \partial/\partial_\mu$),

$$ds^2 = \underline{g}_{\mu\nu}(u) du^\mu du^\nu. \tag{14.2}$$

The parameter ℓ is a very small length–scale, compared to experimental scales $L_{\exp} \sim 10^{-17}$ accessible at present. This defines the so–called low energy, or α'–expansion. A perturbation theory may be carried out as usual by letting $u = u_0 + \ell u_{[1]}$ for some reference configuration u_0 and considering expansions of the fields $\underline{g}, B$ and Φ, for instance,

$$\underline{g}_{\mu\nu}(u) = \underline{g}_{\mu\nu}(u_0) + \ell\partial_\alpha \underline{g}_{\mu\nu}(u_0)u^\alpha_{[1]} + \frac{1}{2}\ell^2\partial_\alpha\partial_\beta \underline{g}_{\mu\nu}(u_0)u^\alpha_{[1]}u^\beta_{[1]} + ... \qquad (14.3)$$

This reveals that the quantum field theory defined by the action (14.1) is with an infinite number of couplings; the independent couplings of this theory correspond to the successive derivatives of the fields $\underline{g}, B$ and Φ at the expansion point u_0. Following an analysis of the general structure of the Weyl dependence of Green functions in the quantum field theory, standard regularizations schemes (see, for instance, Refs. [15]) and conditions of vanishing of Weyl anomalies, computing the β–functions, one derive the low energy string effective actions and field equations.

Anholonomic frame transforms of background metrics

Extending the general relativity principle to the string theory, we should consider that the string dynamics in the background of fields $\underline{g}, B$ and Φ and possible another ones, defined in the low energy limit by certain effective actions and moving equations, does not depend on changing of systems of coordinates, $u^{\alpha'} \to u^{\alpha'}(u^\alpha)$, for a fixed local basis (equivalently, system, frame, or vielbein) of reference, $e_\alpha(u)$, on spacetime M (for which, locally, $u = u^\alpha e_\alpha = u^{\alpha'} e_{\alpha'}$, $e_{\alpha'} = \partial u^\alpha/\partial u^{\alpha'} e_\alpha$, usually one considers local coordinate bases when $e_\alpha = \partial/\partial u^\alpha$) as well the string dynamics should not depend on changing of frames like $e_{\underline{\alpha}} \to e_{\underline{\alpha}}^{\ \alpha}(u)\, e_\alpha$, parametrized by non–degenerated matrices $e_{\underline{\alpha}}^{\ \alpha}(u)$.

Let us remember some details connected with the geometry of moving frames in (pseudo) Riemannian spaces [6] and discuss its applications in string theory, where the orthonormal frames were introduced with the aim to eliminate non–trivial dependencies on the metric $\underline{g}_{\mu\nu}$ and on the background field u_0^μ which appears in elaboration of the covariant background expansion method for the nonlinear sigma models [15, 21]. Such orthonormal frames, in the framework of a $SO(1, k-1)$ like gauge theory are stated by the conditions

$$\begin{aligned} \underline{g}_{\mu\nu}(u) &= e_\mu^{\ \underline{\mu}}(u)\, e_\nu^{\ \underline{\nu}}(u)\, \eta_{\underline{\mu}\underline{\nu}}, \\ e_\mu^{\ \underline{\mu}} e^\nu_{\ \underline{\mu}} &= \delta^\nu_\mu, \quad e_\mu^{\ \underline{\mu}} e^\mu_{\ \underline{\nu}} = \delta^{\underline{\mu}}_{\underline{\nu}}, \end{aligned} \qquad (14.4)$$

where $\eta_{\underline{\mu}\underline{\nu}} = diag\,(-1,+1,...,+1)$ is the flat Minkowski metric and $\delta_{\mu}^{\nu}, \delta_{\underline{\nu}}^{\underline{\mu}}$ are Kronecker's delta symbols. One considers the covariant derivative D_μ with respect to an affine connection Γ and a corresponding spin connection $\omega_{\mu\ \underline{\beta}}^{\ \underline{\alpha}}$ for which the frame $e_{\mu}^{\ \underline{\mu}}$ is covariantly constant,

$$D_\mu e_{\nu}^{\ \underline{\alpha}} \equiv \partial_\mu e_{\nu}^{\ \underline{\alpha}} - \Gamma^{\alpha}_{\ \mu\nu} e_{\alpha}^{\ \underline{\alpha}} + \omega_{\mu\ \underline{\beta}}^{\ \underline{\alpha}} e_{\alpha}^{\ \underline{\beta}} = 0.$$

One also uses the covariant derivative

$$\mathcal{D}_\mu e_{\nu}^{\ \underline{\alpha}} = D_\mu e_{\nu}^{\ \underline{\alpha}} + \frac{1}{2} H_{\mu\nu}^{\ \ \ \rho} e_{\rho}^{\ \underline{\alpha}} \qquad (14.5)$$

including the torsion tensor $H_{\mu\nu\rho}$ which is the field strength of the field $B_{\nu\rho}$, given by $H = dB$, or, in component notation,

$$H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\nu B_{\rho\mu}. \qquad (14.6)$$

All tensors may be written with respect to an orthonormal frame basis, for instance,

$$H_{\mu\nu\rho} = e_{\mu}^{\ \underline{\mu}} e_{\nu}^{\ \underline{\nu}} e_{\rho}^{\ \underline{\rho}} H_{\underline{\mu}\underline{\nu}\underline{\rho}}$$

and

$$\mathcal{R}_{\mu\nu\rho\sigma} = e_{\mu}^{\ \underline{\mu}} e_{\nu}^{\ \underline{\nu}} e_{\rho}^{\ \underline{\rho}} e_{\sigma}^{\ \underline{\sigma}} R_{\underline{\mu}\underline{\nu}\underline{\rho}\underline{\sigma}},$$

where the curvature $\mathcal{R}_{\mu\nu\rho\sigma}$ of the connection $\mathcal{D}_\mu$, defined as

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu)\xi^\rho \doteqdot [\mathcal{D}_\mu \mathcal{D}_\nu]\xi^\rho = H^{\sigma}_{\ \mu\nu} \mathcal{D}_\sigma \xi^\rho + \mathcal{R}^{\rho}_{\ \sigma\mu\nu}\xi^\sigma,$$

can be expressed in terms of the Riemannian tensor $R_{\mu\nu\rho\sigma}$ and the torsion tensor $H^{\sigma}_{\ \mu\nu}$,

$$\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2} D_\rho H_{\sigma\mu\nu} - \frac{1}{2} D_\sigma H_{\rho\mu\nu} + \frac{1}{4} H_{\rho\mu\alpha} H_{\sigma\nu}^{\ \ \ \alpha} - \frac{1}{4} H_{\sigma\mu\alpha} H_{\rho\nu}^{\ \ \ \alpha}.$$

Let us consider a generic off–diagonal metric, a non-degenerated matrix of dimension $k \times k$ with the coefficients $\underline{g}_{\mu\nu}(u)$ defined with respect to a local coordinate frame like in (14.2). This metric can transformed into a block $(n \times n) \oplus (m \times m)$ form, for $k = n+m$,

$$\underline{g}_{\mu\nu}(u) \rightarrow \{g_{ij}(u), h_{ab}\,(u)\}$$

if we perform a frame map with the vielbeins

$$\begin{array}{rcl} e_{\mu}^{\ \underline{\mu}}(u) & = & \left(\begin{array}{cc} e_{i}^{\ \underline{i}}(x^j,y^a) & N_{i}^{a}(x^j,y^a) e_{a}^{\ \underline{a}}(x^j,y^a) \\ 0 & e_{a}^{\ \underline{a}}(x^j,y^a) \end{array} \right) \\ e^{\mu}_{\ \underline{\nu}}(u) & = & \left(\begin{array}{cc} e^{i}_{\ \underline{i}}(x^j,y^a) & -N_{k}^{a}(x^j,y^a) e^{k}_{\ \underline{i}}(x^j,y^a) \\ 0 & e^{a}_{\ \underline{a}}(x^j,y^a) \end{array} \right) \end{array} \qquad (14.7)$$

which conventionally splits the spacetime into two subspaces: the first subspace is parametrized by coordinates x^i provided with indices of type $i, j, k, ...$ running values from 1 to n and the second subspace is parametrized by coordinates y^a provided with indices of type $a, b, c, ...$ running values from 1 to m. This splitting is induced by the coefficients $N_i^a(x^j, y^a)$. For simplicity, we shall write the local coordinates as $u^\alpha = (x^i, y^a)$, or $u = (x, y)$.

The coordinate bases $\partial_\alpha = (\partial_i, \partial_a)$ and theirs duals $d^\alpha = du^\alpha = (d^i = dx^i, d^a = dy^a)$ are transformed under maps (14.7) as

$$\partial_\alpha \rightarrow e_{\underline{\alpha}} = e^\alpha_{\ \underline{\alpha}}(u)\partial_\alpha, d^\alpha \rightarrow e^{\underline{\alpha}} = e_\alpha^{\ \underline{\alpha}}(u)d^\alpha,$$

or, in 'N–distinguished' form,

$$\begin{aligned} e_{\underline{i}} &= e^i_{\ \underline{i}}\partial_i - N_k^a e^k_{\ \underline{i}}\partial_a, e_{\underline{a}} = e^a_{\ \underline{a}}\partial_a, & (14.8)\\ e^{\underline{i}} &= e_i^{\ \underline{i}}d^i,\ e^{\underline{a}} = N_i^a e_a^{\ \underline{a}}d^i + e_a^{\ \underline{a}}d^a. & (14.9)\end{aligned}$$

The quadratic line element (14.2) may be written equivalently in the form

$$ds^2 = g_{\underline{ij}}(x, y)e^{\underline{i}}e^{\underline{j}} + h_{\underline{ab}}(x, y)e^{\underline{a}}e^{\underline{b}} \quad (14.10)$$

with the metric $\underline{g}_{\mu\nu}(u)$ parametrized in the form

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & h_{ab}N_i^a \\ h_{ab}N_j^b & h_{ab} \end{bmatrix}. \quad (14.11)$$

If we choose $e_i^{\ \underline{i}}(x^j, y^a) = \delta_i^{\ \underline{i}}$ and $e_a^{\ \underline{a}}(x^j, y^a) = \delta_a^{\ \underline{a}}$, we may not distinguish the 'underlined' and 'non–underlined' indices. The operators (14.8) and (14.9) transform respectively into the operators of 'N–elongated' partial derivatives and differentials

$$\begin{aligned} e_i &= \delta_i = \partial_i - N_i^a\partial_a, e_a = \partial_a, & (14.12)\\ e^i &= d^i,\ e^a = \delta^a = d^a + N_i^a d^i & \end{aligned}$$

(which means that the anholonomic frames (14.8) and (14.9) generated by vielbein transforms (14.7) are, in general, anholonomic; see the respective formulas (13.2), (13.3) and (13.4) in the Appendix) and the quadratic line element (14.10) trasforms in a d–metric element (see (13.6) in the Appendix).

The physical treatment of the vielbein transforms (14.7) and associated N–coefficients depends on the types of constraints (equivalently, anholonomies) we impose on the string dynamics and/or on the considered curved background. There were considered different possibilities:

- Ansatz of type (14.11) were used in Kaluza–Klein gravity [35], as well in order to describe toroidal Kaluza–Klein reductions in string theory (see, for instance, [25])). The coefficients N_i^a, usually written as A_i^a, are considered as the potentials of some, in general, non–Abelian gauge fields, which in such theories are generated by a corresponding compactification. In this case, the coordinates x^i can be used for the four dimensional spacetime and the coordinates y^a are for extra dimensions.

- Parametrizations of type (14.11) were considered in order to elaborate an unified approach on vector/tangent bundles to Finsler geometry and its generalizations [29, 28, 4, 40, 47, 41, 42, 43]. The coefficients N_i^a were supposed to define a nonlinear connection (N–connection) structure in corresponding (super) bundles and the metric coefficients $g_{ij}(u)$ and $g_{ab}(u)$ were taken for a corresponding Finsler metric, or its generalizations (see formulas (14.127), (13.33), (13.34), (13.35) and related discussions in Appendix). The coordinates x^i were defined on base manifolds and the coordinates y^a were used for fibers of bundles.

- In a series of papers [45, 44, 49, 46, 53, 54, 52] the concept of N–connection was introduced for (pseudo) Riemannian spaces provided with off–diagonal metrics and/or anholonomic frames. In a such approach the coefficients N_i^a are associated to an anholonomic frame structure describing a gravitational and matter fields dynamics with mixed holonomic and anholonomic variables. The coordinates x^i are defined with respect to the subset of holonomic frame vectors, but y^a are given with respect to the subset of anholonomic, N–ellongated, frame vectors. It was proven that by using vielbein transforms of type (14.7) the off–diagonal metrics could be diagonalized and, for a very large class of ansatz of type (14.11), with the coefficients depending on 2,3 or 4 coordinate variables, it was shown that the corresponding vacuum and non–vacuum Einstein equations may be integrated in general form. This allowed an explicit construction of new classes of exact solutions parametrized by off–diagonal metrics with some anholonomically deformed symmetries. Two new and very surprising conclusions were those that the Finsler like (and another type) anisotropies may be modelled even in the framework of the general relativity theory and its higher/lower dimension modifications, as some exact solutions of the Einstein equations, and that the anholonomic frame method is very efficient for constructing such solutions.

There is an important property of the off–diagonal metrics $\underline{g}_{\mu\nu}$ (14.11) which does not depend on the type of space (a pseudo–Riemannian manifold, or a vector/tangent bundle) this metric is given. With respect to the coordinate frames it is defined a unique torsionless and metric compatible linear connection derived as the usual Christoffel symbols (or the Levi Civita connection). If anholonomic frames are introduced into consideration, we can define an infinite number of metric connections constructed from the coefficients of off–diagonal metrics and induced by the anholonomy coefficients (see formulas (3.10) and (13.10) and the related discussion from Appendix); this property is also mentioned in the monograph [30] (pages 216, 223, 261) for anholonomic frames but without any particularities related to associated N–connection structures. In this case there is an infinite number of metric compatible linear connections, constructed from metric and vielbein coefficients, all of them having non–trivial torsions and transforming into the usual Christoffel symbols for $N_i^a \to 0$ and $m \to 0$. For off–diagonal metrics considered, and even diagonalized, with respect to anholonomic frames and associated N–connections, we can not select a linear connection being both torsionless and metric. The problem of establishing of a physical linear connection structure constructed from metric/frame coefficients is to be solved together with that of fixing of a system of reference on a curved spacetime which is not a pure dynamical task but depends on the type of prescribed constraints, symmetries and boundary conditions are imposed on interacting fields and/or string dynamics.

In our further consideration we shall suppose that both a metric $\underline{g}_{\mu\nu}$ (equivalently, a set $\{g_{ij}, g_{ab}, N_i^a\}$) and metric linear connection $\Gamma^{\alpha}{}_{\beta\gamma}$, i.e. satisfying the conditions $D_\alpha g_{\alpha\beta} = 0$, exist in the background spacetime. Such spaces will be called locally anisotropic (equivalently, anolonomic) because the anholonomic frames structure imposes locally a kind of anisotropy with respective constraints on string and effective string dynamics. For such configurations the torsion, induced as an anholonomic frame effect, vanishes only with respect coordinate frames. Here we note that in the string theory there are also another type of torsion contributions to linear connections like $H^{\sigma}{}_{\mu\nu}$, see formula (14.5).

Anholonomic background field quantization method

We revise the perturbation theory around general field configurations for background spaces provided with anholonomic frame structures (14.8) and (14.9), $\delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_a)$ and $\delta^\alpha = (d^i, \delta^a = d^a + N_i^a d^i)$, with associated N–connections, N_i^a, and $\{g_{ij}, h_{ab}\}$ (14.10) adapted to such structures (distinguished metrics, or d–metrics, see formula (13.6)). The linear connection in such locally anisotropic backgrounds is considered to be compatible both to the metric and N–connection structure (for simplicity, being a d–connection or an anholonomic variant of Levi Civita connection, both with nonvanishing torsion, see formulas (13.8), (3.10), (13.10), and (13.13), and related discussions in the Appendix). The general rule for the tensorial calculus on a space provided with

N–connection structure is to split indices $\alpha, \beta, ...$ into "horozontal", $i, j, ...$, and "vertical", $a, b, ...$, subsets and to apply for every type of indices the corresponding operators of N–adapted partial and covariant derivations.

The anisotropic sigma model is to be formulated by anholonomic transforms of the metric, $\underline{g}_{\mu\nu} \rightarrow \{g_{ij}, h_{ab}\}$, partial derivatives and differentials, $\partial_\alpha \rightarrow \delta_\alpha$ and $d^\alpha \rightarrow \delta^a$, volume elements, $d\mu_g \rightarrow \delta\mu_g$ in the action (14.1)

$$S = S_{g_N,B} + S_\Phi + S_U, \tag{14.13}$$

with

$$\begin{aligned} S_{g_N,B}[u,g] &= \frac{1}{8\pi l^2} \int\limits_{\Sigma} \delta\mu_g \{ g^{AB} \left[\partial_A x^i \partial_B x^j g_{ij}(x,y) + \partial_A x^a \partial_B x^b h_{ab}(x,y) \right] \\ &\quad + \varepsilon^{AB} \partial_A u^\mu \partial_B u^\nu B_{\mu\nu}(u) \}, \\ S_\Phi[u,g] &= \frac{1}{2\pi} \int\limits_{\Sigma} \delta\mu_g R_g \Phi(u),\ S_U[u,g] = \frac{1}{4\pi} \int\limits_{\Sigma} \delta\mu_g U(u), \end{aligned}$$

where the coefficients $B_{\mu\nu}$ are computed for a two–form $B = B_{\mu\nu}\delta u^\mu \wedge \delta u^\nu$.

The perturbation theory has to be developed by changing the usual partial derivatives into N–elongated ones, for instance, the decomposition (14.3) is to be written

$$\underline{g}_{\mu\nu}(u) = \underline{g}_{\mu\nu}(u_0) + \ell \delta_\alpha \underline{g}_{\mu\nu}(u_0) u^\alpha_{[1]} + \ell^2 \delta_\alpha \delta_\beta \underline{g}_{\mu\nu}(u_0) u^\alpha_{[1]} u^\beta_{[1]} + \ell^2 \delta_\beta \delta_\alpha \underline{g}_{\mu\nu}(u_0) u^\alpha_{[1]} u^\beta_{[1]} + ...,$$

where we should take into account the fact that the operators δ_α do not commute but satisfy certain anholonomy relations (see (13.4) in Appendix).

The action (14.13) is invariant under the group of diffeomorphisms on Σ and M (on spacetimes provided with N–connections the diffeomorphisms may be adapted to such structures) and posses a $U(1)_B$ gauge invariance, acting by $B \rightarrow B + \delta\gamma$ for some $\gamma \in \Omega^{(1)}(M)$, where $\Omega^{(1)}$ denotes the space of 1–forms on M. Wayl's conformal transformations of Σ leave $S_{g_N,B}$ invariant but result in anomalies under quantization. S_Φ and S_U fail to be conformal invariant even classically. We discuss the renormalization of quantum filed theory defined by the action (14.13) for general fields $g_{ij}, h_{ab}, N^a_\mu, B_{\mu\nu}$ and Φ. We shall not discus in this work the effects of the tachyon field.

The string corrections to gravity (in both locally isotropic and locally anisotropic cases) may be computed following some regularizatons schemes preserving the classical symmetries and determining the general structure of the Weyl dependence of Green functions specified by the action (14.13) in terms of fixed background fields $g_{ij}, h_{ab}, N^a_\mu, B_{\mu\nu}$ and Φ. One can consider un–normalized correlation functions of operators $\phi_1, ..., \phi_p$, instead of points $\xi_1, ..., \xi_p \in \Sigma$ [15].

By definition of the stress tensor T_{AB}, under conformal transforms on the two dimensional hypersurface, $g_{[2]} \rightarrow \exp[2\delta\sigma] g_{[2]}$ with support away from $\xi_1, ..., \xi_p$, we have

$$\triangle_\sigma < \phi_1 ... \phi_p >_{g_{[2]}} = \frac{1}{2\pi} \int_\Sigma \delta\mu_g \triangle\sigma < T_A{}^A \phi_1 ... \phi_p >_{g_{[2]}},$$

when assuming throughout that correlation functions are covariant under the diffeomorphisms on Σ, $\bigtriangledown^A T_{AB} = 0$. The value $T_A{}^A$ receives contributions from the explicit conformal non–invariance of S_Φ, from conformal (Weyl) anomalies which are local functions of u, i.e. dependent on u and on finite order derivatives on u, and polynomial in the derivatives of u. For spaces provided with N–connection structures we should consider N–elongated partial derivatives, choose a N–adapted linear connection structure with some coefficients $\Gamma^\alpha{}_{\mu\nu}$ (for instance the Levi Civita connection (3.10), or d–connection (13.8)). The basic properties of $T_A{}^A$ are the same as for trivial values of N_i^a [15], which allows us to write directly that

$$\begin{aligned} T_A{}^A &= g^{AB}[\partial_A x^i \partial_B x^j \beta_{ij}^{g,N}(x,y) + \partial_A x^i \partial_B y^b \beta_{ib}^{g,N}(x,y) + \partial_A y^a \partial_B x^j \beta_{aj}^{g,N}(x,y) \\ &+ \partial_A y^a \partial_B y^b \beta_{ab}^{g,N}(x,y) + \varepsilon^{AB} \partial_A u^\alpha \partial_B u^\beta \beta_{\alpha\beta}^B(x,y) + \beta^\Phi(x,y) R_g, \end{aligned}$$

where the functions $\beta_{\alpha\beta}^{g,N} = \{\beta_{ij}^{g,N}, \beta_{ab}^{g,N}\}, \beta_{\alpha\beta}^B$ and $\beta^\Phi(x,y)$ are called beta functions. On general grounds, the expansions of β–functions are of type

$$\beta(x,y) = \sum_{r=0}^{\infty} \ell^{2r} \beta^{[2r]}(x,y).$$

One considers expanding up to and including terms with two derivatives on the fields including expansions up to order $r = 0$ of $\beta_{\alpha\beta}^{g,N}$ and $\beta_{\alpha\beta}^B$ and orders $s = 0, 2$ for β^Φ. In this approximation, after cumbersome but simple calculations (similar to those given in [15], in our case on locally anisotropic backgrounds)

$$\begin{aligned} \beta_{ij}^{g,N} &= a_{1[1]} R_{ij} + a_{2[1]} g_{ij} + a_{3[1]} g_{ij} \widehat{R} + a_{4[1]} H_{i\rho\sigma}^{[N]} H_j^{[N]\rho\sigma} + a_{5[1]} g_{ij} H_{\rho\sigma\tau}^{[N]} H^{[N]\rho\sigma\tau} \\ &+ a_{6[1]} D_i D_j \Phi + a_{7[1]} g_{ij} D^2 \Phi + a_{8[1]} g_{ij} D^\rho \Phi D_\rho \Phi, \end{aligned}$$

$$\begin{aligned} \beta_{ib}^{g,N} &= a_{1[2]} R_{ib} + a_{4[2]} H_{i\rho\sigma}^{[N]} H_b^{[N]\rho\sigma} + a_{6[2]} D_i D_b \Phi, \\ \beta_{aj}^{g,N} &= a_{1[3]} R_{aj} + a_{4[3]} H_{a\rho\sigma}^{[N]} H_j^{[N]\rho\sigma} + a_{6[3]} D_a D_j \Phi, \end{aligned} \quad (14.14)$$

$$\begin{aligned} \beta_{ab}^{g,N} &= a_{1[4]} S_{ab} + a_{2[4]} h_{ab} + a_{3[4]} h_{ab} S + a_{4[4]} H_{a\rho\sigma}^{[N]} H_b^{[N]\rho\sigma} + a_{5[4]} h_{ab} H_{\rho\sigma\tau}^{[N]} H^{[N]\rho\sigma\tau} \\ &+ a_{6[4]} D_a D_b \Phi + a_{7[4]} h_{ab} D^2 \Phi + a_{8[4]} h_{ab} D^\rho \Phi D_\rho \Phi, \end{aligned}$$

$$\begin{aligned}
\beta^{B}_{\alpha\beta} &= b_1 D^{\lambda} H^{[N]}_{\lambda\mu\nu} + b_2 (D^{\lambda}\Phi) H^{[N]}_{\lambda\mu\nu}, \\
\beta^{\Phi} &= c_0 + \ell^2 \left[c_{1[1]} \widehat{R} + c_{1[2]} S + c_2 D^2 \Phi + c_3 \left(D^{\lambda}\Phi \right) D_{\lambda}\Phi + c_4 H^{[N]}_{\rho\sigma\tau} H^{[N]\rho\sigma\tau} \right],
\end{aligned}$$

where $R_{\alpha\beta} = \{R_{ij}, R_{ib}, R_{aj}, S_{ab}\}$ and $\overleftarrow{R} = \{\widehat{R}, S\}$ are given respectively by the formulas (13.15) and (13.16) and the B–strength $H^{[N]}_{\lambda\mu\nu}$ is computed not by using partial derivatives, like in (14.6), but with N–adapted partial derivatives,

$$H^{[N]}_{\mu\nu\rho} \equiv \delta_{\mu} B_{\nu\rho} + \delta_{\nu} B_{\rho\mu} + \delta_{\nu} B_{\rho\mu}. \tag{14.15}$$

The formulas for β–functions (14.14) are adapted to the N–connection structure being expressed via invariant decompositions for the Ricci d–tensor and curvature scalar; every such invariant object was provided with proper constants. In order to have physical compatibility with the case $N \to 0$ we should take

$$\begin{aligned}
a_{z[1]} &= a_{z[2]} = a_{z[3]} = a_{z[4]} = a_z, \; z = 1, 2, ..., 8; \\
c_{1[1]} &= c_{1[2]} = c_1,
\end{aligned}$$

where a_z and c_1 are the same as in the usual string theory, computed from the 1– and 2–loop ℓ–dependence of graphs ($a_2 = 0$, $a_6 = 1$, $a_7 = a_8 = 0$ and $b_2 = 1/2$, $c_3 = 2$) and by using the background field method (in order to define the values a_1, a_3, a_4, a_5, b_1 and c_1, c_2, c_4).

14.2.2 Low energy string anholonomic field equations

The effective action, as the generating functional for 1–particle irreducible Feynman diagrams in terms of a functional integral, can be obtained following the background quantization method adapted, in our constructions, to sigma models on spacetimes with N–connection structure. On such spaces, we can also make use of the Riemannian coordinate expansion, but taking into account that the coordinates are defined with respect to N–adapted bases and that the covariant derivative D is of type (13.8), (3.10) or (13.10), i. e. is d–covariant, defined by a d–connection.

For two infinitesimally closed points $u^{\mu}_0 = u^{\mu}(\tau_0)$ and $u^{\mu}(\tau)$, with τ being a parameter on a curve connected the points, we denote $\zeta^{\alpha} = du^{\alpha}/d\tau_{|0}$ and write $u^{\mu} = e^{\ell\zeta} u^{\mu}_0$. We can consider diffeomorphism invariant d–covariant expansions of d–tensors in powers of

ℓ, for instance,

$$\begin{aligned}\Phi(u) &= \Phi(u_0) + \ell D_\alpha[\Phi(u)\zeta^\alpha]_{|u=u_0} + \frac{\ell^2}{2} D_\alpha D_\beta[\Phi(u)\zeta^\alpha\zeta^\beta]_{|u=u_0} + o\left(\ell^3\right),\\ A_{\alpha\beta}(u) &= A_{\alpha\beta}(u_0) + \ell D_\alpha[A_{\alpha\beta}(u)\zeta^\alpha]_{|u=u_0} + \frac{\ell^2}{2}\{D_\alpha D_\beta[A_{\mu\nu}(u)\zeta^\alpha\zeta^\beta \\ &\quad -\frac{1}{3}R^{[N]\rho}_{\ \alpha\mu\beta}(u)A_{\rho\nu}(u) - \frac{1}{3}R^{[N]\rho}_{\ \alpha\nu\beta}(u)A_{\rho\mu}(u)]\}_{|u=u_0} + o\left(\ell^3\right),\end{aligned}$$

where the Riemannian curvature d–tensor $R^{[N]\rho}_{\ \alpha\mu\beta} = \{R^{.i}_{h.jk}, R^{.a}_{b.jk}, P^{.i}_{j.ka}, P^{.c}_{b.ka}, S^{.i}_{j.bc}, S^{.a}_{b.cd}\}$ has the invariant components given by the formulas (13.14) from Appendix. Putting such expansions in the action for the nonlinear sigma model (14.13), we obtain the decomposition

$$S_{g_N,B}[u,g] = S_{g_N,B}[u_0,g] + \ell\int_\Sigma \delta\mu_g \zeta^\beta S_\beta[u_0,g] + \overline{S}[u,\zeta,g],$$

where S_β is given by the variation

$$S_\beta[u_0,g] = (\det|g|)^{-1/2}\frac{\triangle S[e^\chi u_0,g]}{\triangle\chi^\beta}\ |_{\chi=0}$$

and the last term $\overline{S}$ is an expansion on ℓ,

$$\overline{S} = \overline{S}_{[0]} + \ell\overline{S}_{[1]} + \ell^2\overline{S}_{[2]} + o\left(\ell^3\right),$$

with

$$\begin{aligned}\overline{S}_{[0]} &= \frac{1}{8\pi}\int_\Sigma \delta\mu_g\{g^{AB}[g_{ij}(u_0)\mathcal{D}^*_A\zeta^i\mathcal{D}^*_B\zeta^j + h_{ab}(u_0)\mathcal{D}^*_A\zeta^a\mathcal{D}^*_B\zeta^b] \qquad (14.16)\\ &\quad +\mathcal{R}^{[N]}_{\mu\nu\rho\sigma}(u_0)[g^{AB} - \varepsilon^{AB}]\partial_A u_0^\mu\partial_B u_0^\rho\zeta^\nu\zeta^\sigma\},\\ \overline{S}_{[1]} &= \frac{1}{24\pi}\int_\Sigma \delta\mu_g H^{[N]}_{\mu\nu\rho}\varepsilon^{AB}\zeta^\mu\mathcal{D}^*_A\zeta^\nu\mathcal{D}^*_B\zeta^\rho,\\ \overline{S}_{[2]} &= \frac{1}{8\pi}\int_\Sigma \delta\mu_g\{\frac{g^{AB}}{3}R^{[N]}_{\mu\nu\rho\sigma}\zeta^\nu\zeta^\rho\mathcal{D}^*_A\zeta^\mu\mathcal{D}^*_B\zeta^\sigma \\ &\quad -\frac{\varepsilon^{AB}}{2}\mathcal{R}^{[N]}_{\mu\nu\rho\sigma}\zeta^\nu\zeta^\rho\mathcal{D}^*_A\zeta^\mu\mathcal{D}^*_B\zeta^\sigma + 2D_\alpha D_\beta\Phi(u_0)\zeta^\alpha\zeta^\beta R_g\}.\end{aligned}$$

The operator $\mathcal{D}^*_A\zeta^\nu$ from (14.16) is defined according the rule

$$\mathcal{D}^*_A\zeta^\nu = D^*_A\zeta^\nu + \frac{1}{2}H^{[N]\sigma}_{\ \ \mu\rho}g_{AB}\varepsilon^{BC}\partial_C u^\mu\zeta^\rho,$$

with D^*_A being the covariant derivative on $T^*\Sigma\otimes TM$ pulled back to Σ by the map u^α and acting as

$$D^*_A\partial_B u^\nu = \bigtriangledown_A\partial_B u^\nu + \Gamma^\alpha_{\ \mu\nu}\partial_B u^\nu\partial_A u^\nu,$$

with a h– and v–invariant decomposition $\Gamma^\alpha_{\ \beta\gamma} = \{L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}\}$, see (13.8) from Appendix, and the operator $\mathcal{R}^{[N]}_{\mu\nu\rho\sigma}$ is computed as

$$\mathcal{R}^{[N]}_{\mu\nu\rho\sigma} = R^{[N]}_{\mu\nu\rho\sigma} + \frac{1}{2}D_\rho H^{[N]}_{\sigma\mu\nu} - \frac{1}{2}D_\sigma H^{[N]}_{\rho\mu\nu} + \frac{1}{4}H^{[N]}_{\rho\mu\alpha}H^{[N]\alpha}_{\sigma\nu} - \frac{1}{4}H^{[N]}_{\sigma\mu\alpha}H^{[N]\alpha}_{\rho\nu}.$$

A comparative analysis of the expansion (14.16) with a similar one for $N = 0$ from the usual nonlinear sigma model (see, for instance, [15]) define the 'geometric d–covariant rule': we may apply the same formulas as in the usual covariant expansions but with that difference that 1) the usual spacetime partial derivatives and differentials are substituted by N–elongated ones; 2) the Christoffell symbols of connection are changed into certain d–connection ones, of type (13.8), (3.10) or (13.10); 3) the torsion $H^{[N]}_{\sigma\mu\nu}$ is computed via N–elongated partial derivatives as in (14.15) and 4) the curvature $R^{[N]}_{\mu\nu\rho\sigma}$ is split into horizontal–vertical, in brief, h–v–invariant, components according the the formulas (13.14). The geometric d–covariant rule allows us to transform directly the formulas for spacetime backgrounds with metrics written with respect to coordinate frames into the respective formulas with N–elongated terms and splitting of indices into h– and v–subsets.

Low energy string anisotropic field equations and effective action

Following the geometric d–covariant rule we may apply the results of the holonomic sigma models in order to define the coefficients a_1, a_3, a_4, a_5, b_1 and c_1, c_2, c_4 of beta functions (14.14) and to obtain the following equations of (in our case, anholonomic) string dynamics,

$$\begin{aligned}
2\beta^{g,N}_{ij} &= R_{ij} - \frac{1}{4}H^{[N]}_{i\rho\sigma}H^{[N]\rho\sigma}_{j} + 2D_iD_j\Phi = 0,\\
2\beta^{g,N}_{ib} &= R_{ib} - \frac{1}{4}H^{[N]}_{i\rho\sigma}H^{[N]\rho\sigma}_{b} + 2D_iD_b\Phi = 0,
\end{aligned}$$

$$
\begin{aligned}
2\beta_{aj}^{g,N} &= R_{aj} - \frac{1}{4}H_{a\rho\sigma}^{[N]}H_{j}^{[N]\rho\sigma} + 2D_aD_j\Phi = 0, \\
2\beta_{ab}^{g,N} &= S_{ab} - \frac{1}{4}H_{a\rho\sigma}^{[N]}H_{b}^{[N]\rho\sigma} + 2D_aD_b\Phi = 0,
\end{aligned}
$$

$$
\begin{aligned}
2\beta_{\alpha\beta}^{B} &= -\frac{1}{2}D^{\lambda}H_{\lambda\mu\nu}^{[N]} + (D^{\lambda}\Phi)H_{\lambda\mu\nu}^{[N]} = 0, \qquad (14.17) \\
2\beta^{\Phi} &= \frac{n+m-26}{3} + \ell^2\left[\frac{1}{12}H_{\rho\sigma\tau}^{[N]}H^{[N]\rho\sigma\tau} - \widehat{R} - S - 4D^2\Phi + 4\left(D^{\lambda}\Phi\right)D_{\lambda}\Phi\right] = 0,
\end{aligned}
$$

where $n+m$ denotes the total dimension of a spacetime with n holonomic and m anholonomic variables. It should be noted that $\beta^{g,N} = \beta^{B} = 0$ imply the condition that $\beta^{\Phi} = const$, which is similar to the holonomic strings. The only way to satisfy $\beta^{\Phi} = 0$ with integers n and m is to take $n+m = 26$.

The equations (14.17) are similar to the Einstein equations for the locally anisotropic gravity (see (13.18) in Appendix) with the matter energy–momentum d–tensor defined from the string theory. From this viewpoint the fields $B_{\alpha\beta}$ and Φ can be viewed as certain matter fields and the effective field equations (14.17) can be derived from action

$$
S\left(g_{ij}, h_{ab}, N_i^a, B_{\mu\nu}, \Phi\right) = \frac{1}{2\kappa^2}\int \delta^{26}u\sqrt{|\det g_{\alpha\beta}|}e^{-2\Phi}\left[\widehat{R} + S + 4(D\Phi)^2 - \frac{1}{12}H^2\right], \qquad (14.18)
$$

where κ is a constant and, for instance, $D\Phi = D_{\alpha}\Phi$, $H^2 = H_{\mu}H^{\mu}$ and the critical dimension $n+m = 26$ is taken. For $N \to 0$ and $m \to 0$ the metric $g_{\alpha\beta}$ is called the string metric. We shall call $g_{\alpha\beta}$ the string d–metric for nontrivial values of N.

Instead of action (14.18), a more standard action, for arbitrary dimensions, can be obtained via a conformal transform of d–metrics of type (13.6),

$$
g_{\alpha\beta} \to \widetilde{g}_{\alpha\beta} = e^{-4\Phi/(n+m-2)}g_{\alpha\beta}.
$$

The action in d–metric $\widetilde{g}_{\alpha\beta}$ (by analogy with the locally isotropic backgrounds we call it the Einstein d–metric) is written

$$
\begin{aligned}
S\left(\widetilde{g}_{ij}, \widetilde{h}_{ab}, N_i^a, B_{\mu\nu}, \Phi\right) &= \frac{1}{2\kappa^2}\int \delta^{26}u\sqrt{|\det \widetilde{g}_{\alpha\beta}|}[\widetilde{\widehat{R}} + \widetilde{S} \\
&+ \frac{4}{n+m-2}(D\Phi)^2 - \frac{1}{12}e^{-8\Phi/(n+m-2)}H^2].
\end{aligned}
$$

This action, for $N \to 0$ and $m \to 0$, is known in supergravity theory as a part of Chapline–Manton action, see Ref. [15] and for the so–called locally anisotropic supergravity, [43, 47]. When we deal with superstirngs, the susperstring calculations to the mentioned orders give the same results as the bosonic string except the dimension. For anholonomic backrounds we have to take into account the nontrivial contributions of N_i^a and splitting into h– and v–parts.

Ahnolonomic Einstein and Finsler gravity from string theory

It is already known that the B–field can be used for generation of different types of noncommutative geometries from string theories (see original results and reviews in Refs. [57, 13, 9, 38, 50]). Under certain conditions such B–field configurations may result in different variants of geometries with local anistropy like anholonomic Riemannian geometry, Finsler like spaces and their generalizations. There is also an alternative possibility when locally anisotropic interactions are modelled by anholonomic frame fields with arbitrary B–field contributions. In this subsection, we investigate both type of anisotropic models contained certain low energy limits of string theory.

B–fields and anholonomic Einstein–Finsler structures

The simplest way to generate an anholonomic structure in a low energy limit of string theory is to consider a background metric $g_{\mu\nu} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ab} \end{pmatrix}$ with symmetric Christoffel symbols $\{^{\alpha}_{\beta\gamma}\}$ and such $B_{\mu\nu}$, with corresponding $H^{[N]}_{\mu\nu\rho}$ from (14.15), as there are the nonvanishing values $H^{[N]\rho}_{\mu\nu} = \{H^{[N]a}_{ij}, H^{[N]a}_{bj} = -H^{[N]a}_{jb}\}$. The next step is to consider a covariant operator $\mathcal{D}_\mu = D^{\{\}}_\mu + \frac{1}{2}H^{[N]\rho}_{\mu\nu}$ (14.5), where $\frac{1}{2}H^{[N]\rho}_{\mu\nu}$ is identified with the torsion (13.11). This way the torsion $H^{[N]}_{\mu\nu\rho}$ is associated to an aholonomic frame structure with non–trivial $W^a_{ij} = \delta_i N^a_j - \delta_j N^a_i$, $W^b_{ai} = -W^b_{ia} = -\partial_a N^b_i$ (14.108), when $B_{\mu\nu}$ is parametrized in the form $B_{\mu\nu} = \{B_{ij} = -B_{ji}, B_{bj} = -B_{jb}\}$ by identifying

$$g_{\mu\nu}W^{\nu}_{\gamma\beta} = \delta_\mu B_{\gamma\beta},$$

i. .e.

$$h_{ca}W^a_{ij} = \partial_c B_{ij} \text{ and } h_{ca}W^a_{bj} = \partial_c B_{bj}. \tag{14.19}$$

Introducing the formulas for the anholonomy coefficients (14.108) into (14.19), we find some formulas relating partial derivatives $\partial_\alpha N^a_j$ and the coefficients N^a_j with partial derivatives of $\{B_{ij}, B_{bj}\}$,

$$\begin{aligned} h_{ca}\left(\partial_i N^a_j - N^b_i\partial_b N^a_j - \partial_j N^a_i + N^b_j \partial_b N^a_i\right) &= \partial_c B_{ij}, \\ -h_{ca}\partial_b N^a_j &= \partial_c B_{bj}. \end{aligned} \tag{14.20}$$

So, given any data (h_{ca}, N_i^a) we can define from the system of first order partial derivative equations (14.20) the coefficients B_{ij} and B_{bj}, or, inversely, from the data (h_{ca}, B_{ij}, B_{bj}) we may construct some non–trivial values N_j^a. We note that the metric coefficients g_{ij} and the B–field components $B_{ab} = -B_{ba}$ could be arbitrary ones, in the simplest approach we may put $B_{ab} = 0$.

The formulas (14.20) define the conditions when a B–field may be transformed into a N–connection structure, or inversely, a N–connection can be associated to a B–field for a prescribed d–metric structure h_{ca}, (13.6).

The next step is to decide what type of d–connection we consider on our background spacetime. If the values $\{g_{ij}, h_{ca}\}$ and $W_{\gamma\beta}^{\nu}$ (defined by N_i^b as in (14.108), but also induced from $\{B_{ij}, B_{bj}\}$ following (14.20)) are introduced in formulas (3.10) we construct a Levi Civita d–connection $\mathcal{D}_\mu$ with nontrivial torsion induced by anholonomic frames with associated nonlinear connection structure. This spacetime is provided with a d–metric (13.6), $g_{\alpha\beta} = \{g_{ij}, h_{ca}\}$, which is compatible with $\mathcal{D}_\mu$, i. e. $\mathcal{D}_\mu g_{\alpha\beta} = 0$. The coefficients of $\mathcal{D}_\mu$ with respect to anholonomic frames (13.2) and (13.3), $\Gamma_{\beta\gamma}^{\bigtriangledown\tau}$, can be computed in explicit form by using formulas (3.10). It is proven in the Appendix that on spacetimes provided with anholonomic structures the Levi Civita connection is not a priority one being both metric and torsion vanishing. We can construct an infinite number of metric connections, for instance, the canonical d–connection with the coefficients (13.8), or, equivalently, following formulas (13.10), to substitute from the coefficients (3.10) the values $\frac{1}{2} g^{ik} \Omega_{jk}^a h_{ca}$, where the coefficients of N–connection curvature are defined by N_i^a as in (5.7). In general, all such type of linear connections are with nontrivial torsion because of anholonomy coefficients.

We may generate by B–fields an anholonomic (pseudo) Riemannian geometry if (for given values of $g_{\alpha\beta} = \{g_{ij}, h_{ca}\}$ and N_i^a, satisfying the conditions (14.20)) the metric is considered in the form (13.32) with respect to coordinate frames, or, equivalently, in the form (13.6) with respected to N–adapted frame (13.3). The metric has to satisfy the gravitational field equations (13.18) for the Einstein gravity of arbitrary dimensions with holonomic–anholonomic variables, or the equations (14.17) if the gravity with anholonomic constraints is induced in a low energy string dynamics. We emphasize that the Ricci d–tensor coefficients from β–functions (14.17) should be computed by using the formulas (13.15), derived from those for d–curvatures (13.14) and for d–torsions (13.13) for a chosen variant of d–connection coefficients, for instance, (13.8) or (3.10).

We note here that a number of particular ansatz of form (13.32) were considered in Kaluza–Klein gravity [35] for different type of compactifications. In Refs. [45, 46, 53, 54, 52] there were constructed and investigated a number exact solutions with off–diagonal metrics and anholonomic frames with associated N–connection structures in the Einstein gravity of different dimensions (see also the Section 2.5).

Now, we discuss the possibility to generate a Finsler geometry from string theory. We note that the standard definition of Finsler quadratic form

$$g_{ij}^{[F]} = (1/2)\partial^2 F/\partial y^i \partial y^j$$

is considered to be positively definite (see (14.127) in Appendix). There are different possibilities to include Finsler like structures in string theories. For instance, we can consider quadratic forms with non–constant signatures and to generate (pseudo) Finsler geometries [similarly to (pseudo) Euclidean/Riemannian metrics], or, as a second approach, to consider some embedding of Finsler d–metrics (13.34) of signature $(++...+)$ into a 26 dimensional pseudo–Riemannian anholonomic background with signature $(-++...+)$. In the last case, a particular class of Finsler background d–metrics may be chosen in the form

$$G^{[F]} = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + g_{i'j'}^{[F]}(x,y) dx^{i'} \otimes dx^{j'} + g_{i'j'}^{[F]}(x,y) \delta y^{i'} \otimes \delta y^{j'} \quad (14.21)$$

where $i', j', ...$run values $1, 2, .., n' \leq 12$ for bosonic strings. The coefficients $g_{i'j'}^{[F]}$ are of type (14.127) or may take the value $+\delta_{i'j'}$ for some values of $i \neq i', j \neq j'$. We may consider some static Finsler backgrounds if $g_{i'j'}^{[F]}$ do not depend on coordinates (x^0, x^1), but, in general, we are not imposed to restrict ourselves only to such constructions. The N–coefficients from $\delta y^{i'} = dy^{i'} + N_{j'}^{i'} dx^{i'}$ must be of the form (13.33) if we wont to generate in the low energy string limit a Finsler structure with Cartan nonlinear connection (there are possible different variants of nonlinear and distinguished nonlinear connections, see details in Refs. [34, 29, 4] and Appendix).

Let us consider in details how a Finsler metric can be included in a low energy string dynamics. We take a Finsler metric F which generate the metric coefficients $g_{i'j'}^{[F]}$ and the N–connection coefficients $N_{j'}^{[F]i'}$, respectively, via formulas (14.127) and (13.33). The Cartan's N–connection structure $N_{j'}^{[F]i'}$ may be induced by a B–field if there are some nontrivial values, let us denote them $\{B_{ij}^{[F]}, B_{bj}^{[F]}\}$, which satisfy the conditions (14.20). This way the B–field is expressed via a Finsler metric $F(x,y)$ and induces a d–metric (14.21). This Finsler structure follows from a low energy string dynamics if the Ricci d–tensor $R_{\alpha\beta} = \{R_{ij}, R_{ia}, R_{ai}, R_{ab}\}$ (13.15) and the torsion

$$H_{\mu\nu}^{[N]\rho} = \{H_{ij}^{[N]a}, H_{bj}^{[N]a} = -H_{jb}^{[N]a}\}$$

related with $N_{j'}^{[F]i'}$ as in (14.19), all computed for d–metric (14.21) are solutions of the motion equations (14.17) for any value of the dilaton field Φ. In the Section 2.5 we shall consider an explicit example of a string–Finsler metric.

Here it should be noted that instead of a Finsler structure, in a similar manner, we may select from a string locally anisotropic dynamics a Lagrange structure if the metric coefficients $g_{i'j'}$ are generated by a Lagrange function $L(x,y)$ (13.35). The N–connection may be an arbitrary one, or of a similar Cartan form. We omit such constructions in this paper.

Anholonomic Einstein–Finsler structures for arbitrary B–fields

Locally anisotropic metrics may be generated by anholonomic frames with associated N–connections which are not induced by some B–field configurations.

For an anholonomic (pseudo) Riemannian background we consider an ansatz of form (13.32) which by anholonomic transform can be written as an equivalent d–metric (13.6). The coefficients N_i^a and $B_{\mu\nu}$ are related only via the string motion equations (14.17) which must be satisfied by the Ricci d–tensor (13.15) computed, for instance, for the canonical d–connection (13.8).

A Finsler like structure, not induced directly by B–fields, may be emphasized if the d-metric is taken in the form (14.21), but the values $\delta y^{i'} = dy^{i'} + N_{j'}^{i'} dx^{i'}$ being elongated by some $N_{j'}^{i'}$ are not obligatory constrained by the conditions (14.20). Of course, the Finsler metric F and $B_{\mu\nu}$ are not completely independent; these fields must be chosen as to generate a solution of string–Finsler equations (14.17).

In a similar manner we can model as some alternative low energy limits of the string theory, with corresponding nonlinear sigma models, different variants of spacetime geometries with anholonomic and N–connection structures, derived on manifold or vector bundles when the metric, linear and N–connection structures are proper for a Lagrange, generalized Lagrange or anholonomic Riemannian geometry [29, 34, 4, 24, 45, 44, 46, 51, 53].

14.3 Superstrings and Anisotropic Supergravity

The bosonic string theory, from which in the low energy limits we may generate different models of anholonomic Riemannian–Finsler gravity, suffers from at least four major problems: 1) there are tachyonic states which violates the physical causality and divergence of transitions amplitudes; 2) there are not included any fermionic states transforming under a spinor representation of the spacetime Lorentz group; 3) it is not clear why Yang–Mills gauge particles arise in both type of closed and open string theories and to what type of strings should be given priority; 4) experimentally there are 4 dimensions and not 26 as in the bosonic string theory: it must be understood why the remaining dimensions are almost invisible.

The first three problems may be resolved by introducing certain additional dynamical degrees of freedom on the string worldsheet which results in fermionic string states in the physical Hibert space and modifies the critical dimension of spacetime. One tries to solve the forth problem by developing different models of compactification.

There are distinguished five, consistent, tachyon free, spacetime supersymmetric string theories in flat Minkowski spacetime (see, for instance, [15, 25] for basic results and references on types I, IIA, IIB, Heterotic $Spin(32)/Z_2$ and Heterotic $E_8 \times E_8$ string theories). The (super) string and (super) gravity theories in geralized Finsler like, in general, supersymmetric backgrounds provided with N–connection structure, and corresponding anisotropic superstring perturbation theories, were investigated in Refs. [41, 43, 47]. The goal of this Section is to illustrate how anholonomic type structures arise in the low energy limits of the mentioned string theories if the backgrounds are considered with certain anholonomic frame and off–diagonal metric structures. We shall consider the conditions when generalized Finsler like geometries arise in (super) string theories.

We would like to start with the example of the two–dimensional $\mathcal{N} = 1$ supergravity coupled to the dimension 1 superfields, containing a bosonic coordinate X^μ and two fermionic coordinates, one left–moving ψ^μ and one right moving $\overline{\psi}^\mu$ (we use the symbol $\mathcal{N}$ for the supersimmetric dimension which must be not confused with the symbol N for a N–connection structure). We note that the two dimensional $\mathcal{N} = 1$ supergravity multiplet contains the metric and a gravitino χ_A.In order to develop models in backgrounds distinguished by a N–connection structure, we have to consider splitting into h– and v–components, i. e. to write $X^\mu = (X^i, X^a)$ and $\psi^\mu = (\psi^i, \psi^a), \overline{\psi}^\mu = \left(\overline{\psi}^i, \overline{\psi}^a\right).$ The spinor differential geometry on anisotropic spacetimes provided with N–connections (in brief, d–spinor geometry) was developed in Refs. [40, 55]. Here we shall present only the basic formulas, emphasizing the fact that the coefficients of d–spinors have the usual spinor properties on separated h– (v-) subspaces.

The simplest distinguished superstring model can be developed from an analog of the bosonic Polyakov action,

$$\begin{aligned} S_P &= \frac{1}{4\pi\alpha'}\int_{\Sigma} \delta\mu_g \{ g^{AB}\left[\partial_A X^i \partial_B X^j g_{ij} + \partial_A X^a \partial_B X^b h_{ab}\right] \\ &+\frac{i}{2}[\psi^k \gamma^A \partial_A \psi^k + \psi^a \gamma^A \partial_A \psi^a] + \frac{i}{2}\left(\chi_A \gamma^B \gamma^A \psi^k\right)\left(\partial_B X^k - \frac{i}{4}\chi_B \psi^k\right) \\ &+\frac{i}{2}\left(\chi_A \gamma^B \gamma^A \psi^a\right)\left(\partial_B X^a - \frac{i}{4}\chi_B \psi^a\right)\} \end{aligned} \quad (14.22)$$

being invariant under transforms (i. e. being $\mathcal{N}=1$ left–moving $(1,0)$ supersymmetric)

$$\begin{aligned} \triangle g_{AB} &= i\epsilon\left(\gamma_A \chi_B + \gamma_B \chi_A\right), \ \triangle\, \chi_A = 2\bigtriangledown_A \epsilon, \\ \triangle X^i &= i\epsilon\psi^i, \ \triangle\, \psi^k = \gamma^A\left(\partial_A X^k - \frac{i}{2}\chi_A \psi^k\right)\epsilon, \ \triangle\, \overline{\psi}^i = 0, \\ \triangle X^a &= i\epsilon\psi^a, \ \triangle\, \psi^a = \gamma^A\left(\partial_A X^a - \frac{i}{2}\chi_A \psi^a\right)\epsilon, \ \triangle\, \overline{\psi}^a = 0, \end{aligned}$$

where the gamma matrices γ_A and the covariant differential operator $\bigtriangledown_A$ are defined on the two dimensional surface, ϵ is a left–moving Majorana–Weyl spinor. There is also a similar right–moving $(0,1)$ supersymmetry involving a right moving Majorana–Weyl spinor $\overline{\epsilon}$ and the fermions $\overline{\psi}^\mu$ which means that the model has a $(1,1)$ supersymmetry. The superconformal gauge for the action (14.22) is defined as

$$g_{AB} = e^{\Phi}\delta_{AB}, \ \chi_A = \gamma_A \zeta,$$

for a constant Majorana spinor ζ. This action has also certain matter like supercurents $i\psi^\mu \partial X^\mu$ and $i\overline{\psi}^\mu \overline{\partial} X^\mu$.

We remark that the so–called distinguished gamma matrices (d–matrices), $\gamma^\alpha = (\gamma^i, \gamma^a)$ and related spinor calculus are derived from γ–decompositions of the h– and v–components of d–metrics $g^{\alpha\beta} = \{g^{ij}, h^{ab}\}$ (13.6)

$$\gamma^i\gamma^j + \gamma^j\gamma^i = -2g^{ij}, \ \gamma^a\gamma^b + \gamma^b\gamma^a = -2h^{ab},$$

see details in Refs. [40, 55].

In the next subsections we shall distinguish more realistic superstring actions than (14.22) following the geometric d–covariant rule introduced in subsection 14.2.2, when the curved spacetime geometric objects like metrics, connections, tensors, spinors, ... as well the partial and covariant derivatives and differentials are decomposed in invariant h– and v–components, adapted to the N–connection structure. This will allow us to extend directly the results for superstring low energy isotropic actions to backgrounds with local anisotropy.

14.3.1 Locally anisotropic supergravity theories

We indicate that many papers on supergravity theories in various dimensions are reprinted in a set of two volumes [36]. The bulk of supergravity models contain locally anisotropic configurations which can be emphasized by some vielbein transforms (14.7) and metric anzatz (14.11) with associated N–connection. For corresponding parametrizations of the d–metric coefficients, $g_{\alpha\beta}(u) = \{g_{ij}, h_{ab}\}$, N–connection, $N_i^a(x,y)$, and d–connection, $\Gamma^{\alpha}{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$, with possible superspace generalizations, we can generate (pseudo) Riemannian off–diagonal metrics, Finsler or Lagrange (super) geometries. In this subsection, we analyze the anholonomic frame transforms of some supergravity actions which can be coupled to superstring theory.

We note that the field components will be organized according to multiplets of $Spin\,(1,10)$. We shall use 10 dimensional spacetime indices $\alpha,\beta... = 0,1,2,...,9$ or 11 dimensional ones $\overline{\alpha},\overline{\beta}... = 0,1,2,...,9,10$. The coordinate u^{10} could be considered as a compactified one, or distinguished in a non–compactified manner, by the N–connection structure. There is a general argument [31] is that 11 is the largest possible dimension in which supersymmetric multiplets can exist with spin less, or equal to 2, with a single local supersymmetry. We write this as $n+m=11$, which points to possible splittings of indices like $\overline{\alpha} = \left(\overline{i},\overline{a}\right)$ where $\overline{i}$ and $\overline{a}$ run respectively n and m values. A consistent superstring theory holds if $n+m=10$. In this case, indices are to be decomposed as $\alpha=(i,a)$. For simplicity, we shall consider that a metric tensor in $n+m=11$ dimensions decomposes as $g_{\overline{\alpha}\overline{\beta}}\left(u^{\mu},u^{10}\right) \rightarrow g_{\overline{\alpha}\overline{\beta}}\left(u^{\mu}\right)$ and that in low energy approximation the fields are locally anisotropically interacting and independent on u^{10}. The antisymmetric rank 3 tensor is taken to decompose as $A_{\overline{\alpha}\overline{\beta}\overline{\gamma}}\left(u^{\mu},u^{10}\right) \rightarrow A_{\overline{\alpha}\overline{\beta}\overline{\gamma}}(u^{\mu})$. A fitting with superstring theory is to be obtained if $\left(A^{[3]}_{\alpha\beta\gamma}, B_{\mu\nu}\right) \rightarrow A_{\overline{\alpha}\overline{\beta}\overline{\gamma}}$ and consider for spinors "dilatino" fields $\left(\chi_{\mu}{}^{\tau},\lambda_{\tau}\right) \rightarrow \chi_{\overline{\mu}}{}^{\tau}$, see, for instance, Refs. [15] for details on couplings of supergravity and low energy superstrings.

$\mathcal{N}=1, n+m=11$ anisotropic supergravity

The field content of $\mathcal{N}=1$ and 11 dimensional supergravity is given by $g_{\overline{\alpha}\overline{\beta}}$ (graviton), $A_{\overline{\alpha}\overline{\beta}\overline{\gamma}}$ (U(1) gauge fields) and $\chi^{\alpha}_{\overline{\mu}}$ (gravitino). The dimensional reduction is stated by $g_{\alpha 10} = g_{10\alpha} = A^{[1]}_{\alpha}$ and $g_{10\ 10} = e^{-2\Phi}$, where the coefficients are given with respect to an N–elongated basis. We suppose that an effective action

$$S(g_{ij},h_{ab},N^a_i,B_{\mu\nu},\Phi) = \frac{1}{2\kappa^2}\int \delta\mu_{[g,h]}e^{-2\Phi}\left[-\widehat{R} - S + 4(D\Phi)^2 - \frac{1}{12}H^2\right],$$

is to be obtained if the values $A_{\alpha}^{[1]}, A_{\alpha\beta\gamma}^{[3]}, \chi_{\mu}{}^{\tau}, \lambda_{\tau}$ vanish. For $N \to 0, m \to 0$ this action results from the so–called NS sector of the superstring theory, being related to the sigma model action (14.18). A full $\mathcal{N} = 1$ and 11 dimensional locally anisotropic supergravity can be constructed similarly to the locally isotropic case [11] but considering that $H^{[N]} = \delta B$ and $F^{[N]} = \delta A$ are computed as differential forms with respect to N–elongated differentials (13.3),

$$
\begin{aligned}
S\left(g_{ij}, h_{ab}, N_i^a, A_\alpha, \chi\right) &= -\frac{1}{2\kappa^2}\int \delta\mu_{[g,h]}[\widehat{R} + S - \frac{\kappa^2}{12}F^2 + \kappa^2 \overline{\chi}_{\overline{\mu}}\Gamma^{\overline{\mu\nu\lambda}} D_{\overline{\nu}}\chi_{\overline{\lambda}} \quad (14.23)\\
&+\frac{\sqrt{2}\kappa^3}{384}\left(\overline{\chi}_{\overline{\mu}}\Gamma^{\overline{\mu\nu\rho\sigma\tau\lambda}}\overline{\chi}_{\overline{\lambda}} + 12\Gamma^{\overline{\rho\nu\sigma}}\chi^{\overline{\tau}}\right)\left(F + \widehat{F}\right)_{\overline{\nu\rho\sigma\tau}}]\\
&-\frac{\sqrt{2}\kappa}{81\times 56}\int A\wedge F\wedge F,
\end{aligned}
$$

where $\Gamma^{\overline{\mu\nu\rho\sigma\tau\lambda}} = \Gamma^{[\overline{\mu}}\Gamma^{\overline{\nu}}...\Gamma^{\overline{\lambda}]}$ is the standard notation for gamma matrices for 11 dimensional spacetimes, the field $\widehat{F} = F + \chi$–terms and $D_{\overline{\nu}}$ is the covariant derivative with respect to $\frac{1}{2}(\omega + \widehat{\omega})$ where

$$
\widehat{\omega}_{\overline{\mu\alpha\beta}} = \omega_{\overline{\mu\alpha\beta}} + \frac{1}{8}\chi^{\overline{\nu}}\Gamma_{\overline{\nu\mu\alpha\beta\rho}}\chi^{\overline{\rho}}
$$

with $\omega_{\overline{\mu\alpha\beta}}$ being the spin connection determined by its equation of motion. We put the same coefficients in the action (14.23) as in the locally isotropic case as to have compatibility for such limits. Every object (tensors, connections, connections) has a N–distinguished invariant character with indices split into h– and v–subsets. For simplicity we omit here further decompositions of fields with splitting of indices.

Type IIA anisotropic supergravity

The action for a such model can be deduced from (14.23) if $A_{\alpha\beta\gamma} = \kappa^{1/4}A_{\alpha\beta\gamma}^{[3]}$ and $A_{\alpha\beta 10} = \kappa^{-1}B_{\alpha\beta}$ with further h– and v– decompositions of indices. The bosonic part of the type IIA locally anisotropic supergravity is described by

$$
\begin{aligned}
S\left(g_{ij}, h_{ab}, N_i^a, \Phi, A^{(1)}, A^{(3)}\right) &= -\frac{1}{2\kappa^2}\int \delta\mu_{[g,h]}\{e^{-2\Phi}[\widehat{R} + S - 4(D\Phi)^2 + \frac{1}{12}H^2]\\
&+\sqrt{\kappa}G_{[A]} + \frac{\sqrt{\kappa}}{12}F^2 - \frac{\kappa^{-3/2}}{288}\int B\wedge F\wedge F\}, \quad (14.24)
\end{aligned}
$$

with $G_{[A]} = \delta A^{(1)}, H = \delta B$ and $F = \delta A^{(3)}$. This action may be written directly from the locally isotropic analogous following the d–covariant geometric rule.

Type IIB, n+m=10, $\mathcal{N}=2$ anisotropic supergravity

In a similar manner, geometrically, for d–objects, we may compute possible anholonomic effects from an action describing a model of locally anisotropic supergravity with a super Yang–Mills action (the bosonic part)

$$S_{IIB} = -\frac{1}{2\kappa^2}\int \delta\mu_{[g,h]} e^{-2\Phi}[\widehat{R} + S + 4(D\Phi)^2 - \frac{1}{12}\widetilde{H}^2 - \frac{1}{4}F^{\widehat{a}}_{\mu\nu}F^{\widehat{a}\mu\nu}], \quad (14.25)$$

when the super–Yang–Mills multiplet is stated by the action

$$S_{YM} = \frac{1}{\kappa}\int \delta\mu_{[g,h]} e^{-2\Phi}[-\frac{1}{4}F^{\widehat{a}}_{\mu\nu}F^{\widehat{a}\mu\nu} - \frac{1}{2}\overline{\psi}^{\widehat{a}}\Gamma^{\mu}D_{\mu}\psi^{\widehat{a}}].$$

In these actions

$$A = A^{\widehat{a}}_{\mu}t^{\widehat{a}}\delta u^{\mu}$$

is the gauge d–field of $E_8 \times E_8$ or $Spin\,(32)\,/Z_2$ group (with generators $t^{\widehat{a}}$ labelled by the index $\widehat{a}$), having the strength

$$F = \delta A + g_F A \wedge A = \frac{1}{2}F^{\widehat{a}}_{\mu\nu}t^{\widehat{a}}\delta u^{\mu} \wedge \delta u^{\nu},$$

g_F being the coupling constant, and ψ is the gaugino of $E_8 \times E_8$ or $Spin\,(32)\,/Z_2$ group (details on constructions of locally anisotropic gauge and spinor theories can be found in Refs. [51, 50, 47, 55, 48, 40]). The action with B–field strength in (14.25) is defined as follows

$$\widetilde{H} = \delta B - \frac{\kappa}{\sqrt{2}}\omega_{CS}\,(A)\,,$$

for

$$\omega_{CS}\,(A) = tr\left(A \wedge \delta A + \frac{2}{3}g_F A \wedge A \wedge A\right).$$

Such constructions conclude in a theory with $S_{IIB} + S_{YM}+$ fermionic terms with anholonomies and $\mathcal{N}=1$ supersymmetry.

Finally, we emphasize that the actions for supersymmetric anholonomic models can considered in the framework of (super) geometric formulation of supergravities in $n+m = 10$ and 11 dimensions on superbundles provided with N–connection structure [41, 43, 47].

14.3.2 Superstring effective actions and anisotropic toroidal compactifications

The supergravity actions presented in the previous subsection can be included in different supersymmetric string theories which emphasize anisotropic effects if spacetimes provided with N–connection structure are considered. In this subsection we analyze a model with toroidal compactification when the background is locally anisotropic. In order to obtain four–dimensional (4D) theories, the simplest way is to make use of the Kaluza–Klein idea: to propose a model when some of the dimensions are curled–up into a compact manifold, the rest of dimensions living only for non–compact manifold. Our aim is to show that in result of toroidal compactifications the resulting 4D theory could be locally anisotropic.

The action (14.25) can be obtained also as a 10 dimensional heterotic string effective action (in the locally isotropic variant see, for instance, Ref. [25])

$$(\alpha')^{8} S_{10-n'-m'} = \int \delta^{10}u\sqrt{|g_{\alpha\beta}|}e^{-\Phi'}[\widehat{R}+S+(D\Phi')^2-\frac{1}{12}\widetilde{H}^2-\frac{1}{4}F^{\widehat{a}}_{\mu\nu}F^{\widehat{a}\mu\nu}]+o(\alpha'), \quad (14.26)$$

where we redefined $2\Phi \rightarrow \Phi'$, use the string constant α' and consider the (n', m') as the (holonomic, anholonomic) dimensions of the compactified spacetime (as a particular case we can consider $n' + m' = 4$, or $n' + m' < 10$ for any brane configurations. Let us use parametrizations of indices and of vierbeinds: Greek indices $\alpha, \beta, ...\mu...$ run values for a 10 dimensional spacetime and split as $\alpha = (\alpha', \widehat{\alpha}), \beta = \left(\beta', \widehat{\beta}\right), ...$ when primed indices $\alpha', \beta', ...\mu'...$ run values for compactified spacetime and split into h- and v–components like $\alpha' = (i', a'), \beta' = (j', b'), ...$; the frame coefficients are split as

$$e_{\mu}^{\underline{\mu}}(u) = \begin{pmatrix} e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'}) & A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'})e_{\widehat{\alpha}}^{\underline{\widehat{\alpha}}}(u^{\beta'}) \\ 0 & e_{\widehat{\alpha}}^{\underline{\widehat{\alpha}}}(u^{\beta'}) \end{pmatrix}$$

where $e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'})$, in their turn, are taken in the form (14.7),

$$e_{\alpha'}^{\underline{\alpha}'}(u^{\beta'}) = \begin{pmatrix} e_{i'}^{\underline{i}'}(x^{j'}, y^{a'}) & N_{i'}^{a'}(x^{j'}, y^{a'})e_{a'}^{\underline{a}'}(x^{j'}, y^{a'}) \\ 0 & e_{a'}^{\underline{a}'}(x^{j'}, y^{a'}) \end{pmatrix}. \quad (14.27)$$

For the metric we have the recurrent ansatz

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{\alpha'\beta'}(u^{\beta'}) + N_{\alpha'}^{\widehat{\alpha}}(u^{\beta'})N_{\beta'}^{\widehat{\beta}}(u^{\beta'})h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'})N_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) \\ h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'})N_{\beta'}^{\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) \end{bmatrix}.$$

where

$$g_{\alpha'\beta'} = \left[\begin{array}{cc} g_{i'j'}(u^{\beta'}) + N_{i'}^{a'}(u^{\beta'}) N_{j'}^{b'}(u^{\beta'}) h_{a'b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'}) N_{i'}^{a'}(u^{\beta'}) \\ h_{a'b'}(u^{\beta'}) N_{j'}^{b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'}) \end{array} \right]. \tag{14.28}$$

The part of action (14.26) containing the gravity and dilaton terms becomes

$$\begin{aligned} (\alpha')^{n'+m'} S_{n'+m'}^{heterotic} &= \int \delta^{n'+m'} u \sqrt{|g_{\alpha\beta}|} e^{-\phi} [\widehat{R}' + S' + (\delta_{\mu'}\phi)(\delta^{\mu'}\phi) \\ &+ \frac{1}{4}(\delta_{\mu'} h_{\widehat{\alpha}\widehat{\beta}})(\delta^{\mu'} h^{\widehat{\alpha}\widehat{\beta}}) - \frac{1}{4} h_{\widehat{\alpha}\widehat{\beta}} F_{\mu'\nu'}^{[A]\widehat{\alpha}} F^{[A]\widehat{\beta}\mu'\nu'}], \end{aligned} \tag{14.29}$$

where $\phi = \Phi' - \frac{1}{2}\log\left(\det|h_{\widehat{\alpha}\widehat{\beta}}|\right)$ and $F_{\mu'\nu'}^{[A]\widehat{\alpha}} = \delta_{\mu'} A_{\nu'}^{\widehat{\alpha}} - \delta_{\nu'} A_{\mu'}^{\widehat{\alpha}}$ and the h- and v–components of the induced scalar curvature, respectively, $\widehat{R}'$ and S' (see formula (13.16) in Appendix) are primed in order to point that these values are for the lower dimensional space. The antisymmetric tensor part may be decomposed in the form

$$\begin{aligned} -\frac{1}{12}\int \delta^{10} u \sqrt{|g_{\alpha\beta}|} e^{-\Phi'} H^{\mu\nu\rho} H_{\mu\nu\rho} &= -\frac{1}{4}\int \delta^{n'+m'} u \sqrt{|g_{\alpha'\beta'}|} e^{-\phi} \times \\ & [H^{\mu'\widehat{\alpha}\widehat{\beta}} H_{\mu'\widehat{\alpha}\widehat{\beta}} + H^{\mu'\nu'\widehat{\beta}} H_{\mu'\nu'\widehat{\beta}} + \frac{1}{3} H^{\mu'\nu'\rho'} H_{\mu'\nu'\rho'}], \end{aligned} \tag{14.30}$$

where, for instance,

$$H_{\mu'\widehat{\alpha}\widehat{\beta}} = e_{\mu'}^{\underline{\mu}'} e_{\underline{\mu}'}^{\mu} H_{\mu\widehat{\alpha}\widehat{\beta}}$$

and we have considered $H_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = 0$. In a similar manner we can decompose the action for gauge fields $\widehat{A}_\mu^I$ with index $I = 1, ..., 32$,

$$\int \delta^{10} u \sqrt{|g_{\alpha\beta}|} e^{-\Phi'} \sum_{I=1}^{16} \widehat{F}^{I,\mu\nu} \widehat{F}_{\mu\nu}^I = \int \delta^{n'+m'} u \sqrt{|g_{\alpha'\beta'}|} e^{-\phi} \sum_{I=1}^{16} [\widehat{F}^{I,\mu'\nu'} \widehat{F}_{\mu'\nu'}^I + 2\widehat{F}^{I,\mu'\widehat{\nu}} \widehat{F}_{\mu'\widehat{\nu}}^I], \tag{14.31}$$

with

$$\begin{aligned} Y_{\widehat{\alpha}}^I &= A_{\widehat{\alpha}}^I,\ A_{\alpha'}^I = \widehat{A}_{\alpha'}^I - Y_{\widehat{\alpha}}^I A_\mu^{\widehat{\alpha}}, \\ \widehat{F}_{\mu'\nu'}^I &= F_{\mu'\nu'}^I + Y_{\widehat{\alpha}}^I F_{\mu'\nu'}^{[A]\widehat{\alpha}},\ \widehat{F}_{\mu'\widehat{\nu}}^I = \delta_{\mu'} Y_{\widehat{\alpha}}^I,\ \widehat{F}_{\mu'\nu'}^I = \delta_{\mu'} A_{\nu'}^I - \delta_{\nu'} A_{\mu'}^I, \end{aligned}$$

where the scalars $Y_{\widehat{\alpha}}^I$ coming from the ten–dimensional vectors should be associated to a normal Higgs phenomenon generating a mass matrix for the gauge fields. Thy are related to the fact that a non–Abelian gauge field strength contains nonlinear terms not being certain derivatives of potentials.

After a straightforward calculus of the actions' components (14.29), (14.30) and (14.31) (for locally isotropic gauge theories and strings, see a similar calculus, for instance, in Refs. [25]), putting everything together, we can write the $n'+m'$ dimensional action including anholonomic interactions in the form

$$S^{heterotic}_{n'+m'} = \int \delta^{n'+m'}u\sqrt{|g_{\alpha'\beta'}|}e^{-\phi}[\widehat{R}' + S' + (\delta_{\mu'}\phi)(\delta^{\mu'}\phi) - \frac{1}{12}H^{\mu'\nu'\rho'}H_{\mu'\nu'\rho'} -\frac{1}{4}(M^{-1})_{\overline{I}\overline{J}}F^{\overline{I}}_{\mu'\nu'}F^{\overline{J}\mu'\nu'} + \frac{1}{8}Tr\left(\delta_{\mu'}M\delta^{\mu'}M^{-1}\right)], \quad (14.32)$$

where $\overleftarrow{R}' = \widehat{R}' + S'$ is the d–scalar curvature of type (13.16) induced after toroidal compactification, the $(2p+16)\times(2p+16)$ dimensional symmetric matrix M has the structure

$$M = \begin{pmatrix} \underline{g}^{-1} & \underline{g}^{-1}C & \underline{g}^{-1}Y^t \\ C^t\underline{g}^{-1} & \underline{g} + C^t\underline{g}^{-1}C + Y^tY & C^t\underline{g}^{-1}Y^t + Y^t \\ Y\underline{g}^{-1} & Y\underline{g}^{-1}C + Y & I_{16} + Y\underline{g}^{-1}Y^t \end{pmatrix}$$

with the block sub–matrices

$$\underline{g} = \left(\underline{g}_{\alpha\beta}\right), C = \left(C_{\widehat{\alpha}\widehat{\beta}} = B_{\widehat{\alpha}\widehat{\beta}} - \frac{1}{2}Y^I_{\widehat{\alpha}}Y^I_{\widehat{\beta}}\right), Y = \left(Y^I_{\widehat{\alpha}}\right),$$

for which I_{16} is the 16 dimensional unit matrix; for instance, Y^t denotes the transposition of the matrix Y. The dimension p satisfies the condition $n'+m'-p=16$ relevant to the heterotic string describing p left–moving bosons and $n'+m'$ right–moving ones with m' constrained degrees of freedom. To have good modular properties $p-n'-m'$ should be a multiple of eight. The indices $\overline{I},\overline{J}$ run values $1,2,...(2p+16)$. The action (14.32) describes a heterotic string effective action with local anisotropies (contained in the values $\widehat{R}', S'$ and $\delta_{\mu'}$) induced by the fact that the dynamics of the right–moving bosons are subjected to certain constraints. The induced metric $g_{\alpha'\beta'}$ is of type (13.6) given with respect to an N–elongated basis (13.3) (in this case, primed), $\delta_{\mu'} = \partial_{\mu'} + N_{\mu'}$. For $N^{a'}_{i'} \to 0$ and m', i. e. for a subclass of effective backgrounds with block $n'\times n' \oplus m'\times m'$ metrics $g_{\alpha'\beta'}$, the action (14.32) transforms in the well known isotropic form (see, for instance, formula (C22), from the Appendix C in Ref. [25], from which following the 'geometric d–covariant rule' we could write down directly (14.32); this is a more formal approach which hides the physical meaning and anholonomic character of the components (14.29), (14.30) and (14.31)).

14.3.3 4D NS–NS anholonomic field equations

As a matter of principle, compactifications of all type in (super) string theory can be performed in such ways as to include anholonomic frame effects as in the previous subsection. The simplest way to define anisotropic generalizations or such models is to apply the 'geometric d–covariant rule' when the tensors, spinors and connections are changed into theirs corresponding N–distinguished analogous. As an example, we write down here the anholonomic variant of the toroidally compactified (from ten to four dimensions) NS–NS action (we write in brief NS instead of Neveu–Schwarz) [37],

$$S=\int \delta^4 u\sqrt{|g_{\alpha'\beta'}|}e^{-\varphi}[\widehat{R}'+S'+(\delta_{\mu'}\phi)(\delta^{\mu'}\phi)-\frac{1}{2}(\delta_{\mu'}\beta)(\delta^{\mu'}\beta)-\frac{1}{2}e^{2\varphi}(\delta_{\mu'}\sigma)(\delta^{\mu'}\sigma)], \quad (14.33)$$

for a d–metric parametrized as

$$\delta s^2=-\epsilon\delta(x^{0'})^2+g_{\underline{\alpha'}\underline{\beta'}}\delta u^{\underline{\alpha'}}\delta u^{\underline{\beta'}}+e^{\beta/\sqrt{3}}\delta_{\widehat{\alpha}\widehat{\gamma}}\delta u^{\widehat{\alpha}}\delta u^{\widehat{\beta}},$$

where, for instance, $u^{\alpha'}=(x^{0'},u^{\underline{\alpha'}}),\underline{\alpha'}=1,2,3$ and $\widehat{\alpha},\widehat{\beta},...=4,5,...9$ are indices of extra dimension coordinates, $\epsilon=\pm1$ depending on signature (in usual string theory one takes $x^{0'}=t$ and $\epsilon=-1$), the modulus field β is normalized in such a way that it becomes minimally coupled to gravity in the Einstein d–frame, σ is a pseudo–scalar axion d–field, related with the anti–symmetric strength,

$$H^{\alpha'\beta'\gamma'}(u^{\alpha'})=\varepsilon^{\alpha'\beta'\gamma'\tau'}e^{\varphi(u^{\alpha'})}D_{\tau'}\sigma(u^{\alpha'}),$$

$\varepsilon^{\alpha'\beta'\gamma'\tau'}$ being completely antisymmetric and $\varphi(u^{\alpha'})=\Phi'(u^{\alpha'})-\sqrt{3}\beta(u^{\alpha'})$, with $\Phi'(u^{\alpha'})$ taken as in (14.26).

We can derive certain locally anisotropic field equations from the action (14.33) by varying with respect to N–adapted frames for massless excitations of $g_{\alpha'\beta'},B_{\alpha'\beta'},\beta$ and φ, which are given by

$$2\left[R_{\mu'\nu'}-\frac{1}{2}\left(\widehat{R}'+S'\right)g_{\mu'\nu'}\right]=\frac{1}{2}H_{\mu'\lambda'\tau'}H_{\nu'}{}^{\lambda'\tau'}-H^2g_{\mu'\nu'}+ \quad (14.34)$$

$$\left(\delta_{\mu'}^{\lambda'}\delta_{\nu'}^{\tau'}-\frac{1}{2}g_{\mu'\nu'}g^{\lambda'\tau'}\right)D_{\lambda'}\beta D_{\tau'}\beta-g_{\mu'\nu'}(D\varphi)^2+2\left(g_{\mu'\nu'}g^{\lambda'\tau'}-\delta_{\mu'}^{\lambda'}\delta_{\nu'}^{\tau'}\right)D_{\lambda'}D_{\tau'}\varphi = 0,$$

$$D_{\mu'}\left(e^{-\varphi}H^{\mu'\nu'\lambda'}\right) = 0,$$

$$D_{\mu'}\left(e^{-\varphi}D^{\mu'}\beta\right) = 0,$$

$$2D_{\mu'}D^{\mu'}\varphi=-\widehat{R}'-S'+(D\varphi)^2+\frac{1}{2}(D\beta)^2+\frac{1}{12}H^2 = 0,$$

where $H^2 = H_{\mu'\lambda'\tau'}H^{\mu'\ \lambda'\tau'}$ and, for instance, $(D\varphi)^2 = D_{\mu'}\varphi D^{\mu'}\varphi$. We may select a consistent solution of these field equations when the internal space is static with $D_{\mu'}\beta = 0$.

The equations (14.34) can be decomposed in invariant h– and v–components like the Einstein d–equations (13.18) (we omit a such trivial calculus). We recall [15] that the NS–NS sector is common to both the heterotic and type II string theories and is comprised of the dilaton, graviton and antisymmetric two–form potential. The obtained equations (14.34) define respective anisotropic string corrections to the anholonomic Einstein gravity.

14.3.4 Distinguishing anholonomic Riemannian–Finsler (super) gravities

There are two classes of general anisotropies contained in supergravity and superstring effective actions:

- Generic local anisotropies contained in the higher dimension (11, for supergravity models, or 10, for superstring models) which can be also induced in lower dimension after compactification (like it was considered for actions (14.23), (14.24), (14.25) and (14.26)).

- Local anisotropies which are in induced on the lower dimensional spacetime (for instance, actions (14.32) and (14.33) and respective field equations).

All types of general supergravity/superstring anisotropies may be in their turn to be distinguished to be of "pure" B–field origin, of "pure" anholonomic frame origin with arbitrary B–field, or of a mixed type when local anisotropies are both induced in a nonlinear form by both anholonomic (super) vielbeins and B–field (like we considered in subsection 14.2.2 for bosonic strings). In explicit form, a model of locally anisotropic superstring corrected gravity is to be constructed following the type of parametrizations we establish for the N–coefficients, d–metrics and d–connections.

For instance, if we choose the frame ansatz (14.27) and corresponding metric ansatz (14.28) with general coefficients $g_{i'j'}(x^{j'}, y^{c'}), h_{a'b'}(x^{j'}, y^{a'})$ and $N_{i'}^{a'}(x^{j'}, y^{a'})$ satisfying the effective field equations (14.34) (containing also the fields $H_{\mu'\lambda'\tau'}, \varphi$ and β) we define an anholonomic gravity model corrected by toroidally compactified (from ten to four dimensions) NS–NS superstring model. In four and five dimensional Einstein/ Kaluza–Klein gravities, there were constructed a number of anisotropic black hole, wormhole, solitonic, spinor waive and Taub/NUT metrics [45, 49, 46, 53, 52]; in section 2.5 we shall consider some generalizations to string gravity.

Another possibility is to impose the condition that $g_{i'j'}, h_{a'b'}$ and $N_{i'}^{a'}$ are of Finsler type, $g_{i'j'}^{[F]} = h_{i'j'}^{[F]} = \partial^2 F^2 / 2\partial y^i \partial y^j$ (14.127) and $N_j^{[F]i}(x,y) = \partial \left[c_{lk}^{\iota}(x,y) y^l y^k \right] / 4\partial y^j$ (13.33), with an effective d–metric (13.34). If a such set of metric/N–connection coefficients can found as a solution of some string gravity equations, we may construct a lower dimensional Finsler gravity model induced from string theory (it depends of what kind of effective action, (14.32) or (14.33), we consider). Instead of a Finsler gravity we may search for a Lagrange model of string gravity if the d–metric coefficients are taken in the form (13.35).

We conclude this section by a remark that we may construct various type of anholonomic Riemannian and generalized Finsler/Lagrange string gravity models, with anisotropies in higher and/or lower dimensions by prescribing corresponding parametrizations for g_{ij}, h_{ab} and N_i^a (for 'higher' anisotropies) and $g_{i'j'}, h_{a'b'}$ and $N_{i'}^{a'}$ (for 'lower' anisotropies). The anholonomic structures may be of mixed type, for instance, in some dimensions being of Finsler configuration, in another ones being with anholonomic Riemannian metric, in another one of Lagrange type and different combinations and generalizations, see explicit examples in Section 2.5.

14.4 Noncommutative Anisotropic Field Interactions

We define the noncommutative field theory in a new form when spacetimes and configuration spaces are provided with some anholonomic frame and associated N–connection structures. The equations of motions are derived from functional integrals in a usual manner but considering N–elongated partial derivatives and differentials.

14.4.1 Basic definitions and conventions

The basic concepts on noncommutative geometry are outlined here in a somewhat pedestrian way by emphasizing anholonomic structures. More rigorous approaches on mathematical aspects of noncommutative geometry may be found in Refs. [8, 19, 16, 18], physical versions are given in Refs. [13, 57, 9, 38] (the review [50] is a synthesis of results on noncommutative geometry, N–connections and Finsler geometry, Clifford structures and anholonomic gauge gravity based on monographs [18, 47, 55, 29]).

As a fundamental ingredient we use an associative, in general, noncommutative algebra $\mathcal{A}$ with a product of some elements $a, b \in \mathcal{A}$ denoted $ab = a \cdot b$, or in the conotation to noncommutative spaces, written as a "star" product $ab = a \star b$. Every element $a \in \mathcal{A}$

corresponds to a configuration of a classical complex scalar field on a "space" M, a topological manifold, which (in our approach) can be enabled with a N–connection structure. This associated noncommutative algebra generalize the algebra of complex valued functions $\mathcal{C}(M)$ on a manifold M (for different theories we may consider instead M a tangent bundle TM, or a vector bundle $E\left(M\right)$). We consider that all functions referring to the algebra $\mathcal{A},$ denoted as $\mathcal{A}\left(M\right)$, arising in physical considerations are of necessary smooth class (continuous, smooth, subjected to certain bounded conditions etc.).

Matrix algebras and noncommutativity

As the most elementary examples of noncommutative algebras, which are largely applied in quantum field theory and noncommutative geometry, one considers the algebra $Mat_k(\mathbb{C})$ of complex $k\times k$ matrices and the algebra $Mat_k\left(\mathcal{C}(M)\right)$ of $k\times k$ matrices whose matrix elements are elements of $\mathcal{C}(M)$. The last algebra may be also defined as a tensor product,

$$Mat_k\left(\mathcal{C}(M)\right) = Mat_k(\mathbb{C})\otimes\mathcal{C}(M).$$

The last construction is easy to be generalized for arbitrary noncommutative algebra $\mathcal{A}$ as

$$Mat_k\left(\mathcal{A}\right) = Mat_k(\mathbb{C})\otimes\mathcal{A},$$

which is just the algebra of $k\times k$ matrices with elements in $\mathcal{A}$. The algebra $Mat_k\left(\mathcal{A}\right)$ admits an authomorphism group $GL(k,\mathbb{C})$ with the action defined as $a\rightarrow\varsigma^{-1}a\varsigma$, for $a\in\mathcal{A},\varsigma\in GL(k,\mathbb{C})$. One considers the subgroup $U\left(k\right)\subset GL(k,\mathbb{C})$ which is preserved by hermitian conjugations, $a\rightarrow a^+$, and reality conditions, $a=a^+$. To define the hermitian conjugation, for which the hermitian matrices $a=a^+$ have real eigenvalues, it is considered that $\left(a^+\right)^+=a$ and $\left(ca\right)^+=c^*a^+$, for $c\in\mathbb{C}$ and c^* being the complex conjugated element of c, i. e. it defined an antiholomorphic involution.

Noncommutative Euclidean space $\mathbb{R}^k_\theta$

Another simple example of a noncommutative space is the 'noncommutative Euclidean space' $\mathbb{R}^k_\theta$ defined by all complex linear combinations of products of variables $x=\{x^j\}$ satisfying

$$[x^j,x^l]=x^jx^l-x^lx^j=i\theta^{jl}, \qquad (14.35)$$

where i is the complex 'imaginary' unity and θ^{jl} are real constants treated as some noncommutative parameters or a "Poison tensor" by analogy to the Poison bracket in quantum mechanics where the commutator [...] of hermitian operators is antihermitian.

A set of partial derivatives $\partial_j = \partial/\partial x^i$ on $\mathbb{R}^k_\theta$ can be defined by postulating the relations

$$\begin{aligned} \partial_j x^n &= \delta^n_j, \\ [\partial_j, \partial_n] &= -i\Xi_{jn} \end{aligned} \qquad (14.36)$$

where Ξ_{jn} may be zero, but in general is non–trivial if we wont to incorporate some additional magnetic fields or anholonomic relations. A simplified noncommutative differential calculus can be constructed if $\Xi_{jn} = -\left(\theta^{-1}\right)_{jn}$.

The metric structure on $\mathbb{R}^k_\theta$ is stated by a constant symmetric tensor η_{nj} for which $\partial_j \eta_{nj} = 0$.

Infinitesimal translations $x^j \to x^j + a^j$ on $\mathbb{R}^k_\theta$ are defined as actions on functions φ of type $\triangle\varphi = a^j \partial_j \varphi$. Because the coordinates are non–commuting there are formally defined inner derivations as

$$\partial_j \varphi = \left[-i\left(\theta^{-1}\right)_{jn} x^n, \varphi\right] \qquad (14.37)$$

which result in exponential global tanslations

$$\varphi\left(x^j + \epsilon^j\right) = e^{-i\theta_{lj}\epsilon^l x^j} \varphi\left(x^j\right) e^{i\theta_{lj}\epsilon^l x^j}.$$

In order to understand the symmetries of the space $\mathbb{R}^k_\theta$ it is better to write the metric and Poisson tensor in the forms

$$\begin{aligned} ds^2 &= \sum_{A=1}^{r} dz_A d\overline{z}_A + \sum_B dy_B^2, \\ &= dq_A^2 + dp_A^2 + dy_B^2; \end{aligned} \qquad (14.38)$$

$$\theta = \frac{1}{2}\sum_{A=1} \theta_A \partial_{z_A} \wedge \partial_{\overline{z}_A},\ \theta_A > 0,$$

where $z_A = q_A + ip_A$ and $\overline{z}_A = q_A - ip_A$ are some convenient complex coordinates for which there are satisfied the commutation rules

$$\begin{aligned} [y_A, y_B] &= [y_B, q_A] = [y_B, p_A] = 0, \\ [q_A, p_B] &= i\theta_A \delta_{AB}. \end{aligned}$$

Now, it is obvious that for fixed types of metric and Poisson structures (14.38) there are two symmetry groups on $\mathbb{R}^k_\theta$, the group of rotations, denoted $O(k)$, and the group of invariance of the form θ, denoted $Sp(2r)$.

The noncommutative derivative and integral

In order to elaborate noncommutative field theories in terms of an associative noncommutative algebra $\mathcal{A}$, additionally to the derivatives ∂_j we need an integral $\int Tr$ which following the examples of noncommutative matrix spaces must contain also the "trace" operator. In this case we can not separate the notations of trace and integral.

It should be noted here that the role of derivative ∂_j can be played by any sets of elements $d_j \in \mathcal{A}$ which some formal derivatives as $\partial_j A = [d_j, A]$, for $A \in \mathcal{A}$; derivations written in this form are called as inner derivations while those which can not written in this form are referred to as outer derivations.

The general derivation and integration operations are defined as some general dual linear operators satisfying certain formal properties: 1) the Leibnitz rule of the derivative, $\partial_j(AB) = \partial_j(A)B + A(\partial_j B)$; 2) the integral of the trace of a total derivative is zero, $\int Tr \partial_j A = 0$; 3) the integral of the trace of a commutator is zero, $\int Tr[A, B] = 0$, for any $A, B \in \mathcal{A}$. For some particular classes of functions in some noncommutative models the condition 2) and/or 3) may be violated, see details and discussion in Ref. [13].

Given a noncommutative space induced by some relations (14.35), the algebra of functions on $\mathbb{R}^k$ is deformed on $\mathbb{R}^k_\theta$ such that

$$\begin{aligned} f(x) \star \varphi(x) &= e^{\frac{i}{2}\theta^{jk}\frac{\partial}{\partial \xi^j}\frac{\partial}{\partial \varsigma^k}} f(x+\xi)\,\varphi(x+\zeta)_{|\xi=\varsigma=0} \\ &= f\varphi + \frac{i}{2}\theta^{jk}\partial_j f \partial_k \varphi + o\left(\theta^2\right), \end{aligned} \qquad (14.39)$$

which define the Moyal bracket (product), or star product ($\star$–product), of functions which is associative compatible with integration in the sense that for matrix valued functions f and φ that vanish rapidly enough at infinity we can integrate by parts in the integrals

$$\int Tr\ f \star \varphi = \int Tr\ \varphi \star f.$$

In a more rigorous operator form the star multiplication is defined by considering a space M_θ, locally covered by coordinate carts with noncommutative coordinates (14.35), and choosing a linear map S from M_θ to $\mathcal{C}(M)$, called the "symbol" of the operator, when $\widehat{f} \to S\left[\widehat{f}\right]$. This way, the original operator multiplication is expressed in terms of the star product of symbols as

$$\widehat{f}\widehat{\varphi} = S^{-1}\left[S\left[\widehat{f}\right] \star S\left[\widehat{\varphi}\right]\right].$$

It should be noted that there could be many valid definitions of S, corresponding to different choices of operator ordering prescription for S^{-1}. One writes, for simplicity, $\int Tr\ f \star \varphi = \int Tr\ f\varphi$ in some special cases.

14.4.2 Anholonomic frames and noncommutative spacetimes

One may consider that noncommutative relations for coordinates and partial derivatives (14.35) and (14.36) are introduced by specific form of anholonomic relations (13.4) for some formal anholonomic frames of type (13.2) and/or (13.3) (see Appendix) when anholonomy coefficients are complex and depend nonlinearly on frame coefficients. We shall not consider in this work the method of complex nonlinear operator anholonomic frames with associated nonlinear connection structure, containing as particular cases various type of Finsler/Cartan and Lagrange/Hamilton geometries in complexified form, which could consist in a general complex geometric formalism for noncommutative theories but we shall restrict our analysis to noncommutative spaces for which the coordinates and partial derivatives are distinguished by a N–connection structure into certain holonomic and anholonomic subsets which generalize the N–elongated commutative differential calculus (considered in the previous Sections) to a variant of both $N-$ and θ–deformed one.

In order to emphasize the N–connection structure on respective spaces we shall write $M_\theta^N, TM_\theta^N, E_\theta^M(M_\theta)$, $\mathcal{C}(M^N), \mathcal{A}^N$ and $\mathcal{A}\left(M^N\right)$. For a space M^N provided with N–connection structure, the matrix algebras considered in the previous subsection may be denoted $Mat_k\left(\mathcal{C}(M^N)\right)$ and $Mat_k\left(\mathcal{A}^N\right)$.

Noncommutative anholonomic derivatives

We introduce splitting of indices, $\alpha = (i,a)$, $\beta = (j,b)$, ..., and coordinates, $u^\alpha = (x^i, y^a)$, ..., into 'horizontal' and 'vertical' components for a space M_θ (being in general a manifold, tangeng/vector bundle, or their duals, or higher order models [28, 40, 43, 47, 55]. The derivatives ∂_i satisfying the conditions (14.36) must be changed into some N–elongated ones if both anholnomy and noncommutative structures are introduced into consideration.

In explicit form, the anholonomic analogous of (14.35) is stated by a set of coordinates $u^\alpha = (x^i, y^a)$ satisfying the relations

$$[u^\alpha, u^\beta] = i\Theta^{\alpha\beta}, \tag{14.40}$$

with $\Theta^{\alpha\beta} = \left(\Theta^{ij}, \Theta^{ab}\right)$ parametrized as to have a noncommutative structure locally adapted to the N–connection, and the analogouses of (14.36) redefined for operators (13.2) as

$$\begin{aligned} \delta_\alpha u^\beta &= \delta_\alpha^\beta, \text{ for } \delta_\alpha = (\delta_i = \partial_i - N_i^a \partial_a, \partial_b), \\ [\delta_\alpha, \delta_\beta] &= -i\Xi_{\alpha\beta}, \end{aligned} \tag{14.41}$$

where $\Xi_{\alpha\beta} = -\left(\Theta^{-1}\right)_{\alpha\beta}$ for a simplified N–elongated noncommutative differential calculus. We emphasize that if the vielbein transforms of type (14.7) and frames of type (14.8) and (14.9) are considered, the values $\Theta^{\alpha\beta}$ and $\Xi_{\alpha\beta}$ could be some complex functions depending on variables u^β including also the anholonomy contributions of N_i^a. In particular cases, they my constructed by some anholonomic frame transforms from some constant real tensors.

An anholonomic noncommutative Euclidean space $\mathbb{R}^{n+m}_{N,\theta}$ is defined as a usual one of dimension $k = n + m$ for which a N–connection structure is prescribed by coefficients $N_i^a(x, y)$ which states an N–elongated differential calculus. The d–metric $\eta_{\alpha\beta} = (\eta_{ij}, \eta_{ab})$ and Poisson d–tensor $\Theta^{\alpha\beta} = \left(\Theta^{ij}, \Theta^{ab}\right)$ are introduced via vielbein transforms (14.7) depending on N–coefficients of the corresponding constant values contained in (14.35) and (14.38). As a matter of principle such noncommutative spaces are already curved.

The interior derivative (14.37) is to be extended on $\mathbb{R}^{n+m}_{N,\theta}$ as

$$\delta_\alpha \varphi = \left[-i\left(\Theta^{-1}\right)_{\alpha\beta} u^\beta, \varphi\right].$$

In a similar form, by introducing operators δ_α instead of ∂_α, we can generalize the Moyal product (14.39) for anisotropic spaces:

$$\begin{aligned} f(x) \star \varphi(x) &= e^{\frac{i}{2}\Theta^{\alpha\beta}\frac{\delta}{\partial\xi^\alpha}\frac{\delta}{\partial\varsigma^\beta}} f(x+\xi)\,\varphi(x+\zeta)_{|\xi=\varsigma=0} \\ &= f\varphi + \frac{i}{2}\Theta^{\alpha\beta}\delta_\alpha f \delta_\beta \varphi + o\left(\theta^2\right). \end{aligned}$$

For elaborating of perturbation and scattering theory, the more useful basis is the plane wave basis, which for anholonomic noncommutative Euclidean spaces, consists of eigenfunctions of the derivatives

$$\delta_\alpha e^{ipu} = ip_\alpha e^{ipu}, pu = p_\alpha u^\alpha.$$

In this basis, the integral can be defined as

$${}^N\!\!\int Tr\ e^{ipu} = \delta_{p,0}$$

where the symbol $\int Tr$ is enabled with the left upper index N in order to emphasize that integration is to be performed on a N–deformed space (we shall briefly call this as "N–integration") and the delta function may be interpreted as usually (its value at zero represents the volume of physical space, in our case, N–deformed). There is a specific multiplication low with respect to the plane wave basis: for instance, by operator reordering,

$$e^{ipu}\cdot e^{ip'u}=e^{-\frac{1}{2}\Theta^{\alpha\beta}p_\alpha p'_\beta}\ e^{i(p+p')u},$$

when $\Theta^{\alpha\beta}p_\alpha p'_\beta$ may be written as $p\times p'\equiv\Theta^{\alpha\beta}p_\alpha p'_\beta=p\times_\Theta p'$. There is another example of multiplication, when the N–elongated partial derivative is involved,

$$e^{ipu}\cdot f(u)\cdot e^{-ipu}=e^{-\Theta^{\alpha\beta}p_\alpha\delta_\beta}f(u)=f\left(u^\beta-\Theta^{\alpha\beta}p_\alpha\right),$$

which shows that multiplication by a plane wave in anholonomic noncommutative Euclidean space translates and N–deform a general function by $u^\beta\to u^\beta-\Theta^{\alpha\beta}p_\alpha$. This exhibits both the nonlocality and anholonomy of the theory and preserves the principles that large momenta lead to large nonlocality which can be also locally anisotropic.

Noncommutative anholonomic torus

Let us define the concept of noncommutative anholonomic torus, $\mathbf{T}^{n+m}_{N,\theta}$, i. e. the algebra of functions on a noncommutative torus with some splitting of coordinates into holonomic and anholonomic ones. We note that a function f on a anholonomic torus $\mathbf{T}^{n+m}_{N}$ with N–decomposition is a function on $\mathbb{R}^{n+m}_N$ which satisfies a periodicity condition, $f(u^\alpha)=f(u^\alpha+2\pi z^\alpha)$ for d–vectors z^α with integer coordinates. Then the noncommutative extension is to define $\mathbf{T}^{n+m}_{N,\theta}$ as the algebra of all sums of products of arbitrary integer powers of the set of distinguished $n+m$ variables $U_\alpha=(U_i,U_j)$ satisfying

$$U_\alpha U_\beta=e^{-i\Theta^{\alpha\beta}}U_\beta U_\alpha. \tag{14.42}$$

The variables U_α are taken instead of e^{iu^α} for plane waves and the derivation of a Weyl algebra from (14.40) is possible if we take

$$\begin{aligned}[\delta_\alpha U_\beta] &= i\delta_{\alpha\beta}U_\beta,\\ {}^N\!\!\int Tr\ U_1^{z_1}...U_{n+m}^{z_{n+m}} &= \delta_{\overrightarrow{z},0}.\end{aligned}$$

In addition to the usual topological aspects for nontrivial values of N–connection there is much more to say in dependence of the fact what type of topology is induced by the N–connection curvature. We omit such consideration in this paper. The introduced in this subsection formulas and definitions transform into usual ones from noncommutative geometry if $N,m\to 0$.

14.4.3 Anisotropic field theories and anholonomic symmetries

In a formal sense, every field theory, commutative or noncommutative, can be anholonomically transformed by changing partial derivatives into N–elongated ones and redefining the integrating measure in corresponding Lagrangians. We shall apply this rule to noncommutative scalar, gauge and Dirac fields and make them to be locally anisotropic and to investigate their anholonomic symmetries.

Locally anisotropic matrix scalar field theory

A generic matrix locally anisotropic matrix scalar field theory with a hermitian matrix valued field $\phi(u) = \phi^{+}(u)$ and anholonomically N–deformed Euclidean action

$$S = {}^{N}\int \delta^{n+m}u\sqrt{|g_{\alpha\beta}|}\left[\frac{1}{2}g^{\alpha\beta}Tr\ \delta_{\alpha}\phi\ \delta_{\beta}\phi + V(\phi)\right]$$

where $V(\phi)$ is polynomial in variable $\phi, g_{\alpha\beta}$ is a d–metric of type (13.6) and δ_{α} are N–elongated partial derivatives (13.2). It is easy to check that if we replace the matrix algebra by a general associative noncommutative algebra $\mathcal{A}$, the standard procedure of derivation of motion equations, classical symmetries from Noether's theorem and related physical considerations go through but with N–elongated partial derivatives and N–integration: The field equations are

$$g^{\alpha\beta}\delta_{\alpha}\ \delta_{\beta}\phi = \frac{\partial V(\phi)}{\partial\phi}$$

and the conservation laws

$$\delta_{\alpha}J^{\alpha} = 0$$

for the current J^{α} is associated to a symmetry $\triangle\phi(\epsilon,\phi)$ determined by the N–adapted variational procedure, $\triangle S = {}^{N}\int Tr\ J^{\alpha}\delta_{\alpha}\epsilon$. We emphasize that these equations are obtained according the prescription that we at the first stage perform a usual variational calculus then we change the usual derivatives and differentials into N–elongated ones. If we treat the N–connection as an object which generates and associated linear connection with corresponding curvature we have to introduce into the motion equations and conservation laws necessary d–covariant objects curvature/torsion terms.

We may define the momentum operator

$$P_{\alpha} = -i\left(\Theta^{-1}\right)_{\alpha\beta}\ {}^{N}\int Tr\ u^{\beta}T^{0},$$

which follows from the anholonomic transform of the restricted stress–energy tensor' constructed from the Noether procedure with symmetries $\triangle\phi = i[\phi, \epsilon]$ resulting in

$$T^{\alpha} = ig^{\alpha\beta}[\phi, \delta_{\beta}\phi].$$

We chosen the simplest possibility to define for noncommutative scalar fields certain energy–momentum values and their anholonomic deformations. In general, in noncommutatie field theory one introduced more conventional stress–energy tensors [1].

Locally anisotropic noncommutative gauge fields

Some models of locally anisotropic Yang–Mills and gauge gravity noncommutative theories are analyzed in Refs. [48, 50]. Here we say only the basic facts about such theories with possible supersymmetry but not concerning points of gauge gravity.

Anholonomic Yang–Mills actions and MSYM model

A gauge field is introduced as a one form A_{α} having each component taking values in $\mathcal{A}$ and satisfying $A_{\alpha} = A_{\alpha}^{+}$ and curvature (equivalently, field strength)

$$F_{\alpha\beta} = \delta_{\alpha}A_{\beta} - \delta_{\beta}A_{\alpha} + i\left[A_{\alpha}A_{\beta}\right]$$

with gauge locally anisotropic transformation laws,

$$\triangle F_{\alpha\beta} = i\left[F_{\alpha\beta}, \epsilon\right] \text{ for } \triangle A_{\alpha} = \delta_{\alpha}\epsilon + i\left[A_{\alpha}, \epsilon\right]. \tag{14.43}$$

Now we can introduce the noncommutative locally anisotropic Yang–Mills action

$$S = -\frac{1}{4g_{YM}}{}^{N}\int Tr\ F^{2}$$

which describes the N–anholonomic dynamics of the gauge field A_{α}. Coupling to matter field can be introduced in a standard way by using N–elongated partial derivatives δ_{α},

$$\bigtriangledown_{\alpha}\varphi = \delta_{\alpha}\varphi + i\left[A_{\alpha}, \varphi\right].$$

Here we note that by using $Mat_{Z}\left(\mathcal{A}\right)$ we can construct both noncommutative and anisotropic analog of $U\left(Z\right)$ gauge theory, or, by introducing supervariables adapted to N–connections [43] and locally anisotropic spinors [40, 55], we can generate supersymmetric Yang–Mills theories. For instance, the maximally supersymmetric Yang–Mills

(MSYM) Lagrangian in ten dimensions with $\mathcal{N} = 4$ can be deduced in anisotropic form, by corresponding dimensional reductions and anholonomic constraints, as

$$S = {}^{N}\int \delta^{10}u \ Tr \ \left(F^2_{\alpha\beta} + i\overline{\chi}\overrightarrow{\bigtriangledown}\chi \right)$$

where χ is a 16 component adjoint Majorana–Weyl fermion and the spinor d–covariant derivative operator $\overrightarrow{\bigtriangledown}$ is written by using N–anholonomic frames.

The emergence of locally anisotropic spacetime

It is well known that spacetime translations may arise from a gauge group transforms in noncommutative gauge theory (see, for instance, Refs. [13]). If the same procedure is reconsidered for N–elongated partial derivatives and distinguished noncommutative parameters, we can write

$$\delta A_\alpha = v^\beta \delta_\beta A_\alpha$$

as a gauge transform (14.43) when the parameter ϵ is expressed as

$$\epsilon = v^\alpha \left(\Theta^{-1}\right)_{\alpha\beta} u^\beta = v^i \left(\Theta^{-1}\right)_{jk} x^k + v^a \left(\Theta^{-1}\right)_{ab} x^b,$$

which generates

$$\triangle A_\alpha = v^\beta \delta_\beta A_\alpha + v^\beta \left(\Theta^{-1}\right)_{\alpha\beta}.$$

This way the spacetime anholonomy is induced by a noncommutative gauge anisotropy. For another type of functions $\epsilon(u)$, we may generate another spacetime locally anisotropic transforms. For instance, we can generate a Poisson bracket $\{\varphi, \epsilon\}$ with N–elongated derivatives,

$$\triangle\varphi = i\left[\varphi, \epsilon\right] = \Theta^{\alpha\beta}\delta_\alpha\varphi\delta_\beta\epsilon + o\left(\delta^2_\alpha\varphi\delta^2_\beta\epsilon\right) \rightarrow \{\varphi, \epsilon\}$$

which proves that at leading order the locally anisotropic gauge transforms preserve the locally anisotropic noncommutative structure of parameter $\Theta^{\alpha\beta}$.

Now, we demonstrate that the Yang–Mills action may be rewritten as a "matrix model" action even the spacetime background is N–deformed. This is another side of unification of noncommutative spacetime and gauge field with anholonomically deformed symmetries. We can absorb a inner derivation into a vector potential by associating the covariant operator $\bigtriangledown_\alpha = \delta_\alpha + iA_\alpha$ to connection operators in $\mathbb{R}^{n+m}_{N,\theta}$,

$$\bigtriangledown_\alpha\varphi \rightarrow \left[C_\alpha, \varphi\right]$$

for

$$C_\alpha = \left(-i\Theta^{-1}\right)_{\alpha\beta} u^\beta + iA_\alpha. \quad (14.44)$$

As in usual noncommutative gauge theory we introduce the "covariant coordinates" but distinguished by the N–connection,

$$Y^{\alpha} = u^{\alpha} + \Theta^{\alpha\beta} A_{\beta}\left(u\right).$$

For invertible $\Theta^{\alpha\beta}$, one considers another notation, $Y^{\alpha} = i\Theta^{\alpha\beta}C_{\beta}$. Such transforms allow to express $F_{\alpha\beta} = i\left[\bigtriangledown_{\alpha}, \bigtriangledown_{\beta}\right]$ as

$$F_{\alpha\beta} = i\left[C_{\alpha}, C_{\beta}\right] - \left(\Theta^{-1}\right)_{\alpha\beta}$$

for which the Yang–Mills action transform into a matrix relation,

$$\begin{aligned} S &= {}^{N}Tr \sum_{\alpha,\beta} \left(\acute{\imath}\left[C_{\alpha}, C_{\beta}\right] - \left(\Theta^{-1}\right)_{\alpha\beta} \right)^{2} \qquad (14.45) \\ &= {}^{N}Tr\{ \left[\acute{\imath}\left[C_{k}, C_{j}\right] - \left(\Theta^{-1}\right)_{kj} \right] \left[\acute{\imath}\left[C^{k}, C^{j}\right] - \left(\Theta^{-1}\right)^{kj} \right] \\ &\quad + \left[\acute{\imath}\left[C_{a}, C_{b}\right] - \left(\Theta^{-1}\right)_{ab} \right] \left[\acute{\imath}\left[C^{a}, C^{b}\right] - \left(\Theta^{-1}\right)^{ab} \right] \} \end{aligned}$$

where we emphasize the N–distinguished components.

The noncommutative Dirac d–operator

If we consider multiplications $a \cdot \psi$ with $a \in \mathcal{A}$ on a Dirac spinor ψ, we can have two different physics depending on the orders of such multiplications we consider, $a\psi$ or ψa. In order to avoid infinite spectral densities, in the locally isotropic noncommutative gauge theory, one writes the Dirac operator as

$$\overrightarrow{\bigtriangledown}\psi = \gamma^{i}\left(\overrightarrow{\bigtriangledown}_{i}\psi - \psi\partial_{i}\right) = 0.$$

In the locally anisotropic case we have to introduce N–elongated partial derivatives,

$$\begin{aligned} \overrightarrow{\bigtriangledown}\psi &= \gamma^{\alpha}\left(\overrightarrow{\bigtriangledown}_{\alpha}\psi - \psi\delta_{\alpha}\right) \\ &= \gamma^{i}\left(\overrightarrow{\bigtriangledown}_{i}\psi - \psi\delta_{i}\right) + \gamma^{a}\left(\overrightarrow{\bigtriangledown}_{a}\psi - \psi\delta_{a}\right) = 0 \end{aligned}$$

and use a d–covariant spinor calculus [40, 55].

The N–adapted stress–energy tensor

The action (14.45) produces a stress–energy d–tensor

$$T_{\alpha\beta}(p) = \sum_{\gamma} \int_0^1 ds \ {}^N\!\int Tr \ e^{isp_\tau Y^\tau} [C_\alpha, C_\gamma] \ e^{i(1-s)p_\tau Y^\tau} [C_\beta, C_\gamma]$$

as a Noether current derived by the variation $C_\alpha \to C_\alpha + a_\alpha(p) e^{isp_\tau Y^\tau}$. This d–tensor has a property of conservation,

$$p_\tau \Theta^{\tau\lambda} T_{\lambda\beta}(p) = 0$$

for the solutions of field equations and seem to be a more natural object in string theory, which admits an anholonomic generalizations by "distinguishing of indices".

The anholonomic Seiberg–Witten map

There are two different types of gauge theories: commutative and noncommutative ones. They my be related by the so–called Seiberg–Witten map [38] which explicitly transforms a noncommutative vector potential to a conventional Yang–Mills vector potential. This map can be generalized in gauge gravity and for locally anisotropic gravity [48, 50]. Here we define the Seiberg–Witten map for locally anisotropic gauge fields with N–elongated partial derivatives.

The idea is that if there exists a standard, but locally anisotropic, Yang–Mills potential A_α with gauge transformation laws parametrized by the parameter ϵ like in (14.43), a noncommutative gauge potential $\widehat{A}_\alpha(A_\alpha)$ with gauge transformation parameter $\widehat{\epsilon}(A,\epsilon)$, when

$$\widehat{\triangle}_{\widehat{\epsilon}} \widehat{A}_\alpha = \delta_\alpha \widehat{\epsilon} + i\left(\widehat{A}_\alpha \star \widehat{\epsilon} - \widehat{\epsilon} \star \widehat{A}_\alpha\right),$$

should satisfy the equation

$$\widehat{A}(A) + \widehat{\triangle}_{\widehat{\epsilon}} \widehat{A}(A) = \widehat{A}(A + \triangle_\epsilon A), \tag{14.46}$$

where, for simplicity, the indices were omitted. This is the Seiberg–Witten equation which, in our case, contains N–adapted operators δ_α (13.2) and d–vector gauge potentials, respectively, $\widehat{A}_\alpha = \left(\widehat{A}_i, \widehat{A}_a\right)$ and $A_\alpha = (A_i, A_a)$. To first order in $\Theta^{\alpha\beta} = \triangle\Theta^{\alpha\beta}$, the equation (14.46) can be solved in a usual way, by related respectively the potentials and transformation parameters,

$$\begin{aligned} \widehat{A}_\alpha(A_\alpha) - A_\alpha &= -\frac{1}{4} \triangle \Theta^{\beta\lambda} \left[A_\beta(\delta_\lambda A_\alpha + F_{\lambda\alpha}) + (\delta_\lambda A_\alpha + F_{\lambda\alpha}) A_\beta\right] + o(\triangle\Theta^2), \\ \widehat{\epsilon}(A,\epsilon) - \epsilon &= \frac{1}{4} \triangle \Theta^{\beta\lambda} (\delta_\beta \epsilon \ A_\lambda + A_\lambda \delta_\beta \epsilon) + o(\triangle\Theta^2), \end{aligned}$$

from which we can also find a first order relation for the field strength,

$$\begin{aligned}\widehat{F}_{\lambda\alpha}-F_{\lambda\alpha} &= \frac{1}{2}\triangle\,\Theta^{\beta\tau}\left(F_{\lambda\beta}F_{\alpha\tau}+F_{\alpha\tau}F_{\lambda\beta}\right)\\ &\quad -A_{\beta}\left(\bigtriangledown_{\tau}F_{\lambda\alpha}+\delta_{\tau}F_{\lambda\alpha}\right)-\left(\bigtriangledown_{\tau}F_{\lambda\alpha}+\delta_{\tau}F_{\lambda\alpha}\right)A_{\beta}+o(\triangle\Theta^{2}).\end{aligned}$$

By a recurrent procedure the solution of (14.46) can be constructed in all orders of $\triangle\Theta^{\alpha\beta}$ as in the locally isotropic case (see details on recent supersymmetric generalizations in Refs. [27] which can be transformed at least in a formal form into certain anisotropic analogs following the d–covariant geometric rule.

14.5 Anholonomy and Noncommutativity: Relations to String/ M–Theory

The aim of this Section is to discuss how both noncommutative and locally anisotropic field theories arise from string theory and M–theory. The first use of noncommutative geometry in string theory was suggested by E. Witten (see Refs. [57, 38] for details and developments). Noncommutativity is natural in open string theory: interactions of open strings with two ends contains formal similarities to matrix multiplication which explicitly results in noncommutative structures. In other turn, matrix noncommutativity is contained in off–diagonal metrics and anholonomic vielbeins with associated N–connection and anholonomic relations (see (13.4) and related details in Appendix) which are used in order to develop locally anisotropic geometries and field theories. We emphasize that the constructed exact solutions with off–diagonal metrics in general relativity and extra dimension gravity together with the existence of a string field framework strongly suggest that noncommutative locally anisotropic structures have a deep underlying significance in such theories [45, 46, 53, 54, 48, 50, 42, 43].

14.5.1 Noncommutativity and anholonomy in string theory

In this subsection, we will analyze strings in curved spacetimes with constant coefficients $\{g_{ij},h_{ab}\}$ of d–metric (13.6) (the coefficients $N_i^a\left(x^k,y^a\right)$ are not constant and the off–diagonal metric (14.11) has a non–trivial curvature tensor). With respect to N–adapted frames (13.2) and (13.3) the string propagation is like in constant Neveu–Schwarz constant B–field and with Dp–branes. We work under the conditions of string and brane theory which results in noncommutative geometry [57] but the background under consideration here is an anholonomic one. The B–field is a like constant magnetic field which is polarized by the N–connection structure. The rank of the matrix

$B_{\alpha\beta}$ is denoted $k = n + m = 11 \leq p + 1$, where $p \geq 10$ is a constant. For a target space, defined with respect to anholonomic frames, we will assume that $B_{0\beta} = 0$ with "0" the time direction (for a Euclidean signature, this condition is not necessary). We can similarly consider another dimensions than 11, or to suppose that some dimensions are compactified. We can pick some torus like coordinates, in general anholonomic, by certain conditions, $u^\alpha \sim u^\alpha + 2\pi k^\alpha$. For simplicity, we parametrize $B_{\alpha\beta} = const \neq 0$ for $\alpha, \beta = 1, ..., k$ and $g_{\alpha\beta} = 0$ for $\alpha = 1, ..., r, \beta \neq 1, ..., k = n + m$ with a further distinguishing of indices

There are two possibilities of writing out the worldsheet action,

$$
\begin{aligned}
S &= \frac{1}{4\pi\alpha'} \int_\Sigma \delta\underline{\mu}_g \left(g_{\underline{\alpha}\underline{\beta}} \partial_A u^{\underline{\alpha}} \partial^A u^{\underline{\beta}} - 2\pi\alpha' i B_{\underline{\alpha}\underline{\beta}} \varepsilon^{AB} \partial_A u^{\underline{\alpha}} \partial_B u^{\underline{\beta}} \right) \qquad (14.47) \\
&= \frac{1}{4\pi\alpha'} \int_\Sigma \delta\underline{\mu}_g g_{\underline{\alpha}\underline{\beta}} \partial_A u^{\underline{\alpha}} \partial^A u^{\underline{\beta}} - \frac{i}{2} \int_{\partial\Sigma} \delta\underline{\mu}_g B_{\underline{\alpha}\underline{\beta}} \; u^{\underline{\alpha}} \partial_{\tan} u^{\underline{\beta}}; \\
&= \frac{1}{4\pi\alpha'} \int_\Sigma \delta\mu_g (g_{ij} \partial_A x^i \partial^A x^j + h_{ab} \partial_A y^a \partial^A y^b \\
&\quad - 2\pi\alpha' i B_{ij} \varepsilon^{AB} \partial_A x^i \partial_B x^j - 2\pi\alpha' i B_{ab} \varepsilon^{AB} \partial_A y^a \partial_B y^b) \\
&= \frac{1}{4\pi\alpha'} \int_\Sigma \delta\mu_g \left(g_{ij} \partial_A x^i \partial^A x^j + h_{ab} \partial_A y^a \partial^A y^b \right) \\
&\quad - \frac{i}{2} \int_{\partial\Sigma} \delta\mu_g B_{ij} \; x^i \partial_{\tan} x^j - \frac{i}{2} \int_{\partial\Sigma} \delta\mu_g B_{ab} \; y^a \partial_{\tan} y^b,
\end{aligned}
$$

where the first variant is written by using metric ansatz $g_{\underline{\alpha}\underline{\beta}}$ (14.11) but the second variant is just the term $S_{g_N,B}$ from action (14.13) with d–metric (13.6) and different boundary conditions and $\partial_{\tan}$ is the tangential derivative along the worldwheet boundary $\partial\Sigma$. We emphasize that the values g_{ij}, h_{ab} and B_{ij}, B_{ab}, given with respect to N–adapted frames are constant, but the off–diagonal $g_{\underline{\alpha}\underline{\beta}}$ and $B_{\underline{\alpha}\underline{\beta}}$, in coordinate base, are some functions on (x, y). The worldsheet Σ is taken to be with Euclidean signature (for a Lorentzian wolrdsheet the complex i should be omitted multiplying B).

The equation of motion of string in anholonomic constant background define respective anholonomic, N–adapted boundary conditions. For coordinated α along the Dp–branes they are

$$
\begin{aligned}
g_{\alpha\beta} \partial_{norm} u^\beta + 2\pi i \alpha' B_{\alpha\beta} \partial_{\tan} u^\beta &= \qquad (14.48) \\
g_{ij} \partial_{norm} x^j + h_{ab} \partial_{norm} y^b + 2\pi i \alpha' B_{ij} \partial_B x^j - 2\pi\alpha' i B_{ab} \partial_{\tan} y^b |_{\partial\Sigma} &= 0,
\end{aligned}
$$

where ∂_{norm} is a normal derivative to $\partial\Sigma$. By transforms of type $g_{\underline{\alpha}\underline{\beta}} = e^\alpha_{\ \underline{\alpha}}(u) e^\beta_{\ \underline{\beta}}(u) g_{\alpha\beta}$ and $B_{\underline{\alpha}\underline{\beta}} = e^\alpha_{\ \underline{\alpha}}(u) e^\beta_{\ \underline{\beta}}(u) B_{\alpha\beta}$ we can remove these boundary conditions into a holonomic off–diagonal form which is more difficult to investigate. With respect to N–adapted frames (with non–underlined indices) the analysis is very similar to the case constant values of the metric and B–field. For $B = 0$, the boundary conditions (14.48) are Neumann ones. If B has the rank $r = p$ and $B \to \infty$ (equivalently, $g_{\alpha\beta} \to 0$ along the spactial directions of the brane, the boundary conditions become of Dirichlet type). The effect of all such type conditions and their possible interpolations can be investigated as in the usual open string theory with constant B–field but, in this subsection, with respect to N–adapted frames.

For instance, we can suppose that Σ is a disc, conformally and anholonomically mapped to the upper half plane with complex variables z and $\overline{z}$ and $Im\ z \geq 0$. The propagator with such boundary conditions is the same as in [20] with coordinates redefined to anholonomic frames,

$$
\begin{aligned}
< x^i(z)x^j(z') > &= -\alpha'[g^{ij}\log\frac{|z-z'|}{|z-\overline{z}'|} + H^{ij}\log|z-\overline{z}'|^2 \\
&\quad + \frac{1}{2\pi\alpha'}\Theta^{ij}\log\frac{|z-\overline{z}'|}{|\overline{z}-z'|} + Q^{ij}], \\
< y^a(z)y^b(z') > &= -\alpha'[h^{ab}\log\frac{|z-z'|}{|z-\overline{z}'|} + H^{ab}\log|z-\overline{z}'|^2 \\
&\quad + \frac{1}{2\pi\alpha'}\Theta^{ab}\log\frac{|z-\overline{z}'|}{|\overline{z}-z'|} + Q^{ab}],
\end{aligned}
$$

where the coefficients are correspondingly computed,

$$
H_{ij} = g_{ij} - (2\pi\alpha')^2\left(Bg^{-1}B\right)_{ij},\ H_{ab} = h_{ab} - (2\pi\alpha')^2\left(Bg^{-1}B\right)_{ab}, \qquad (14.49)
$$

$$
\begin{aligned}
H^{ij} &= \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_{[sym]} = \left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{ij}, \\
H^{ab} &= \left(\frac{1}{h+2\pi\alpha' B}\right)^{ij}_{[sym]} = \left(\frac{1}{h+2\pi\alpha' B}h\frac{1}{h-2\pi\alpha' B}\right)^{ij},
\end{aligned}
$$

$$
\begin{aligned}
\Theta^{ij} &= 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}_{[antisym]} = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{ij}, \\
\Theta^{ab} &= 2\pi\alpha'\left(\frac{1}{g+2\pi\alpha' B}\right)^{ab}_{[antisym]} = -(2\pi\alpha')^2\left(\frac{1}{g+2\pi\alpha' B}g\frac{1}{g-2\pi\alpha' B}\right)^{ab},
\end{aligned}
$$

with $[sym]$ and $[antisym]$ prescribing, respectively, the symmetric and antisymmetric parts of a matrix and constants Q^{ij} and Q^{ab} (in general, depending on B, but not on z or z') do to not play an essential role which allows to set them to a convenient value. The last two terms are signed–valued (if the branch cut of the logarithm is taken in lower half plane) and the rest ones are manifestly sign–valued.

Restricting our considerations to the open string vertex operators and interactions with real $z = \tau$ and $z = \tau'$, evaluating at boundary points of Σ for a convenient value of $D^{\alpha\beta}$, the propagator (in non–distinguished form) becomes

$$< u^{\alpha}(\tau)u^{\beta}(\tau') >= -\alpha' H^{\alpha\beta} \log (\tau - \tau')^2 + \frac{i}{2}\Theta^{\alpha\beta}\epsilon(\tau - \tau')$$

for $\epsilon(\tau - \tau')$ being 1 for $\tau > \tau'$ and -1 for $\tau < \tau'$. The d–tensor $H_{\alpha\beta}$ defines the effective metric seen by the open string subjected to some anholonomic constraints being constant with respect to N–adapted frames. Working as in conformal field theory, one can compute commutators of operators from the short distance behavior of operator products (by interpreting time ordering as operator ordering with time τ) and find that the coordinate commutator

$$\left[u^{\alpha}(\tau), u^{\beta}(\tau)\right] = i\Theta^{\alpha\beta}$$

which is just the relation (14.40) for noncommutative coordinates with constant noncommutativity parameter $\Theta^{\alpha\beta}$ distinguished by a N–connection structure.

In a similar manner we can introduce gauge fields and consider worldsheet supersymmetry together with noncommutative relations with respect to N–adapted frames. This results in locally anisotropic modifications of the results from [57] via anholonomic frame transforms and distinguished tensor and noncommutative calculus (we omit here the details of such calculations).

We emphasize that even the values $H^{\alpha\beta}$ and $\Theta^{\alpha\beta}$ (14.49) are constant with respect to N–adapted frames the anholonomic noncommutative string configurations are characterized by locally anisotropic values $H^{\underline{\alpha}\underline{\beta}}$ and $\theta^{\underline{\alpha}\underline{\beta}}$ which are defined with respect to coordinate frames as

$$H^{\underline{\alpha}\underline{\beta}} = e_{\alpha}^{\ \underline{\alpha}}(u)e_{\beta}^{\ \underline{\beta}}(u)H^{\alpha\beta} \text{ and } \theta^{\underline{\alpha}\underline{\beta}} = e_{\alpha}^{\ \underline{\alpha}}(u)e_{\beta}^{\ \underline{\beta}}(u)\theta^{\alpha\beta}$$

with $e_{\alpha}^{\ \underline{\alpha}}(u)$ (14.7) defined by N_i^a as in (14.12), i. e.

$$e_i^{\ \underline{i}} = \delta_i^{\ \underline{i}},\ e_i^{\ \underline{a}} = -N_i^{\underline{a}}(u),\ e_a^{\ \underline{a}} = \delta_a^{\underline{a}},\ e_a^{\ \underline{i}} = 0.$$

Now, we make use of he standard relation between world–sheet correlation function of vertex operators, the S–matrix for string scattering and effective actions which can reproduce this low energy string physics [15] but generalizing them for anholonomic structures. We consider that operators in the bulk of the world–sheet correspond to closed strings, while operators on the boundary correspond to open strings and thus fields which propagate on the world volume of a D–brane. The basic idea is that each local world–sheet operator $V_s(z)$ corresponds to an interaction with a spacetime field $\varphi_s(z)$ which results in the effective Lagrangian

$$\int \delta^{p+1}u\sqrt{|\det g_{\alpha\beta}|}\ {}^{N}Tr\ \varphi_1\varphi_2...\varphi_s$$

which is computed by integrating on z_s following the prescribed order for the correlation function

$$\left\langle \int dz_1 V_1(z_1) \int dz_2 V_2(z_2) ... \int dz_s V_s(z_s) \right\rangle$$

on a world–sheet Σ with disk topology, with operators V_s as successive points z_s on the boundary $\partial\Sigma$. The integrating measure is constructed from N–elongated values and coefficients of d–metric. In the leading limit of the S–matrix with vertex operators only for the massless fields we reproduce a locally anisotropic variant of the MSYM effective action which describes the physics of a D–brane with arbitrarily large but anisotropically and slowly varying field strength,

$$S_{BNI}^{[anh]} = \frac{1}{g_s l_s (2\pi l_s)^p} \int \delta^{p+1}u\sqrt{|\det(g_{\alpha\beta} + 2\pi l_s^2(B+F))|} \qquad (14.50)$$

where g_s is the string coupling, the constant l_s is the usual one from D–brane theory and $g_{\alpha\beta}$ is the induced d–metric on the brane world–volume. The action (14.50) is just the Nambu–Born–Infeld (NBI) action [20] but defined for d–metrics and d–tensor fields with coefficients computed with respect to N–adapted frames.

14.5.2 Noncommutative anisotropic structures in M(atrix) theory

For an introduction to M–theory, we refer to [32, 15]. Throughout this subsection we consider M–theory as to be not completely defined but with a well–defined quantum gravity theory with the low energy spectrum of the 11 dimensional supergravity theory [11], containing solitonic "branes", the 2–brane, or supermembrane, and five–branes and that from M–theory there exists connections to the superstring theories. Our claim is that in the low energy limits the noncommutative structures are, in general, locally anisotropic.

The simplest way to derive noncommutativity from M–theory is to start with a matrix model action such in subsection 14.4.3 and by introducing operators of type C_α (14.44) and actions (14.45). For instance, we can consider the action for maximally supersymmetric quantum mechanics, i. e. a trivial case with $p = 0$ of MSYM, when

$$S = \int \delta t \ \ {}^N Tr \sum_{\alpha=1}^{9} (D_t X)^2 - \sum_{\alpha<\beta} \left[X^\alpha, X^\beta\right]^2 + \chi^+ (D_t + \Gamma_\alpha X^\alpha)\chi, \qquad (14.51)$$

where $D_t = \delta/\partial t + iA_0$ with d–derivative (13.2) with varying A_0 which introduces constraints in physical states because of restriction of unitary symmetry. This action is written in anholonomic variables and generalizes the approach of entering the M–theory as a regularized form of the actions for the supermembranes [56]. In this interpretation the the compact eleventh dimension does not disappear and the M–theory is to be considered as to be anisotropically compactified on a light–like circle.

In order to understand how anisotropic torus compactifications may be performed (see subsection 14.4.2) we use the general theory of D–branes on quotient spaces [39]. We consider $U_\alpha = \gamma(\beta_\alpha)$ for a set of generators of $\mathbb{Z}^{n+m}$ with $\mathcal{A} = Mat_{n+m}(\mathbb{C})$ which satisfy the equations

$$U_\alpha^{-1} X^\beta U_\alpha = X^\beta + \delta_\alpha^\beta 2\pi R_\alpha$$

having solutions of type

$$X_\beta = -i\delta/\partial\sigma^\beta + A_\beta$$

for A_β commuting with U_α and indices distinguished by a N–connection structure as $\alpha = (i,a), \beta = (j,b)$. For such variables the action (14.51) leads to a locally anisotropic MSYM on $T^{n+m} \times \mathbb{R}$. Of course, this construction admits a natural generalization for variables U_α satisfying relations (14.40) for noncommutative locally anisotropic tori which leads to noncommutative anholonomic gauge theories [48, 50]. In original form this type of noncommutativity was introduced in M–theory (without anisotropies) in Ref. [9].

The anisotropic noncommutativity in M–theory can related to string model via nontrivial components $C_{\alpha\beta-}$ of a three–form potential ("-" denotes the compact light–like direction). This potential has as a background value if the M(atrix) theory is treated as M–theory on a light–like circle as in usual isotropic models. In the IIA string interpretation of $C_{\alpha\beta-}$ as a Neveu–Schwarz B–field which minimally coupled to the string world–sheet, we obtain the action (14.47) compactified on a $\mathbb{R} \times T^{n+m}$ spacetime where torus has constant d–metric and B–field coefficients.

14.6 Anisotropic Gravity on Noncommutative D–Branes

We develop a model of locally anisotropic gravity on noncommutative D–branes (see Refs. [2] for a locally isotropic variant). We investigate what kind of deformations of the low energy effective action of closed strings are induced in the presence of constant background antisymmetric field (or it anholonomic transforms) and/or in the presence of generic off–diagonal metric and associated nonlinear connection terms. It should be noted that there were proposed and studied different models of nocommutative deformations of gravity [7], which were not derived from string theory but introduced "ad hoc". Anholonomic and/or gauge transforms in noncommutative gravity were considered in Refs. [50, 48]. In this Section, we illustrate how such gravity models with generic anisotropy and noncommutativity can be embedded in D–brane physics.

We can compute the tree level bosonic string scattering amplitude of two massless closed string off a noncommutative D–brane with locally anisotropic contributions by considering boundary conditions and correlators stated with respect to anholonomic frames. By using the 'geometric d–covariant rule' of changing the tensors, spinors and connections into theirs corresponding N–distinguished d–objects we derive the locally anisotropic variant of effective actions in a straightforword manner.

For instance, the action which describes this amplitude to order of the string constant $(\alpha')^0$ is just the so–called DBI and Einstein–Hilbert action. With respect to the Einstein N–emphasized frame the DBI action is

$$S^{[0]}_{D-brane} = -T_p \int \delta^{p+1}u\ e^{-\Phi}\sqrt{\left|\det\left(e^{-\gamma\Phi}g_{\alpha\beta} + \mathcal{B}_{\alpha\beta} + f_{\alpha\beta}\right)\right|} \qquad (14.52)$$

where $g_{\alpha\beta}$ is the induced metric on the D–brane, $\mathcal{B}_{\alpha\beta} = B_{\alpha\beta} - 2\kappa b_{\alpha\beta}$ is the pull back of the antisymmetric d–field B being constant with respect to N–adapted frames along D–brane, $f_{\alpha\beta}$ is the gauge d–field strength and $\gamma = -4/\left(n+m-2\right)$ and the constant $T_p = \mathcal{C}(\alpha')^2/C\kappa^2$ is taken as in Ref. [2] for usual D–brane theory (this allow to obtain in a limit the Einstein–Hilber action in the bulk). There are used such parametrizations of indices:

$$\mu', \nu', ... = 0, ..., 25; \mu' = \left(\mu, \hat{\mu}\right);\ \hat{\mu}, \hat{\nu}... = p+1, ..., 25; \hat{\mu} = \left(\hat{\imath}, \hat{a}\right)$$

where i takes n–dimensional 'horizontal' values and a takes m–dimensional 'vertical' being used for a D–brane localized at $u^{p+1}, ...u^{25}$ with the boundary conditions given with respect to a N–adapted frame,

$$g_{\alpha\beta}\left(\partial - \overline{\partial}\right)U^{\alpha} + B_{\alpha\beta}\left(\partial + \overline{\partial}\right)U^{\alpha}_{|z=\overline{z}} = 0,$$

which should be distinguished in h- and v–components, and the two point correlator of string anholonomic coordinates $U^{\alpha'}(z,\overline{z})$ on the D–brane is

$$< \ U_{\hat{\mu}}^{\alpha'} U_{\hat{\nu}}^{\beta'} \ > \ = -\frac{\alpha'}{2}\{g^{\alpha'\beta'} \log\left[\left(z_{\hat{\mu}}-z_{\hat{\nu}}\right)\left(\overline{z}_{\hat{\mu}}-\overline{z}_{\hat{\nu}}\right)\right] + D^{\alpha'\beta'} \log\left(z_{\hat{\mu}}-\overline{z}_{\hat{\nu}}\right) + D^{\beta'\alpha'} \log\left(\overline{z}_{\hat{\mu}}-z_{\hat{\nu}}\right)\}$$

where

$$D^{\beta\alpha} = 2\left(\frac{1}{\eta+B}\right)^{\alpha\beta} - \eta^{\alpha\beta},\ D^{\hat{\mu}\hat{\nu}} = -\delta^{\hat{\mu}\hat{\nu}},\ D^{\beta'}_{\ \alpha'} D^{\nu'\alpha'} = \eta^{\beta'\alpha'}$$

for constant $\eta^{\alpha'\beta'}$ given with respect to N–adapted frames.

The scattering amplitude of two closed strings off a D–brane is computed as the integral

$$A = g_c^2\ e^{-\lambda}\int d^2 z_{\underline{1}}\ d^2 z_{\underline{2}}\ < V\left(z_{\underline{1}},\overline{z}_{\underline{1}}\right) V\left(z_{\underline{2}},\overline{z}_{\underline{2}}\right)>, \tag{14.53}$$

for g_c being the closed string coupling constant, λ being the Euler number of the world sheet and the vertex operators for the massless closed strings with the momenta $k_{\underline{i}\mu'} = \left(k_{\underline{i}i'}, k_{\underline{i}a'}\right)$ and polarizations $\epsilon_{\mu'\nu'}$ (satisfying the conditions $\epsilon_{\mu'\nu'} k_{\underline{i}}^{\mu'} = \epsilon_{\mu'\nu'} k_{\underline{i}}^{\nu'} = 0$ and $k_{\underline{i}\mu'} k_{\underline{i}}^{\mu'} = 0$ taken as

$$V\left(z_i,\overline{z}_i\right) = \epsilon_{\mu'\nu'}\ \ D^{\nu'}_{\ \alpha'} : \partial X^{\mu'}\left(z_{\underline{i}}\right) \exp\left[ik_{\underline{i}} X\left(z_{\underline{i}}\right)\right] : \ \ : \overline{\partial} X^{\alpha'}\left(\overline{z}_{\underline{i}}\right) \exp\left[ik_{\underline{i}\beta'} D^{\beta'}_{\ \tau'} X^{\tau'}\left(z_{\underline{i}}\right)\right] : .$$

Calculation of such calculation functions can be performed as in usual string theory with that difference that the tensors and derivatives are distinguished by N–connections.

Decomposing the metric $g_{\alpha\beta}$ as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2\kappa\chi_{\alpha\beta}$$

where $\eta_{\alpha\beta}$ is constant (Minkowski metric but with respect to N–adapted frames) and $\chi_{\alpha\beta}$ could be of (pseudo) Riemannian or Finsler like type. Action (14.52) can be written to the first order of χ,

$$S_{D-brane}^{[0]} = -\kappa T_p c\int \delta^{p+1}u\ \chi_{\alpha\beta} Q^{\alpha\beta}, \tag{14.54}$$

where

$$Q^{\alpha\beta} = \frac{1}{2}\left(\eta^{\alpha\beta} + D^{\alpha\beta}\right) \tag{14.55}$$

and $c = \sqrt{|\det(\eta_{\alpha\beta} + B_{\alpha\beta})|}$, which exhibits a source for locally anisotropic gravity on D–brane,

$$T_{\chi}^{\alpha\beta} = -\frac{1}{2} T_p \kappa C \left(\eta^{\alpha\beta} + D_{(S)}^{\beta\alpha} \right),$$

for $D_{(S)}^{\beta\alpha}$ denoting symmetrization of the matrix $D^{\beta\alpha}$. This way we reproduce the same action as in superstring theory [22] but in a manner when anholonomic effects and anisotropic scattering can be included.

Next order terms on α' in the string amplitude are included by the term

$$\begin{aligned} S_{bulk}^{[1]} &= \frac{\alpha'}{8\kappa^2} \int \delta^{26} u' \; e^{\gamma\Phi} \sqrt{|g_{\mu'\nu'}|} [R_{h'i'j'k'} R^{h'i'j'k'} + R_{a'b'j'k'} R^{a'b'j'k'} + P_{j'i'k'a'} P^{j'i'k'a'} \\ & + P_{c'd'k'a'} P^{c'd'k'a'} + S_{j'i'b'c'} S^{j'i'b'c'} + S_{d'e'b'c'} S^{d'e'b'c'} \\ & -4 \left(R_{i'j'} R^{\iota' j'} + R_{i'a'} R^{i'a'} + P_{a'i'} P^{a'i'} + R_{a'b'} R^{a'b'} \right) + (g^{i'j'} R_{i'j'} + h^{a'b'} S_{a'b'})^2] \end{aligned}$$

where the indices are split as $\mu' = (i', a')$ and we use respectively the d–curvatures (13.14), Ricci d–tensors (13.15) and d–scalars (13.16). Splitting of "primed' indices reduces to splitting of D–brane values.

The DBI action on D–brane (14.53) is defined with a gauge field strength

$$f_{\alpha\beta} = \delta_\alpha a_\beta - \delta_\beta a_\alpha$$

and with the induced metric

$$g_{\alpha\beta} = \delta_\alpha X^{\mu'} \delta_\beta X_{\mu'}$$

expanded around the flat space in the static gauge $U^\mu = u^\mu$,

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\chi_{\mu\nu} + 2\kappa \left(\chi_{\hat{\mu}\mu} \delta_\nu U^{\hat{\mu}} + \chi_{\hat{\mu}\nu} \delta_\mu U^{\hat{\mu}} \right) + \delta_\mu U^{\hat{\mu}} \delta_\nu U_{\hat{\mu}} + 2\kappa \chi_{\hat{\mu}\hat{\nu}} \delta_\mu U^{\hat{\mu}} \delta_\nu U^{\hat{\nu}}.$$

In order to describe D–brane locally anisotropic processes in the first order in α' we need to add a new term to the DBI as follow,

$$S^1 = -\frac{\alpha' T_p}{2} \int \delta^{p+1} u \sqrt{|\det q_{\mu\nu}|} \{ \; R_{\alpha\beta\gamma\tau} q^{\alpha\tau} - \left(\Psi_{\alpha\gamma}^{\hat{\mu}} \Psi_{\hat{\mu}\beta\tau} - \Psi_{\alpha\tau}^{\hat{\mu}} \Psi_{\hat{\mu}\beta\gamma} \right) \tilde{q}^{\alpha\tau} \} \tilde{q}^{\beta\gamma} \quad (14.56)$$

where $q_{\mu\nu} = \eta_{\mu\nu} + B_{\mu\nu} + f_{\mu\nu}$, $q^{\mu\nu}$ is the inverse of $q_{\mu\nu}$, $\tilde{q}_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu} + f_{\mu\nu}$, $\tilde{q}^{\mu\nu}$ is the inverse of $\tilde{q}_{\mu\nu}$, the curvature d–tensor $R_{\alpha\beta\gamma\tau}$ is constructed from the induced d–metric by using the canonical d–connection (see (13.14) and (13.8)) and

$$\Psi_{\alpha\beta}^{\hat{\mu}} = \kappa \left(-\delta^{\hat{\mu}} \chi_{\alpha\beta} + \delta_\alpha \chi^{\hat{\mu}}{}_{\beta} + \delta_\beta \chi^{\hat{\mu}}{}_{\alpha} \right) + \delta_\alpha \delta_\beta U^{\hat{\mu}}.$$

The action (14.56) can be related to the Einstein–Hilbert action on the D–brane if the the B–field is turned off. To see this we consider the field $Q^{\alpha\beta} = \eta^{\alpha\beta}$ (14.55) which reduces (up to some total d–derivatives, which by the momentum conservation relation have no effects in scattering amplitudes, and ignoring gauge fields because they do not any contraction with gravitons because of antisymmetry of $f_{\alpha\beta}$) to

$$S^{[1]}_{D-brane} = -\frac{\alpha' T_p}{2} \int \delta^{p+1} u \sqrt{|\det g_{\mu\nu}|} \left(\widehat{R} + S + \Psi^{\hat{\mu}\ \alpha}_{\ \alpha} \Psi_{\hat{\mu}\beta}^{\ \ \ \beta} - \Psi^{\hat{\mu}}_{\alpha\beta} \Psi_{\hat{\mu}}^{\ \ \alpha\beta} \right) \qquad (14.57)$$

were $\widehat{R}$ and S are computed as d–scalar objects (13.16) and by following the relation at $\mathcal{O}(\chi^2)$,

$$\sqrt{|\det \eta_{\mu\nu}|} R_{\alpha\beta\gamma\tau} \eta^{\alpha\tau} g^{\beta\gamma} = \sqrt{|\det g_{\mu\nu}|} R_{\alpha\beta\gamma\tau} g^{\alpha\gamma} g^{\beta\tau} + \text{total d–derivatives.}$$

The action (14.57) transforms into the Einstein–Hilbert action as it was proven for the locally isotropic D–brane theory [10] for vanishing N–connections and trivial vertical (anisotropic) dimensions.

In conclusion, it has been shown in this Section that the D–brane dynamics can be transformed into a locally anisotropic one, which in low energy limits contains different models of generalized Lagrange/ Finsler or anholonomic Riemannian spacetimes, by introducing corresponding anholonomic frames with associated N–connection structures and d–metric fields (like (13.33) and (13.35) and (13.34)).

14.7 Exact Solutions: Noncommutative and/ or Locally Anisotropic Structures

In the previous sections we demonstrated that locally anisotropic noncommutative geometric structures are hidden in string/ M–theory. Our aim here is to construct and analyze four classes of exact solutions in string gravity with effective metrics possessing generic off–diagonal terms which for associated anholonomic frames and N–connections can be extended to commutative or noncommutative string configurations.

14.7.1 Black ellipsoids from string gravity

A simple string gravity model with antisymmetric two form potential field $H^{\alpha'\beta'\gamma'}$, for constant dilaton ϕ, and static internal space, β, is to be found for the NS–NS sector which is common to both the heterotic and type II string theories [26]. The equations (14.34) reduce to

$$R_{\mu'\nu'} = \frac{1}{4} H_{\mu'\lambda'\tau'} H_{\nu'}{}^{\lambda'\tau'}, \qquad (14.58)$$
$$D_{\mu'} H^{\mu'\ \lambda'\tau'} = 0,$$

for

$$H_{\mu'\nu'\lambda'} = \delta_{\mu'} B_{\nu'\lambda'} + \delta_{\lambda'} B_{\mu'\nu'} + \delta_{\nu'} B_{\lambda'\mu'}.$$

If $H_{\mu'\nu'\lambda'} = \sqrt{|g_{\mu'\nu'}|}\epsilon_{\mu'\nu'\lambda'}$, we obtain the vacuum equations for the gravity with cosmological constant λ,

$$R_{\mu'\nu'} = \lambda g_{\mu'\nu'}, \qquad (14.59)$$

for $\lambda = 1/4$ where $R_{\mu'\nu'}$ is the Ricci d–tensor (13.15), with "primed" indices emphasizing that the geometry is induced after a topological compactification.

For an ansatz of type

$$\delta s^2 = g_1(dx^1)^2 + g_2(dx^2)^2 + h_3\left(x^{i'}, y^3\right)(\delta y^3)^2 + h_4\left(x^{i'}, y^3\right)(\delta y^4)^2, \qquad (14.60)$$
$$\delta y^3 = dy^3 + w_{i'}\left(x^{k'}, y^3\right)dx^{i'}, \quad \delta y^4 = dy^4 + n_{i'}\left(x^{k'}, y^3\right)dx^{i'},$$

for the d–metric (13.6) the Einstein equations (14.59) are written (see [49, 45] for details on computation)

$$R_1^1 = R_2^2 = -\frac{1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1^{''} - \frac{g_1^{'}g_2^{'}}{2g_2} - \frac{(g_1^{'})^2}{2g_1}] = \lambda, \qquad (14.61)$$
$$R_3^3 = R_4^4 = -\frac{\beta}{2h_3h_4} = \lambda, \qquad (14.62)$$
$$R_{3i'} = -w_{i'}\frac{\beta}{2h_4} - \frac{\alpha_{i'}}{2h_4} = 0, \qquad (14.63)$$
$$R_{4i'} = -\frac{h_4}{2h_3}\left[n_{i'}^{**} + \gamma n_{i'}^{*}\right] = 0, \qquad (14.64)$$

where the indices take values $i', k' = 1, 2$ and $a', b' = 3, 4$. The coefficients of equations (14.61) - (14.64) are given by

$$\alpha_i = \partial_i h_4^* - h_4^* \partial_i \ln\sqrt{|h_3h_4|}, \qquad \beta = h_4^{**} - h_4^*[\ln\sqrt{|h_3h_4|}]^*, \qquad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_3^*}{h_3}. \qquad (14.65)$$

The various partial derivatives are denoted as $a^\bullet = \partial a/\partial x^1, a^{'} = \partial a/\partial x^2, a^* = \partial a/\partial y^3$. This system of equations (14.61)–(14.64) can be solved by choosing one of the ansatz functions (*e.g.* $g_1(x^i)$ or $g_2(x^i)$) and one of the ansatz functions (*e.g.* $h_3(x^i, y^3)$ or $h_4(x^i, y^3)$) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way we can generate a lost of different solutions, but we impose the condition that the initial, arbitrary choice of the ansatz functions is "physically interesting" which means that one wants to make this original choice so that the generated final solution yield a well behaved metric.

In references [46] it is proved that for

$$\begin{aligned} g_1 &= -1, \quad g_2 = r^2(\xi)\, q(\xi), \\ h_3 &= -\eta_3(\xi,\varphi)\, r^2(\xi)\sin^2\theta, \\ h_4 &= \eta_4(\xi,\varphi)\, h_{4[0]}(\xi) = 1 - \frac{2\mu}{r} + \varepsilon\frac{\Phi_4(\xi,\varphi)}{2\mu^2}, \end{aligned} \tag{14.66}$$

with coordinates $x^1 = \xi = \int dr\sqrt{1-2m/r+\varepsilon/r^2}, x^2 = \theta, y^3 = \varphi, y^4 = t$ (the (r,θ,φ) being usual radial coordinates), the ansatz (14.60) is a vacuum solution with $\lambda = 0$ of the equations (14.59) which defines a black ellipsoid with mass μ, eccentricity ε and gravitational polarizations $q(\xi), \eta_3(\xi,\varphi)$ and $\Phi_4(\xi,\varphi)$. Such black holes are certain deformations of the Schwarzschild metrics to static configurations with ellipsoidal horizons which is possible if generic off–diagonal metrics and anholonomic frames are considered. In this subsection we show that the data (14.66) can be extended as to generate exact black ellipsoid solutions with nontrivial cosmological constant $\lambda = 1/4$ which can be imbedded in string theory.

At the first step, we find a class of solutions with $g_1 = -1$ and $g_2 = g_2(\xi)$ solving the equation (14.61), which under such parametrizations transforms to

$$g_2^{\bullet\bullet} - \frac{(g_2^{\bullet})^2}{2g_2} = 2g_2\lambda.$$

With respect to the variable $Z = (g_2)^2$ this equation is written as

$$Z^{\bullet\bullet} + 2\lambda Z = 0$$

which can be integrated in explicit form, $Z = Z_{[0]}\sin\left(\sqrt{2\lambda}\xi + \xi_{[0]}\right)$, for some constants $Z_{[0]}$ and $\xi_{[0]}$ which means that

$$g_2 = -Z_{[0]}^2\sin^2\left(\sqrt{2\lambda}\xi + \xi_{[0]}\right) \tag{14.67}$$

parametrize a class of solution of (14.61) for the signature $(-,-,-,+)$. For $\lambda \to 0$ we can approximate $g_2 = r^2(\xi)\, q(\xi) = -\xi^2$ and $Z_{[0]}^2 = 1$ which has compatibility with the data (14.66). The solution (14.67) with cosmological constant (of string or non–string origin) induces oscillations in the "horozontal" part of the d–metric.

The next step is to solve the equation (14.62),

$$h_4^{**} - h_4^{*}[\ln\sqrt{|h_3 h_4|}]^{*} = -2\lambda h_3 h_4.$$

For $\lambda = 0$ a class of solution is given by any $\widehat{h}_3$ and $\widehat{h}_4$ related as

$$\widehat{h}_3 = \eta_0 \left[\left(\sqrt{|\hat{h}_4|}\right)^{*}\right]^2$$

for a constant η_0 chosen to be negative in order to generate the signature $(-,-,-,+)$. For non–trivial λ, we may search the solution as

$$h_3 = \widehat{h}_3\left(\xi,\varphi\right)\ q_3\left(\xi,\varphi\right) \text{ and } h_4 = \widehat{h}_4\left(\xi,\varphi\right), \qquad (14.68)$$

which solves (14.62) if $q_3 = 1$ for $\lambda = 0$ and

$$q_3 = \frac{1}{4\lambda}\left[\int \frac{\hat{h}_3 \hat{h}_4}{\hat{h}_4^{*}} d\varphi\right]^{-1} \text{ for } \lambda \neq 0.$$

Now it is easy to write down the solutions of equations (14.63) (being a linear equation for $w_{i'}$) and (14.64) (after two integrations of $n_{i'}$ on φ),

$$w_{i'} = \varepsilon \widehat{w}_{i'} = -\alpha_{i'}/\beta, \qquad (14.69)$$

were $\alpha_{i'}$ and β are computed by putting (14.68) into corresponding values from (14.65) (we chose the initial conditions as $w_{i'} \to 0$ for $\varepsilon \to 0$) and

$$n_1 = \varepsilon \widehat{n}_1\left(\xi,\varphi\right)$$

where

$$\begin{aligned}
\widehat{n}_1\left(\xi,\varphi\right) &= n_{1[1]}\left(\xi\right) + n_{1[2]}\left(\xi\right)\int d\varphi\ \eta_3\left(\xi,\varphi\right)/\left(\sqrt{|\eta_4\left(\xi,\varphi\right)|}\right)^3, \eta_4^{*} \neq 0; \qquad (14.70)\\
&= n_{1[1]}\left(\xi\right) + n_{1[2]}\left(\xi\right)\int d\varphi\ \eta_3\left(\xi,\varphi\right), \eta_4^{*} = 0;\\
&= n_{1[1]}\left(\xi\right) + n_{1[2]}\left(\xi\right)\int d\varphi/\left(\sqrt{|\eta_4\left(\xi,\varphi\right)|}\right)^3, \eta_3^{*} = 0;
\end{aligned}$$

with the functions $n_{k[1,2]}\left(\xi\right)$ to be stated by boundary conditions.

We conclude that the set of data $g_1 = -1$, with non–trivial $g_2(\xi), h_3, h_4, w_{i'}, n_1$ stated respectively by (14.67), (14.68), (14.69), (14.70) define a black ellipsoid solution with explicit dependence on cosmological constant λ, i. e. a d–metric (14.60), which can be induced from string theory for $\lambda = 1/4$. The stability of such string static black ellipsoids can be proven exactly as it was done in Refs. [46] for the vanishing cosmological constant.

14.7.2 2D Finsler structures in string theory

There are some constructions which prove that two dimensional (2D) Finsler structures can be embedded into the Einstein's theory of gravity [12]. Here we analyze the conditions when such Finsler configurations can be generated from string theory. The aim is to include a 2D Finsler metric (14.127) into a d–metric (13.6) being an exact solution of the string corrected Einstein's equations (14.59).

If

$$h_{a'b'} = \frac{1}{2}\frac{\partial^2 F^2(x^{i'}, y^{c'})}{\partial y^{a'} \partial y^{b'}}$$

for $i', j', ... = 1, 2$ and $a', b', ... = 3, 4$ and following the homogeneity conditions for Finsler metric, we can write

$$F\left(x^{i'}, y^3, y^4\right) = y^3 f\left(x^{i'}, s\right)$$

for $s = y^4/y^3$ with $f\left(x^{i'}, s\right) = F\left(x^{i'}, 1, s\right)$, that

$$\begin{aligned} h_{33} &= \frac{s^2}{2}(f^2)^{**} - s(f^2)^{*} + f^2, \\ h_{34} &= -\frac{s^2}{2}(f^2)^{**} + \frac{1}{2}(f^2)^{*}, \\ h_{44} &= \frac{1}{2}(f^2)^{**}, \end{aligned} \tag{14.71}$$

in this subsection we denote $a^* = \partial a / \partial s$. The condition of vanishing of the off–diagonal term h_{34} gives us the trivial case, when $f^2 \simeq s^2 ... + ... s + ...$, i. e. Riemannian 2D metrics, so we can not include some general Finsler coefficients (14.71) directly into a diagonal d–metric ansatz (14.60). There is also another problem related with the Cartan's N–connection (13.33) being computed directly from the coefficients (14.71) generated by a function f^2 : all such values substituted into the equations (14.62) - (14.64) result in systems of nonlinear equations containing the 6th and higher derivatives of f on s which is very difficult to deal with.

We can include 2D Finsler structures in the Einstein and string gravity via additional 2D anholnomic frame transforms,

$$h_{ab} = e_a^{a'}\left(x^{i'}, s\right)\ e_b^{b'}\left(x^{i'}, s\right)\ h_{a'b'}\left(x^{i'}, s\right)$$

where $h_{a'b'}$ are induced by a Finsler metric f^2 as in (14.71) and h_{ab} may be diagonal, $h_{ab} = diag[h_a]$. We also should consider an embedding of the Cartan's N–connection into a more general N–connection, $N_{b'}^{a'} \subset N_{i'}^{a'}$, via transforms $N_{i'}^{a'} = \hat{e}_{i'}^{b'}\left(x^{i'}, s\right) N_{b'}^{a'}$ where $\hat{e}_{i'}^{b'}\left(x^{i'}, s\right)$ are some additional frame transforms in the off–diagonal sector of the ansatz (14.11). Such way generated metrics,

$$\begin{aligned} \delta s^2 &= g_{i'}(dx^{i'})^2 + e_a^{a'}\ e_a^{b'} h_{a'b'}(\delta y^a)^2, \\ \delta y^a &= dy^a + \hat{e}_{i'}^{b'} N_{b'}^{a'} dx^{i'} \end{aligned}$$

may be constrained by the condition to be an exact solution of the Einstein equations with (or not) certain string corrections. As a matter of principle, any string black ellipsoid configuration (of the type examined in the previous subsection) can be related to a 2D Finsler configuration for corresponding coefficients $e_a^{a'}$ and $\hat{e}_{i'}^{b'}$. An explicit form of anisotropic configuration is to be stated by corresponding boundary conditions and the type of anholonomic transforms. Finally, we note that instead of a 2D Finsler metric (14.127) we can use a 2D Lagrange metric (13.35).

14.7.3 Moving soliton–black hole string configurations

In this subsection, we consider that the primed coordinates are 5D ones obtained after a string compactification background for the NS–NS sector being common to both the heterotic and type II string theories. The $u^{\alpha'} = (x^{i'}, y^{a'})$ are split into coordinates x^i, with indices $i', j', k'... = 1, 2, 3$, and coordinates $y^{a'}$, with indices $a', b', c', ... = 4, 5$. Explicitly the coordinates are of the form

$$x^{i'} = (x^1 = \chi, \quad x^2 = \phi = \ln \widehat{\rho}, \quad x^3 = \theta) \quad \text{and} \quad y^{a'} = \left(y^4 = v, \qquad y^5 = p\right),$$

where χ is the 5th extra–dimensional coordinate and $\widehat{\rho}$ will be related with the 4D Schwarzschild coordinate. We analyze a metric interval written as

$$ds^2 = \Omega^2(x^{i'}, v)\hat{g}_{\alpha'\beta'}\left(x^{i'}, v\right) du^{\alpha'} du^{\beta'}, \tag{14.72}$$

were the coefficients $\hat{g}_{\alpha'\beta'}$ are parametrized by the ansatz

$$\left[\begin{array}{ccccc} g_1+(w_1{}^2+\zeta_1{}^2)h_4+n_1{}^2h_5 & (w_1w_2+\zeta_1\zeta_2)h_4+n_1n_2h_5 & (w_1w_3+\zeta_1\zeta_3)h_4+n_1n_3h_5 & (w_1+\zeta_1)h_4 & n_1h_5 \\ (w_1w_2+\zeta_1\zeta_2)h_4+n_1n_2h_5 & g_2+(w_2{}^2+\zeta_2{}^2)h_4+n_2{}^2h_5 & (w_2w_3++\zeta_2\zeta_3)h_4+n_2n_3h_5 & (w_2+\zeta_2)h_4 & n_2h_5 \\ (w_1w_3+\zeta_1\zeta_3)h_4+n_1n_3h_5 & (w_2w_3++\zeta_2\zeta_3)h_4+n_2n_3h_5 & g_3+(w_3{}^2+\zeta_3{}^2)h_4+n_3{}^2h_5 & (w_3+\zeta_3)h_4 & n_3h_5 \\ (w_1+\zeta_1)h_4 & (w_2+\zeta_2)h_4 & (w_3+\zeta_3)h_4 & h_4 & 0 \\ n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5 \end{array}\right] \quad (14.73)$$

The metric coefficients are necessary class smooth functions of the form:

$$\begin{array}{rcl} g_1 &=& \pm 1, \qquad g_{2,3}=g_{2,3}(x^2,x^3), \qquad h_{4,5}=h_{4,5}(x^{i'},v)=\eta_{4,5}(x^i,v)h_{4,5[0]}(x^{k'}), \\ w_{i'} &=& w_{i'}(x^{k'},v), \qquad n_{i'}=n_{i'}(x^{k'},v), \qquad \zeta_{i'}=\zeta_{i'}(x^{k'},v), \qquad \Omega=\Omega(x^{i'},v). \end{array} \quad (14.74)$$

The quadratic line element (14.72) with metric coefficients (14.73) can be diagonalized by anholonmic transforms,

$$\delta s^2=\Omega^2(x^{i'},v)[g_1(dx^1)^2+g_2(dx^2)^2+g_3(dx^3)^2+h_4(\hat{\delta}v)^2+h_5(\delta p)^2], \quad (14.75)$$

with respect to the anholonomic co–frame $\left(dx^{i'},\hat{\delta}v,\delta p\right)$, where

$$\hat{\delta}v=dv+(w_{i'}+\zeta_{i'})dx^{i'}+\zeta_5\delta p \qquad \text{and} \qquad \delta p=dp+n_{i'}dx^{i'} \quad (14.76)$$

which is dual to the frame $\left(\hat{\delta}_{i'},\partial_4,\hat{\partial}_5\right)$, where

$$\hat{\delta}_{i'}=\partial_{i'}-(w_{i'}+\zeta_{i'})\partial_4+n_{i'}\partial_5, \qquad \hat{\partial}_5=\partial_5-\zeta_5\partial_4. \quad (14.77)$$

The simplest way to compute the nontrivial coefficients of the Ricci tensor for the (14.75) is to do this with respect to anholonomic bases (14.76) and (14.77) (see details in [49, 53]), which reduces the 5D vacuum Einstein equations to the following system (in this paper containing a non–trivial cosmological constant):

$$\begin{array}{rcl} \frac{1}{2}R_1^1=R_2^2=R_3^3=-\frac{1}{2g_2g_3}[g_3^{\bullet\bullet}-\frac{g_2^{\bullet}g_3^{\bullet}}{2g_2}-\frac{(g_3^{\bullet})^2}{2g_3}+g_2^{''}-\frac{g_2^{'}g_3^{'}}{2g_3}-\frac{(g_2^{'})^2}{2g_2}] &=& \lambda, (14.78) \\ R_4^4=R_5^5=-\frac{\beta}{2h_4h_5} &=& \lambda, (14.79) \\ R_{4i'}=-w_{i'}\frac{\beta}{2h_5}-\frac{\alpha_{i'}}{2h_5} &=& 0, (14.80) \\ R_{5i'}=-\frac{h_5}{2h_4}\left[n_{i'}^{**}+\gamma n_{i'}^{*}\right] &=& 0, (14.81) \end{array}$$

with the conditions that

$$\Omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}), \qquad (14.82)$$

and ζ_i satisfies the equations

$$\partial_{i'}\Omega - (w_{i'} + \zeta_{i'})\Omega^* = 0, \qquad (14.83)$$

The coefficients of equations (14.78) - (14.81) are given by

$$\alpha_{i'} = \partial_i h_5^* - h_5^* \partial_{i'} \ln \sqrt{|h_4 h_5|}, \qquad \beta = h_5^{**} - h_5^*[\ln \sqrt{|h_4 h_5|}]^*, \qquad \gamma = \frac{3h_5^*}{2h_5} - \frac{h_4^*}{h_4}. \ (14.84)$$

The various partial derivatives are denoted as $a^\bullet = \partial a/\partial x^2, a^{'} = \partial a/\partial x^3, a^* = \partial a/\partial v$.

The system of equations (14.78)–(14.81), (14.82) and (14.83) can be solved by choosing one of the ansatz functions ($e.g.$ $h_4\left(x^{i'}, v\right)$ or $h_5\left(x^{i'}, v\right)$) to take some arbitrary, but physically interesting form. Then the other ansatz functions can be analytically determined up to an integration in terms of this choice. In this way one can generate many solutions, but the requirement that the initial, arbitrary choice of the ansatz functions be "physically interesting" means that one wants to make this original choice so that the final solution generated in this way yield a well behaved solution. To satisfy this requirement we start from well known solutions of Einstein's equations and then use the above procedure to deform this solutions in a number of ways as to include it in a string theory. In the simplest case we derive 5D locally anisotropic string gravity solutions connected to the the Schwarzschild solution in *isotropic spherical coordinates* [17] given by the quadratic line interval

$$ds^2 = \left(\frac{\widehat{\rho}-1}{\widehat{\rho}+1}\right)^2 dt^2 - \rho_g^2 \left(\frac{\widehat{\rho}+1}{\widehat{\rho}}\right)^4 \left(d\widehat{\rho}^2 + \widehat{\rho}^2 d\theta^2 + \widehat{\rho}^2 \sin^2\theta d\varphi^2\right). \qquad (14.85)$$

We identify the coordinate $\widehat{\rho}$ with the re–scaled isotropic radial coordinate, $\widehat{\rho} = \rho/\rho_g$, with $\rho_g = r_g/4$; ρ is connected with the usual radial coordinate r by $r = \rho\left(1 + r_g/4\rho\right)^2$; $r_g = 2G_{[4]}m_0/c^2$ is the 4D Schwarzschild radius of a point particle of mass m_0; $G_{[4]} = 1/M^2_{P[4]}$ is the 4D Newton constant expressed via the Planck mass $M_{P[4]}$ (in general, we may consider that $M_{P[4]}$ may be an effective 4D mass scale which arises from a more fundamental scale of the full, higher dimensional spacetime); we set $c = 1$.

The metric (14.85) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass m_0. It has a singularity for $r = 0$ and a spherical horizon at $r = r_g$, or at $\widehat{\rho} = 1$ in the re–scaled isotropic coordinates. This solution is parametrized by a diagonal metric given with respect to holonomic coordinate frames. This spherically symmetric solution can be deformed in various interesting ways using the anholonomic frames method.

Vacuum gravitational 2D solitons in 4D Einstein vacuum gravity were originally investigated by Belinski and Zakharov [5]. In Refs. [52] 3D solitonic configurations were constructed on anisotropic Taub-NUT backgrounds. Here we show that 3D solitonic/black hole configurations can be embedded into the 5D locally anisotropic string gravity.

3D solitonic deformations in string gravity

The simplest way to construct a solitonic deformation of the off–diagonal metric in equation (14.73) is to take one of the "polarization" factors η_4, η_5 from (14.74) or the ansatz function $n_{i'}$ as a solitonic solution of some particular non-linear equation. The rest of the ansatz functions can then be found by carrying out the integrations of equations (14.78)– (14.83).

As an example of this procedure we suggest to take $\eta_5(r,\theta,\chi)$ as a soliton solution of the Kadomtsev–Petviashvili (KdP) equation or (2+1) sine-Gordon (SG) equation (Refs. [58] contain the original results, basic references and methods for handling such non-linear equations with solitonic solutions). In the KdP case $\eta_5(v,\theta,\chi)$ satisfies the following equation

$$\eta_5^{**} + \epsilon\left(\dot{\eta}_5 - 6\eta_5\eta_5' + \eta_5'''\right)' = 0, \qquad \epsilon = \pm 1, \tag{14.86}$$

while in the most general SG case $\eta_5(v,\chi)$ satisfies

$$\pm\eta_5^{**} \mp \ddot{\eta}_5 = \sin(\eta_5). \tag{14.87}$$

For simplicity, we can also consider less general versions of the SG equation where η_5 depends on only one (*e.g.* v and x_1) variable. We use the notation $\eta_5 = \eta_5^{KP}$ or $\eta_5 = \eta_5^{SG}$ ($h_5 = h_5^{KP}$ or $h_5 = h_5^{SG}$) depending on if (η_5) (h_5) satisfies equation (14.86), or (14.87) respectively.

For a stated solitonic form for $h_5 = h_5^{KP,SG}$, h_4 can be found from

$$h_4 = h_4^{KP,SG} = h_{[0]}^2\left[\left(\sqrt{|h_5^{KP,SG}(x^{i'},v)|}\right)^*\right]^2 \tag{14.88}$$

where $h_{[0]}$ is a constant. By direct substitution it can be shown that equation (14.88) solves equation (14.79) with β given by (14.84) when $h_5^* \neq 0$ but $\lambda = 0$. If $h_5^* = 0$, then $\hat{h}_4$ is an arbitrary function $\hat{h}_4(x^{i'},v)$. In either case we will denote the ansatz function determined in this way as $\hat{h}_4^{KP,SG}$ although it does not necessarily share the solitonic character of $\hat{h}_5$. Substituting the values $\hat{h}_4^{KP,SG}$ and $\hat{h}_5^{KP,SG}$ into γ from equation (14.65) gives, after two v integrations of equation (14.64), the ansatz functions $n_{i'} = n_{i'}^{KP,SG}(v,\theta,\chi)$. Solutions with $\lambda \neq 0$ can be generated similarly as in (14.68) by redefining

$$h_4 = \widehat{h}_4 \left(x^{i'}, v \right) \ q_4 \left(x^{i'}, v \right) \text{ and } h_5 = \widehat{h}_5 \left(x^{i'}, v \right),$$

which solves (14.79) if $q_4 = 1$ for $\lambda = 0$ and

$$q_4 = \frac{1}{4\lambda} \left[\int \frac{\hat{h}_5 \hat{h}_4}{\hat{h}_5^*} dv \right]^{-1} \text{ for } \lambda \neq 0. \tag{14.89}$$

Here, for simplicity, we may set $g_2 = -1$ but

$$g_3 = -Z_{[0]}^2 \sin^2 \left(\sqrt{2\lambda} x^3 + \xi_{[0]} \right), \ Z_{[0]}, \xi_{[0]} = const, \tag{14.90}$$

parametrize a class of solution of (14.78) for the signature $(-,-,-,-,+)$ like we constructed the solution (14.67). In ref. [52, 53] it was shown how to generate solutions using 2D solitonic configurations for g_2 or g_3.

The main conclusion to be formulated here is that the ansatz (14.73), when treated with anholonomic frames, has a degree of freedom that allows one to pick one of the ansatz functions (η_4 , η_5 , or $n_{i'}$) to satisfy some 3D solitonic equation. Then in terms of this choice all the other ansatz functions can be generated up to carrying out some explicit integrations and differentiations. In this way it is possible to build exact solutions of the 5D string gravity equations with a solitonic character.

Solitonically propagating string black hole backgrounds

The Schwarzschild solution is given in terms of the parameterization in (14.73) by

$$\begin{aligned} g_1 &= \pm 1, \qquad g_2 = g_3 = -1, \qquad h_4 = h_{4[0]}(x^{i'}), \qquad h_5 = h_{5[0]}(x^{i'}), \\ w_{i'} &= 0, \qquad n_{i'} = 0, \qquad \zeta_{i'} = 0, \qquad \Omega = \Omega_{[0]}(x^{i'}), \end{aligned}$$

with

$$h_{4[0]}(x^i) = \frac{b(\phi)}{a(\phi)}, \qquad h_{5[0]}(x^{i'}) = -\sin^2\theta, \qquad \Omega_{[0]}^2(x^{i'}) = a(\phi) \tag{14.91}$$

or alternatively, for another class of solutions,

$$h_{4[0]}(x^{i'}) = -\sin^2\theta, \qquad h_{5[0]}(x^{i'}) = \frac{b(\phi)}{a(\phi)}, \tag{14.92}$$

were

$$a(\phi) = \rho_g^2 \frac{\left(e^{\phi}+1\right)^4}{e^{2\phi}} \qquad \text{and} \qquad b(\phi) = \frac{\left(e^{\phi}-1\right)^2}{\left(e^{\phi}+1\right)^2}, \tag{14.93}$$

Putting this together gives

$$ds^2 = \pm d\chi^2 - a(\phi)\left(d\lambda^2 + d\theta^2 + \sin^2\theta d\varphi^2\right) + b\left(\phi\right) dt^2 \tag{14.94}$$

which represents a trivial embedding of the 4D Schwarzschild metric (14.85) into the 5D spacetime. We now want to deform anisotropically the coefficients of (14.94) in the following way

$$\begin{aligned} h_{4[0]}(x^{i'}) &\rightarrow h_4(x^{i'},v) = \eta_4\left(x^{i'},v\right) h_{4[0]}(x^{i'}),\\ h_{5[0]}(x^{i'}) &\rightarrow h_5(x^{i'},v) = \eta_5\left(x^{i'},v\right) h_{5[0]}(x^{i'}),\\ \Omega^2_{[0]}(x^{i'}) &\rightarrow \Omega^2(x^{i'},v) = \Omega^2_{[0]}(x^{i'})\Omega^2_{[1]}(x^{i'},v). \end{aligned}$$

The factors $\eta_{i'}$ and $\Omega^2_{[1]}$ can be interpreted as re-scaling or "renormalizing" the original ansatz functions. These gravitational "polarization" factors – $\eta_{4,5}$ and $\Omega^2_{[1]}$ – generate non–trivial values for $w_{i'}(x^{i'},v), n_{i'}(x^{i'},v)$ and $\zeta_{i'}(x^{i'},v)$, via the vacuum equations (14.78)– (14.83). We shall also consider more general nonlinear polarizations which can not be expresses as $h \sim \eta h_{[0]}$ and show how the coefficients $a(\phi)$ and $b(\phi)$ of the Schwarzschild metric can be polarized by choosing the original, arbitrary ansatz function to be some 3D soliton configuration.

The horizon is defined by the vanishing of the coefficient $b\left(\phi\right)$ from equation (14.93). This occurs when $e^{\phi} = 1$. In order to create a solitonically propagating black hole we define the function $\tau = \phi - \tau_0\left(\chi, v\right)$, and let $\tau_0\left(\chi, v\right)$ be a soliton solution of either the 3D KdP equation (14.86), or the SG equation (14.87). This redefines $b\left(\phi\right)$ as

$$b\left(\phi\right) \rightarrow B\left(x^{i'},v\right) = \frac{e^{\tau}-1}{e^{\phi}+1}.$$

A class of 5D string gravity metrics can be constructed by parametrizing $h_4 = \eta_4\left(x^{i'},v\right)$ $h_{4[0]}(x^{i'})$ and $h_5 = B\left(x^{i'},v\right)/a\left(\phi\right)$, or inversely, $h_4 = B\left(x^{i'},v\right)/a\left(\phi\right)$ and $h_5 = \eta_5\left(x^{i'},v\right)$ $h_{5[0]}(x^{i'})$. The polarization $\eta_4\left(x^{i'},v\right)$ (or $\eta_5\left(x^{i'},v\right)$) is determined from equation (14.88) with the factor q_4 (14.89) included in h^2,

$$|\eta_4\left(x^{i'},v\right) h_{4(0)}(x^{i'})| = h^2\left[\left(\sqrt{\left|\frac{B\left(x^{i'},v\right)}{a\left(\phi\right)}\right|}\right)^*\right]^2$$

or

$$\left|\frac{B\left(x^{i},v\right)}{a\left(\phi\right)}\right| = h^2 h_{5(0)}(x^{i'})\left[\left(\sqrt{|\eta_5\left(x^{i'},v\right)|}\right)^{*}\right]^2.$$

The last step in constructing of the form for these solitonically propagating black hole solutions is to use h_4 and h_5 in equation (14.64) to determine $n_{k'}$

$$\begin{aligned} n_{k'} &= n_{k'[1]}(x^{i'}) + n_{k'[2]}(x^{i'})\int \frac{h_4}{(\sqrt{|h_5|})^3}dv, \qquad h_5^{*}\neq 0; \\ &= n_{k'[1]}(x^{i'}) + n_{k'[2]}(x^{i'})\int h_4 dv, \qquad h_5^{*}=0; \\ &= n_{k'[1]}(x^{i'}) + n_{k'[2]}(x^{i'})\int \frac{1}{(\sqrt{|h_5|})^3}dv, \qquad h_4^{*}=0, \end{aligned} \tag{14.95}$$

where $n_{k[1,2]}\left(x^{i'}\right)$ are set by boundary conditions.

The simplest version of the above class of solutions are the so–called t–solutions (depending on t–variable), defined by a pair of ansatz functions, $\left[B\left(x^{i'},t\right),h_{5(0)}\right]$, with $h_5^{*}=0$ and $B\left(x^{i'},t\right)$ being a 3D solitonic configuration. Such solutions have a spherical horizon when $h_4=0$, *i.e.* when $\tau=0$. This solution describes a propagating black hole horizon. The propagation occurs via a 3D solitonic wave form depending on the time coordinate, t, and on the 5^{th} coordinate χ. The form of the ansatz functions for this solution (both with trivial and non-trivial conformal factors) is

$$\begin{aligned} t\text{–solutions} &: (x^1=\chi, \quad x^2=\phi, \quad x^3=\theta, \quad y^4=v=t, \quad y^5=p=\varphi), \\ g_1 &= \pm 1, g_2=-1, g_3=-Z_{[0]}^2\sin^2\left(\sqrt{2\lambda}x^3+\xi_{[0]}\right), \tau=\phi-\tau_0\left(\chi,t\right), \\ h_4 &= B/a(\phi), h_5=h_{5(0)}(x^{i'})=-\sin^2\theta, \omega=\eta_5=1, B\left(x^{i'},t\right)=\frac{e^{\tau}-1}{e^{\phi}+1}, \\ w_{i'} &= \zeta_{i'}=0, \qquad n_{k'}\left(x^{i'},t\right)=n_{k'[1]}(x^{i'})+n_{k'[2]}(x^{i'})\int B\left(x^{i'},t\right)dt, \end{aligned} \tag{14.96}$$

where q_4 is chosen to preserve the condition $w_{i'}=\zeta_{i'}=0$.

As a simple example of the above solutions we take τ_0 to satisfy the SG equation $\partial_{\chi\chi}\tau_0-\partial_{tt}\tau_0=\sin(\tau_0)$. This has the standard propagating kink solution

$$\tau_0(\chi,t)=4\tan^{-1}\left[\pm\gamma(\chi-Vt)\right]$$

where $\gamma=(1-V^2)^{-1/2}$ and V is the velocity at which the kink moves into the extra dimension χ. To obtain the simplest form of this solution we also take $n_{k'[1]}(x^{i'})=$

$n_{k'[2]}(x^{i'}) = 0$. This example can be easily extended to solutions with a non-trivial conformal factor Ω that gives an exponentially suppressing factor, $\exp[-2k|\chi|]$, see details in Ref. [53]. In this manner one has an effective 4D black hole which propagates from the 3D brane into the non-compact, but exponentially suppressed extra dimension, χ.

The solution constructed in this subsection describes propagating 4D Schwarzschild black holes in a bulk 5D spacetime obtained from string theory. The propagation arises from requiring that certain of the ansatz functions take a 3D soliton form. In the simplest version of these propagating solutions the parameters of the ansatz functions are constant, and the horizons are spherical. It can be also shown that such propagating solutions could be formed with a polarization of the parameters and/or deformation of the horizons, see the non–string case in [53].

14.7.4 Noncommutative anisotropic wormholes and strings

Let us construct and analyze an exact 5D solution of the string gravity which can also considered as a noncommutative structure in string theory. The d–metric ansatz is taken in the form

$$\begin{aligned}
\delta s^2 &= g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta y^4)^2 + h_5(\delta y^5)^2, \\
\delta y^4 &= dy^4 + w_{k'}\left(x^{i'}, v\right) dx^{k'}, \delta y^5 = dy^5 + n_{k'}\left(x^{i'}, v\right) dx^{k'}; i', k' = 1,2,3,
\end{aligned} \tag{14.97}$$

where

$$\begin{aligned}
g_1 &= 1, \quad g_2 = g_2(r), \quad g_3 = -a(r), \\
h_4 &= \hat{h}_4 = \widehat{\eta}_4\left(r,\theta,\varphi\right) h_{4[0]}(r), \quad h_5 = \hat{h}_5 = \widehat{\eta}_5\left(r,\theta,\varphi\right) h_{5[0]}(r,\theta)
\end{aligned} \tag{14.98}$$

for the parametrization of coordinate of type

$$x^1 = t, x^2 = r, x^3 = \theta, y^4 = v = \varphi, y^5 = p = \chi \tag{14.99}$$

where t is the time coordinate, (r,θ,φ) are spherical coordinates, χ is the 5th coordinate; φ is the anholonomic coordinate; for this ansatz there is not considered the dependence of d–metric coefficients on the second anholonomic coordinate χ. The data

$$\begin{aligned}
g_1 &= 1,\ \hat{g}_2 = -1,\ g_3 = -a(r), \\
h_{4[0]}(r) &= -r_0^2 e^{2\psi(r)},\ \eta_4 = 1/\kappa_r^2\left(r,\theta,\varphi\right),\ h_{5[0]} = -a\left(r\right)\sin^2\theta,\ \eta_5 = 1, \\
w_1 &= \widehat{w}_1 = \omega\left(r\right),\ w_2 = \widehat{w}_2 = 0, w_3 = \ \widehat{w}_3 = n\cos\theta/\kappa_n^2\left(r,\theta,\varphi\right), \\
n_1 &= \widehat{n}_1 = 0,\ n_{2,3} = \widehat{n}_{2,3} = n_{2,3[1]}\left(r,\theta\right)\int \ln\left|\kappa_r^2\left(r,\theta,\varphi\right)\right| d\varphi
\end{aligned} \tag{14.100}$$

for some constants r_0 and n and arbitrary functions $a(r), \psi(r)$ and arbitrary vacuum gravitational polarizations $\kappa_r(r,\theta,\varphi)$ and $\kappa_n(r,\theta,\varphi)$ define an exact vacuum 5D solution of Kaluza–Klein gravity [54] describing a locally anisotropic wormhole with elliptic gravitational vacuum polarization of charges,

$$\frac{q_0^2}{4a(0)\kappa_r^2} + \frac{Q_0^2}{4a(0)\kappa_n^2} = 1,$$

where $q_0 = 2\sqrt{a(0)}\sin\alpha_0$ and $Q_0 = 2\sqrt{a(0)}\cos\alpha_0$ are respectively the electric and magnetic charges and $2\sqrt{a(0)}\kappa_r$ and $2\sqrt{a(0)}\kappa_n$ are ellipse's axes.

The first aim in this subsection is to prove that following the ansatz (14.97) we can construct locally anisotropic wormhole metrics in string gravity as solutions of the system of equations (14.78) - (14.81) with redefined coordinates as in (14.99). Having the vacuum data (14.100) we may generalize the solution for a nontrivial cosmological constant following the method presented in subsection 14.7.3, when the new solutions are represented

$$h_4 = \widehat{h}_4\left(x^{i'}, v\right)\ q_4\left(x^{i'}, v\right) \text{ and } h_5 = \widehat{h}_5\left(x^{i'}, v\right), \tag{14.101}$$

with $\widehat{h}_{4,5}$ taken as in (14.98) which solves (14.79) if $q_4 = 1$ for $\lambda = 0$ and

$$q_4 = \frac{1}{4\lambda}\left[\int \frac{\hat{h}_5(r,\theta,\varphi)\,\hat{h}_4(r,\theta,\varphi)}{\hat{h}_5^*(r,\theta,\varphi)} d\varphi\right]^{-1} \text{ for } \lambda \neq 0.$$

This q_4 can be considered as an additional polarization to η_4 induced by the cosmological constant λ. We state $g_2 = -1$ but

$$g_3 = -\sin^2\left(\sqrt{2\lambda}\theta + \xi_{[0]}\right),$$

which give of solution of (14.78) with signature $(+,-,-,-,-)$ which is different from the solution (14.67). A non–trivial q_4 results in modification of coefficients (14.84),

$$\begin{aligned}
\alpha_{i'} &= \hat{\alpha}_{i'} + \alpha_{i'}^{[q]},\ \beta = \hat{\beta} + \beta^{[q]},\ \gamma = \hat{\gamma} + \gamma^{[q]}, \\
\hat{\alpha}_{i'} &= \partial_i \hat{h}_5^* - \hat{h}_5^* \partial_{i'} \ln\sqrt{|\hat{h}_4 \hat{h}_5|}, \qquad \hat{\beta} = \hat{h}_5^{**} - \hat{h}_5^*[\ln\sqrt{|\hat{h}_4\hat{h}_5|}]^*, \qquad \hat{\gamma} = \frac{3\hat{h}_5^*}{2\hat{h}_5} - \frac{\hat{h}_4^*}{\hat{h}_4} \\
\alpha_{i'}^{[q]} &= -h_5^* \partial_{i'} \ln\sqrt{|q_4|}, \qquad \beta^{[q]} = -h_5^*[\ln\sqrt{|q_4|}]^*, \qquad \gamma^{[q]} = -\frac{q_4^*}{q_4},
\end{aligned}$$

which following formulas (14.80) and (14.81) result in additional terms to the N–connection coefficients, i. e.

$$w_{i'} = \widehat{w}_{i'} + w_{i'}^{[q]} \text{ and } n_{i'} = \widehat{n}_{i'} + n_{i'}^{[q]}, \tag{14.102}$$

with $w_{i'}^{[q]}$ and $n_{i'}^{[q]}$ computed by using respectively $\alpha_{i'}^{[q]}, \beta^{[q]}$ and $\gamma^{[q]}$.

The N–connection coefficients (14.102) can be transformed partially into a B–field with $\{B_{i'j'}, B_{b'j'}\}$ defined by integrating the conditions (14.20), i. e.

$$B_{i'j'} = B_{i'j'[0]}\left(x^{k'}\right) + \int h_4 \delta_{[i'} w_{j']} d\varphi,\ B_{4j'} = B_{4j'[0]}\left(x^{k'}\right) + \int h_4 w_{j'}^{*} d\varphi, \tag{14.103}$$

for some arbitrary functions $B_{i'j'[0]}\left(x^{k'}\right)$ and $B_{4j'[0]}\left(x^{k'}\right)$. The string background corrections are presented via nontrivial $w_{i'}^{[q]}$ induced by $\lambda = 1/4$. The formulas (14.103) consist the second aim of this subsection: to illustrate how a a B–field inducing noncommutativity may be related with a N–connection inducing local anisotropy. This is an explicit example of locally anisotropic noncommutative configuration contained in string theory. For the considered class of wormhole solutions the coefficients $n_{i'}$ do not contribute into the noncommutative configuration, but, in general, following (14.19), they can be also related to noncommutativity.

14.8 Comments and Questions

In this paper, we have developed the method of anholonomic frames and associated nonlinear connections from a viewpoint of application in noncommutative geometry and string theory. We note in this retrospect that several futures connecting Finsler like generalizations of gravity and gauge theories, which in the past were considered ad hoc and sophisticated, actually have a very natural physical and geometric interpretation in the noncommutative and D–brane picture in string/M–theory. Such locally anisotropic and/ or noncommutative configurations are hidden even in general relativity and its various Kaluza–Klein like and supergravity extension. To emphasize them we have to consider off–diagonal metrics which can be diagonalized in result of certain anholonomic frame transforms which induce also nonlinear connection structures in the curved spacetime, in general, with noncompactified extra dimensions.

On general grounds, it could be said the the appearance of noncommutative and Finsler like geometry when considering B–fields, off–diagonal metrics and anholonomic frames (all parametrized, in general, by noncommutative matrices) is a natural thing. Such implementations in the presence of D–branes and matrix approaches to M–theory were proven here to have explicit realizations and supported by six background constructions elaborated in this paper:

First, both the local anisotropy and noncommutativity can be derived from considering string propagation in general manifolds and bundles and in various low energy string limits. This way the anholonomic Einstein and Finsler generalized gravity models are generated from string theory.

Second, the anholonomic constructions with associated nonlinear connection geometry can be explicitly modelled on superbundles which results in superstring effective actions with anholonomic (super) field equations which can be related to various superstring and supergravity theories.

Third, noncommutative geometries and associated differential calculi can be distinguished in anholonomic geometric form which allows formulation of locally anisotropic field theories with anholonomic symmetries.

Forth, anholonomy and noncommutativity can be related to string/M–theory following consequently the matrix algebra and geometry and/or associated to nonlinear connections noncommutative covariant differential calculi.

Fifth, different models of locally anisotropic gravity with explicit limits to string and Einstein gravity can be realized on noncommutative D–branes.

Sixth, the anholonomic frame method is a very powerful one in constructing and investigating new classes of exact solutions in string and gravity theories; such solutions contain generic noncommutativity and/or local anisotropy and can be parametrized as to describe locally anisotropic black hole configurations, Finsler like structures, anisoropic solitonic and moving string black hole metrics, or noncommutative and anisotropic wormhole structures which may be derived in Einstein gravity and/or its Kaluza–Klein and (super) string generalizations.

The obtained in this paper results have a recent confirmation in Ref. [33] where the spacetime noncommutativity is obtained in string theory with a constant off–diagonal metric background when an appropriate form is present and one of the spatial direction has Dirichlet boundary conditions. We note that in Refs. [49, 46, 52, 53, 54] we constructed exact solutions in the Einstein and extra dimension gravity with off–diagonal metrics which were diagonalized by anholonomic transforms to effective spacetimes with noncommutative structure induced by anholonomic frames. Those results were extended to noncommutative geometry and gauge gravity theories, in general, containing local anisotropy, in Refs. [48, 50]. The low energy string spacetime with noncommutativity constructed in subsection 7.4 of this work is parametrized by an off–diagonal metric which is a very general (non–constant) pseudo–Riemannian one defining an exact solution in string gravity.

Finally, our work raises a number of other interesting questions:

1. What kind of anholonomic quantum noncommutative structures are hidden in string theory and gravity; how such constructions are to be modelled by modern geometric methods.

2. How, in general, to relate the commutative and noncommutative gauge models of (super) gravity with local anisotropy directly to string/M–theory.

3. What kind of quantum structure is more naturally associated to string gravity and how to develop such anisotropic generalizations.

4. To formulate a nonlinear connection theory in quantum bundles and relate it to various Finsler like quantum generalizations.

5. What kind of Clifford structures are more natural for developing a unified geometric approach to anholonomic noncommutative and quantum geometry following in various perturbative limits and non–perturbative sectors of string/M–theory and when a such geometry is to be associated to D–brane configurations.

6. To construct new classes of exact solutions with generic anisotropy and noncommutativity and analyze theirs physical meaning and possible applications.

We hope to address some of these questions in future works.

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14.9 Appendix: Anholonomic Frames and N–Connections

We outline the basic definitions and formulas on anholonomic frames and associated nonlinear connection (N–connection) structures on vector bundles [29] and (pseudo) Riemannian manifolds [45, 49]. The Einstein equations are written in mixed holonomlic--anholonomic variables. We state the conditions when locally anisotropic structures (Finsler like and another type ones) can be modelled in general relativity and its extra dimension generalizations. This Abstract contains the necessary formulas in coordinate form taken from a geometric paper under preparation together with a co-author.

14.9.1 The N–connection geometry

The concept of N–connection came from Finsler geometry (as a set of coefficients it is present in the works of E. Cartan [6], then it was elaborated in a more explicit fashion by A. Kawaguchi [23]). The global definition of N–connections in commutative spaces is due to W. Barthel [3]. The geometry of N–connections was developed in details for vector, covector and higher order bundles [29, 28, 4], spinor bundles [40, 55] and superspaces and superstrings [41, 47, 42] with recent applications in modern anisotropic kinetics and theormodynamics [44] and elaboration of new methods of constructing exact off–diagonal solutions of the Einstein equations [45, 49]. The concept of N–connection can be extended in a similar manner from commutative to noncommutative spaces if a differential calculus is fixed on a noncommutative vector (or covector) bundle or another type of quantum manifolds [50].

N–connections in vector bundles and (pseudo) Riemannian spaces

Let us consider a vector bundle $\xi = (E, \mu, M)$ with typical fibre $\mathbb{R}^m$ and the map

$$\mu^T : TE \to TM$$

being the differential of the map $\mu : E \to M$. The map μ^T is a fibre–preserving morphism of the tangent bundle (TE, τ_E, E) to E and of tangent bundle (TM, τ, M) to M. The kernel of the morphism μ^T is a vector subbundle of the vector bundle (TE, τ_E, E). This kernel is denoted (VE, τ_V, E) and called the vertical subbundle over E. By

$$i : VE \to TE$$

it is denoted the inclusion mapping when the local coordinates of a point $u \in E$ are written $u^\alpha = (x^i, y^a)$, where the values of indices are $i, j, k, ... = 1, 2, ..., n$ and $a, b, c, ... = 1, 2, ..., m$.

A vector $X_u \in TE$, tangent in the point $u \in E$, is locally represented

$$\left(x, y, X, \widetilde{X}\right) = \left(x^i, y^a, X^i, X^a\right),$$

where $(X^i) \in \mathbb{R}^n$ and $(X^a) \in \mathbb{R}^m$ are defined by the equality

$$X_u = X^i \partial_i + X^a \partial_a$$

$[\partial_\alpha = (\partial_i, \partial_a)$ are usual partial derivatives on respective coordinates x^i and $y^a]$. For instance, $\mu^T \left(x, y, X, \widetilde{X}\right) = (x, X)$ and the submanifold VE contains elements of type

$\left(x, y, 0, \widetilde{X}\right)$ and the local fibers of the vertical subbundle are isomorphic to $\mathbb{R}^m$. Having $\mu^T(\partial_a) = 0$, one comes out that ∂_a is a local basis of the vertical distribution $u \to V_u E$ on E, which is an integrable distribution.

A nonlinear connection (in brief, N–connection) in the vector bundle $\xi = (E, \mu, M)$ is the splitting on the left of the exact sequence

$$0 \to VE \to TE/VE \to 0,$$

i. e. a morphism of vector bundles $N : TE \to VE$ such that $C \circ i$ is the identity on VE.

The kernel of the morphism N is a vector subbundle of (TE, τ_E, E), it is called the horizontal subbundle and denoted by (HE, τ_H, E). Every vector bundle (TE, τ_E, E) provided with a N–connection structure is Whitney sum of the vertical and horizontal subbundles, i. e.

$$TE = HE \oplus VE. \tag{14.104}$$

It is proven that for every vector bundle $\xi = (E, \mu, M)$ over a compact manifold M there exists a nonlinear connection [29].

Locally a N–connection N is parametrized by a set of coefficients $\left\{N_i^a(u^\alpha) = N_i^a(x^j, y^b)\right\}$ which transform as

$$N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a$$

under coordinate transforms on the vector bundle $\xi = (E, \mu, M)$,

$$x^{i'} = x^{i'}\left(x^i\right) \text{ and } y^{a'} = M_a^{a'}(x) y^a.$$

The well known class of linear connections consists a particular parametization of the coefficients N_i^a when

$$N_i^a(x^j, y^b) = \Gamma_{bi}^a(x^j) y^b$$

are linear on variables y^b.

If a N–connection structure is associated to local frame (basis, vielbein) on ξ, the operators of local partial derivatives $\partial_\alpha = (\partial_i, \partial_a)$ and differentials $d^\alpha = du^\alpha = (d^i = dx^i,$ $d^a = dy^a)$ should be elongated as to adapt the local basis (and dual basis) structure to the Whitney decomposition of the vector bundle into vertical and horizontal subbundles, (14.104):

$$\partial_\alpha = (\partial_i, \partial_a) \to \delta_\alpha = \left(\delta_i = \partial_i - N_i^b \partial_b, \partial_a\right), \tag{14.105}$$

$$d^\alpha = \left(d^i, d^a\right) \to \delta^\alpha = \left(d^i, \delta^a = d^a + N_i^b d^i\right). \tag{14.106}$$

The transforms can be considered as some particular case of frame transforms of type

$$\partial_\alpha \to \delta_\alpha = e^\beta_\alpha \partial_\beta \text{ and } d^\alpha \to \delta^\alpha = (e^{-1})^\alpha_\beta \delta^\beta,$$

$e^\beta_\alpha (e^{-1})^\gamma_\beta = \delta^\gamma_\alpha$, when the vielbein coefficients e^β_α are constructed by using the Kronecker symbols δ^b_a, δ^i_j and N^b_i.

The bases δ_α and δ^α satisfy, in general, some anholonomy conditions, for instance,

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W^\gamma_{\alpha\beta} \delta_\gamma, \qquad (14.107)$$

where $W^\gamma_{\alpha\beta}$ are called the anholonomy coefficients. An explicit calculus of commutators of operators (14.105) shows that there are the non–trivial values:

$$W^a_{ij} = R^a_{ij} = \delta_i N^a_j - \delta_j N^a_i,\ W^b_{ai} = -W^b_{ia} = -\partial_a N^b_i. \qquad (14.108)$$

Tensor fields on a vector bundle $\xi = (E, \mu, M)$ provided with N–connection structure N (we subject such spaces with the index N, ξ_N) may be decomposed in N–adapted form with respect to the bases δ_α and δ^α, and their tensor products. For instance, for a tensor of rang (1,1) $T = \{T_\alpha{}^\beta = \left(T_i{}^j, T_i{}^a, T_b{}^j, T_a{}^b\right)\}$ we have

$$T = T_\alpha{}^\beta \delta^\alpha \otimes \delta_\beta = T_i{}^j d^i \otimes \delta_i + T_i{}^a d^i \otimes \partial_a + T_b{}^j \delta^b \otimes \delta_j + T_a{}^b \delta^a \otimes \partial_b. \qquad (14.109)$$

Every N–connection with coefficients N^b_i generates also a linear connection on ξ_N as $\Gamma^{(N)\gamma}_{\alpha\beta} = \{N^a_{bi} = \partial N^a_i(x,y)/\partial y^b\}$ which defines a covariant derivative

$$D^{(N)}_\alpha A^\beta = \delta_\alpha A^\beta + \Gamma^{(N)\beta}_{\alpha\gamma} A^\gamma.$$

Another important characteristic of a N–connection is its curvature $\Omega = \{\Omega^a_{ij}\}$ with the coefficients

$$\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j = \partial_j N^a_i - \partial_i N^a_j + N^b_i N^a_{bj} - N^b_j N^a_{bi}. \qquad (14.110)$$

In general, on a vector bundle we may consider arbitrary linear connections and metric structures adapted to the N–connection decomposition into vertical and horizontal subbundles (one says that such objects are distinguished by the N–connection, in brief, d–objects, like the d-tensor (14.109), d–connection, d–metric:

- The coefficients of linear d–connections $\Gamma = \{\Gamma^\beta_{\alpha\gamma} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^b_{ac}\right)\}$ are defined for an arbitrary covariant derivative D on ξ being adapted to the N–connection structure as $D_{\delta_\alpha}(\delta_\beta) = \Gamma^\gamma_{\beta\alpha}\delta_\gamma$ with the coefficients being invariant under horizontal and vertical decomposition

$$D_{\delta_i}(\delta_j) = L^k_{ji}\delta_k,\ D_{\delta_i}(\partial_a) = L^b_{ai}\partial_b,\ D_{\partial_c}(\delta_j) = C^k_{jc}\delta_k,\quad D_{\partial_c}(\partial_a) = C^b_{ac}\partial_b.$$

The operator of covariant differentiation D splits into the horizontal covariant derivative $D^{[h]}$, stated by the coefficients $\left(L^i_{jk}, L^a_{bk}\right)$, for instance, and the operator of vertical covariant derivative $D^{[v]}$, stated by the coefficients $\left(C^i_{jc}, C^b_{ac}\right)$. For instance, for $A = A^i\delta_i + A^a\partial_a = A_i\partial^i + A_a\delta^a$ one holds the d–covariant derivation rules,

$$\begin{aligned}
D^{[h]}_i A^k &= \delta_i A^k + L^k_{ij}A^j,\ D^{[h]}_i A^b = \delta_i A^b + L^b_{ic}A^c,\\
D^{[h]}_i A_k &= \delta_i A_k - L^j_{ik}A_j, D^{[h]}_i A_b = \delta_i A_b - L^c_{ib}A_c,\\
D^{[v]}_a A^k &= \partial_a A^k + C^k_{aj}A^j,\ D^{[v]}_a A^b = \partial_a A^b + C^b_{ac}A^c,\\
D^{[v]}_a A_k &= \partial_a A_k - C^j_{ak}A_j, D^{[v]}_a A_b = \partial_a A_b - C^c_{ab}A_c.
\end{aligned}$$

- The d–metric structure $G = g_{\alpha\beta}\delta^a \otimes \delta^b$ which has the invariant decomposition as $g_{\alpha\beta} = (g_{ij}, g_{ab})$ following from

$$G = g_{ij}(x,y)d^i \otimes d^j + g_{ab}(x,y)\delta^a \otimes \delta^b. \tag{14.111}$$

We may impose the condition that a d–metric $g_{\alpha\beta}$ and a d–connection $\Gamma^\beta_{\alpha\gamma}$ are compatible, i. e. there are satisfied the conditions

$$D_\gamma g_{\alpha\beta} = 0. \tag{14.112}$$

With respect to the anholonomic frames (14.105) and (14.106), there is a linear connection, called the canonical distinguished linear connection, which is similar to the metric connection introduced by the Christoffel symbols in the case of holonomic bases, i. e. being constructed only from the metric components and satisfying the metricity conditions (14.112). It is parametrized by the coefficients, $\Gamma^\alpha{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$ where

$$\begin{aligned}
L^i{}_{jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \qquad (14.113)\\
L^a{}_{bk} &= \partial_b N^a_k + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N^d_k - h_{db}\partial_c N^d_k\right),\\
C^i{}_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\ C^a{}_{bc} = \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right).
\end{aligned}$$

Instead of this connection one can consider on ξ another types of linear connections which are/or not adapted to the N–connection structure (see examples in [29]).

D–torsions and d–curvatures:

The anholonomic coefficients $W^{\gamma}_{\ \alpha\beta}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^{\alpha}_{\ \beta\gamma}\delta_\alpha$, where

$$T^{\alpha}_{\ \beta\gamma} = \Gamma^{\alpha}_{\ \beta\gamma} - \Gamma^{\alpha}_{\ \gamma\beta} + W^{\alpha}_{\ \beta\gamma}, \tag{14.114}$$

and for the curvature tensor, $R(\delta_\tau, \delta_\gamma)\delta_\beta = R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha$, where

$$R_{\beta\ \gamma\tau}^{\ \alpha} = \delta_\tau \Gamma^{\alpha}_{\ \beta\gamma} - \delta_\gamma \Gamma^{\alpha}_{\ \beta\tau} + \Gamma^{\varphi}_{\ \beta\gamma}\Gamma^{\alpha}_{\ \varphi\tau} - \Gamma^{\varphi}_{\ \beta\tau}\Gamma^{\alpha}_{\ \varphi\gamma} + \Gamma^{\alpha}_{\ \beta\varphi}W^{\varphi}_{\ \gamma\tau}. \tag{14.115}$$

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible (horizontal–vertical) h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components,
$T^{\alpha}_{\ \beta\gamma} = \{T^{i}_{\ jk}, C^{i}_{\ ja}, S^{a}_{\ bc}, T^{a}_{\ ij}, T^{a}_{\ bi}\}$, where

$$\begin{aligned} T^{i}_{.jk} &= T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{ja} = C^{i}_{.ja}, \quad T^{i}_{aj} = -C^{i}_{ja}, \\ T^{i}_{.ja} &= 0, \quad T^{a}_{.bc} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}, \\ T^{a}_{.ij} &= -\Omega^{a}_{ij}, \quad T^{a}_{.bi} = \partial_b N^{a}_{i} - L^{a}_{.bi}, \quad T^{a}_{.ib} = -T^{a}_{.bi} \end{aligned} \tag{14.116}$$

(the d–torsion is computed by substituting the h–v–components of the canonical d–connection (14.113) and anholonomy coefficients (14.107) into the formula for the torsion coefficients (14.114)).

We emphasize that with respect to anholonomic frames the torsion is not zero even for symmetric connections with $\Gamma^{\alpha}_{\ \beta\gamma} = \Gamma^{\alpha}_{\ \gamma\beta}$ because the anholonomy coefficients $W^{\alpha}_{\ \beta\gamma}$ are contained in the formulas for the torsion coefficients (14.114). By straightforward computations we can prove that for nontrivial N–connection curvatures, $\Omega^{a}_{ij} \neq 0$, even the Levi–Civita connection for the metric (14.111) contains nonvanishing torsion coefficients. Of course, the torsion vanishes if the Levi–Civita connection is defined as the usual Christoffel symbols with respect to the coordinate frames, (∂_i, ∂_a) and (d^i, ∂^a); in this case the d–metric (14.111) is redefined into, in general, off–diagonal metric containing products of N^{a}_{i} and h_{ab}.

The curvature d–tensor has the following irreducible, non-vanishing, h–v–components
$R_{\beta\ \gamma\tau}^{\ \alpha} = \{R^{.i}_{h.jk}, R^{.a}_{b.jk}, P^{.i}_{j.ka}, P^{.c}_{b.ka}, S^{.i}_{j.bc}, S^{.a}_{b.cd}\}$, where

$$\begin{aligned}
R_{h.jk}^{.i} &= \delta_k L_{.hj}^i - \delta_j L_{.hk}^i + L_{.hj}^m L_{mk}^i - L_{.hk}^m L_{mj}^i - C_{.ha}^i \Omega_{.jk}^a, \\
R_{b.jk}^{.a} &= \delta_k L_{.bj}^a - \delta_j L_{.bk}^a + L_{.bj}^c L_{.ck}^a - L_{.bk}^c L_{.cj}^a - C_{.bc}^a \Omega_{.jk}^c, \\
P_{j.ka}^{.i} &= \partial_a L_{.jk}^i + C_{.jb}^i T_{.ka}^b - (\delta_k C_{.ja}^i + L_{.lk}^i C_{.ja}^l - L_{.jk}^l C_{.la}^i - L_{.ak}^c C_{.jc}^i), \\
P_{b.ka}^{.c} &= \partial_a L_{.bk}^c + C_{.bd}^c T_{.ka}^d - (\delta_k C_{.ba}^c + L_{.dk}^c C_{.ba}^d - L_{.bk}^d C_{.da}^c - L_{.ak}^d C_{.bd}^c), \\
S_{j.bc}^{.i} &= \partial_c C_{.jb}^i - \partial_b C_{.jc}^i + C_{.jb}^h C_{.hc}^i - C_{.jc}^h C_{hb}^i, \\
S_{b.cd}^{.a} &= \partial_d C_{.bc}^a - \partial_c C_{.bd}^a + C_{.bc}^e C_{.ed}^a - C_{.bd}^e C_{.ec}^a
\end{aligned} \tag{14.117}$$

(the d–curvature components are computed in a similar fashion by using the formula for curvature coefficients (14.115)).

Einstein equations in d–variables

In this subsection we write and analyze the Einstein equations on spaces provided with anholonomic frame structures and associated N–connections.

The Ricci tensor $R_{\beta\gamma} = R_{\beta\ \gamma\alpha}^{\ \alpha}$ has the d–components

$$R_{ij} = R_{i.jk}^{.k}, \quad R_{ia} = -{}^2P_{ia} = -P_{i.ka}^{.k}, R_{ai} = {}^1P_{ai} = P_{a.ib}^{.b}, \quad R_{ab} = S_{a.bc}^{.c}. \tag{14.118}$$

In general, since ${}^1P_{ai} \neq {}^2P_{ia}$, the Ricci d-tensor is non-symmetric (we emphasize that this could be with respect to anholonomic frames of reference because the N–connection and its curvature coefficients, N_i^a and $\Omega_{.jk}^a$, as well the anholonomy coefficients $W^\alpha_{\beta\gamma}$ and d–torsions $T^\alpha_{\beta\gamma}$ are contained in the formulas for d–curvatures (14.115)). The scalar curvature of the metric d–connection, $\overleftarrow{R} = g^{\beta\gamma} R_{\beta\gamma}$, is computed

$$\overleftarrow{R} = G^{\alpha\beta} R_{\alpha\beta} = \widehat{R} + S, \tag{14.119}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

By substituting (14.118) and (14.119) into the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \tag{14.120}$$

where κ and $\Upsilon_{\alpha\beta}$ are respectively the coupling constant and the energy–momentum tensor we obtain the h-v-decomposition by N–connection of the Einstein equations

$$\begin{aligned}
R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right) g_{ij} &= \kappa \Upsilon_{ij}, \\
S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right) h_{ab} &= \kappa \Upsilon_{ab}, \\
{}^1P_{ai} = \kappa \Upsilon_{ai},\ {}^2P_{ia} &= \kappa \Upsilon_{ia}.
\end{aligned} \tag{14.121}$$

The definition of matter sources with respect to anholonomic frames is considered in Refs. [40, 47, 29].

The vacuum locally anisotropic gravitational field equations, in invariant h– v–components, are written

$$R_{ij} = 0, S_{ab} = 0,^1 P_{ai} = 0, \ ^2P_{ia} = 0. \qquad (14.122)$$

We emphasize that general linear connections in vector bundles and even in the (pseudo) Riemannian spacetimes have non–trivial torsion components if off–diagonal metrics and anholomomic frames are introduced into consideration. This is a "pure" anholonomic frame effect: the torsion vanishes for the Levi–Civita connection stated with respect to a coordinate frame, but even this metric connection contains some torsion coefficients if it is defined with respect to anholonomic frames (this follows from the w–terms in (3.10)). For the (pseudo) Riemannian spaces we conclude that the Einstein theory transforms into an effective Einstein–Cartan theory with anholonomically induced torsion if the general relativity is formulated with respect to general frame bases (both holonomic and anholonomic).

The N–connection geometry can be similarly formulated for a tangent bundle TM of a manifold M (which is used in Finsler and Lagrange geometry [29]), on cotangent bundle T^*M and higher order bundles (higher order Lagrange and Hamilton geometry [28]) as well in the geometry of locally anisotropic superspaces [41], superstrings [43], anisotropic spinor [40] and gauge [51] theories or even on (pseudo) Riemannian spaces provided with anholonomic frame structures [55].

14.9.2 Anholonomic Frames in Commutative Gravity

We introduce the concepts of generalized Lagrange and Finsler geometry and outline the conditions when such structures can be modelled on a Riemannian space by using anholnomic frames.

Different classes of commutative anisotropic spacetimes are modelled by corresponding parametriztions of some compatible (or even non–compatible) N–connection, d–connection and d–metric structures on (pseudo) Riemannian spaces, tangent (or cotangent) bundles, vector (or covector) bundles and their higher order generalizations in their usual manifold, supersymmetric, spinor, gauge like or another type approaches (see Refs. [45, 28, 29, 4, 40, 51, 47, 55]).

Anholonomic structures on Riemannian spaces

We note that the N–connection structure may be defined not only in vector bundles but also on (pseudo) Riemannian spaces [45]. In this case the N–connection is an object completely defined by anholonomic frames, when the coefficients of frame transforms, $e_\alpha^\beta(u^\gamma)$, are parametrized explicitly via certain values $\left(N_i^a, \delta_i^j, \delta_b^a\right)$, where δ_i^j and δ_b^a are the Kronecker symbols. By straightforward calculations we can compute that the coefficients of the Levi–Civita metric connection

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = g\left(\delta_\alpha, \bigtriangledown_\gamma \delta_\beta\right) = g_{\alpha\tau}\Gamma^{\bigtriangledown\tau}_{\beta\gamma},$$

associated to a covariant derivative operator $\bigtriangledown$, satisfying the metricity condition $\bigtriangledown_\gamma g_{\alpha\beta} = 0$ for $g_{\alpha\beta} = (g_{ij}, h_{ab})$ and

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_\beta g_{\alpha\gamma} + \delta_\gamma g_{\beta\alpha} - \delta_\alpha g_{\gamma\beta} + g_{\alpha\tau}W^\tau_{\gamma\beta} + g_{\beta\tau}W^\tau_{\alpha\gamma} - g_{\gamma\tau}W^\tau_{\beta\alpha}\right], \qquad (14.123)$$

are given with respect to the anholonomic basis (14.106) by the coefficients

$$\Gamma^{\bigtriangledown\tau}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc} + \frac{1}{2}g^{ik}\Omega^a_{jk}h_{ca}, C^a{}_{bc}\right) \qquad (14.124)$$

when $L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}$ and Ω^a_{jk} are respectively computed by the formulas (14.113) and (14.110). A specific property of off–diagonal metrics is that they can define different classes of linear connections which satisfy the metricity conditions for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection. This result was originally obtained by A. Kawaguchi [23] (Details can be found in Ref. [29], see Theorems 5.4 and 5.5 in Chapter III, formulated for vector bundles; here we note that similar proofs hold also on manifolds enabled with anholonomic frames associated to a N–connection structure).

With respect to anholonomic frames, we can not distinguish the Levi–Civita connection as the unique one being both metric and torsionless. For instance, both linear connections (14.113) and (14.124) contain anholonomically induced torsion coefficients, are compatible with the same metric and transform into the usual Levi–Civita coefficients for vanishing N–connection and "pure" holonomic coordinates. This means that to an off–diagonal metric in general relativity one may be associated different covariant differential calculi, all being compatible with the same metric structure (like in the non–commutative geometry, which is not a surprising fact because the anolonomic frames satisfy by definition some non–commutative relations (14.107)). In such cases we have to select a particular type of connection following some physical or geometrical arguments, or to impose some conditions when there is a single compatible linear connection constructed only from the metric and N–coefficients. We note that if $\Omega^a_{jk} = 0$ the connections (14.113) and (14.124) coincide, i. e. $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\bigtriangledown\alpha}_{\beta\gamma}$.

If an anholonomic (equivalently, anisotropic) frame structure is defined on a (pseudo) Riemannian space of dimension $(n+m)$ space, the space is called to be an anholonomic (pseudo) Riemannian one (denoted as $V^{(n+m)}$). By fixing an anholonomic frame basis and co–basis with associated N–connection $N_i^a(x,y)$, respectively, as (14.105) and (14.106), one splits the local coordinates $u^\alpha = (x^i, y^a)$ into two classes: the fist class consists from n holonomic coordinates, x^i, and the second class consists from m anholonomic coordinates, y^a. The d–metric (14.111) on $V^{(n+m)}$,

$$G^{[R]} = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b \qquad (14.125)$$

written with respect to a usual coordinate basis $du^\alpha = (dx^i, dy^a)$,

$$ds^2 = \underline{g}_{\alpha\beta}(x,y)\, du^\alpha du^\beta$$

is a generic off–diagonal Riemannian metric parametrized as

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b g_{ab} & h_{ab} N_i^a \\ h_{ab} N_j^b & h_{ab} \end{array} \right]. \qquad (14.126)$$

Such type of metrics were largely investigated in the Kaluza–Klein gravity [35], but also in the Einstein gravity [45]. An off–diagonal metric (14.126) can be reduced to a block $(n \times n) \oplus (m \times m)$ form (g_{ij}, g_{ab}), and even effectively diagonalized in result of a superposition of anholonomic N–transforms. It can be defined as an exact solution of the Einstein equations. With respect to anholonomic frames, in general, the Levi–Civita connection obtains a torsion component (14.123). Every class of off–diagonal metrics can be anholonomically equivalent to another ones for which it is not possible to a select the Levi–Civita metric defied as the unique torsionless and metric compatible linear connection. The conclusion is that if anholonomic frames of reference, which authomatically induce the torsion via anholonomy coefficients, are considered on a Riemannian space we have to postulate explicitly what type of linear connection (adapted both to the anholonomic frame and metric structure) is chosen in order to construct a Riemannian geometry and corresponding physical models. For instance, we may postulate the connection (14.124) or the d–connection (14.113). Both these connections are metric compatible and transform into the usual Christoffel symbols if the N–connection vanishes, i. e. the local frames became holonomic. But, in general, anholonomic frames and off–diagonal Riemannian metrics are connected with anisotropic configurations which allow, in principle, to model even Finsler like structures in (pseudo) Riemannian spaces [44, 45].

Finsler geometry and its almost Kahlerian model

The modern approaches to Finsler geometry are outlined in Refs. [34, 29, 28, 4, 47, 55]. Here we emphasize that a Finsler metric can be defined on a tangent bundle TM with local coordinates $u^\alpha = (x^i, y^a \rightarrow y^i)$ of dimension $2n$, with a d–metric (14.111) for which the Finsler metric, i. e. the quadratic form

$$g_{ij}^{[F]} = h_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} \tag{14.127}$$

is positive definite, is defined in this way: 1) A Finsler metric on a real manifold M is a function $F : TM \rightarrow \mathbb{R}$ which on $\widetilde{TM} = TM\backslash\{0\}$ is of class C^∞ and F is only continuous on the image of the null cross–sections in the tangent bundle to M. 2) $F(x, \chi y) = \chi F(x, y)$ for every $\mathbb{R}_+^*$. 3) The restriction of F to $\widetilde{TM}$ is a positive function. 4) $rank\left[g_{ij}^{[F]}(x,y)\right] = n.$

The Finsler metric $F(x, y)$ and the quadratic form $g_{ij}^{[F]}$ can be used to define the Christoffel symbols (not those from the usual Riemannian geometry)

$$c_{jk}^{\iota}(x,y) = \frac{1}{2} g^{ih}\left(\partial_j g_{hk}^{[F]} + \partial_k g_{jh}^{[F]} - \partial_h g_{jk}^{[F]}\right),$$

where $\partial_j = \partial/\partial x^j$, which allows us to define the Cartan nonlinear connection as

$$N_j^{[F]i}(x,y) = \frac{1}{4}\frac{\partial}{\partial y^j}\left[c_{lk}^{\iota}(x,y) y^l y^k\right] \tag{14.128}$$

where we may not distinguish the v- and h- indices taking on TM the same values.

In Finsler geometry there were investigated different classes of remarkable Finsler linear connections introduced by Cartan, Berwald, Matsumoto and other ones (see details in Refs. [34, 29, 4]). Here we note that we can introduce $g_{ij}^{[F]} = g_{ab}$ and $N_j^i(x,y)$ in (14.111) and construct a d–connection via formulas (14.113).

A usual Finsler space $F^n = (M, F(x,y))$ is completely defined by its fundamental tensor $g_{ij}^{[F]}(x,y)$ and Cartan nonlinear connection $N_j^i(x,y)$ and its chosen d–connection structure. But the N–connection allows us to define an almost complex structure I on TM as follows

$$I(\delta_i) = -\partial/\partial y^i \text{ and } I\left(\partial/\partial y^i\right) = \delta_i$$

for which $I^2 = -1$.

The pair $\left(G^{[F]}, I\right)$ consisting from a Riemannian metric on TM,

$$G^{[F]} = g_{ij}^{[F]}(x,y)dx^i \otimes dx^j + g_{ij}^{[F]}(x,y)\delta y^i \otimes \delta y^j \tag{14.129}$$

and the almost complex structure I defines an almost Hermitian structure on $\widetilde{TM}$ associated to a 2–form

$$\theta = g_{ij}^{[F]}(x,y)\delta y^i \wedge dx^j.$$

This model of Finsler geometry is called almost Hermitian and denoted H^{2n} and it is proven [29] that is almost Kahlerian, i. e. the form θ is closed. The almost Kahlerian space $K^{2n} = \left(\widetilde{TM}, G^{[F]}, I\right)$ is also called the almost Kahlerian model of the Finsler space F^n.

On Finsler (and their almost Kahlerian models) spaces one distinguishes the almost Kahler linear connection of Finsler type, $D^{[I]}$ on $\widetilde{TM}$ with the property that this covariant derivation preserves by parallelism the vertical distribution and is compatible with the almost Kahler structure $\left(G^{[F]}, I\right)$, i.e.

$$D_X^{[I]} G^{[F]} = 0 \text{ and } D_X^{[I]} I = 0$$

for every d–vector field on $\widetilde{TM}$. This d–connection is defined by the data

$$\Gamma = \left(L_{jk}^i, L_{bk}^a = 0, C_{ja}^i = 0, C_{bc}^a \rightarrow C_{jk}^i\right)$$

with L_{jk}^i and C_{jk}^i computed as in the formulas (14.113) by using $g_{ij}^{[F]}$ and N_j^i from (14.128).

We emphasize that a Finsler space F^n with a d–metric (14.129) and Cartan's N–connection structure (14.128), or the corresponding almost Hermitian (Kahler) model H^{2n}, can be equivalently modelled on a Riemannian space of dimension $2n$ provided with an off–diagonal Riemannian metric (14.126). From this viewpoint a Finsler geometry is a corresponding Riemannian geometry with a respective off–diagonal metric (or, equivalently, with an anholonomic frame structure with associated N–connection) and a corresponding prescription for the type of linear connection chosen to be compatible with the metric and N–connection structures.

Lagrange and generalized Lagrange geometry

Lagrange spaces were introduced in order to geometrize the fundamental concepts in mechanics [24] and investigated in Refs. [29] (see [40, 51, 41, 43, 47, 55] for their spinor, gauge and supersymmetric generalizations).

A Lagrange space $L^n = (M, L(x,y))$ is defined as a pair which consists of a real, smooth n–dimensional manifold M and regular Lagrangian $L : TM \to \mathbb{R}$. Similarly as for Finsler spaces one introduces the symmetric d–tensor field

$$g_{ij}^{[L]} = h_{ab} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}. \qquad (14.130)$$

So, the Lagrangian $L(x,y)$ is like the square of the fundamental Finsler metric, $F^2(x,y)$, but not subjected to any homogeneity conditions.

In the rest me can introduce similar concepts of almost Hermitian (Kahlerian) models of Lagrange spaces as for the Finsler spaces, by using the similar definitions and formulas as in the previous subsection, but changing $g_{ij}^{[F]} \to g_{ij}^{[L]}$.

R. Miron introduced the concept of generalized Lagrange space, GL–space (see details in [29]) and a corresponding N–connection geometry on TM when the fundamental metric function $g_{ij} = g_{ij}(x,y)$ is a general one, not obligatory defined as a second derivative from a Lagrangian as in (14.130). The corresponding almost Hermitian (Kahlerian) models of GL–spaces were investigated and applied in order to elaborate generalizations of gravity and gauge theories [29, 51].

Finally, a few remarks on definition of gravity models with generic local anisotropy on anholonomic Riemannian, Finsler or (generalized) Lagrange spaces and vector bundles. So, by choosing a d-metric (14.111) (in particular cases (14.125), or (14.129) with $g_{ij}^{[F]}$, or $g_{ij}^{[L]}$) we may compute the coefficients of, for instance, d–connection (14.113), d–torsion (14.116) and (14.117) and even to write down the explicit form of Einstein equations (14.121) which define such geometries. For instance, in a series of works [44, 45, 55] we found explicit solutions when Finsler like and another type anisotropic configurations are modelled in anisotropic kinetic theory and irreversible thermodynamics and even in Einstein or low/extra–dimension gravity as exact solutions of the vacuum (14.121) and nonvacuum (14.122) Einstein equations. From the viewpoint of the geometry of anholonomic frames is not much difference between the usual Riemannian geometry and its Finsler like generalizations. The explicit form and parametrizations of coefficients of metric, linear connections, torsions, curvatures and Einstein equations in all types of mentioned geometric models depends on the type of anholomic frame relations and compatibility metric conditions between the associated N–connection structure and linear connections we fixed. Such structures can be correspondingly picked up from a noncommutative functional model, for instance, from some almost Hermitian structures over projective modules and/or generalized to some noncommutative configurations [50].

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Chapter 15

Nonholonomic Clifford Structures and Noncommutative Riemann–Finsler Geometry

Abstract [1]

We survey the geometry of Lagrange and Finsler spaces and discuss the issues related to the definition of curvature of nonholonomic manifolds enabled with nonlinear connection structure. It is proved that any commutative Riemannian geometry (in general, any Riemann–Cartan space) defined by a generic off–diagonal metric structure (with an additional affine connection possessing nontrivial torsion) is equivalent to a generalized Lagrange, or Finsler, geometry modelled on nonholonomic manifolds. This results in the problem of constructing noncommutative geometries with local anisotropy, in particular, related to geometrization of classical and quantum mechanical and field theories, even if we restrict our considerations only to commutative and noncommutative Riemannian spaces. We elaborate a geometric approach to the Clifford modules adapted to nonlinear connections, to the theory of spinors and the Dirac operators on nonholonomic spaces and consider possible generalizations to noncommutative geometry. We argue that any commutative Riemann–Finsler geometry and generalizations my be derived from noncommutative geometry by applying certain methods elaborated for Riemannian spaces but extended to nonholonomic frame transforms and manifolds provided with nonlinear connection structure.

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15.1 Introduction

The goal of this work is to provide a better understanding of the relationship between the theory of nonholonomic manifolds with associated nonlinear connection structure, locally anisotropic spin configurations and Dirac operators on such manifolds and noncommutative Riemann–Finsler and Lagrange geometry. The latter approach is based on geometrical modelling of mechanical and classical field theories (defined, for simplicity, by regular Lagrangians in analytic mechanics and Finsler like anisotropic structures) and gravitational, gauge and spinor field interactions in low energy limits of string theory. This allows to apply the Serre–Swan theorem and think of vector bundles as projective modules, which, for our purposes, are provided with nonlinear connection (in brief, N–connection) structure and can be defined as a nonintegrabele (nonholonomic) distribution into conventional horizontal and vertical submodules. We relay on the theory of Clifford and spinor structures adapted to N–connections which results in locally anisotropic (Finsler like, or more general ones defined by more general nonholonomic frame structures) Dirac operators. In the former item, it is the machinery of noncommutative geometry to derive distance formulas and to consider noncommutative extensions of Riemann–Finsler and Lagrange geometry and related off–diagonal metrics in gravity theories.

In [76] it was proposed that an equivalent reformulation of the general relativity theory as a gauge model with nonlinear realizations of the affine, Poincare and/or de Sitter groups allows a standard extension of gravity theories in the language of noncommutative gauge fields. The approach was developed in [77] as an attempt to generalize the A. Connes' noncommutative geometry [17] to spaces with generic local anisotropy. The nonlinear connection formalism was elaborated for projective module spaces and the Dirac operator associated to metrics in Finsler geometry and some generalizations [69, 72] (such as Sasaki type lifts of metrics to the tangent bundles and vector bundle analogs) were considered as certain examples of noncommutative Finsler geometry. The constructions were synthesized and revised in connection to ideas about appearance of both noncommutative and Finsler geometry in string theory with nonvanishing B–field and/or anholonomic (super) frame structures [66, 18, 2, 1, 65, 78, 70, 73] and in supergravity and gauge gravity [8, 30, 89, 90, 22]. In particular, one has considered hidden noncommutative and Finsler like structures in general relativity and extra dimension gravity [62, 85, 79, 81, 82].

In this work, we confine ourselves to the classical aspects of Lagrange–Finsler geometry (sprays, nonlinear connections, metric and linear connection structures and almost complex structure derived from from a Lagrange or Finsler fundamental form) in order to generalize the doctrine of the "spectral action" and the theory of distance in noncommutative geometry which is an extension of the previous results [17]. For a complete information on modern noncommutative geometry and physics, we refer the reader to [39, 44, 34, 29, 20, 38], see a historical sketch in Ref. [34] as well the aspects related to quantum group theory [47, 45, 32] (here we note that the first quantum group Finsler structure was considered in [92]). The theory of Dirac operators and related Clifford analysis is a subject of various investigations in modern mathematics and mathematical physics [48, 49, 59, 61, 26, 11, 12, 14, 15, 67, 9] (see also a relation to Finsler geometry [91] and an off–diagonal "non" Kaluza–Klein compactified ansatz, but without N–connection counstructions [13]). [2] For an exposition spelling out all the details of proofs and important concepts preliminary undertaken on the subjects elaborated in our works, we refer to proofs and quotations in Refs. [56, 57, 80, 83, 82, 77, 78, 29, 63, 43, 50].

This paper consists of two heterogeneous parts:

The first (commutative) contains an overview of the Lagrange and Finsler geometry and the off–diagonal metric and nonholonomic frame geometry in gravity theories. In Section 2, we formulate the N–connection geometry for arbitrary manifolds with tangent bundles admitting splitting into conventional horizontal and vertical subspaces. We illustrate how regular Lagrangians induce natural semispray, N–connection, metric and almost complex structures on tangent bundles and discuss the relation between Lagrange and Finsler geometry and theirs generalizations. We formulate six most important Results 3.57–15.2.6 demonstrating that the geometrization of Lagrange mechanics and the geometric models in gravity theories with generic off–diagonal metrics and nonholonomic frame structures are rigorously described by certain generalized Finsler geometries, which can be modelled equivalently both on Riemannian manifold and Riemann–Cartan nonholonomic manifolds. This give rise to the Conclusion 15.2.1 stating that a rigorous geometric study of nonholonmic frame and related metric, linear connection and spin structures in both commutative and noncommutative Riemann geometries requests the elaboration of noncommutative Lagrange–Finsler geometry. Then, in Section 3, we consider the theory of linear connections on N–anholonomic manifolds (i. e. on manifolds with nonholonomic structure defined by N–connections). We construct in explicit form

[2]The theory of N–connections should not be confused with nonlinear gauge theories and nonlinear realizations of gauge groups.

the curvature tensor of such spaces and define the Einstein equations for N–adapted linear connection and metric structures.

The second (noncommutative) part starts with Section 4 where we define noncommutative N–anholonomic spaces. We consider the example of noncommutative gauge theories adapted to the N–connection structure. Section 5 is devoted to the geometry of nonholonomic Clifford–Lagrange structures. We define the Clifford–Lagrange modules and Clifford N–anholonomic bundles being induced by the Lagrange quadratic form and adapted to the corresponding N–connection. Then we prove the **Main Result 1,** of this work, (Theorem 15.5.3), stating that any regular Lagrangian and/or N–connection structure define naturally the fundamental geometric objects and structures (such as the Clifford–Lagrange module and Clifford d–modules, the Lagrange spin structure and d–spinors) for the corresponding Lagrange spin manifold and/or N–anholonomic spinor (d–spinor) manifold. We conclude that the Lagrange mechanics and off–diagonal gravitational interactions (in general, with nontrivial torsion and nonholonomic constraints) can be adequately geometrized as certain Lagrange spin (N–anholonomic) manifolds.

In Section 6, we link up the theory of Dirac operators to nonholonomic structures and spectral triples. We prove that there is a canonical spin d–connection on the N–anholonomic manifolds generalizing that induced by the Levi–Civita to the naturally ones induced by regular Lagrangians and off–diagonal metrics. We define the Dirac d–operator and the Dirac–Lagrange operator and formulate the **Main Result 2** (Theorem 15.6.4) arguing that such N–adapted operators can be induced canonically by almost Hermitian spin operators. The concept of distinguished spin triple is introduced in order to adapt the constructions to the N–connection structure. Finally, the **Main Result 3,** Theorem 15.6.4, is devoted to the definition, main properties and computation of distance in noncommutative spaces defined by N–anholonomic spin varieties. In these lecture notes, we only sketch in brief the ideas of proofs of the Main Results: the details will be published in our further works.

15.2 Lagrange–Finsler Geometry and Nonholonomic Manifolds

This section presents some basic facts from the geometry of nonholonomic manifolds provided with nonlinear connection structure [69, 53, 54, 19, 55, 93]. The constructions and methods are inspired from the Lagrange–Finsler geometry and generalizations [27, 10, 64, 56, 3, 52, 6, 57, 71, 73, 4, 58, 94] and gravity models on metric–affine spaces provided with generic off–diagonal metric, nonholonomic frame and affine connection structures [74, 87, 85, 81, 80, 83] (such spaces, in general, possess nontrivial torsion and nonmetricity).

15.2.1 Preliminaries: Lagrange–Finsler metrics

Let us consider a nondegenerate bilinear symmetric form $q(u,v)$ on a n–dimensional real vector space V^n. With respect to a basis $\{e_i\}_{i=1}^n$ for V^n, we express

$$q(u,v) \doteq q_{ij}u^i v^j$$

for any vectors $u = u^i e_i,\ v = v^i e_i \in V^n$ and q_{ij} being a nodegenerate symmetric matrix (the Einstein's convention on summing on repeating indices is adopted). This gives rise to the Euclidean inner product

$$u\rfloor v \doteq q_E(u,v),$$

if q_{ij} is positive definite, and to the Euclidean norm

$$|\cdot| \doteq \sqrt{q_E(u,u)}$$

defining an Euclidean space $(V^n, |\cdot|)$. Every Euclidean space is linearly isometric to the standard Euclidean space $\mathbb{R}^n = (R^n, |\cdot|)$ if $q_{ij} = diag[1,1,...,1]$ with standard Euclidean norm, $|\ y\ | \doteq \sqrt{\sum_{i=1}^n |y^i|^2}$, for any $y = (y^i) \in R^n$, where R^n denotes the n–dimensional canonical real vector space.

There are also different types of quadratic forms/norms then the Euclidean one:

Definition 15.2.1. *A Lagrange fundamental form $q_L(u,v)$ on vector space V^n is defined by a Lagrange functional $L:\ V^n \to \mathbb{R}$, with*

$$q_{L(y)}(u,v) \doteq \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[L(y + su + tu)]_{|s=t=0} \qquad (15.1)$$

which is a C^∞–function on $V^n\backslash\{0\}$ and nondegenerate for any nonzero vector $y \in V^n$ and real parameters s and t.

Having taken a basis $\{e_i\}_{i=1}^n$ for V^n, we transform $L = L(y^i e_i)$ in a function of $(y^i) \in R^n$.

The Lagrange norm is $|\cdot|_L \doteq \sqrt{q_L(u,u)}$.

Definition 15.2.2. *A Minkowski space is a pair (V^n, F) where the Minkowski functional F is a positively homogeneous of degree two Lagrange functional with the fundamental form (15.1) defined for $L = F^2$ satisfying $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$ and $y \in V^n$.*

The Minkowski norm is defined by $|\cdot|_F \doteq \sqrt{q_F(u,u)}$.

Definition 15.2.3. *The Lagrange (or Minkowski) metric fundamental function is defined*

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}(y) \tag{15.2}$$

(or

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(y)\ ). \tag{15.3}$$

Remark 15.2.1. *If L is a Lagrange functional on R^n (it could be also any functional of class C^∞) with local coordinates $(y^2, y^3, ..., y^n)$, it also defines a singular Minkowski functional*

$$F(y) = [y^1 L(\frac{y^2}{y^1}, ..., \frac{y^n}{y^1})]^2 \tag{15.4}$$

which is of class C^∞) on $R^n\backslash\{y^1 = 0\}$.

The Remark 15.2.1 states that the Lagrange functionals are not essentially more general than the Minkowski functionals [94]. Nevertheless, we must introduce more general functionals if we extend our considerations in relativistic optics, string models of gravity and the theory of locally anisotropic stochastic and/or kinetic processes [57, 74, 87, 85, 81, 75].

Let us consider a base manifold M, $dimM = n$, and its tangent bundle (TM, π, M) with natural surjective projection $\pi : TM \to M$. From now on, all manifolds and geometric objects are supposed to be of class $\mathcal{C}^\infty$. We write $\widetilde{TM} = TM\backslash\{0\}$ where $\{0\}$ means the null section of the map π.

A differentiable Lagrangian $L(x, y)$ is defined by a map $L : (x, y) \in TM \to L(x, y) \in \mathbb{R}$ of class $\mathcal{C}^\infty$ on $\widetilde{TM}$ and continuous on the null section $0 : M \to TM$ of π. For any point $x \in M$, the restriction $L_x \doteq L_{|T_xM}$ is a Lagrange functional on T_xM (see Definition 15.2.1). For simplicity, in this work we shall consider only regular Lagrangians with nondegenerated Hessians,

$$^{(L)}g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \tag{15.5}$$

when $rank\,|g_{ij}| = n$ on $\widetilde{TM}$, which is a Lagrange fundamental quadratic form (15.2) on T_xM. In our further considerations, we shall write $M_{(L)}$ if would be necessary to emphasize that the manifold M is provided in any its points with a quadratic form (15.5).

Definition 15.2.4. *A Lagrange space is a pair* $L^n = [M, L(x,y)]$ *with the metric form* ${}^{(L)}g_{ij}(x,y)$ *being of constant signature over* $\widetilde{TM}$.

Definition 15.2.5. *A Finsler space is a pair* $F^n = [M, F(x,y)]$ *where* $F_{|x}(y)$ *defines a Minkowski space with metric fundamental function of type (15.3).*

The notion of Lagrange space was introduced by J. Kern [37] and elaborated in details by the R. Miron's school on Finsler and Lagrange geometry, see Refs. [56, 57], as a natural extension of Finsler geometry [27, 10, 64, 3, 52, 6, 4, 94] (see also Refs.[71, 73], on Lagrange–Finsler supergeometry, and Refs. [76, 77, 78], on some examples of noncommutative locally anisotropic gravity and string theory).

15.2.2 Nonlinear connection geometry

We consider two important examples when the nonlinear connection (in brief, N–connection) is naturally defined in Lagrange mechanics and in gravity theories with generic off–diagonal metrics and nonholonomic frames.

Geometrization of mechanics: some important results

The Lagrange mechanics was geometrized by using methods of Finsler geometry [56, 57] on tangent bundles enabled with a corresponding nonholonomic structure (non-integrable distribution) defining a canonical N–connection.[3] By straightforward calculations, one proved the results:

Result 15.2.1. *The Euler–Lagrange equations*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0 \tag{15.6}$$

where $y^i = \frac{dx^i}{d\tau}$ *for* $x^i(\tau)$ *depending on parameter* τ, *are equivalent to the "nonlinear" geodesic equations*

$$\frac{d^2x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^j}{d\tau}) = 0 \tag{15.7}$$

[3] We cite a recent review [41] on alternative approaches to geometric mechanics and geometry of classical fields related to investigation of the geometric properties of Euler–Lagrange equations for various type of nonholonomic, singular or higher order systems. In the approach developed by R. Miron's school [56, 57, 58], the nonlinear connection and fundamental geometric structures are derived in general form from the Lagrangian and/or Hamiltonian: the basic geometric constructions are not related to the particular properties of certain systems of partial differential equations, symmetries and constraints of mechanical and field models.

where

$$2G^i(x,y) = \frac{1}{2}\ {}^{(L)}g^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^k}y^k - \frac{\partial L}{\partial x^i}\right) \qquad (15.8)$$

with ${}^{(L)}g^{ij}$ *being inverse to (15.5).*

Result 15.2.2. *The coefficients* $G^i(x,y)$ *from (15.8) define the solutions of both type of equations (15.6) and (15.7) as paths of the canonical semispray*

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$$

and a canonical N–connection structure on $\widetilde{TM}$,

$${}^{(L)}N^i_j = \frac{\partial G^i(x,y)}{\partial y^i}, \qquad (15.9)$$

induced by the fundamental Lagrange function $L(x,y)$ *(see Section 15.2.3 on exact definitions and main properties).*

Result 15.2.3. *The coefficients* ${}^{(L)}N^i_j$ *defined by a Lagrange (Finsler) fundamental function induce a global splitting on* TTM, *a Whitney sum,*

$$TTM = hTM \oplus vTM$$

as a nonintegrable distribution (nonholonomic, or equivalently, anholonomic structure) into horizontal (h) and vertical (v) subspaces parametrized locally by frames (vielbeins) $\mathbf{e}_\nu = (e_i, e_a)$, *where*

$$e_i = \frac{\partial}{\partial x^i} - N^a_i(u)\frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \qquad (15.10)$$

and the dual frames (coframes) $\vartheta^\mu = (\vartheta^i, \vartheta^a)$, *where*

$$\vartheta^i = dx^i \text{ and } \vartheta^a = dy^a + N^a_i(u)dx^i. \qquad (15.11)$$

The vielbeins (15.10) and (15.11) are called N–adapted (co) frames. We omitted the label (L) and used vertical indices $a, b, c, ...$ for the N–connection coefficients in order to be able to use the formulas for arbitrary N–connections). We also note that we shall use 'boldfaced' symbols for the geometric objects and spaces adapted/ enabled to N–connection structure. For instance, we shall write in brief $\mathbf{e} = (e,\ ^\star e)$ and $\vartheta = (\vartheta,\ ^\star\vartheta)$, respectively, for

$$\mathbf{e}_\nu = (e_i,\ ^\star e_k) = (e_i,\ e_a) \text{ and } \vartheta^\mu = (\vartheta^i,\ ^\star\vartheta^k) = (\vartheta^i, \vartheta^a).$$

The vielbeins (15.10) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W^\gamma_{\alpha\beta} \mathbf{e}_\gamma \tag{15.12}$$

with (antisymmetric) nontrivial anholonomy coefficients $W^b_{ia} = \partial_a N^b_i$ and $W^a_{ji} = \Omega^a_{ij}$ where

$$\Omega^a_{ij} = \delta_{[j} N^a_{i]} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}. \tag{15.13}$$

In order to preserve a relation with our previous denotations [74, 69, 73], we note that $\mathbf{e}_\nu = (e_i, e_a)$ and $\vartheta^\mu = (\vartheta^i, \vartheta^a)$ are, respectively, the former $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$ and $\delta^\mu = \delta u^\mu = (dx^i, \delta y^a)$ which emphasize that the operators (15.10) and (15.11) define, correspondingly, certain 'N–elongated' partial derivatives and differentials which are more convenient for calculations on spaces provided with nonholonomic structure.

Result 15.2.4. *On $\widetilde{TM}$, there is a canonical metric structure ${}^{(L)}\mathbf{g} = [g, {}^\star g]$,*

$${}^{(L)}\mathbf{g} = {}^{(L)}g_{ij}(x,y)\ \vartheta^i \otimes \vartheta^j + {}^{(L)}g_{ij}(x,y)\ {}^\star\vartheta^i \otimes {}^\star\vartheta^j \tag{15.14}$$

constructed as a Sasaki type lift from M.[4]

We note that a complete geometrical model of Lagrange mechanics or a well defined Finsler geometry can be elaborated only by additional assumptions about a linear connection structure, which can be adapted, or not, to a defined N–connection (see Section 15.3.1).

Result 15.2.5. *The canonical N–connection (15.9) induces naturally an almost complex structure $\mathbf{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, where χ denotes the module of vector fields on $\widetilde{TM}$,*

$$\mathbf{F}(e_i) = {}^\star e_i \text{ and } \mathbf{F}({}^\star e_i) = -e_i,$$

when

$$\mathbf{F} = {}^\star e_i \otimes \vartheta^i - e_i \otimes {}^\star\vartheta^i \tag{15.15}$$

satisfies the condition $\mathbf{F} \rfloor \mathbf{F} = -\mathbf{I}$, i. e. $F^\alpha{}_\beta F^\beta{}_\gamma = -\delta^\alpha_\gamma$, where δ^α_γ is the Kronecker symbol and "$\rfloor$" denotes the interior product.

[4] In Refs. [94, 58], it was suggested to use lifts with h- and v–components of type ${}^{(L)}\mathbf{g} = (g_{ij}, g_{ij}a/\parallel y \parallel)$ where $a = const$ and $\parallel y \parallel = g_{ij}y^i y^j$ in order to elaborate more physical extensions of the general relativity to the tangent bundles of manifolds. In another turn, such modifications are not necessary if we model Lagrange–Finsler structures by exact solutions with generic off–diagonal metrics in Einstein and/or gravity [74, 87, 85, 81, 83, 82, 75]. For simplicity, in this work, we consider only lifts of metrics of type (15.14).

The last result is important for elaborating an approach to geometric quantization of mechanical systems modelled on nonholonomic manifolds [25] as well for definition of almost complex structures derived from the real N–connection geometry related to nonholonomic (anisotropic) Clifford structures and spinors in commutative [69, 72, 86, 88, 84] and noncommutative spaces [76, 77, 78].

N–connections in gravity theories

For nonholonomic geometric models of gravity and string theories, one does not consider the bundle $\widetilde{TM}$ but a general manifold $\mathbf{V},\ dim\mathbf{V}=n+m$, which is a (pseudo) Riemannian space or a certain generalization with possible torsion and nonmetricity fields. A metric structure is defined on $\mathbf{V}$, with the coefficients stated with respect to a local coordinate basis $du^{\alpha}=\left(dx^{i},dy^{a}\right)$, [5]

$$\mathbf{g}=\underline{g}_{\alpha\beta}(u)du^{\alpha}\otimes du^{\beta}$$

where

$$\underline{g}_{\alpha\beta}=\left[\begin{array}{cc} g_{ij}+N_{i}^{a}N_{j}^{b}h_{ab} & N_{j}^{e}h_{ae} \\ N_{i}^{e}h_{be} & h_{ab} \end{array}\right]. \tag{15.16}$$

A metric, for instance, parametrized in the form (15.16), is generic off–diagonal if it can not be diagonalized by any coordinate transforms. Performing a frame transform with the coefficients

$$\mathbf{e}_{\alpha}^{\ \underline{\alpha}}(u) = \left[\begin{array}{cc} e_{i}^{\ \underline{i}}(u) & N_{i}^{b}(u)e_{b}^{\ \underline{a}}(u) \\ 0 & e_{a}^{\ \underline{a}}(u) \end{array}\right], \tag{15.17}$$

$$\mathbf{e}^{\beta}_{\ \underline{\beta}}(u) = \left[\begin{array}{cc} e^{i}_{\ \underline{i}}(u) & -N_{k}^{b}(u)e^{k}_{\ \underline{i}}(u) \\ 0 & e^{a}_{\ \underline{a}}(u) \end{array}\right], \tag{15.18}$$

we write equivalently the metric in the form

$$\mathbf{g}=\mathbf{g}_{\alpha\beta}\left(u\right)\vartheta^{\alpha}\otimes\vartheta^{\beta}=g_{ij}\left(u\right)\vartheta^{i}\otimes\vartheta^{j}+h_{ab}\left(u\right)\ {}^{\star}\vartheta^{a}\otimes\ {}^{\star}\vartheta^{b}, \tag{15.19}$$

where $g_{ij}\doteqdot\mathbf{g}\left(e_{i},e_{j}\right)$ and $h_{ab}\doteqdot\mathbf{g}\left(e_{a},e_{b}\right)$ and

$$\mathbf{e}_{\alpha}=\mathbf{e}_{\alpha}^{\ \underline{\alpha}}\partial_{\underline{\alpha}}\text{ and }\vartheta^{\beta}=\mathbf{e}^{\beta}_{\ \underline{\beta}}du^{\underline{\beta}}.$$

are vielbeins of type (15.10) and (15.11) defined for arbitrary $N_{i}^{b}(u)$. We can consider a special class of manifolds provided with a global splitting into conventional "horizontal" and "vertical" subspaces (15.20) induced by the "off–diagonal" terms $N_{i}^{b}(u)$ and prescribed type of nonholonomic frame structure.

[5]the indices run correspondingly the values $i,j,k,...=1,2,...,n$ and $a,b,c,...=n+1,n+2,...,n+m.$

If the manifold $\mathbf{V}$ is (pseudo) Riemannian, there is a unique linear connection (the Levi–Civita connection) $\bigtriangledown$, which is metric, $\bigtriangledown\mathbf{g} = \mathbf{0}$, and torsionless, $^{\bigtriangledown}T = 0$. Nevertheless, the connection $\bigtriangledown$ is not adapted to the nonintegrable distribution induced by $N_i^b(u)$. In this case, for instance, in order to construct exact solutions parametrized by generic off–diagonal metrics, or for investigating nonholonomic frame structures in gravity models with nontrivial torsion, it is more convenient to work with more general classes of linear connections which are N–adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients $W_{\alpha\beta}^{\gamma}$ (15.12).

For a splitting of a (pseudo) Riemannian–Cartan space of dimension $(n+m)$ (under certain constraints, we can consider (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modelled by N–anholonomic structures as exact solutions of gravitational field equations [74, 87, 85, 81], see also Refs. [83, 82] for exact solutions with nonmetricity. One holds [80] the

Result 15.2.6. *The geometry of any Riemannian space of dimension $n+m$ where $n, m \geq 2$ (we can consider $n, m = 1$ as special degenerated cases), provided with off–diagonal metric structure of type (15.16) can be equivalently modelled, by vielbein transforms of type (15.17) and (15.18) as a geometry of nonholonomic manifold enabled with N–connection structure $N_i^b(u)$ and 'more diagonalized' metric (15.19).*

For particular cases, we present the

Remark 15.2.2. *For certain special conditions when $n = m$, $N_i^b = {}^{(L)}N_i^b$ (15.9) and the metric (15.19) is of type (15.14), a such Riemann space of even dimension is 'nonholonomically' equivalent to a Lagrange space (for the corresponding homogeneity conditions, see Definition 15.2.2, one obtains the equivalence to a Finsler space).*

Roughly speaking, by prescribing corresponding nonholonomic frame structures, we can model a Lagrange, or Finsler, geometry on a Riemannian manifold and, inversely, a Riemannian geometry is 'not only a Riemannian one' but also could be a generalized Finsler one. It is possible to define similar constructions for the (pseudo) Riemannian spaces. This is a quite surprising result if to compare it with the "superficial" interpretation of the Finsler geometry as a nonlinear extension, 'more sophisticate' on the tangent bundle, of the Riemannian geometry.

It is known the fact that the first example of Finsler geometry was considered in 1854 in the famous B. Riemann's hability thesis (see historical details and discussion in Refs. [94, 4, 57, 80]) who, for simplicity, restricted his considerations only to the curvatures defined by quadratic forms on hypersurfaces. Sure, for B. Riemann, it was unknown the fact that if we consider general (nonholonomic) frames with associated nonlinear connections (the E. Cartan's geometry, see Refs. in [10]) and off–diagonal metrics, the Finsler geometry may be derived naturally even from quadratic metric forms being adapted to the N–connection structure.

More rigorous geometric constructions involving the Cartan–Miron metric connections and, respectively, the Berwald and Chern–Rund nonmetric connections in Finsler geometry and generalizations, see more details in subsection 15.3.1, result in equivalence theorems to certain types of Riemann–Cartan nonholonomic manifolds (with nontirvial N–connection and torsion) and metric–affine nonholonomic manifolds (with additional nontrivial nonmetricity structures) [80].

This Result 15.2.6 give rise to an important:

Conclusion 15.2.1. *To study generalized Finsler spinor and noncommutative geometries is necessary even if we restrict our considerations only to (non) commutative Riemannian geometries.*

For simplicity, in this work we restrict our considerations only to certain Riemannian commutative and noncommutative geometries when the N–connection and torsion are defined by corresponding nonholonomic frames.

15.2.3 N–anholonomic manifolds

Now we shall demonstrate how general N–connection structures define a certain class of nonholonomic geometries. In this case, it is convenient to work on a general manifold $\mathbf{V}$, $\dim \mathbf{V} = n+m$, with global splitting, instead of the tangent bundle $\widetilde{TM}$. The constructions will contain those from geometric mechanics and gravity theories, as certain particular cases.

Let $\mathbf{V}$ be a $(n+m)$–dimensional manifold. It is supposed that in any point $u \in \mathbf{V}$ there is a local distribution (splitting) $\mathbf{V}_u = M_u \oplus V_u$, where M is a $n-$dimensional subspace and V is a m–dimensional subspace. The local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) may be written in the form $u = (x, y)$, or $u^{\alpha} = (x^i, y^a)$. We denote by $\pi^{\top} : T\mathbf{V} \rightarrow TM$ the differential of a map $\pi : V^{n+m} \rightarrow V^n$ defined by fiber preserving morphisms of the tangent bundles $T\mathbf{V}$ and TM. The kernel of $\pi^{\top}$ is just the vertical subspace $v\mathbf{V}$ with a related inclusion mapping $i : v\mathbf{V} \rightarrow T\mathbf{V}$.

Definition 15.2.6. *A nonlinear connection (N–connection)* $\mathbf{N}$ *on a manifold* $\mathbf{V}$ *is defined by the splitting on the left of an exact sequence*

$$0 \rightarrow v\mathbf{V} \stackrel{i}{\rightarrow} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0,$$

i. e. by a morphism of submanifolds $\mathbf{N} : \ T\mathbf{V} \rightarrow v\mathbf{V}$ *such that* $\mathbf{N} \circ \mathbf{i}$ *is the unity in* $v\mathbf{V}$.

In an equivalent form, we can say that a N–connection is defined by a splitting to subspaces with a Whitney sum of conventional h–subspace, $(h\mathbf{V})$, and v–subspace, $(v\mathbf{V})$,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \tag{15.20}$$

where $h\mathbf{V}$ is isomorphic to M. This generalizes the splitting considered in Result 15.2.3.

Locally, a N–connection is defined by its coefficients $N_i^a(u)$,

$$\mathbf{N} = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a}. \tag{15.21}$$

The well known class of linear connections consists a particular subclass with the coefficients being linear on y^a, i. e. $N_i^a(u) = \Gamma^a_{bj}(x)y^b$.

Any N–connection also defines a N–connection curvature

$$\mathbf{\Omega} = \frac{1}{2}\Omega^a_{ij} d^i \wedge d^j \otimes \partial_a,$$

with N–connection curvature coefficients given by formula (15.12).

Definition 15.2.7. *A manifold* $\mathbf{V}$ *is called N–anholonomic if on the tangent space* $T\mathbf{V}$ *it is defined a local (nonintegrable) distribution (15.20), i. e.* $T\mathbf{V}$ *is enabled with a N–connection (15.21) inducing a vielbein structure (15.10) satisfying the nonholonomy relations (15.12) (such N–connections and associated vielbeins may be general ones or any derived from a Lagrange/ Finsler fundamental function).*

We note that the boldfaced symbols are used for the spaces and geometric objects provided/adapted to a N–connection structure. For instance, a vector field $\mathbf{X} \in T\mathbf{V}$ is expressed $\mathbf{X} = (X \equiv {}^{-}X, {}^{\star}X)$, or $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i e_i + X^a e_a$, where $X = {}^{-}X = X^i e_i$ and ${}^{\star}X = X^a e_a$ state, respectively, the irreducible (adapted to the N–connection structure) h– and v–components of the vector (which following Refs. [56, 57] is called a distinguished vectors, in brief, d–vector). In a similar fashion, the geometric objects on $\mathbf{V}$ like tensors, spinors, connections, ... are respectively defined and called d–tensors, d–spinors, d–connections if they are adapted to the N–connection splitting.[6]

Definition 15.2.8. *A d–metric structure on N–anholonomic manifold* $\mathbf{V}$ *is defined by a symmetric d–tensor field of type* $\mathbf{g} = [g, {}^{\star}h]$ *(15.19).*

[6]In order to emphasize h– and v–splitting of any d–objects $\mathbf{Y}, \mathbf{g}$, ... we shall write the irreducible components as $\mathbf{Y} = ({}^{-}Y, {}^{\star}Y)$, $\mathbf{g} = ({}^{-}g, {}^{\star}g)$ but we shall omit "$-$" or "$\star$" if the simplified denotations will not result in ambiguities.

For any fixed values of coordinates $u = (x, y) \in \mathbf{V}$ a d–metric it defines a symmetric quadratic d–metric form,

$$\mathbf{q}(\mathbf{x}, \mathbf{y}) \doteq g_{ij} x^i x^j + h_{ab} y^a y^b, \qquad (15.22)$$

where the $n + m$–splitting is defined by the N–connection structure and $\mathbf{x} = x^i e_i + x^a e_a,\ \mathbf{y} = y^i e_i + y^a e_a \in V^{n+m}$.

Any d–metric is parametrized by a generic off–diagonal matrix (15.16) if the coefficients are redefined with respect to a local coordinate basis (for corresponding parametrizations of the the data $[g, h, N]$ such ansatz model a geometry of mechanics, or a Finsler like structure, in a Riemann–Cartan–Weyl space provided with N–connection structure [80, 83]; for certain constraints, there are possible models derived as exact solutions in Einsten gravity and noncommutative generalizations [74, 81, 82]).

Remark 15.2.3. *There is a special case when* $\dim \mathbf{V} = n + n,\ h_{ab} \to g_{ij}$ *and* $N_i^a \to N^j{}_i$ *in (15.19), which models locally, on* $\mathbf{V}$*, a tangent bundle structure. We denote a such space by* $\widetilde{\mathbf{V}}_{(n,n)}$*. If the N–connection and d–metric coefficients are just the canonical ones for the Lagrange (Finsler) geometry (see, respectively, formulas (15.9) and (15.14)), we model such locally anisotropic structures not on a tangent bundle* TM *but on a N–anholonomic manifold of dimension* $2n$*.*

We present some historical remarks on N–connections and related subjects: The geometrical aspects of the N–connection formalism has been studied since the first papers of E. Cartan [10] and A. Kawaguchi [35, 36] (who used it in component form for Finsler geometry). Then one should be mentioned the so called Ehressman connection [23]) and the work of W. Barthel [5] where the global definition of N–connection was given. In monographs [56, 57, 58], the N–connection formalism was elaborated in details and applied to the geometry of generalized Finsler–Lagrange and Cartan–Hamilton spaces, see also the approaches [42, 40, 24].It should be noted that the works related to nonholonomic geometry and N–connections have appeared many times in a rather dispersive way when different schools of authors from geometry, mechanics and physics have worked many times not having relation with another. We cite only some our recent results with explicit applications in modern mathematical physics and particle and string theories: N–connection structures were modelled on Clifford and spinor bundles [69, 72, 88, 86], on superbundles and in some directions of (super) string theory [71, 73], as well in noncommutative geometry and gravity [76, 77, 78]. The idea to apply the N–connections formalism as a new geometric method of constructing exact solutions in gravity theories was suggested in Refs. [74, 75] and developed in a number of works, see for instance, Ref. [87, 85, 81]).

15.3 Curvature of N–anholonomic Manifolds

The geometry of nonholonomic manifolds has a long time history of yet unfinished elaboration: For instance, in the review [93] it is stated that it is probably impossible to construct an analog of the Riemannian tensor for the general nonholonomic manifold. In a more recent review [55], it is emphasized that in the past there were proposed well defined Riemannian tensors for a number of spaces provided with nonholonomic distributions, like Finsler and Lagrange spaces and various type of theirs higher order generalizations, i. e. for nonholonomic manifolds possessing corresponding N–connection structures. As some examples of first such investigations, we cite the works [54, 53, 19]. In this section we shall construct in explicit form the curvature tensor for the N–anholonomic manifolds.

15.3.1 Distinguished connections

On N–anholonomic manifolds, the geometric constructions can be adapted to the N–connection structure:

Definition 15.3.9. *A distinguished connection (d–connection)* $\mathbf{D}$ *on a manifold* $\mathbf{V}$ *is a linear connection conserving under parallelism the Whitney sum (15.20) defining a general N–connection.*

The N–adapted components $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ of a d-connection $\mathbf{D}_{\alpha} = (\delta_{\alpha}\rfloor\mathbf{D})$ are defined by the equations $\mathbf{D}_{\alpha}\delta_{\beta} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\delta_{\gamma}$, or

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\,(u) = (\mathbf{D}_{\alpha}\delta_{\beta})\rfloor\delta^{\gamma}. \qquad (15.23)$$

In its turn, this defines a N–adapted splitting into h– and v–covariant derivatives, $\mathbf{D} = D + {}^{\star}D$, where $D_k = \left(L^i_{jk}, L^a_{bk}\right)$ and ${}^{\star}D_c = \left(C^i_{jk}, C^a_{bc}\right)$ are introduced as corresponding h- and v–parametrizations of (15.23),

$$L^i_{jk} = (\mathbf{D}_k e_j)\rfloor\vartheta^i, \quad L^a_{bk} = (\mathbf{D}_k e_b)\rfloor\vartheta^a, \; C^i_{jc} = (\mathbf{D}_c e_j)\rfloor\vartheta^i, \quad C^a_{bc} = (\mathbf{D}_c e_b)\rfloor\vartheta^a.$$

The components $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ completely define a d–connection $\mathbf{D}$ on a N–anholonomic manifold $\mathbf{V}$.

The simplest way to perform computations with d–connections is to use N–adapted differential forms like $\mathbf{\Gamma}^{\alpha}_{\beta} = \mathbf{\Gamma}^{\alpha}_{\beta\gamma}\vartheta^{\gamma}$ with the coefficients defined with respect to N–elongate bases (15.11) and (15.10).

The torsion of d–connection $\mathbf{D}$ is defined by the usual formula

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteqdot \mathbf{D}_X\mathbf{D}_Y - \mathbf{D}_Y\mathbf{D}_X - [\mathbf{X}, \mathbf{Y}].$$

Theorem 15.3.1. *The torsion* $\mathbf{T}^{\alpha} \doteqdot \mathbf{D}\vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma^{\alpha}_{\beta} \wedge \vartheta^{\beta}$ *of a d–connection has the irreducible h- v– components (d–torsions)with N–adapted coefficients*

$$\begin{aligned} T^{i}_{\ jk} &= L^{i}_{\ [jk]},\ T^{i}_{\ ja} = -T^{i}_{\ aj} = C^{i}_{\ ja},\ T^{a}_{\ ji} = \Omega^{a}_{\ ji}, \\ T^{a}_{\ bi} &= T^{a}_{\ ib} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - L^{a}_{\ bi},\ T^{a}_{\ bc} = C^{a}_{\ [bc]}. \end{aligned} \qquad (15.24)$$

Proof. By a straightforward calculation we can verify the formulas. □

The Levi–Civita linear connection $\bigtriangledown = \{{}^{\bigtriangledown}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}\}$, with vanishing both torsion and nonmetricity, is not adapted to the global splitting (15.20). One holds:

Proposition 15.3.1. *There is a preferred, canonical d–connection structure,* $\widehat{\mathbf{D}}$, *on N–anholonomic manifold* $\mathbf{V}$ *constructed only from the metric and N–connection coefficients* $[g_{ij}, h_{ab}, N^{a}_{i}]$ *and satisfying the metricity conditions* $\widehat{\mathbf{D}}\mathbf{g} = 0$ *and* $\widehat{T}^{i}_{\ jk} = 0$ *and* $\widehat{T}^{a}_{\ bc} = 0$.

Proof. By straightforward calculations with respect to the N–adapted bases (15.11) and (15.10), we can verify that the connection

$$\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = {}^{\bigtriangledown}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} + \widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} \qquad (15.25)$$

with the deformation d–tensor

$$\widehat{\mathbf{P}}^{\alpha}_{\beta\gamma} = (P^{i}_{jk} = 0, P^{a}_{bk} = \frac{\partial N^{a}_{k}}{\partial y^{b}}, P^{i}_{jc} = -\frac{1}{2} g^{ik} \Omega^{a}_{\ kj} h_{ca}, P^{a}_{bc} = 0)$$

satisfies the conditions of this Proposition. It should be noted that, in general, the components $\widehat{T}^{i}_{\ ja}$, $\widehat{T}^{a}_{\ ji}$ and $\widehat{T}^{a}_{\ bi}$ are not zero. This is an anholonomic frame (or, equivalently, off–diagonal metric) effect. □

With respect to the N–adapted frames, the coefficients $\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$ are computed:

$$\begin{aligned} \widehat{L}^{i}_{jk} &= \frac{1}{2} g^{ir} \left(\frac{\delta g_{jr}}{\partial x^{k}} + \frac{\delta g_{kr}}{\partial x^{j}} - \frac{\delta g_{jk}}{\partial x^{r}} \right), \\ \widehat{L}^{a}_{bk} &= \frac{\partial N^{a}_{k}}{\partial y^{b}} + \frac{1}{2} h^{ac} \left(\frac{\delta h_{bc}}{\partial x^{k}} - \frac{\partial N^{d}_{k}}{\partial y^{b}} h_{dc} - \frac{\partial N^{d}_{k}}{\partial y^{c}} h_{db} \right), \\ \widehat{C}^{i}_{jc} &= \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^{c}}, \\ \widehat{C}^{a}_{bc} &= \frac{1}{2} h^{ad} \left(\frac{\partial h_{bd}}{\partial y^{c}} + \frac{\partial h_{cd}}{\partial y^{b}} - \frac{\partial h_{bc}}{\partial y^{d}} \right). \end{aligned} \qquad (15.26)$$

The d–connection (15.26) defines the 'most minimal' N–adapted extension of the Levi–Civita connection in order to preserve the metricity condition and to have zero torsions on the h– and v–subspaces (the rest of nonzero torsion coefficients are defined by the condition of compatibility with the N–connection splitting).

Remark 15.3.4. *The canonical d–connection* $\widehat{\mathbf{D}}$ *(15.26) for a local modelling of a* $\widetilde{TM}$ *space on* $\widetilde{\mathbf{V}}_{(n,n)}$ *is defined by the coefficients* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^{i}_{jk}, \widehat{C}^{i}_{jk})$ *with*

$$\widehat{L}^{i}_{jk} = \frac{1}{2} g^{ir} \left(\frac{\delta g_{jr}}{\partial x^{k}} + \frac{\delta g_{kr}}{\partial x^{j}} - \frac{\delta g_{jk}}{\partial x^{r}} \right), \widehat{C}^{i}_{jk} = \frac{1}{2} g^{ir} \left(\frac{\partial g_{jr}}{\partial y^{k}} + \frac{\partial g_{kr}}{\partial y^{j}} - \frac{\partial g_{jk}}{\partial y^{r}} \right) \qquad (15.27)$$

computed with respect to N–adapted bases (15.10) and (15.11) when $\widehat{L}^{i}_{jk}$ *and* $\widehat{C}^{i}_{jk}$ *define respectively the canonical h– and v–covariant derivations.*

Various models of Finsler geometry and generalizations were elaborated by using different types of d–connections which satisfy, or not, the compatibility conditions with a fixed d–metric structure (for instance, with a Sasaki type one). Let us consider the main examples:

Example 15.3.1. *The Cartan's d–connection [10] with the coefficients (15.27) was defined by some generalized Christoffel symbols with the aim to have a 'minimal' torsion and to preserve the metricity condition. This approach was developed for generalized Lagrange spaces and on vector bundles provided with N–connection structure [56, 57] by introducing the canonical d–connection (15.26). The direction emphasized metric compatible and N–adapted geometric constructions.*

An alternative class of Finsler geometries is concluded in monographs [4, 94]:

Example 15.3.2. *It was the idea of C. C. Chern [16] (latter also proposed by H. Rund [64]) to consider a d–connection* ${}^{[Chern]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^{i}_{jk}, C^{i}_{jk} = 0)$ *and to work not on a tangent bundle* TM *but to try to 'keep maximally' the constructions on the base manifold* M*. The Chern d–connection, as well the Berwald d–connection* ${}^{[Berwald]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (L^{i}_{jk} = \frac{\partial N^{i}_{k}}{\partial y^{j}}, C^{i}_{jk} = 0)$ *[7], are not subjected to the metricity conditions.*

We note that the constructions mentioned in the last example define certain 'non-metric geometries' (a Finsler modification of the Riemann–Cartan–Weyl spaces). For the Chern's connection, the torsion vanishes but there is a nontrivial nonmetricity. A detailed study and classification of Finsler–affine spaces with general nontrivial N–connection, torsion and nonmetricity was recently performed in Refs. [80, 83, 82]. Here we also note that we may consider any linear connection can be generated by deformations of type

$$\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} + \mathbf{P}^{\alpha}_{\beta\gamma}. \tag{15.28}$$

This splits all geometric objects into canonical and post-canonical pieces which results in N–adapted geometric constructions.

In order to define spinors on generalized Lagrange and Finsler spaces [69, 72, 86, 88] the canonical d–connection and Cartan's d–connection were used because the metric compatibility allows the simplest definition of Clifford structures locally adapted to the N–connection. This is also the simplest way to define the Dirac operator for generalized Finsler spaces and to extend the constructions to noncommutative Finsler geometry [76, 77, 78]. The geometric constructions with general metric compatible affine connection (with torsion) are preferred in modern gravity and string theories. Nevertheless, the geometrical and physical models with generic nonmetricity also present certain interest [28, 80, 83, 82] (see also [46] where nonmetricity is considered to be important in quantum group co gravity). In such cases, we can use deformations of connection (15.28) in order to 'deform', for instance, the spinorial geometric constructions defined by the canonical d–connection and to transform them into certain 'nonmetric' configurations.

15.3.2 Curvature of d–connections

The curvature of a d–connection $\mathbf{D}$ on an N–anholonomic manifold is defined by the usual formula

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} \doteqdot \mathbf{D}_X\mathbf{D}_Y\mathbf{Z} - \mathbf{D}_Y\mathbf{D}_X\mathbf{Z} - \mathbf{D}_{[X,X]}\mathbf{Z}.$$

By straightforward calculations we prove:

Theorem 15.3.2. *The curvature $\mathcal{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}_{\beta} = d\mathbf{\Gamma}^{\alpha}_{\beta} - \mathbf{\Gamma}^{\gamma}_{\beta} \wedge \mathbf{\Gamma}^{\alpha}_{\gamma}$ of a d–connection $\mathcal{D} \doteq \mathbf{\Gamma}^{\alpha}_{\gamma}$ has the irreducible h- v– components (d–curvatures) of $\mathbf{R}^{\alpha}{}_{\beta\gamma\delta}$,*

$$\begin{aligned}
R^{i}{}_{hjk} &= e_k L^{i}{}_{hj} - e_j L^{i}{}_{hk} + L^{m}{}_{hj} L^{i}{}_{mk} - L^{m}{}_{hk} L^{i}{}_{mj} - C^{i}{}_{ha}\Omega^{a}{}_{kj}, \\
R^{a}{}_{bjk} &= e_k L^{a}{}_{bj} - e_j L^{a}{}_{bk} + L^{c}{}_{bj} L^{a}{}_{ck} - L^{c}{}_{bk} L^{a}{}_{cj} - C^{a}{}_{bc}\Omega^{c}{}_{kj}, \\
R^{i}{}_{jka} &= e_a L^{i}{}_{jk} - D_k C^{i}{}_{ja} + C^{i}{}_{jb} T^{b}{}_{ka}, \\
R^{c}{}_{bka} &= e_a L^{c}{}_{bk} - D_k C^{c}{}_{ba} + C^{c}{}_{bd} T^{c}{}_{ka}, \\
R^{i}{}_{jbc} &= e_c C^{i}{}_{jb} - e_b C^{i}{}_{jc} + C^{h}{}_{jb} C^{i}{}_{hc} - C^{h}{}_{jc} C^{i}{}_{hb}, \\
R^{a}{}_{bcd} &= e_d C^{a}{}_{bc} - e_c C^{a}{}_{bd} + C^{e}{}_{bc} C^{a}{}_{ed} - C^{e}{}_{bd} C^{a}{}_{ec}.
\end{aligned} \tag{15.29}$$

Remark 15.3.5. *For an N–anholonomic manifold $\widetilde{\mathbf{V}}_{(n,n)}$ provided with N–sympletic canonical d–connection $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\alpha\beta}$, see (15.27), the d–curvatures (15.29) reduces to three irreducible components*

$$\begin{aligned} R^{i}{}_{hjk} &= e_k L^{i}{}_{hj} - e_j L^{i}{}_{hk} + L^{m}{}_{hj} L^{i}{}_{mk} - L^{m}{}_{hk} L^{i}{}_{mj} - C^{i}{}_{ha} \Omega^{a}{}_{kj}, \\ R^{i}{}_{jka} &= e_a L^{i}{}_{jk} - D_k C^{i}{}_{ja} + C^{i}{}_{jb} T^{b}{}_{ka}, \\ R^{a}{}_{bcd} &= e_d C^{a}{}_{bc} - e_c C^{a}{}_{bd} + C^{e}{}_{bc} C^{a}{}_{ed} - C^{e}{}_{bd} C^{a}{}_{ec} \end{aligned} \tag{15.30}$$

where all indices $i, j, k...$ and $a, b, ..$ run the same values but label the components with respect to different h– or v–frames.

Contracting respectively the components of (15.29) and (15.30) we prove:

Corollary 15.3.1. *The Ricci d–tensor $\mathbf{R}_{\alpha\beta} \doteqdot \mathbf{R}^{\tau}{}_{\alpha\beta\tau}$ has the irreducible h- v–components*

$$R_{ij} \doteqdot R^{k}{}_{ijk}, \quad R_{ia} \doteqdot -R^{k}{}_{ika}, \ R_{ai} \doteqdot R^{b}{}_{aib}, \ R_{ab} \doteqdot R^{c}{}_{abc}, \tag{15.31}$$

for a general N–holonomic manifold $\mathbf{V}$, and

$$R_{ij} \doteqdot R^{k}{}_{ijk}, \quad R_{ia} \doteqdot -R^{k}{}_{ika}, \quad R_{ab} \doteqdot R^{c}{}_{abc}, \tag{15.32}$$

for an N–anholonomic manifold $\widetilde{\mathbf{V}}_{(n,n)}$.

Corollary 15.3.2. *The scalar curvature of a d–connection is*

$$\overleftarrow{\mathbf{R}} \doteqdot \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \tag{15.33}$$

defined by the "pure" h– and v–components of (15.32).

Corollary 15.3.3. *The Einstein d–tensor is computed $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}$.*

For physical applications, the Riemann, Ricci and Einstein d–tensors can be computed for the canonical d–connection. We can redefine the constructions for arbitrary d–connections by using the corresponding deformation tensors like in (15.28), for instance,

$$\mathcal{R}^{\alpha}{}_{\beta} = \widehat{\mathcal{R}}^{\alpha}{}_{\beta} + \mathbf{D}\mathcal{P}^{\alpha}{}_{\beta} + \mathcal{P}^{\alpha}{}_{\gamma} \wedge \mathcal{P}^{\gamma}{}_{\beta} \tag{15.34}$$

for $\mathcal{P}^{\alpha}_{\beta} = \mathbf{P}^{\alpha}_{\beta\gamma}\vartheta^{\gamma}$. A set of examples of such deformations are analyzed in Refs. [80, 83, 82].

15.4 Noncommutative N–Anholonomic Spaces

In this section, we outline how the analogs of basic objects in commutative geometry of N–anholonomic manifolds, such as vector/tangent bundles, N– and d–connections can be defined in noncommutative geometry [77, 78]. We note that the A. Connes' functional analytic approach [17] to the noncommutative topology and geometry is based on the theory of noncommutative C^*–algebras. Any commutative C^*–algebra can be realized as the C^*–algebra of complex valued functions over locally compact Hausdorff space. A noncommutative C^*–algebra can be thought of as the algebra of continuous functions on some 'noncommutative space' (see main definitions and results in Refs. [17, 29, 39, 44]).

The starting idea of noncommutative geometry is to derive the geometric properties of "commutative" spaces from their algebras of functions characterized by involutive algebras of operators by dropping the condition of commutativity (see the Gelfand and Naimark theorem [31]). A space topology is defined by the algebra of commutative continuous functions, but the geometric constructions request a differentiable structure. Usually, one considers a differentiable and compact manifold M, $dimN = n$ (there is an open problem how to include in noncommutative geometry spaces with indefinite metric signature like pseudo–Euclidean and pseudo–Riemannian ones). In order to construct models of commutative and noncommutative differential geometries it is more or less obvious that the class of algebras of smooth functions, $\mathcal{C} \doteq C^{\infty}(M)$ is more appropriate. If M is a smooth manifold, it is possible to reconstruct this manifold with its smooth structure and the attached objects (differential forms, etc...) by starting from $\mathcal{C}$ considered as an abstract (commutative) unity $*$–algebra with involution. As a set M can be identified with the set of characters of $\mathcal{C}$, but its differential structure is connected with the abundance of derivations of $\mathcal{C}$ which identify with the smooth vector fields on M. There are two standard constructions: 1) when the vector fields are considered to be the derivations of $\mathcal{C}$ (into itself) or 2) one considers a generalization of the calculus of differential forms which is the Kahler differential calculus (see, details in Lectures [21]). The noncommutative versions of differential geometry may be elaborated if the algebra of smooth complex functions on a smooth manifold is replaced by a noncommutative associative unity complex $*$–algebra $\mathcal{A}$.

The geometry of commutative gauge and gravity theories is derived from the notions of connections (linear and nonlinear ones), metrics and frames of references on manifolds and vector bundle spaces. The possibility of extending such theories to some noncommutative models is based on the Serre–Swan theorem [68] stating that there is a complete equivalence between the category of (smooth) vector bundles over a smooth compact space (with bundle maps) and the category of porjective modules of finite type over commutative algebras and module morphisms. So, the space $\Gamma(E)$ of smooth sections of a vector bundle E over a compact space is a projective module of finite type over the algebra $C(M)$ of smooth functions over M and any finite projective $C(M)$–module can be realized as the module of sections of some vector bundle over M. This construction may be extended if a noncommutative algebra $\mathcal{A}$ is taken as the starting ingredient: the noncommutative analogue of vector bundles are projective modules of finite type over $\mathcal{A}$. This way one developed a theory of linear connections which culminates in the definition of Yang–Mills type actions or, by some much more general settings, one reproduced Lagrangians for the Standard model with its Higgs sector or different type of gravity and Kaluza–Klein models (see, for instance, Refs [17, 44]).

15.4.1 Modules as bundles

A vector space $\mathcal{E}$ over the complex number field $\mathbb{C}$ can be defined also as a right module of an algebra $\mathcal{A}$ over $\mathbb{C}$ which carries a right representation of $\mathcal{A}$, when for every map of elements $\mathcal{E} \times \mathcal{A} \ni (\eta, a) \rightarrow \eta a \in \mathcal{E}$ one hold the properties

$$\lambda(ab) = (\lambda a)b,\ \lambda(a+b) = \lambda a + \lambda b,\ (\lambda + \mu)a = \lambda a + \mu a$$

for every $\lambda, \mu \in \mathcal{E}$ and $a, b \in \mathcal{A}$.

Having two $\mathcal{A}$–modules $\mathcal{E}$ and $\mathcal{F}$, a morphism of $\mathcal{E}$ into $\mathcal{F}$ is any linear map $\rho : \mathcal{E} \rightarrow \mathcal{F}$ which is also $\mathcal{A}$–linear, i. e. $\rho(\eta a) = \rho(\eta) a$ for every $\eta \in \mathcal{E}$ and $a \in \mathcal{A}$.

We can define in a similar (dual) manner the left modules and theirs morphisms which are distinct from the right ones for noncommutative algebras $\mathcal{A}$. A bimodule over an algebra $\mathcal{A}$ is a vector space $\mathcal{E}$ which carries both a left and right module structures. The bimodule structure is important for modelling of real geometries starting from complex structures. We may define the opposite algebra $\mathcal{A}^o$ with elements a^o being in bijective correspondence with the elements $a \in \mathcal{A}$ while the multiplication is given by $a^o b^o = (ba)^o$ .A right (respectively, left) $\mathcal{A}$–module $\mathcal{E}$ is connected to a left (respectively right) $\mathcal{A}^o$–module via relations $a^o \eta = \eta a^o$ (respectively, $a\eta = \eta a$). One introduces the enveloping algebra $\mathcal{A}^\varepsilon = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^o$; any $\mathcal{A}$–bimodule $\mathcal{E}$ can be regarded as a right [left] $\mathcal{A}^\varepsilon$–module by setting $\eta\,(a \otimes b^o) = b\eta a \quad [(a \otimes b^o)\,\eta = a\eta b]$.

For a (for instance, right) module $\mathcal{E}$, we may introduce a family of elements $(e_t)_{t\in T}$ parametrized by any (finite or infinite) directed set T for which any element $\eta \in \mathcal{E}$ is expressed as a combination (in general, in more than one manner) $\eta = \sum_{t\in T} e_t a_t$ with $a_t \in \mathcal{A}$ and only a finite number of non vanishing terms in the sum. A family $(e_t)_{t\in T}$ is free if it consists from linearly independent elements and defines a basis if any element $\eta \in \mathcal{E}$ can be written as a unique combination (sum). One says a module to be free if it admits a basis. The module $\mathcal{E}$ is said to be of finite type if it is finitely generated, i. e. it admits a generating family of finite cardinality.

Let us consider the module $\mathcal{A}^K \doteqdot \mathbb{C}^K \otimes_{\mathbb{C}} \mathcal{A}$. The elements of this module can be thought as K–dimensional vectors with entries in $\mathcal{A}$ and written uniquely as a linear combination $\eta = \sum_{t=1}^{K} e_t a_t$ were the basis e_t identified with the canonical basis of $\mathbb{C}^K$. This is a free and finite type module. In general, we can have bases of different cardinality. However, if a module $\mathcal{E}$ is of finite type there is always an integer K and a module surjection $\rho : \mathcal{A}^K \to \mathcal{E}$ with a base being a image of a free basis, $\epsilon_j = \rho(e_j); j = 1, 2, ..., K$.

We say that a right $\mathcal{A}$–module $\mathcal{E}$ is projective if for every surjective module morphism $\rho : \mathcal{M} \to \mathcal{N}$ splits, i. e. there exists a module morphism $s : \mathcal{E} \to \mathcal{M}$ such that $\rho \circ s = id_{\mathcal{E}}$. There are different definitions of porjective modules (see Ref. [39] on properties of such modules). Here we note the property that if a $\mathcal{A}$–module $\mathcal{E}$ is projective, there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ (being a priory projective) such that $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$.

For the right $\mathcal{A}$–module $\mathcal{E}$ being projective and of finite type with surjection $\rho : \mathcal{A}^K \to \mathcal{E}$ and following the projective property we can find a lift $\widetilde{\lambda} : \mathcal{E} \to \mathcal{A}^K$ such that $\rho \circ \widetilde{\lambda} = id_{\mathcal{E}}$. There is a proof of the property that the module $\mathcal{E}$ is projective of finite type over $\mathcal{A}$ if and only if there exists an idempotent $p \in End_{\mathcal{A}} \mathcal{A}^K = M_K(\mathcal{A})$, $p^2 = p$, the $M_K(\mathcal{A})$ denoting the algebra of $K \times K$ matrices with entry in $\mathcal{A}$, such that $\mathcal{E} = p\mathcal{A}^K$. We may associate the elements of $\mathcal{E}$ to K–dimensional column vectors whose elements are in $\mathcal{A}$, the collection of which are invariant under the map p, $\mathcal{E} = \{\xi = (\xi_1, ..., \xi_K); \xi_j \in \mathcal{A},\ p\xi = \xi\}$. For simplicity, we shall use the term finite projective to mean projective of finite type.

15.4.2 Nonlinear connections in projective modules

The nonlinear connection (N–connection) for noncommutative spaces can be defined similarly to commutative spaces by considering instead of usual vector bundles theirs noncommutative analogs defined as finite projective modules over noncommutative algebras [77]. The explicit constructions depend on the type of differential calculus we use for definition of tangent structures and theirs maps. In this subsection, we shall consider such projective modules provided with N–connection which define noncommutative analogous both of vector bundles and of N–anholonomic manifolds (see Definition 15.2.7).

In general, one can be defined several differential calculi over a given algebra $\mathcal{A}$ (for a more detailed discussion within the context of noncommutative geometry, see Refs. [17, 44]). For simplicity, in this work we consider that a differential calculus on $\mathcal{A}$ is fixed, which means that we choose a (graded) algebra $\Omega^*(\mathcal{A}) = \cup_p \Omega^p(\mathcal{A})$ giving a differential structure to $\mathcal{A}$. The elements of $\Omega^p(\mathcal{A})$ are called p–forms. There is a linear map d which takes p–forms into $(p+1)$–forms and which satisfies a graded Leibniz rule as well the condition $d^2 = 0$. By definition $\Omega^0(\mathcal{A}) = \mathcal{A}$.

The differential df of a real or complex variable on a N–anholonomic manifold $\mathbf{V}$

$$\begin{aligned} df &= \delta_i f \ dx^i + \partial_a f \ \delta y^a, \\ \delta_i f &= \partial_i f - N_i^a \partial_a f \ , \ \delta y^a = dy^a + N_i^a dx^i, \end{aligned}$$

where the N–elongated derivatives and differentials are defined respectively by formulas (15.10) and (15.11), in the noncommutative case is replaced by a distinguished commutator (d–commutator)

$$\overline{d}f = [F, f] = \left[F^{[h]}, f\right] + \left[F^{[v]}, f\right]$$

where the operator $F^{[h]}$ $(F^{[v]})$ acts on the horizontal (vertical) projective submodule and this operator is defined by a fixed differential calculus $\Omega^*(\mathcal{A}^{[h]})$ $(\Omega^*(\mathcal{A}^{[v]}))$ on the so–called horizontal (vertical) $\mathcal{A}^{[h]}$ $(\mathcal{A}^{[v]})$ algebras. We conclude that in order to elaborated noncommutative versions of N–anholonomic manifolds we need couples of 'horizontal' and 'vertical' operators which reflects the nonholonomic splitting given by the N–connection structure.

Let us consider instead of a N–anholonomic manifold $\mathbf{V}$ an $\mathcal{A}$–module $\mathcal{E}$ being projective and of finite type. For a fixed differential calculus on $\mathcal{E}$ we define the tangent structures $T\mathcal{E}$.

Definition 15.4.10. *A nonlinear connection (N–connection)* $\mathbf{N}$ *on an* $\mathcal{A}$*–module* $\mathcal{E}$ *is defined by the splitting on the left of an exact sequence of finite projective* $\mathcal{A}$*–moduli*

$$0 \rightarrow v\mathcal{E} \xrightarrow{i} T\mathcal{E} \rightarrow T\mathcal{E}/v\mathcal{E} \rightarrow 0,$$

i. e. by a morphism of submanifolds $\mathbf{N} : T\mathcal{E} \rightarrow v\mathcal{E}$ *such that* $\mathbf{N} \circ \mathbf{i}$ *is the unity in* $v\mathcal{E}$*.*

In an equivalent form, we can say that a N–connection is defined by a splitting to projective submodules with a Whitney sum of conventional h–submodule, $(h\mathcal{E})$, and v–submodule, $(v\mathcal{E})$,

$$T\mathcal{E} = h\mathcal{E} \oplus v\mathcal{E}. \tag{15.35}$$

We note that locally $h\mathcal{E}$ is isomorphic to TM where M is a differential compact manifold of dimension n.

The Definition 15.4.10 reconsiders for noncommutative spaces the Definition 15.2.6. In result, we may generalize the concept of 'commutative' N–anholonomic space:

Definition 15.4.11. *A N–anholonomic noncommutative space* $\mathcal{E}_N$ *is an* $\mathcal{A}$*–module* $\mathcal{E}$ *possessing a tangent structure* $T\mathcal{E}$ *defined by a Whitney sum of projective submodules (15.35).*

Such geometric constructions depend on the type of fixed differential calculus, i. e. on the procedure how the tangent spaces are defined.

Remark 15.4.6. *Locally always N–connections exist, but it is not obvious if they could be glued together. In the classical case of vector bundles over paracompact manifolds this is possible [56]. If there is an appropriate partition of unity, a similar result can be proved for finite projective modules. For certain applications, it is more convenient to use the Dirac operator already defined on N–anholonomic manifolds, see Section 15.6.*

In order to understand how the N–connection structure may be taken into account on noncommutative spaces but distinguished from the class of linear gauge fields, we analyze an example:

15.4.3 Commutative and noncommutative gauge d–fields

Let us consider a N–anholonomic manifold $\mathbf{V}$ and a vector bundle $\beta = (B, \pi, \mathbf{V})$ with $\pi : B \to \mathbf{V}$ with a typical k-dimensional vector fiber. In local coordinates a linear connection (a gauge field) in β is given by a collection of differential operators

$$\bigtriangledown_\alpha = D_\alpha + B_\alpha(u),$$

acting on $T\xi_N$ where

$$D_\alpha = \delta_\alpha \pm \Gamma^{\cdot}_{\cdot\alpha} \text{ with } D_i = \delta_i \pm \Gamma^{\cdot}_{\cdot i} \text{ and } D_a = \partial_a \pm \Gamma^{\cdot}_{\cdot a}$$

is a d–connection in $\mathbf{V}$ $(\alpha = 1, 2, ..., n+m)$, with the operator δ_α, being N–elongated as in (15.10), $u = (x, y) \in \xi_N$ and B_α are $k \times k$–matrix valued functions. For every vector field

$$X = X^\alpha(u)\delta_\alpha = X^i(u)\delta_i + X^a(u)\partial_a \in T\mathbf{V}$$

we can consider the operator

$$X^\alpha(u) \bigtriangledown_\alpha (f \cdot s) = f \cdot \bigtriangledown_X s + \delta_X f \cdot s \tag{15.36}$$

for any section $s \in \mathcal{B}$ and function $f \in C^\infty(\mathbf{V})$, where

$$\delta_X f = X^\alpha \delta_\alpha \ \text{ and } \bigtriangledown_{fX} = f \bigtriangledown_X .$$

In the simplest definition we assume that there is a Lie algebra $\mathcal{GL}B$ that acts on associative algebra B by means of infinitesimal automorphisms (derivations). This means that we have linear operators $\delta_X : B \to B$ which linearly depend on X and satisfy

$$\delta_X(a \cdot b) = (\delta_X a) \cdot b + a \cdot (\delta_X b)$$

for any $a, b \in B$. The mapping $X \to \delta_X$ is a Lie algebra homomorphism, i. e. $\delta_{[X,Y]} = [\delta_X, \delta_Y]$.

Now we consider respectively instead of commutative spaces $\mathbf{V}$ and β the finite projective $\mathcal{A}$–module $\mathcal{E}_N$, provided with N–connection structure, and the finite projective $\mathcal{B}$–module $\mathcal{E}_\beta$.

A d–connection $\bigtriangledown_X$ on $\mathcal{E}_\beta$ is by definition a set of linear d–operators, adapted to the N–connection structure, depending linearly on X and satisfying the Leibniz rule

$$\bigtriangledown_X(b \cdot e) = b \cdot \bigtriangledown_X(e) + \delta_X b \cdot e \qquad (15.37)$$

for any $e \in \mathcal{E}_\beta$ and $b \in \mathcal{B}$. The rule (15.37) is a noncommutative generalization of (15.36). We emphasize that both operators $\bigtriangledown_X$ and δ_X are distinguished by the N–connection structure and that the difference of two such linear d–operators, $\bigtriangledown_X - \bigtriangledown'_X$ commutes with action of B on $\mathcal{E}_\beta$, which is an endomorphism of $\mathcal{E}_\beta$. Hence, if we fix some fiducial connection $\bigtriangledown'_X$ (for instance, $\bigtriangledown'_X = D_X$) on $\mathcal{E}_\beta$ an arbitrary connection has the form

$$\bigtriangledown_X = D_X + B_X,$$

where $B_X \in End_B \mathcal{E}_\beta$ depend linearly on X.

The curvature of connection $\bigtriangledown_X$ is a two–form F_{XY} which values linear operator in $\mathcal{B}$ and measures a deviation of mapping $X \to \bigtriangledown_X$ from being a Lie algebra homomorphism,

$$F_{XY} = [\bigtriangledown_X, \bigtriangledown_Y] - \bigtriangledown_{[X,Y]}.$$

The usual curvature d–tensor is defined as

$$F_{\alpha\beta} = [\bigtriangledown_\alpha, \bigtriangledown_\beta] - \bigtriangledown_{[\alpha,\beta]}.$$

The simplest connection on a finite projective $\mathcal{B}$–module $\mathcal{E}_\beta$ is to be specified by a projector $P : \mathcal{B}^k \otimes \mathcal{B}^k$ when the d–operator δ_X acts naturally on the free module $\mathcal{B}^k$. The operator $\bigtriangledown_X^{LC} = P \cdot \delta_X \cdot P$ is called the Levi–Civita operator and satisfy the condition $Tr[\bigtriangledown_X^{LC}, \phi] = 0$ for any endomorphism $\phi \in End_B \mathcal{E}_\beta$. From this identity, and from the fact that any two connections differ by an endomorphism that $Tr[\bigtriangledown_X, \phi] = 0$ for an arbitrary connection $\bigtriangledown_X$ and an arbitrary endomorphism ϕ, that instead of $\bigtriangledown_X^{LC}$ we may consider equivalently the canonical d–connection, constructed only from d-metric and N–connection coefficients.

15.5 Nonholonomic Clifford–Lagrange Structures

The geometry of spinors on generalized Lagrange and Finsler spaces was elaborated in Refs. [69, 72, 86, 88]. It was applied for definition of noncommutative extensions of the Finsler geometry related to certain models of Einstein, gauge and string gravity [71, 77, 78, 74, 87, 84]. Recently, it is was proposed an extended Clifford approach to relativity, strings and noncommutativity based on the concept of "C–space" [11, 12, 14, 15].

The aim of this section is to formulate the geometry of nonholonomic Clifford–Lagrange structures in a form adapted to generalizations for noncommutative spaces.

15.5.1 Clifford d–module

Let $\mathbf{V}$ be a compact N–anholonomic manifold. We denote, respectively, by $T_x\mathbf{V}$ and $T_x^*\mathbf{V}$ the tangent and cotangent spaces in a point $x \in \mathbf{V}$. We consider a complex vector bundle $\tau : E \to \mathbf{V}$ where, in general, both the base $\mathbf{V}$ and the total space E may be provided with N–connection structure, and denote by $\Gamma^\infty(E)$ (respectively, $\Gamma(E)$) the set of differentiable (continuous) sections of E. The symbols $\chi(\mathbf{M}) = \Gamma^\infty(\mathbf{TM})$ and $\Omega^1(\mathbf{M}) \doteq \Gamma^\infty(\mathbf{T}^*\mathbf{M})$ are used respectively for the set of d–vectors and one d–forms on $\mathbf{TM}$.

Clifford–Lagrange functionals

In the simplest case, a generic nonholonomic Clifford structure can be associated to a Lagrange metric on a n–dimensional real vector space V^n provided with a Lagrange quadratic form $L(y) = q_L(y, y)$, see subsection 15.2.1. We consider the exterior algebra $\wedge V^n$ defined by the identity element $\mathbb{I}$ and antisymmetric products $v_{[1]} \wedge ... \wedge v_{[k]}$ with $v_{[1]}, ..., v_{[k]} \in V^n$ for $k \leq dim V^n$ where $\mathbb{I} \wedge v = v$, $v_{[1]} \wedge v_{[2]} = -v_{[2]} \wedge v_{[1]}, ...$

Definition 15.5.12. *The Clifford–Lagrange (or Clifford–Minkowski) algebra is a $\wedge V^n$ algebra provided with a product*

$$uv + vu = 2\,{}^{(L)}g(u, v)\ \mathbb{I} \tag{15.38}$$

$$(or\ uv + vu = 2\,{}^{(F)}g(u, v)\ \mathbb{I}\ ) \tag{15.39}$$

for any $u, v \in V^n$ and ${}^{(L)}g(u, v)$ (or ${}^{(F)}g(u, v)$) defined by formulas (15.2) (or(15.3)).

For simplicity, hereafter we shall prefer to write down the formulas for the Lagrange configurations instead of dubbing of similar formulas for the Finsler configurations.

We can introduce the complex Clifford–Lagrange algebra $\mathbb{C}l_{(L)}(V^n)$ structure by using the complex unity "i", $V_{\mathbb{C}} \doteq V^n + iV^n$, enabled with complex metric

$${}^{(L)}g_{\mathbb{C}}(u, v + iw) \doteq {}^{(L)}g(u, v) + i\ {}^{(L)}g(u, w),$$

which results in certain isomorphisms of matrix algebras (see, for instance, [29]),

$$\begin{aligned} \mathbb{C}l(\mathbb{R}^{2m}) &\simeq M_{2^m}(\mathbb{C}), \\ \mathbb{C}l(\mathbb{R}^{2m+1}) &\simeq M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}). \end{aligned}$$

We omitted the label (L) because such isomorphisms hold true for any quadratic forms.

The Clifford–Lagrange algebra possesses usual properties:

1. On $\mathbb{C}l_{(L)}(V^n)$, it is linearly defined the involution "$*$",
$$(\lambda v_{[1]}...v_{[k]})^* = \overline{\lambda} v_{[1]}...v_{[k]}, \ \forall \lambda \in \mathbb{C}.$$

2. There is a $\mathbb{Z}_2$ graduation,
$$\mathbb{C}l_{(L)}(V^n) = \mathbb{C}l^{+}_{(L)}(V^n) \oplus \mathbb{C}l^{-}_{(L)}(V^n)$$
with $\chi_{(L)}(a) = \pm 1$ for $a \in \mathbb{C}l^{\pm}_{(L)}(V^n)$, where $\mathbb{C}l^{+}_{(L)}(V^n)$, or $\mathbb{C}l^{-}_{(L)}(V^n)$, are defined by products of an odd, or even, number of vectors.

3. For positive definite forms $q_L(u,v)$, one defines the chirality of $\mathbb{C}l_{(L)}(V^n)$,
$$\gamma_{(L)} = (-i)^n e_1 e_2 ... e_n, \ \gamma^2 = \gamma^* \gamma = \mathbb{I}$$
where $\{e_i\}_{i=1}^n$ is an orthonormal basis of V^n and $n = 2n'$, or $= 2n' + 1$.

In a more general case, a nonholonomic Clifford structure is defined by quadratic d–metric form $\mathbf{q}(\mathbf{x},\mathbf{y})$ (15.22) on a $n+m$–dimensional real d–vector space V^{n+m} with the $(n+m)$–splitting defined by the N–connection structure.

Definition 15.5.13. *The Clifford d–algebra is a $\wedge V^{n+m}$ algebra provided with a product*

$$\mathbf{uv} + \mathbf{vu} = 2\mathbf{g}(\mathbf{u},\mathbf{v}) \ \mathbb{I} \tag{15.40}$$

or, equivalently, distinguished into h– and v–products

$$uv + vu = 2g(u,v) \ \mathbb{I}$$

and

$$^{\star}u \ ^{\star}v + \ ^{\star}v \ ^{\star}u = 2 \ ^{\star}h(\ ^{\star}u, \ ^{\star}v) \ \mathbb{I}$$

for any $\mathbf{u} = (u, \ ^{\star}u), \ \mathbf{v} = (v, \ ^{\star}v) \in V^{n+m}$.

Such Clifford d–algebras have similar properties on the h– and v–components as the Clifford–Lagrange algebras. We may define a standard complexification but it should be emphasized that for $n = m$ the N–connection (in particular, the canonical Lagrange N–connection) induces naturally an almost complex structure (15.15) which gives the possibility to define almost complex Clifford d–algebras (see details in [69, 88]).

Clifford–Lagrange and Clifford N–anholonomic structures

A metric on a manifold M is defined by sections of the tangent bundle TM provided with a bilinear symmetric form on continuous sections $\Gamma(TM)$. In Lagrange geometry, the metric structure is of type ${}^{(L)}g_{ij}(x,y)$ (15.2) which allows us to define Clifford–Lagrange algebras $\mathbb{C}l_{(L)}(T_xM)$, in any point $x \in TM$,

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\ {}^{(L)}g_{ij}\ \mathbb{I}.$$

For any point $x \in M$ and fixed $y = y_0$, one exists a standard complexification, $T_xM^{\mathbb{C}} \doteq T_xM + iT_xM$, which can be used for definition of the 'involution' operator on sections of $T_xM^{\mathbb{C}}$,

$$\sigma_1\sigma_2(x) \doteq \sigma_2(x)\sigma_1(x),\ \sigma^*(x) \doteq \sigma(x)^*, \forall x \in M,$$

where "*" denotes the involution on every $\mathbb{C}l_{(L)}(T_xM)$. The norm is defined by using the Lagrange norm, see Definition 15.2.1,

$$\parallel \sigma \parallel_L \doteq sup_{x\in M}\ \{\mid \sigma(x) \mid_L\},$$

which defines a C_L^*–algebra instead of the usual C^*–algebra of $\mathbb{C}l(T_xM)$. Such constructions can be also performed on the cotangent space T_xM, or for any vector bundle E on M enabled with a symmetric bilinear form of class C^∞ on $\Gamma^\infty(E) \times \Gamma^\infty(E)$.

For Lagrange spaces modelled on $\widetilde{TM}$, there is a natural almost complex structure $\mathbf{F}$ (15.15) induced by the canonical N–connection ${}^{(L)}N$, see the Results 15.2.2, 15.2.4 and 15.2.5, which allows also to construct an almost Kahler model of Lagrange geometry, see details in Refs. [56, 57], and to define an Clifford–Kahler d–algebra $\mathbb{C}l_{(KL)}(T_xM)$ [69], for $y = y_0$, being provided with the norm

$$\parallel \sigma \parallel_{KL} \doteq sup_{x\in M}\ \{\mid \sigma(x) \mid_{KL}\},$$

which on T_xM is defined by projecting on x the d–metric ${}^{(L)}\mathbf{g}$ (15.14).

In order to model Clifford–Lagrange structures on $\widetilde{TM}$ and $\widetilde{T^*M}$ it is necessary to consider d–metrics induced by Lagrangians:

Definition 15.5.14. *A Clifford–Lagrange space on a manifold M enabled with a fundamental metric ${}^{(L)}g_{ij}(x,y)$ (15.5) and canonical N–connection ${}^{(L)}N_j^i$ (15.9) inducing a Sasaki type d–metric ${}^{(L)}\mathbf{g}$ (15.14) is defined as a Clifford bundle $\mathbb{C}l_{(L)}(M) \doteq \mathbb{C}l_{(L)}(T^*M)$.*

For a general N–anholonomic manifold $\mathbf{V}$ of dimension $n+m$ provided with a general d–metric structure $\mathbf{g}$ (15.19) (for instance, in a gravitational model, or constructed by conformal transforms and imbedding into higher dimensions of a Lagrange (or Finsler) d–metrics), we introduce

Definition 15.5.15. *A Clifford N–anholonomic bundle on* $\mathbf{V}$ *is defined as* $\mathbb{C}l_{(N)}(\mathbf{V}) \doteq \mathbb{C}l_{(N)}(T^*\mathbf{V})$.

Let us consider a complex vector bundle $\pi : \mathbf{E} \to M$ provided with N–connection structure which can be defined by a corresponding exact chain of subbundles, or non-integrable distributions, like for real vector bundles, see [56, 57] and subsection 15.2.3. Denoting by $V_{\mathbb{C}}^m$ the typical fiber (a complex vector space), we can define the usual Clifford map

$$c: \ \mathbb{C}l(T^*M) \to End(V_{\mathbb{C}}^m)$$

via (by convention, left) action on sections $c(\sigma)\sigma^1(x) \doteq c(\sigma(x))\sigma^1(x)$.

Definition 15.5.16. *The Clifford d–module (distinguished by a N–connection) of a N–anholonomic vector bundle* $\mathbf{E}$ *is defined by the* $C(M)$*–module* $\Gamma(\mathbf{E})$ *of continuous sections in* $\mathbf{E}$,

$$c: \ \Gamma(\mathbb{C}l(M)) \to End(\Gamma(\mathbf{E})).$$

In an alternative case, one considers a complex vector bundle $\pi : \ E \to \mathbf{V}$ on an N–anholonomic space $\mathbf{V}$ when the N–connection structure is given for the base manifold.

Definition 15.5.17. *The Clifford d–module of a vector bundle* E *is defined by the* $C(\mathbf{V})$*–module* $\Gamma(E)$ *of continuous sections in* E,

$$c: \ \Gamma(\mathbb{C}l_{(N)}(\mathbf{V})) \to End(\Gamma(E)).$$

A Clifford d–module with both N–anholonomic total space $\mathbf{E}$ and base space $\mathbf{V}$ with corresponding N–connections (in general, two independent ones, but the N–connection in the distinguished complex vector bundle must be adapted to the N–connection on the base) has to be defined by an "interference" of Definitions 15.5.16 and 15.5.17.

15.5.2 N–anholonomic spin structures

Usually, the spinor bundle on a manifold M, $dim M = n$, is constructed on the tangent bundle by substituting the group $SO(n)$ by its universal covering $Spin(n)$. If a Lagrange fundamental quadratic form ${}^{(L)}g_{ij}(x,y)$ (15.5) is defined on T_x, M we can consider Lagrange–spinor spaces in every point $x \in M$. The constructions can be completed on $\widetilde{TM}$ by using the Sasaki type metric ${}^{(L)}\mathbf{g}$ (15.14) being similar for any type of N–connection and d–metric structure on TM. On general N–anholonomic manifolds $\mathbf{V}, dim\mathbf{V} = n+m$, the distinguished spinor space (in brief, d–spinor space) is to be derived from the d–metric (15.19) and adapted to the N–connection structure. In this case, the group $SO(n+m)$ is not only substituted by $Spin(n+m)$ but with respect to N–adapted frames (15.10) and (15.11) one defines irreducible decompositions to $Spin(n) \oplus Spin(m)$.

Lagrange spin groups

Let us consider a vector space V^n provided with Clifford–Lagrange structures as in subsection 15.5.1. We denote a such space as $V^n_{(L)}$ in order to emphasize that its tangent space is provided with a Lagrange type quadratic form ${}^{(L)}g$. In a similar form, we shall write $\mathbb{C}l_{(L)}(V^n) \equiv \mathbb{C}l(V^n_{(L)})$ if this will be more convenient. A vector $u \in V^n_{(L)}$ has a unity length on the Lagrange quadratic form if ${}^{(L)}g(u,u) = 1$, or $u^2 = \mathbb{I}$, as an element of corresponding Clifford algebra, which follows from (15.38). We define an endomorphism of V^n :

$$\phi^u_{(L)} \doteq \chi_{(L)}(u)vu^{-1} = -uvu = \left(uv - 2\ {}^{(L)}g(u,v)\right)u = u - 2\ {}^{(L)}g(u,v)u$$

where $\chi_{(L)}$ is the $\mathbb{Z}_2$ graduation defined by ${}^{(L)}g$. By multiplication,

$$\phi^{u_1 u_2}_{(L)}(v) \doteq u_2^{-1}u_1^{-1}vu_1u_2 = \phi^{u_2}_{(L)} \circ \phi^{u_1}_{(L)}(v),$$

which defines the subgroup $SO(V^n_{(L)}) \subset O(V^n_{(L)})$. Now we can define [69, 88]

Definition 15.5.18. *The space of complex Lagrange spins is defined by the subgroup* $Spin^c_{(L)}(n) \equiv Spin^c(V^n_{(L)}) \subset \mathbb{C}l(V^n_{(L)})$, *determined by the products of pairs of vectors* $w \in V^{\mathbb{C}}_{(L)}$ *when* $w \doteq \lambda u$ *where* λ *is a complex number of module 1 and* u *is of unity length in* $V^n_{(L)}$.

We note that $ker\phi_{(L)} \cong U(1)$. We can define a homomorphism $\nu_{(L)}$ with values in $U(1)$,

$$\nu_{(L)}(w) = w_{2k}...w_1w_1...w_{2k} = \lambda_1...\lambda_{2k},$$

where $w = w_1...w_{2k} \in Spin^c(V^n_{(L)})$ and $\lambda_i = w_i^2 \in U(1)$.

Definition 15.5.19. *The group of real Lagrange spins* $Spin^c_{(L)}(n) \equiv Spin(V^n_{(L)})$ *is defined by* $ker\ \nu_{(L)}$.

The complex conjugation on $\mathbb{C}l(V^n_{(L)})$ is usually defined as $\overline{\lambda v} \doteq \overline{\lambda} v$ for $\lambda \in \mathbb{C},\ v \in V^n_{(L)}$. So, any element $w \in Spin(V^n_{(L)})$ satisfies the conditions $\overline{w^*} w = w^* w = \mathbb{I}$ and $\overline{w} = w$. If we take $V^n_{(L)} = \mathbb{R}^n$ provided with a (pseudo) Euclidean quadratic form instead of the Lagrange norm, we obtain the usual spin–group constructions from the (pseudo) Euclidean geometry.

Lagrange spinors and d–spinors: Main Result 1

A usual spinor is a section of a vector bundle S on a manifold M when an irreducible representation of the group $Spin(M) \doteq Spin(T^*_x M)$ is defined on the typical fiber. The set of sections $\Gamma(S)$ is a irreducible Clifford module. If the base manifold of type $M_{(L)}$, or is a general N–anholonomic manifold $\mathbf{V}$, we have to define the spinors on such spaces as to be adapted to the respective N–connection structure.

In the case when the base space is of even dimension (the geometric constructions in in this subsection will be considered for even dimensions both for the base and typical fiber spaces), one should consider the so–called Morita equivalence (see details in [29, 50] for a such equivalence between $C(M)$ and $\Gamma(\mathbb{C}l(M))$). One says that two algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita–equivalent if

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B} \text{ and } \mathcal{F} \otimes_{\mathcal{B}} \mathcal{F} \simeq \mathcal{A},$$

respectively, for $\mathcal{B}$– and $\mathcal{A}$–bimodules and $\mathcal{B} - \mathcal{A}$–bimodule $\mathcal{E}$ and $\mathcal{A} - \mathcal{B}$–bimodule $\mathcal{F}$. If we study algebras through theirs representations, we also have to consider various algebras related by the Morita equivalence.

Definition 15.5.20. *A Lagrange spinor bundle $S_{(L)}$ on a manifold M, $dimM = n$, is a complex vector bundle with both defined action of the spin group $Spin(V^n_{(L)})$ on the typical fiber and an irreducible reprezentation of the group $Spin_{(L)}(M) \equiv Spin(M_{(L)}) \doteq Spin(T^*_x M_{(L)})$. The set of sections $\Gamma(S_{(L)})$ defines an irreducible Clifford–Lagrange module.*

The so–called "d–spinors" have been introduced for the spaces provided with N–connection structure [69, 72, 73]:

Definition 15.5.21. *A distinguished spinor (d–spinor) bundle $\mathbf{S} \doteq (S,\ ^{\star}S)$ on an N–anholonomic manifold $\mathbf{V}$, $dim\mathbf{V} = n + m$, is a complex vector bundle with a defined action of the spin d–group $Spin\ \mathbf{V} \doteq Spin(V^n) \oplus Spin(V^m)$ with the splitting adapted to the N–connection structure which results in an irreducible representation $Spin(\mathbf{V}) \doteq Spin(T^*\mathbf{V})$. The set of sections $\Gamma(\mathbf{S}) = \Gamma\ (S) \oplus \Gamma(\ ^{\star}S)$ is an irreducible Clifford d–module.*

The fact that $C(\mathbf{V})$ and $\Gamma(\mathbb{C}l(\mathbf{V}))$ are Morita equivalent can be analyzed by applying in N–adapted form, both on the base and fiber spaces, the consequences of the Plymen's theorem (see Theorem 9.3 in Ref. [29]). This is connected with the possibility to distinguish the $Spin(n)$ (or, correspondingly $Spin(M_{(L)})$, $Spin(V^n) \oplus Spin(V^m)$) an antilinear bijection $J: \ S \ \rightarrow \ S$ (or $J: \ S_{(L)} \ \rightarrow \ S_{(L)}$ and $J: \ \mathbf{S} \ \rightarrow \ \mathbf{S}$) with the properties:

$$\begin{aligned} J(\psi f) &= (J\psi)f \text{ for } f \in C(M)(\text{ or } C(M_{(L)}),\ C(\mathbf{V})); \\ J(a\psi) &= \chi(a)J\psi,\ \text{for } a \in \Gamma^{\infty}(\mathbb{C}l(M))(\text{ or } \Gamma^{\infty}(\mathbb{C}l(M_{(L)})),\ \Gamma^{\infty}(\mathbb{C}l(\mathbf{V})); \\ (J\phi|J\psi) &= (\psi|\phi) \text{ for } \phi,\psi \in S(\text{ or } S_{(L)},\mathbf{S}). \end{aligned} \tag{15.41}$$

Definition 15.5.22. *The spin structure on a manifold M (respectively, on $M_{(L)}$, or on N–anholonomic manifold $\mathbf{V}$) with even dimensions for the corresponding base and typical fiber spaces is defined by a bimodule S (respectively, $M_{(L)}$, or $\mathbf{V}$) obeying the Morita equivalence $C(M) - \Gamma(\mathbb{C}l(M))$ (respectively, $C(M_{(L)}) - \Gamma(\mathbb{C}l(M_{(L)}))$, or $C(\mathbf{V}) - \Gamma(\mathbb{C}l(\mathbf{V})))$ by a corresponding bijections (15.41) and a fixed orientation on M (respectively, on $M_{(L)}$ or $\mathbf{V}$).*

In brief, we may call M ($M_{(L)}$, or $\mathbf{V}$) as a spin manifold (Lagrange spin manifold, or N–anholonomic spin manifold). If any of the base or typical fiber spaces is of odd dimension, we may perform similar constructions by considering $\mathbb{C}l^{+}$ instead of $\mathbb{C}l$.

The considerations presented in this Section consists the proof of the first main Result of this paper (let us conventionally say that it is the 7th one after the Results 15.2.1–15.2.6:

Theorem 15.5.3. (Main Result 1) *Any regular Lagrangian and/or N–connection structure define naturally the fundamental geometric objects and structures (such as the Clifford–Lagrange module and Clifford d–modules, the Lagrange spin structure and d–spinors) for the corresponding Lagrange spin manifold and/or N–anholonomic spinor (d–spinor) manifold.*

We note that similar results were obtained in Refs. [69, 72, 86, 88] for the standard Finsler and Lagrange geometries and theirs higher order generalizations. In a more restricted form, the idea of Theorem 15.5.3 can be found in Ref. [77], where the first models of noncommutative Finsler geometry and related gravity were considered (in a more rough form, for instance, with constructions not reflecting the Morita equivalence).

Finally, in this Section, we can make the

Conclusion 15.5.2. *Any regular Lagrange and/or N–connection structure (the second one being any admissible N–connection in Lagrange–Finsler geometry and their generalizations, or induced by any generic off–diagonal and/ or nonholonomic frame structure)* *define certain, corresponding, Clifford–Lagrange module and/or Clifford d–module and related Lagrange spinor and/or d–spinor structures.*

It is a bit surprizing that a Lagrangian may define not only the fundamental geometric objects of a nonholonomic Lagrange space but also the structure of a naturally associated Lagrange spin manifold. The Lagrange mechanics and off–diagonal gravitational interactions (in general, with nontrivial torsion and nonholonomic constraints) can be completely geometrized on Lagrange spin (N–anholonomic) manifolds.

15.6 The Dirac Operator, Nonholonomy, and Spectral Triples

The Dirac operator for a certain class of (non) commutative Finsler spaces provided with compatible metric structure was introduced in Ref. [77] following previous constructions for the Dirac equations on locally anisotropic spaces [69, 72, 73, 86, 88]. The aim of this Section is to elucidate the possibility of definition of Dirac operators for general N–anholonomic manifolds and Lagrange–Finsler spaces. It should be noted that such geometric constructions depend on the type of linear connections which are used for the complete definition of the Dirac operator. They are metric compatible and N–adapted if the canonical d–connection is used, see Proposition 15.3.1 (we can also use any its deformation which results in a metric compatible d–connection). The constructions can be more sophisticate and nonmetric (with some geometric objects not completely defined on the tangent spaces) if the Chern, or the Berwald d–connection, is considered, see Example 15.3.2.

15.6.1 N–anholonomic Dirac operators

We introduce the basic definitions and formulas with respect to N–adapted frames of type (15.10) and (15.11). Then we shall present the main results in a global form.

Noholonomic vielbeins and spin d–connections

Let us consider a Hilbert space of finite dimension. For a local dual coordinate basis $e^{\underline{i}} \doteq dx^{\underline{i}}$ on a manifold $M,\ dimM = n$, we may respectively introduce certain classes of orthonormalized vielbeins and the N–adapted vielbeins, [7]

$$e^{\hat{i}} \doteq e^{\hat{i}}_{\ \underline{i}}(x,y)\ e^{\underline{i}} \text{ and } e^{i} \doteq e^{i}_{\ \underline{i}}(x,y)\ e^{\underline{i}}, \tag{15.42}$$

where

$$g^{\underline{ij}}(x,y)\ e^{\hat{i}}_{\ \underline{i}}(x,y)e^{\hat{j}}_{\ \underline{j}}(x,y) = \delta^{\hat{i}\hat{j}} \text{ and } g^{\underline{ij}}(x,y)\ e^{i}_{\ \underline{i}}(x,y)e^{j}_{\ \underline{j}}(x,y) = g^{ij}(x,y).$$

We define the the algebra of Dirac's gamma matrices (in brief, h–gamma matrices defined by self–adjoints matrices $M_k(\mathbb{C})$ where $k = 2^{n/2}$ is the dimension of the irreducible representation of $\mathbb{C}l(M)$ for even dimensions, or of $\mathbb{C}l(M)^+$ for odd dimensions) from the relation

$$\gamma^{\hat{i}}\gamma^{\hat{j}} + \gamma^{\hat{j}}\gamma^{\hat{i}} = 2\delta^{\hat{i}\hat{j}}\ \mathbb{I}. \tag{15.43}$$

We can consider the action of $dx^i \in \mathbb{C}l(M)$ on a spinor $\psi \in S$ via representations

$$^{-}c(dx^{\hat{i}}) \doteq \gamma^{\hat{i}} \text{ and } ^{-}c(dx^{i})\psi \doteq \gamma^{i}\psi \equiv e^{i}_{\ \hat{i}}\ \gamma^{\hat{i}}\psi. \tag{15.44}$$

For any type of spaces $T_xM, TM, \mathbf{V}$ possessing a local (in any point) or global fibered structure and, in general, enabled with a N–connection structure, we can introduce similar definitions of the gamma matrices following algebraic relations and metric structures on fiber subspaces,

$$e^{\hat{a}} \doteq e^{\hat{a}}_{\ \underline{a}}(x,y)\ e^{\underline{a}} \text{ and } e^{a} \doteq e^{a}_{\ \underline{a}}(x,y)\ e^{\underline{a}}, \tag{15.45}$$

where

$$g^{\underline{ab}}(x,y)\ e^{\hat{a}}_{\ \underline{a}}(x,y)e^{\hat{b}}_{\ \underline{b}}(x,y) = \delta^{\hat{a}\hat{b}} \text{ and } g^{\underline{ab}}(x,y)\ e^{a}_{\ \underline{a}}(x,y)e^{b}_{\ \underline{b}}(x,y) = h^{ab}(x,y).$$

Similarly, we define the algebra of Dirac's matrices related to typical fibers (in brief, v–gamma matrices defined by self–adjoints matrices $M'_k(\mathbb{C})$ where $k' = 2^{m/2}$ is the dimension of the irreducible representation of $\mathbb{C}l(F)$ for even dimensions, or of $\mathbb{C}l(F)^+$ for odd dimensions, of the typical fiber) from the relation

$$\gamma^{\hat{a}}\gamma^{\hat{b}} + \gamma^{\hat{b}}\gamma^{\hat{a}} = 2\delta^{\hat{a}\hat{b}}\ \mathbb{I}. \tag{15.46}$$

The action of $dy^a \in \mathbb{C}l(F)$ on a spinor $^{\star}\psi \in {}^{\star}S$ is considered via representations

$$^{\star}c(dy^{\hat{a}}) \doteq \gamma^{\hat{a}} \text{ and } ^{\star}c(dy^{a})\ ^{\star}\psi \doteq \gamma^{a}\ ^{\star}\psi \equiv e^{a}_{\ \hat{a}}\ \gamma^{\hat{a}}\ ^{\star}\psi. \tag{15.47}$$

[7](depending both on the base coordinates $x \doteq x^i$ and some "fiber" coordinates $y \doteq y^a$, the status of y^a depends on what kind of models we shall consider: elongated on TM, for a Lagrange space, for a vector bundle, or on a N–anholonomic manifold)

We note that additionally to formulas (15.44) and (15.47) we may write respectively

$$c(dx^{\underline{i}})\psi \doteq \gamma^{\underline{i}}\psi \equiv e^{\underline{i}}_{\ \hat{i}}\ \gamma^{\hat{i}}\psi \text{ and } c(dy^{\underline{a}})\ ^{\star}\psi \doteq \gamma^{\underline{a}}\ ^{\star}\psi \equiv e^{\underline{a}}_{\ \hat{a}}\ \gamma^{\hat{a}}\ ^{\star}\psi$$

but such operators are not adapted to the N–connection structure.

A more general gamma matrix calculus with distinguished gamma matrices (in brief, d–gamma matrices[8]) can be elaborated for N–anholonomic manifolds $\mathbf{V}$ provided with d–metric structure $\mathbf{g} = [g,^{\star} g]$ and for d–spinors $\breve{\psi} \doteq (\psi,\ ^{\star}\psi) \in \mathbf{S} \doteq (S,\ ^{\star}S)$, see the corresponding Definitions 15.2.7, 15.2.8 and 15.5.21. Firstly, we should write in a unified form, related to a d–metric (15.19), the formulas (15.42) and (15.45),

$$e^{\hat{\alpha}} \doteq e^{\hat{\alpha}}_{\ \underline{a}}(u)\ e^{\underline{\alpha}} \text{ and } e^{\alpha} \doteq e^{\alpha}_{\ \underline{\alpha}}(u)\ e^{\underline{\alpha}}, \tag{15.48}$$

where

$$g^{\underline{\alpha\beta}}(u)\ e^{\hat{\alpha}}_{\ \underline{\alpha}}(u)e^{\hat{\beta}}_{\ \underline{\beta}}(u) = \delta^{\hat{\alpha}\hat{\beta}} \text{ and } g^{\underline{\alpha\beta}}(u)\ e^{\alpha}_{\ \underline{\alpha}}(u)e^{\beta}_{\ \underline{\beta}}(u) = g^{\alpha\beta}(u).$$

The second step, is to consider d–gamma matrix relations (unifying (15.43) and (15.46))

$$\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} + \gamma^{\hat{\beta}}\gamma^{\hat{\alpha}} = 2\delta^{\hat{\alpha}\hat{\beta}}\ \mathbb{I}, \tag{15.49}$$

with the action of $du^{\alpha} \in \mathbb{C}l(\mathbf{V})$ on a d–spinor $\breve{\psi} \in\ \mathbf{S}$ resulting in distinguished irreducible representations (unifying (15.44) and (15.47))

$$\mathbf{c}(du^{\hat{\alpha}}) \doteq \gamma^{\hat{\alpha}} \text{ and } \mathbf{c} = (du^{\alpha})\ \breve{\psi} \doteq \gamma^{\alpha}\ \breve{\psi} \equiv e^{\alpha}_{\ \hat{\alpha}}\ \gamma^{\hat{\alpha}}\ \breve{\psi} \tag{15.50}$$

which allows to write

$$\gamma^{\alpha}(u)\gamma^{\beta}(u) + \gamma^{\beta}(u)\gamma^{\alpha}(u) = 2g^{\alpha\beta}(u)\ \mathbb{I}. \tag{15.51}$$

In the canonical representation we can write in irreducible form $\breve{\gamma} \doteq \gamma \oplus\ ^{\star}\gamma$ and $\breve{\psi} \doteq \psi \oplus\ ^{\star}\psi$, for instance, by using block type of h– and v–matrices, or, writing alternatively as couples of gamma and/or h– and v–spinor objects written in N–adapted form,

$$\gamma^{\alpha} \doteq (\gamma^{i}, \gamma^{a}) \text{ and } \breve{\psi} \doteq (\psi,\ ^{\star}\psi). \tag{15.52}$$

The decomposition (15.51) holds with respect to a N–adapted vielbein (15.10). We also note that for a spinor calculus, the indices of spinor objects should be treated as abstract spinorial ones possessing certain reducible, or irreducible, properties depending on the space dimension (see details in Refs. [69, 72, 73, 86, 88]). For simplicity, we

[8] in our previous works [69, 72, 73, 86, 88] we wrote σ instead of γ

shall consider that spinors like $\breve{\psi}, \psi, \ ^{\star}\psi$ and all type of gamma objects can be enabled with corresponding spinor indices running certain values which are different from the usual coordinate space indices. In a "rough" but brief form we can use the same indices $i, j, ..., a, b..., \alpha, \beta, ...$ both for d–spinor and d–tensor objects.

The spin connection $\bigtriangledown^S$ for the Riemannian manifolds is induced by the Levi–Civita connection $\ ^{\bigtriangledown}\Gamma$,

$$\bigtriangledown^S \doteq d - \frac{1}{4} \ ^{\bigtriangledown}\Gamma^i_{\ jk}\gamma_i\gamma^j \ dx^k. \tag{15.53}$$

On N–anholonomic spaces, it is possible to define spin connections which are N–adapted by replacing the Levi–Civita connection by any d–connection (see Definition 15.3.9).

Definition 15.6.23. *The canonical spin d–connection is defined by the canonical d–connection (15.25) as*

$$\widehat{\bigtriangledown}^{\mathbf{S}} \doteq \delta - \frac{1}{4} \ \widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\mu}\gamma_\alpha\gamma^\beta\delta u^\mu, \tag{15.54}$$

where the absolute differential δ acts in N–adapted form resulting in 1–forms decomposed with respect to N–elongated differentials like $\delta u^\mu = (dx^i, \delta y^a)$ (15.11).

We note that the canonical spin d–connection $\widehat{\bigtriangledown}^{\mathbf{S}}$ is metric compatible and contains nontrivial d–torsion coefficients induced by the N–anholonomy relations (see the formulas (15.24) proved for arbitrary d–connection). It is possible to introduce more general spin d–connections $\mathbf{D}^{\mathbf{S}}$ by using the same formula (15.54) but for arbitrary metric compatible d–connection $\mathbf{\Gamma}^{\alpha}_{\ \beta\mu}$.

In a particular case, we can define, for instance, the canonical spin d–connections for a local modelling of a $\widetilde{TM}$ space on $\widetilde{\mathbf{V}}_{(n,n)}$ with the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{C}^i_{jk})$, see formulas (15.27). This allows us to prove (by considering d–connection and d–metric structure defined by the fundamental Lagrange, or Finsler, functions, we put formulas (15.9) and (15.14) into (15.27)):

Proposition 15.6.2. *On Lagrange spaces, there is a canonical spin d–connection (the canonical spin–Lagrange connection),*

$$\widehat{\bigtriangledown}^{(SL)} \doteq \delta - \frac{1}{4} \ ^{(L)}\mathbf{\Gamma}^{\alpha}_{\ \beta\mu}\gamma_\alpha\gamma^\beta\delta u^\mu, \tag{15.55}$$

where $\delta u^\mu = (dx^i, \delta y^k = dy^k + \ ^{(L)}N^k_{\ i} \ dx^i)$.

We emphasize that even regular Lagrangians of classical mechanics without spin particles induce in a canonical (but nonholonomic) form certain classes of spin d–connections like (15.55).

For the spaces provided with generic off–diagonal metric structure (15.16) (in particular, for such Riemannian manifolds) resulting in equivalent N–anholonomic manifolds, it is possible to prove a result being similar to Proposition 15.6.2:

Remark 15.6.7. *There is a canonical spin d–connection (15.54) induced by the off–diagonal metric coefficients with nontrivial N_i^a and associated nonholonomic frames in gravity theories.*

The N–connection structure also states a global h– and v–splitting of spin d–connection operators, for instance,

$$\widehat{\bigtriangledown}^{(SL)} \doteq \delta - \frac{1}{4}\ {}^{(L)}\widehat{L}^{i}{}_{jk}\gamma_i\gamma^j dx^k - \frac{1}{4}\ {}^{(L)}\widehat{C}^{a}{}_{bc}\gamma_a\gamma^b\delta y^c. \qquad (15.56)$$

So, any spin d–connection is a d–operator with conventional splitting of action like $\bigtriangledown^{(\mathbf{S})} \equiv (\ {}^{-}\bigtriangledown^{(\mathbf{S})},\ {}^{\star}\bigtriangledown^{(\mathbf{S})})$, or $\bigtriangledown^{(SL)} \equiv (\ {}^{-}\bigtriangledown^{(SL)},\ {}^{\star}\bigtriangledown^{(SL)})$. For instance, for $\widehat{\bigtriangledown}^{(SL)} \equiv (\ {}^{-}\widehat{\bigtriangledown}^{(SL)},\ {}^{\star}\widehat{\bigtriangledown}^{(SL)})$, the operators ${}^{-}\widehat{\bigtriangledown}^{(SL)}$ and ${}^{\star}\widehat{\bigtriangledown}^{(SL)}$ act respectively on a h–spinor ψ as

$${}^{-}\widehat{\bigtriangledown}^{(SL)}\psi \doteq dx^i\ \frac{\delta\psi}{dx^i} - dx^k\frac{1}{4}\ {}^{(L)}\widehat{L}^{i}{}_{jk}\gamma_i\gamma^j\ \psi \qquad (15.57)$$

and

$${}^{\star}\widehat{\bigtriangledown}^{(SL)}\psi \doteq \delta y^a\ \frac{\partial\psi}{dy^a} - \delta y^c\ \frac{1}{4}\ {}^{(L)}\widehat{C}^{a}{}_{bc}\gamma_a\gamma^b\ \psi$$

being defined by the canonical d–connection (15.27).

Remark 15.6.8. *We can consider that the h–operator (15.57) defines a spin generalization of the Chern's d–connection* ${}^{[Chern]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^{i}_{jk}, C^{i}_{jk} = 0)$, *see Example 15.3.2, which may be introduced as a minimal extension, with Finsler structure, of the spin connection defined by the Levi–Civita connection (15.53) preserving the torsionless condition. This is an example of nonmetric spin connection operator because* ${}^{[Chern]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ *does not satisfy the condition of metric compatibility.*

We can define spin Chern–Finsler structures, considered in the Remark 15.6.8, for any point of an N–anholonomic manifold. There are necessary some additional assumptions in order to completely define such structures (for instance, on the tangent bundle). We can say that this is a deformed nonholonomic spin structure derived from a d–spinor one provided with the canonical spin d–connection by deforming the canonical d–connection in a manner that the horizontal torsion vanishes transforming into a nonmetricity d–tensor. The "nonspinor" aspects of such generalizations of the Riemann–Finsler spaces and gravity models with nontrivial nonmetricity are analyzed in Refs. [83].

Dirac d–operators: Main Result 2

We consider a vector bundle $\mathbf{E}$ on an N–anholonomic manifold $\mathbf{M}$ (with two compatible N–connections defined as h– and v–splitting of $T\mathbf{E}$ and $T\mathbf{M}$)). A d–connection

$$\mathcal{D}: \ \Gamma^{\infty}(\mathbf{E}) \rightarrow \Gamma^{\infty}(\mathbf{E}) \otimes \Omega^{1}(\mathbf{M})$$

preserves by parallelism splitting of the tangent total and base spaces and satisfy the Leibniz condition

$$\mathcal{D}(f\sigma) = f(\mathcal{D}\sigma) + \delta f \otimes \sigma$$

for any $f \in C^{\infty}(\mathbf{M})$, and $\sigma \in \Gamma^{\infty}(\mathbf{E})$ and δ defining an N–adapted exterior calculus by using N–elongated operators (15.10) and (15.11) which emphasize d–forms instead of usual forms on $\mathbf{M}$, with the coefficients taking values in $\mathbf{E}$.

The metricity and Leibniz conditions for $\mathcal{D}$ are written respectively

$$\mathbf{g}(\mathcal{D}\mathbf{X}, \mathbf{Y}) + \mathbf{g}(\mathbf{X}, \mathcal{D}\mathbf{Y}) = \delta[\mathbf{g}(\mathbf{X}, \mathbf{Y})], \tag{15.58}$$

for any $\mathbf{X}, \ \mathbf{Y} \in \chi(\mathbf{M})$, and

$$\mathcal{D}(\sigma\beta) \doteq \mathcal{D}(\sigma)\beta + \sigma\mathcal{D}(\beta), \tag{15.59}$$

for any $\sigma, \beta \in \Gamma^{\infty}(\mathbf{E})$.

For local computations, we may define the corresponding coefficients of the geometric d–objects and write

$$\mathcal{D}\sigma_{\acute{\beta}} \doteq \mathbf{\Gamma}^{\acute{\alpha}}_{\ \acute{\beta}\mu}\ \sigma_{\acute{\alpha}} \otimes \delta u^{\mu} = \mathbf{\Gamma}^{\acute{\alpha}}_{\ \acute{\beta}i}\ \sigma_{\acute{\alpha}} \otimes dx^{i} + \mathbf{\Gamma}^{\acute{\alpha}}_{\ \acute{\beta}a}\ \sigma_{\acute{\alpha}} \otimes \delta y^{a},$$

where fiber "acute" indices, in their turn, may split $\acute{\alpha} \doteq (\acute{i}, \acute{a})$ if any N–connection structure is defined on $T\mathbf{E}$. For some particular constructions of particular interest, we can take $\mathbf{E} = T^{*}\mathbf{V}, = T^{*}V_{(L)}$ and/or any Clifford d–algebra $\mathbf{E} = \mathbb{C}l(\mathbf{V}), \mathbb{C}l(V_{(L)}), ...$ with a corresponding treating of "acute" indices to of d–tensor and/or d–spinor type as well when the d–operator $\mathcal{D}$ transforms into respective d–connection $\mathbf{D}$ and spin d–connections $\widehat{\bigtriangledown}^{\mathbf{S}}$ (15.54), $\widehat{\bigtriangledown}^{(SL)}$ (15.55).... All such, adapted to the N–connections, computations are similar for both N–anholonomic (co) vector and spinor bundles.

The respective actions of the Clifford d–algebra and Clifford–Lagrange algebra (see Definitions 15.5.13 and 15.5.13) can be transformed into maps $\Gamma^{\infty}(\mathbf{S}) \otimes \Gamma^{(}\mathbb{C}l(\mathbf{V}))$ and $\Gamma^{\infty}(S_{(L)}) \otimes \Gamma^{(}\mathbb{C}l(V_{(L)}))$ to $\Gamma^{\infty}(\mathbf{S})$ and, respectively, $\Gamma^{\infty}(S_{(L)})$ by considering maps of type (15.44) and (15.50)

$$\widehat{\mathbf{c}}(\breve{\psi} \otimes \mathbf{a}) \doteq \mathbf{c}(\mathbf{a})\breve{\psi} \text{ and } \widehat{c}(\psi \otimes a) \doteq c(a)\psi.$$

Definition 15.6.24. *The Dirac d–operator (Dirac–Lagrange operator) on a spin N–anholonomic manifold* $(\mathbf{V}, \mathbf{S}, J)$ *(on a Lagrange spin manifold* $(M_{(L)}, S_{(L)}, J))$ *is defined*

$$\begin{aligned} \mathbb{D} &\doteq -i\ (\widehat{\mathbf{c}} \circ \bigtriangledown^{\mathbf{S}}) \\ &= \left({}^{-}\mathbb{D} = -i\ (\ {}^{-}\widehat{c} \circ\ {}^{-}\bigtriangledown^{\mathbf{S}}),\ {}^{\star}\mathbb{D} = -i\ (\ {}^{\star}\widehat{c} \circ\ {}^{\star}\bigtriangledown^{\mathbf{S}}) \right) \end{aligned} \tag{15.60}$$

$$\begin{aligned} (\ {}_{(L)}\mathbb{D} &\doteq -i\ (\widehat{c} \circ \bigtriangledown^{(SL)})\ ) \\ &= \left(\ {}_{(L)}{}^{-}\mathbb{D} = -i(\ {}^{-}\widehat{c} \circ\ {}^{-}\bigtriangledown^{(SL)}),\ {}_{(L)}{}^{\star}\mathbb{D} = -i(\ {}^{\star}\widehat{c} \circ\ {}^{\star}\bigtriangledown^{(SL)}) \right)\ ). \end{aligned} \tag{15.61}$$

Such N–adapted Dirac d–operators are called canonical and denoted $\widehat{\mathbb{D}} = (\ {}^{-}\widehat{\mathbb{D}},\ {}^{\star}\widehat{\mathbb{D}}\ )$ *(* ${}_{(L)}\widehat{\mathbb{D}} = (\ {}_{(L)}{}^{-}\widehat{\mathbb{D}},\ {}_{(L)}{}^{\star}\widehat{\mathbb{D}}\ )$ *) if they are defined for the canonical d–connection (15.26) ((15.27)) and respective spin d–connection (15.54) ((15.55)).*

Now we can formulate the

Theorem 15.6.4. (Main Result 2) *Let* $(\mathbf{V}, \mathbf{S}, J)$ *(* $(M_{(L)}, S_{(L)}, J)$ *be a spin N–anholonomic manifold (spin Lagrange space). There is the canonical Dirac d–operator (Dirac–Lagrange operator) defined by the almost Hermitian spin d–operator*

$$\widehat{\bigtriangledown}^{\mathbf{S}} :\ \Gamma^{\infty}(\mathbf{S}) \rightarrow \Gamma^{\infty}(\mathbf{S}) \otimes \Omega^{1}(\mathbf{V})$$

(spin Lagrange operator

$$\widehat{\bigtriangledown}^{(SL)} :\ \Gamma^{\infty}(S_{(L)}) \rightarrow \Gamma^{\infty}(S_{(L)}) \otimes \Omega^{1}(M_{(L)})\ )$$

commuting with J (15.41) and satisfying the conditions

$$(\widehat{\bigtriangledown}^{\mathbf{S}} \breve{\psi} \mid \breve{\phi})\ + (\breve{\psi} \mid \widehat{\bigtriangledown}^{\mathbf{S}} \breve{\phi})\ = \delta(\breve{\psi} \mid \breve{\phi}) \tag{15.62}$$

and

$$\widehat{\bigtriangledown}^{\mathbf{S}}(\mathbf{c}(\mathbf{a})\breve{\psi})\ = \mathbf{c}(\widehat{\mathbf{D}}\mathbf{a})\breve{\psi} + \mathbf{c}(\mathbf{a})\widehat{\bigtriangledown}^{\mathbf{S}}\breve{\psi}$$

for $\mathbf{a} \in \mathbb{C}l(\mathbf{V})$ *and* $\breve{\psi} \in \Gamma^{\infty}(\mathbf{S})$

$$(\ (\widehat{\bigtriangledown}^{(SL)} \breve{\psi} \mid \breve{\phi})\ + (\breve{\psi} \mid \widehat{\bigtriangledown}^{(SL)} \breve{\phi})\ = \delta(\breve{\psi} \mid \breve{\phi}) \tag{15.63}$$

and

$$\widehat{\bigtriangledown}^{(SL)}(\mathbf{c}(\mathbf{a})\breve{\psi})\ = \mathbf{c}(\widehat{\mathbf{D}}\mathbf{a})\breve{\psi} + \mathbf{c}(\mathbf{a})\widehat{\bigtriangledown}^{(SL)}\breve{\psi}$$

for $\mathbf{a} \in \mathbb{C}l(M_{(L)})$ *and* $\breve{\psi} \in \Gamma^{\infty}(S_{(L)}\ )$ *determined by the metricity (15.58) and Leibnitz (15.59) conditions.*

Proof. We sketch the main ideas of such Proofs. There two ways:

The first one is similar to that given in Ref. [29], Theorem 9.8, for the Levi–Civita connection, see similar considerations in [67]. In our case, we have to extend the constructions for d–metrics and canonical d–connections by applying N–elongated operators for differentials and partial derivatives. The formulas have to be distinguished into h– and v–irreducible components. We are going to present the related technical details in our further publications.

In other turn, the second way, is to argue a such proof is a straightforward consequence of the Result 15.2.6 stating that any Riemannian manifold can be modelled as a N–anholonomic manifold induced by the generic off–diagonal metric structure. If the results from [29] hold true for any Riemannian space, the formulas may be rewritten with respect to any local frame system, as well with respect to (15.10) and (15.11). Nevertheless, on N–anholonomic manifolds the canonical d–connection is not just the Levi–Civita connection but a deformation of type (15.25): we must verify that such deformations results in N–adapted constructions satisfying the metricity and Leibnitz conditions. The existence of such configurations was proven from the properties of the canonical d–connection completely defined from the d–metric and N–connection coefficients. The main difference from the case of the Levi–Civita configuration is that we have a nontrivial torsion induced by the frame nonholonomy. But it is not a problem to define the Dirac operator with nontrivial torsion if the metricity conditions are satisfied. □ □

The canonical Dirac d–operator has very similar properties for spin N–anholonomic manifolds and spin Lagrange spaces. Nevertheless, theirs geometric and physical meaning may be completely different and that why we have written the corresponding formulas with different labels and emphasized the existing differences. With respect to the **Main Result 2**, one holds three important remarks:

Remark 15.6.9. *The first type of canonical Dirac d–operators may be associated to Riemannian–Cartan (in particular, Riemann) off–diagonal metric and nonholonomic frame structures and the second type of canonical Dirac–Lagrange operators are completely induced by a regular Lagrange mechanics. In both cases, such d–operators are completely determined by the coefficients of the corresponding Sasaki type d–metric and the N–connection structure.*

Remark 15.6.10. *The conditions of the Theorem 15.6.4 may be revised for any d–connection and induced spin d–connection satisfying the metricity condition. But, for such cases, the corresponding Dirac d–operators are not completely defined by the d–metric and N–connection structures. We can prescribe certain type of torsions of d–*

connections and, via such 'noncanonical' Dirac operators, we are able to define noncommutative geometries with prescribed d–torsions.

Remark 15.6.11. *The properties (15.62) and (15.63) hold if and only if the metricity conditions are satisfied (15.58). So, for the Chern or Berwald type d–connections which are nonmetric (see Example 15.3.2 and Remark 15.6.8), the conditions of Theorem 15.6.4 do not hold.*

It is a more sophisticate problem to find applications in physics for such nonmetric constructions [9] but they define positively some examples of nonmetric d–spinor and noncommutative structures minimally deformed from the Riemannian (non) commutative geometry to certain Finsler type (non) commutative geometries.

15.6.2 Distinguished spectral triples

The geometric information of a spin manifold (in particular, the metric) is contained in the Dirac operator. For nonholonomic manifolds, the canonical Dirac d–operator has h– and v–irreducible parts related to off–diagonal metric terms and nonholonomic frames with associated structure. In a more special case, the canonical Dirac–Lagrange operator is defined by a regular Lagrangian. So, such Driac d–operators contain more information than the usual, holonomic, ones.

For simplicity, hereafter, we shall formulate the results for the general N–anholonomic spaces, by omitting the explicit formulas and proofs for Lagrange and Finsler spaces, which can be derived by imposing certain conditions that the N–connection, d–connection and d–metric are just those defined canonically by a Lagrangian. We shall only present the Main Result and some important Remarks concerning Lagrange mechanics and/or Finsler structures.

Proposition 15.6.3. *If* $\widehat{\mathbb{D}} = (\ {}^{-}\widehat{\mathbb{D}},\ {}^{\star}\widehat{\mathbb{D}}\ )$ *is the canonical Dirac d–operator then*

$$\begin{aligned} \left[\widehat{\mathbb{D}},\ f\right] &= i\mathbf{c}(\delta f),\ \textit{equivalently,} \\ \left[\ {}^{-}\widehat{\mathbb{D}},\ f\right] + \left[\ {}^{\star}\widehat{\mathbb{D}},\ f\right] &= i\ {}^{-}c(dx^i \frac{\delta f}{\partial x^i}) + i\ {}^{\star}c(\delta y^a \frac{\partial f}{\partial y^a}), \end{aligned}$$

for all $f \in C^{\infty}(\mathbf{V})$.

Proof. It is a straightforward computation following from Definition 15.6.24. □

[9] See Refs. [28] and [80, 83, 82] for details on elaborated geometrical and physical models being, respectively, locally isotropic and locally anisotropic.

The canonical Dirac d–operator and its irreversible h– and v–components have all the properties of the usual Dirac operators (for instance, they are self–adjoint but unbounded). It is possible to define a scalar product on $\Gamma^{\infty}(\mathbf{S})$,

$$< \breve{\psi}, \breve{\phi} >\doteq \int_{\mathbf{V}} (\breve{\psi}|\breve{\phi}) |\nu_{\mathbf{g}}| \tag{15.64}$$

where

$$\nu_{\mathbf{g}} = \sqrt{detg}\ \sqrt{deth}\ dx^1...dx^n\ dy^{n+1}...dy^{n+m}$$

is the volume d–form on the N–anholonomic manifold $\mathbf{V}$.

We denote by

$$\mathcal{H}_N \doteq L_2(\mathbf{V},\mathbf{S}) = \left[\ ^{-}\mathcal{H} = L_2(\mathbf{V},\ ^{-}S),\ ^{\star}\mathcal{H} = L_2(\mathbf{V},\ ^{\star}S)\right] \tag{15.65}$$

the Hilbert d–space obtained by completing $\Gamma^{\infty}(\mathbf{S})$ with the norm defined by the scalar product (15.64).

Similarly to the holonomic spaces, by using formulas (15.60) and (15.54), one may prove that there is a self–adjoint unitary endomorphism $\Gamma^{[cr]}$ of $\mathcal{H}_N$, called "chirality", being a $\mathbb{Z}_2$ graduation of $\mathcal{H}_N$, [10] which satisfies the condition

$$\widehat{\mathbb{D}}\ \Gamma^{[cr]} = -\Gamma^{[cr]}\ \widehat{\mathbb{D}}. \tag{15.66}$$

We note that the condition (15.66) may be written also for the irreducible components $^{-}\widehat{\mathbb{D}}$ and $^{\star}\widehat{\mathbb{D}}$.

Definition 15.6.25. *A distinguished canonical spectral triple (canonical spectral d–triple) $(\mathcal{A},\mathcal{H}_N,\ \widehat{\mathbb{D}})$ for an algebra $\mathcal{A}$ is defined by a Hilbert d–space $\mathcal{H}_N$, a representation of $\mathcal{A}$ in the algebra $\mathcal{B}(\mathcal{H})$ of d–operators bounded on $\mathcal{H}_N$, and by a self–adjoint d–operator $\widehat{\mathbb{D}}$, of compact resolution,[11] such that $[\widehat{\mathbb{D}}, a] \in \mathcal{B}(\mathcal{H})$ for any $a \in \mathcal{A}$.*

Roughly speaking, every canonical spectral d–triple is defined by two usual spectral triples which in our case corresponds to certain h– and v–irreducible components induced by the corresponding h– and v–components of the Dirac d–operator. For such spectral h(v)–triples we can define the notion of KR^n–cycle (KR^m–cycle) and consider respective Hochschild complexes. We note that in order to define a noncommutative geometry the h– and v– components of a canonical spectral d–triples must satisfy some well defined

[10] We use the label $[cr]$ in order to avoid misunderstanding with the symbol Γ used for the connections.

[11] An operator D is of compact resolution if for any $\lambda \in sp(D)$ the operator $(D - \lambda \mathbb{I})^{-1}$ is compact, see details in [50, 29].

Conditions [17, 29] (Conditions 1 - 7, enumerated in [50], section II.4) which states: 1) the spectral dimension, being of order $1/(n+m)$ for a Dirac d–operator, and of order $1/n$ (or $1/m$) for its h– (or v)–components; 2) regularity; 3) finiteness; 4) reality; 5) representation of 1st order; 6) orientability; 7) Poincaré duality. Such conditions can be satisfied by any Dirac operators and canonical Dirac d–operators (in the second case we have to work with d–objects). [12]

Definition 15.6.26. *A spectral d–triple satisfying the mentioned seven Conditions for his h– and v–irreversible components is a real one which defines a (d–spinor) N–anholonomic noncommutative geometry defined by the data* $(\mathcal{A}, \mathcal{H}_N,\ \widehat{\mathbb{D}},\ J,\ \Gamma^{[cr]}\ )$ *and derived for the Dirac d–operator (15.60).*

For a particular case, when the N–distinguished structures are of Lagrange (Finsler) type, we can consider:

Definition 15.6.27. *A spectral d–triple satisfying the mentioned seven Conditions for his h– and v–irreversible components is a real one which defines a Lagrange, or Finsler, (spinor) noncommutative geometry defined by the data* $(\mathcal{A}, \mathcal{H}_{(SL)},\ {}_{(L)}\widehat{\mathbb{D}},\ J,\ \Gamma^{[cr]}\ )$ *and derived for the Dirac d–operator (15.61).*

In Ref. [77], we used the concept of d–algebra $\mathcal{A}_N \doteq (\ {}^{-}\mathcal{A},\ {}^{\star}\mathcal{A})$ which we introduced as a "couple" of algebras for respective h– and v–irreducible decomposition of constructions defined by the N–connection. This is possible if $\mathcal{A}_N \doteq\ {}^{-}\mathcal{A} \oplus\ {}^{\star}\mathcal{A})$, but we can consider arbitrary noncommutative associative algebras $\mathcal{A}$ if the splitting is defined by the Dirac d–operator.

15.6.3 Distance in d–spinor spaces: Main Result 3

We can select N–anholonomic and Lagrange commutative geometries from the corresponding Definitions 15.6.26 and 15.6.27 if we put respectively $\mathcal{A} \doteq C^{\infty}(\mathbf{V})$ and $\mathcal{A} \doteq C^{\infty}(V_{(L)})$ and consider real spectral d–triples. One holds:

Theorem 15.6.5. (Main Result 3) *Let* $(\mathcal{A}, \mathcal{H}_N,\ \widehat{\mathbb{D}},\ J,\ \Gamma^{[cr]}\ )$
(or $(\mathcal{A}, \mathcal{H}_{(SL)},\ {}_{(L)}\widehat{\mathbb{D}},\ J,\ \Gamma^{[cr]}\ )$*) defines a noncommutative geometry being irreducible for*

[12] We omit in this paper the details on axiomatics and related proofs for such considerations: we shall present details and proofs in our further works. Roughly speaking, we are in right to do this because the canonical d–connection and the Sasaki type d–metric for N–anholonomic spaces satisfy the bulk of properties of the metric and connection on the Riemannian space but "slightly" nonholonomically modified).

$\mathcal{A} \doteq C^{\infty}(\mathbf{V})$ (or $\mathcal{A} \doteq C^{\infty}(V_{(L)})$), where $\mathbf{V}$ (or $V_{(L)}$) is a compact, connected and oriented manifold without boundaries, of spectral dimension dim $\mathbf{V} = n+m$ (or dim $V_{(L)} = n+n$). In this case, there are satisfied the conditions:

1. *There is a unique d–metric $\mathbf{g}(\widehat{\mathbb{D}}) = (g, \ ^{\star}g)$ of type ((15.19)) on $\mathbf{V}$ (or of type (15.14) on $V_{(L)}$) with the "nonlinear" geodesic distance defined by*

$$d(u_1, u_2) = \sup_{f \in C(\mathbf{V})} \{f(u_1, u_2)/ \parallel [\mathbb{D}, f] \parallel \leq 1\} \tag{15.67}$$

   (we have to consider $f \in C(V_{(L)})$ and $\ _{(L)}\widehat{\mathbb{D}}$ if we compute $d(u_1, u_2)$ for Lagrange configurations).

2. *The N–anholonomic manifold $\mathbf{V}$ (or Lagrange space $V_{(L)}$) is a spin N–anholonmic space (or a spin Lagrange manifold) for which the operators $\mathbb{D}'$ satisfying $\mathbf{g}(\mathbb{D}') = \mathbf{g}(\widehat{\mathbb{D}})$ define an union of affine spaces identified by the d–spinor structures on $\mathbf{V}$ (we should consider the operators $\ _{(L)}\mathbb{D}'$ satisfying $\ ^{(L)}\mathbf{g}(\ _{(L)}\mathbb{D}') = \ ^{(L)}\mathbf{g}(\ _{(L)}\widehat{\mathbb{D}})$ for the space $V_{(L)}$)).*

3. *The functional $S(\mathbb{D}) \doteq \int |\mathbb{D}|^{-n-m+2}$ defines a quadratic d–form with $(n+m)$–splitting for every affine spaces which is minimal for $\widehat{\mathbb{D}} = \overleftarrow{\mathbb{D}}$ as the Dirac d–operator corresponding to the d–spin structure with the minimum proportional to the Einstein–Hilbert action constructed for the canonical d–connection with the d–scalar curvature $\overleftarrow{\mathbf{R}}$ (15.33),* [13]

$$S(\overleftarrow{\mathbb{D}}) = -\frac{n+m-2}{24} \int_{\mathbf{V}} \overleftarrow{\mathbf{R}} \ \sqrt{g} \ \sqrt{h} \ dx^1...dx^n \ \delta y^{n+1}...\delta y^{n+k}.$$

Proof. In this work, we sketch only the idea and the key points of a such Proof. The Theorem is a generalization for N–anholonomic spaces of a similar one, formulated in Ref. [17], with a detailed proof presented in [29], which seems to be a final almost generally accepted result. There are also alternative considerations, with useful details, in Refs. [63, 43]. For the Dirac d–operators, we have to start with the Proposition 15.6.3 and then to repeat all constructions from [17, 29], both on h– and v–subspaces, in N–adapted form.

[13] The integral for the usual Dirac operator related to the Levi–Civita connection D is computed: $\int |D|^{-n+2} \doteq \frac{1}{2^{[n/2]}\Omega_n} Wres|D|^{-n+2}$, where Ω_n is the integral of the volume on the sphere S^n and $Wres$ is the Wodzicki residu, see details in Theorem 7.5 [29]. On N–anholonomic manifolds, we may consider similar definitions and computations but applying N–elongated partial derivatives and differentials.

The existence of a canonical d–connection structure which is metric compatible and constructed from the coefficients of the d–metric and N–connection structure is a crucial result allowing the formulation and proof of the Main Results 1-3 of this work. Roughly speaking, if the commutative Riemannian geometry can be extracted from a noncommutative geometry, we can also generate (in a similar, but technically more sophisticate form) Finsler like geometries and generalizations. To do this, we have to consider the corresponding parametrizations of the nonholonomic frame structure, off–diagonal metrics and deformations of the linear connection structure, all constructions being adapted to the N–connection splitting. If a fixed d–connection satisfies the metricity conditions, the resulting Lagrange–Finsler geometry belongs to a class of nonholonomic Riemann–Cartan geometries, which (in their turns) are equivalents, related by nonholonomic maps, of Riemannian spaces, see [80, 82]. However, it is not yet clear how to perform a such general proof for nonmetric d–connections (of Berwald or Chern type). We shall present the technical details of such considerations in our further works.

Finally, we emphasize that for the Main Result 3 there is the possibility to elaborate an alternative proof (like for the Main Result 2) by verifying that the basic formulas proved for the Riemannian geometry hold true on N–anholonomic manifolds by a corresponding substitution of the N–elongated differential and partial derivatives operators acting on canonical d–connections and d–metrics. All such constructions are elaborated in N–adapted form by preserving the respective h- and v–irreducible decompositions. □ □

Finally, we can formulate three important conclusions:

Conclusion 15.6.3. *The formula (15.67) defines the distance in a manner as to be satisfied all necessary properties (finitenes, positivity conditions, ...) discussed in details in Ref. [29]. It allows to generalize the constructions for discrete spaces with anisotropies and to consider anisotropic fluctuations of noncommutative geometries [50, 51] (of Finsler type, and more general ones, we omit such constructions in this work). For the nonholonomic configurations we have to work with canonical d–connection and d–metric structures.*

Following the N–connection formalism originally elaborated in the framework of Finsler geometry, we may state:

Conclusion 15.6.4. *In the particular case of the canonical N–connection, d–connection and d–metrics defined by a regular Lagrangian, it is possible a noncommutative geometrization of Lagrange mechanics related to corresponding classes of noncommutative Lagrange–Finsler geometry.*

Such geometric methods have a number of applications in modern gravity:

Conclusion 15.6.5. *By anholonomic frame transforms, we can generate noncommutative Riemann–Cartan and Lagrange–Finsler spaces, in particular exact solutions of the Einstein equations with noncommutative variables* [14]*, by considering N–anholonomic deformations of the Dirac operator.*

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The work summarizes the results communicated in a series of Lectures and Seminars:

1. S. Vacaru, A Survey of (Non) Commutative Lagrange and Finsler Geometry, lecture 2: Noncommutative Lagrange and Finsler Geometry. Inst. Sup. Tecnico, Dep. Math., Lisboa, Portugal, May 19, 2004 (host: P. Almeida).

2. S. Vacaru, A Survey of (Non) Commutative Lagrange and Finsler Geometry, lecture 1: Commutative Lagrange and Finsler Spaces and Spinors. Inst. Sup. Tecnico, Dep. Math., Lisboa, Portugal, May 19, 2004 (host: P. Almeida).

3. S. Vacaru, Geometric Models in Mechanics and Field Theory, lecture at the Dep. Mathematics, University of Cantabria, Santander, Spain, March 16, 2004 (host: F. Etayo).

4. S. Vacaru, Noncommutative Symmetries Associated to the Lagrange and Hamilton Geometry, seminar at the Instituto de Matematicas y Fisica Fundamental, Consejo Superior de Investigationes, Ministerio de Ciencia y Tecnologia, Madrid, Spain, March 10, 2004 (host: M. de Leon).

5. S. Vacaru, Commutative and Noncommutative Gauge Models, lecture at the Department of Experimental Sciences, University of Huelva, Spain, March 5, 2004 (host: M.E. Gomez).

6. S. Vacaru, Noncommutative Finsler Geometry, Gauge Fields and Gravity, seminar at the Dep. Theoretical Physics, University of Zaragoza, Spain, November 19, 2003 (host: L. Boya).

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[14]see examples in Refs. [79, 85, 83, 82, 78]

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