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Adina BALMUŞ

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BIHARMONIC MAPS AND SUBMANIFOLDS

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To my mother and brother, Elena and Sorin-Bogdan,
in appreciation of their love and support

Abstract

Biharmonic maps generalize the notion of harmonic map and are defined as critical points of the bienergy functional.
This work is devoted to the study of biharmonic maps and submanifolds.
The main results are:

- several new methods, inspired by the Baird-Kamissoko method, for constructing proper biharmonic maps starting with harmonic maps;
- a geometric approach to the study of the proper biharmonic curves of the 3-dimensional unit Euclidean sphere $\mathbb{S}^3$, and the study of biharmonic curves on 3-dimensional Berger spheres;
- the study of the type, in the sense of B-Y. Chen, of compact proper biharmonic submanifolds with constant mean curvature in spheres;
- the complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures in space forms;
- the non-existence of proper biharmonic hypersurfaces in $\mathbb{R}^4$ and $\mathbb{H}^4$;
- the classification of compact proper biharmonic hypersurfaces of $\mathbb{S}^4$;
- some rigidity properties for pseudo-umbilical biharmonic submanifolds of codimension 2 and for biharmonic surfaces with parallel mean curvature vector field in $\mathbb{S}^n$;
- the characterization and classification of proper biharmonic products of spheres in spheres;

- the characterization of submanifolds of the Euclidean space with biharmonic Gauss map;
- the construction of examples of hypersurfaces of the Euclidean space with biharmonic Gauss map.

Abstract

Le applicazioni biarmoniche generalizzano la nozione di applicazione armonica e sono definite come punti critici del funzionale bienergia.
Questa tesi è dedicata allo studio delle applicazioni e delle sottovarietà biarmoniche.
I risultati principali sono:

- la determinazione, ispirandoci al metodo di Baird-Kamissoko, di nuovi metodi per la costruzione di applicazioni biarmoniche proprie a partire da applicazioni armoniche;
- un'interpretazione geometrica delle curve biarmoniche nella sfera euclidea 3-dimensionale e la determinazione delle curve biarmoniche delle sfere 3-dimensionali di Berger;
- lo studio del tipo, nel senso di B-Y. Chen, delle sottovarietà biarmoniche compatte con curvatura media costante;
- la classificazione completa delle ipersuperfici biarmoniche con al più due curvature principali distinte nelle forme spaziali;
- la non-esistenza delle ipersuperfici biarmoniche proprie in $\mathbb{R}^4$ e $\mathbb{H}^4$;
- la classificazione delle ipersuperfici biarmoniche proprie di $\mathbb{S}^4$;
- alcune proprietà di rigidità per le sottovarietà biarmoniche pseudo-ombelicali di codimensione 2 e per le superfici biarmoniche con curvatura media parallela in $\mathbb{S}^n$;
- la caratterizzazione e classificazione delle sottovarietà biarmoniche proprie di tipo prodotto di sfere in sfere;
- la caratterizzazione dello sottovarietà dello spazio euclideo con applicazione di Gauss biarmonica;
- la costruzione di esempi notevoli di ipersuperfici con applicazione di Gauss biarmonica.

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Chapter 1

Introduction

The goal of this introduction is, firstly, to present some ideas that encouraged the study of the geometry of biharmonic maps and, secondly, to briefly describe the new results gathered in the present thesis.

Denote by $C^\infty(M,N)$ the space of smooth maps $\varphi : (M,g) \to (N,h)$ between two Riemannian manifolds. A map $\varphi \in C^\infty(M,N)$ is called *harmonic* if it is a critical point of the *energy* functional

$$E : C^\infty(M,N) \to \mathbb{R}, \quad E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 \, v_g,$$

and is characterized by the vanishing of the tension field $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$. If we consider $\operatorname{Imm}(M,N)$ to be the space of Riemannian immersions in (N,h), then a Riemannian immersion $\varphi : (M,\varphi^*h) \to (N,h)$ is called *minimal* if it is a critical point of the *volume* functional

$$V : \operatorname{Imm}(M,N) \to \mathbb{R}, \quad V(\varphi) = \frac{1}{2}\int_M v_{\varphi^*h},$$

and the associated Euler-Lagrange equation is $H = 0$, where H is the mean curvature vector field. Although they are conceptually different, there is a strong connection between these two variational problems: if $\varphi : (M,g) \to (N,h)$ is a Riemannian immersion, then it is a critical point of the energy in $C^\infty(M,N)$ if and only if it is a minimal immersion [53].

One can generalize harmonic maps and minimal immersions by considering the functionals obtained by integrating the squared norm of the tension field or of the mean curvature vector field, respectively. More precisely, *biharmonic maps* are the critical points of the *bienergy* functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g \,,$$

and *Willmore immersions* are the critical points of the Willmore functional

$$W : \mathrm{Imm}(M^2, N) \to \mathbb{R}, \quad W(\varphi) = \int_{M^2} \left(|H|^2 + K\right) v_{\varphi^* h} \,,$$

where K is the sectional curvature of (N, h) restricted to the image of M^2. Although the above variational problems produce natural generalizations of harmonic maps and minimal immersions, biharmonic Riemannian immersions do not recover Willmore immersions, not even when the ambient space is $\mathbb{R}^n$. In $\mathbb{R}^3$ there exist no proper (non-minimal) biharmonic surfaces (see [48]), while all the spheres are Willmore surfaces. Also, the unique compact proper biharmonic surface of $\mathbb{S}^3$ is the hypersphere $\mathbb{S}^2(\frac{1}{\sqrt{2}})$, while all the hyperspheres $\mathbb{S}^2(a)$, $a \in (0, 1)$, are non-trivial Willmore surfaces. An example of proper biharmonic immersion which is not a Willmore immersion was provided by T. Sasahara in the class of Legendre immersions in $\mathbb{S}^5$ (see [106]). As we shall see in Section 6.2, the biharmonic equation for Riemannian immersions in space forms is equivalent to the vanishing of both a tangential and a normal component, while the Willmore equation involves only a normal component. We also point out that the submanifolds with vanishing normal component of the bitension field, which are called biminimal (see [79]), do not recover Willmore immersions.

In his studies on finite type submanifolds (see [43]), B.Y. Chen defined biharmonic submanifolds $M \subset \mathbb{R}^n$ of the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian. By considering the definition of biharmonic maps for Riemannian immersions into the Euclidean space one recovers the notion of biharmonic submanifolds in the sense of B-Y. Chen. Thus biharmonic Riemannian immersions can also be thought of as a generalization of Chen's biharmonic submanifolds. All the results obtained by B-Y. Chen and his collaborators on proper biharmonic submanifolds in Euclidean spaces are non-existence results. Nevertheless, their techniques were adapted and led to classification results for proper biharmonic submanifolds in spheres.

Biharmonic maps have been extensively studied in the last decade and there are two main research directions. On one hand, in differential geometry, a special attention has been payed to the construction of examples and classification results (see, for example, [4, 7, 12, 14, 15, 29, 30, 32, 33, 46, 48, 49, 67, 123, 124]). On the other hand, from the analytic point of view, biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE and the study of their regularity is nowadays a well-developed field (see, for example, [40, 74, 75, 110, 116, 117]).

Complementary results can be found in [9, 10, 21, 35, 66, 89, 114].

We mention some other reasons that encourage the study of biharmonic maps.

The theory of biharmonic maps is an old and rich subject, initially studied due to its implications in the theory of elasticity and fluid mechanics. G.B. Airy and J.C. Maxwell

were the first to study and express plane elastic problems in terms of the biharmonic equation (see [2, 84]). Later on, the theory evolved with the study of polyharmonic functions developed by E. Almansi, T. Levi-Civita, M. Nicolaescu. Biharmonic and polyharmonic functions on Riemannian manifolds were studied by R. Caddeo and L. Vanheke [28, 34], L. Sario et all (see [105]) and others.

The differential geometric aspect of biharmonic submanifolds was also studied in the semi-Riemannian case (see, for example, [46]).

The variational problem associated by considering, for a fixed map, the bienergy functional defined on the set of Riemannian metrics on the domain gave rise to the biharmonic stress-energy tensor (see [80]). This proved to be useful for obtaining new examples of proper biharmonic maps and for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds. There also exists an interesting connection between the biharmonic stress-energy tensor associated to the inclusion of a submanifold in the Euclidean space and the harmonic stress-energy tensor of the corresponding Gauss map.

Proper biharmonic surfaces in the 3-dimensional sphere are also II-minimal surfaces, i.e. critical points of the volume functional associated to the second fundamental form (see [64]).

It was proved that there exists no harmonic map from $\mathbb{T}^2$ to $\mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1 ([54]). The biharmonic maps are expected to exist where harmonic maps do not.

The interest in the theory of biharmonic maps crossed the border of differential geometry and analysis of PDE's. In computational geometry, more precisely in the field of boundary based surface design, the biharmonic Bézier surfaces are studied (see [71, 87, 88]).

The present thesis mainly deals with: the construction of new classes of biharmonic maps; the study and classification of proper biharmonic submanifolds in space forms; the study of submanifolds with proper biharmonic Gauss map in Euclidean spaces.

The first chapter is intended to establish the notations and to recall some basic results which will be used throughout the entire thesis. We focus on calculus on vector bundles and harmonic maps and we closely follow the monographs [8, 50, 115].

In the second chapter we introduce the notion of biharmonic map as it was suggested by J. Eells and J.H. Sampson. We recall the classical biharmonic problem and some of its applications in continuous media mechanics. We present fundamental results and examples of proper biharmonic maps and submanifolds (see [123]).

The third chapter contains new methods for constructing proper biharmonic maps by using harmonic ones. We present the Baird-Kamissoko method (see [7]) and new results that are gathered in [11] and [14].

A natural way for constructing proper biharmonic maps is the following: given a harmonic map $\varphi : (M, g) \to (N, h)$ one can conformally change the metric g, or h, in order to render φ proper biharmonic. This problem represents the object of the first section in the present chapter. The behavior of the bitension field, under the conformal change of the domain metric, was studied by P. Baird and D. Kamissoko in [7] and we present here the main results. A metric $\widetilde{g}$, which renders the identity map $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ biharmonic, was called a *biharmonic metric with respect to* g and it was proved that on Einstein manifolds the conformally equivalent biharmonic metrics are defined only by means of isoparametric functions. We study the biharmonicity of the identity after a conformal change of the codomain metric. The results are similar to those obtained in [7], but we underline the fact that, in general, the biharmonicity of $\overline{\mathbf{1}} : (M, g) \to (M, e^{2\rho}g)$ is not equivalent to the biharmonicity of $\widetilde{\mathbf{1}} : (M, e^{2\rho}g) \to (M, g)$ (see Remark 4.22).

The results gathered in the second section extend the idea of studying the effect of conformal changes of metric to the biharmonic equation by considering the situation of warped products. The first problem consists in the study of the biharmonicity of the inclusion of a Riemannian manifold N into the warped product $M \times_{f^2} N$ and we obtain

Corollary 4.29 ([14]). *The inclusion map $\mathbf{i}_{x_0} : N \to M \times_{f^2} N$ is a proper biharmonic map if and only if x_0 is not a critical point for f^2, but is a critical point for $|\operatorname{grad} f^2|^2$.*

With this setting we obtain new examples of proper biharmonic maps and recover some of the examples first obtained in [29] and [32].

Affine functions play an important role in our study:

Corollary 4.31 ([14]). *Let (M, g) be a Riemannian manifold with a positive non-trivial affine function f^2 and let (N, h) be an arbitrary Riemannian manifold. Then any inclusion $\mathbf{i}_x : N \to M \times_{f^2} N$, $x \in M$, is a proper biharmonic map.*

We then consider the product of two harmonic maps $\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$. By warping the metric on the domain or codomain we lose the harmonicity; nevertheless, under certain conditions on the warping function the product map remains biharmonic. In the case that the product map is the identity map $\overline{\mathbf{1}} : M \times_{f^2} N \to M \times N$ we shall call the warping function, which is a solution of the above problem, a *biharmonic warping function*. We characterize the biharmonicity:

Theorem 4.37 ([14]). *Let $\psi : N \to N$ be a harmonic map and let $f \in C^\infty(M)$ be a positive function. Then $\overline{\phi} = \overline{\mathbf{1}_M \times \psi} : M \times_{f^2} N \to M \times N$ is a proper biharmonic*

map if and only if f is a non-constant solution of

$$\operatorname{trace}_g \nabla^2 \operatorname{grad} \ln f + \operatorname{Ricci}^M(\operatorname{grad} \ln f) + \frac{n}{2} \operatorname{grad}(|\operatorname{grad} \ln f|^2) = 0.$$

In the case that M is an Einstein manifold we show that the isoparametric functions provide examples of biharmonic warping ones:

Proposition 4.44 ([14]). *Let M be an Einstein space. If $\rho \in C^\infty(M)$ is an isoparametric function, then it admits away from its critical points a local reparameterization f which is a biharmonic warping function.*

For the projection $\overline{\pi} : M \times_{f^2} N \to M$ we obtain

Proposition 4.47 ([14]). *The projection $\overline{\pi} : M \times_{f^2} N \to M$ is proper biharmonic if and only if f is a biharmonic warping function.*

We also give the complete classification of the biharmonic warping functions when $M = \mathbb{R}$ (see Example 4.46).

Similar results are obtained when the codomain metric is warped and ψ is harmonic with constant energy density.

In the last section we use the warped product setting to study axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to an n-dimensional space form. We get a general characterization result

Theorem 4.57 ([14]). *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k \geq 0$. Then $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is biharmonic if and only if ρ is a solution of*

$$F'' + \frac{m-1}{t} F' - \frac{2k}{t^2}\big(f'^2(\rho) + f(\rho) f''(\rho)\big) F = 0,$$

where

$$F = \rho'' + \frac{m-1}{t}\rho' - \frac{2k}{t^2} f(\rho) f'(\rho).$$

Then we discuss several examples and, when the target manifold is $\mathbb{R}^n \setminus \{0\}$, we give the complete classification of biharmonic axially symmetric maps. From the former classification we also deduce that the generalised Kelvin transformation $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(y) = y/|y|^\ell$ is a proper biharmonic map if and only if $m = \ell + 2$.

The goal of the fourth chapter is to give a geometric approach for the study of the proper biharmonic curves of the 3-dimensional unit Euclidean sphere $\mathbb{S}^3$, since it was proved in [30] that the proper biharmonic curves of $\mathbb{S}^n$, $n \geq 4$, are essentially the ones of $\mathbb{S}^3$ composed with the totally geodesic embedding of $\mathbb{S}^3$ in $\mathbb{S}^n$. We also study the biharmonic curves on 3-dimensional spheres endowed with distorted metrics called Berger metrics. The new results of this chapter can be found in [12] and [19].

The first section contains general characterization results for biharmonic curves in Riemannian manifolds, which were obtained in [31] by expressing the biharmonic equation in terms of the Frenet frame associated to the curve.

In the next section we present the classification result for proper biharmonic curves of $\mathbb{S}^n$, and by using the stereographic projection we obtain the image in $\mathbb{R}^3$ of the biharmonic curves of $\mathbb{S}^3$.

The last section deals with proper biharmonic curves in Berger spheres. There are several ways to introduce the 3-dimensional Berger metrics; here we choose to define them as distortions of the canonical metric on $\mathbb{S}^3$ constructed by performing a biconformal change of metric with respect to the Hopf fibration. The study of the behavior of biharmonic curves under these distortions comes as a natural question. In this study we use the same techniques used in [33], for the case of the Heisenberg space, which have been afterwards developed for the general case of biharmonic curves in $SO(2)$-isotropic 3-dimensional manifolds (see [31]). We obtain the explicit parametric equations of proper biharmonic curves in Berger spheres (see Theorem 5.19).

If we restrict our interest only to the Legendre proper biharmonic curves we get

Corollary 5.20. *Let $\mathbb{S}^3_\varepsilon$ be a Berger sphere, $\varepsilon \in (0,1)$. Then the parametric equations of all proper biharmonic Legendre curves, parametrized by arc length, of $\mathbb{S}^3_\varepsilon$ are*

$$\gamma(s) = \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\exp(-\mathrm{i}\alpha_1 s)e_1 + \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\exp(\mathrm{i}\alpha_1 s)e_3,$$

where e_1, e_3 are unit orthogonal vectors of $\mathbb{R}^4$ with e_3 orthogonal to Je_1,

$$\alpha_{1,2} = \sqrt{3-2\varepsilon^2 \pm 2\sqrt{(1-\varepsilon^2)(2-\varepsilon^2)}},$$

and J is the canonical complex structure on $\mathbb{R}^4$.

All the known results on proper biharmonic submanifolds in non-positively curved spaces are non-existence results. For example, it was proved (see [45]) that a spherical submanifold $M \subset \mathbb{S}^{n-1}$ cannot be biharmonic in $\mathbb{R}^n$. Also there exist no proper biharmonic curves, submanifolds of finite type, submanifolds of constant mean curvature, pseudo-umbilical submanifolds of dimension $m \neq 4$, hypersurfaces with at most two distinct principal curvatures, conformally flat hypersurfaces in the Euclidean space or proper biharmonic hypersurfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ ([48, 49, 63]). Similar results were obtained (see [30]) for curves, submanifolds of constant mean curvature, pseudo-umbilical submanifolds of dimension $m \neq 4$ in the hyperbolic space or surfaces in $\mathbb{H}^3$.

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

Generalized Chen Conjecture. *Biharmonic submanifolds of a non-positive sectional curvature manifold are minimal,*

encouraged the study of proper biharmonic submanifolds in spheres and other curved spaces [29, 30, 55, 67, 86, 98].

The attempt to classify the biharmonic submanifolds in space forms was initiated in [48] and [29]. The first achievement towards the classification problem is represented by the complete classification of proper biharmonic submanifolds of the 3-dimensional unit Euclidean sphere, obtained in [29]. Then, inspired by the 3-dimensional case, two methods for constructing proper biharmonic submanifolds in $\mathbb{S}^n$ were given in [30].

Although important results and examples were obtained, the classification of proper biharmonic submanifolds in spheres is still an open problem. The fifth chapter is fully devoted to the classification of proper biharmonic submanifolds in space forms and the new results presented here are contained in [15], [16], [17].

In the preliminary section we include definitions and results on some special classes of submanifolds in space forms.

The second section contains fundamental characterization results on proper biharmonic submanifolds of space forms and, in particular, of the Euclidean sphere. We present the biharmonic equation for this particular case, non-existence results for non-positively curved space forms and the main examples of biharmonic submanifolds in spheres. The section also contains some basic information on finite type Euclidean submanifolds. A partial classification result for biharmonic submanifolds in spheres was given in [94]. It was proved that proper biharmonic submanifolds M of constant mean curvature $|H|$ in $\mathbb{S}^n$ satisfy $|H| \in (0,1]$, with $|H| = 1$ if and only if M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. Taking this further, by using this result, we study the type of compact proper biharmonic submanifolds of constant mean curvature in $\mathbb{S}^n$ and prove that, depending on the value of the mean curvature, they are of 1-type or of 2-type as submanifolds of $\mathbb{R}^{n+1}$,

Theorem 6.25 ([15]). *Let M^m be a compact constant mean curvature, $|H|^2 = k$, submanifold in $\mathbb{S}^n$. Then M is proper biharmonic if and only if*

either

(i) $|H|^2 = 1$ *and M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$,*

or

(ii) $|H|^2 = k \in (0,1)$ *and M is a mass-symmetric 2-type submanifold of $\mathbb{R}^{n+1}$ with the eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.*

The third section is devoted to the study of proper biharmonic hypersurfaces with at most two distinct principal curvatures in space forms. We prove

Theorem 6.27 ([15]). *Let M be a hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$. If M is proper biharmonic in $\mathbb{E}^{m+1}(c)$, then it has constant*

mean curvature.

We obtain the full classification of such hypersurfaces in spheres:

Theorem 6.30 ([15]). *Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

A similar classification is obtained for conformally flat biharmonic hypersurfaces in spheres. In contrast, for the hyperbolic space $\mathbb{H}^{m+1}$ we prove a non-existence result.

The problem of the biharmonic hypersurfaces with at most 3 distinct principal curvatures is also investigated. A non-existence result is obtained for hypersurfaces with constant mean curvature and exactly three distinct principal curvatures,

Theorem 6.35 ([16]). *There exist no compact proper biharmonic hypersurfaces of constant mean curvature and three distinct curvatures in the unit Euclidean sphere.*

and the full classification of compact hypersurfaces in $\mathbb{S}^4$ is given,

Theorem 6.38 ([16]). *The only proper biharmonic compact hypersurfaces of $\mathbb{S}^4$ are the hypersphere $\mathbb{S}^3(\frac{1}{\sqrt{2}})$ and the torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.*

We also recover the non-existence result for proper biharmonic hypersurfaces in $\mathbb{R}^4$ and prove that it also holds for hypersurfaces of $\mathbb{H}^4$.

In the fourth section we prove that the pseudo-umbilical biharmonic submanifolds in spheres have constant mean curvature and we give an estimate for their scalar curvature. Then we classify the proper biharmonic pseudo-umbilical submanifolds of codimension 2,

Theorem 6.42 ([15]). *Let M^m be a pseudo-umbilical submanifold of $\mathbb{S}^{m+2}$, $m \neq 4$. Then M is proper biharmonic if and only if it is minimal in $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$.*

We also investigate surfaces with parallel mean curvature vector field in $\mathbb{S}^n$,

Theorem 6.45 ([15]). *Let M^2 be a proper biharmonic surface with parallel mean curvature vector field in $\mathbb{S}^n$. Then M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.*

The last section contains new examples of proper biharmonic submanifolds in spheres. We analyze biharmonic products of spheres, proving

Theorem 6.47 ([17]). *Consider $\mathcal{T} = \mathbb{S}^{n_1}(a_1) \times \mathbb{S}^{n_2}(a_2) \times \ldots \times \mathbb{S}^{n_r}(a_r) \subset \mathbb{S}^{m+r-1} \subset \mathbb{R}^{m+r}$, where $m = \sum_{k=1}^{r} n_k$ and $\sum_{k=1}^{r} a_k^2 = 1$. Then*

(i) *$\mathcal{T}$ is minimal in $\mathbb{S}^{m+r-1}$ if and only if*

$$\frac{a_k^2}{n_k} = \frac{1}{m},$$

for all $k = 1, \ldots, r$.

(ii) *$\mathcal{T}$ is proper biharmonic in $\mathbb{S}^{m+r-1}$ if and only if there exists $p = 1, \ldots, r-1$ such*

that

$$\frac{a_1^2}{n_1} = \ldots = \frac{a_p^2}{n_p} = \frac{1}{2(n_1 + \ldots + n_p)} \neq \frac{1}{m}$$

and

$$\frac{a_{p+1}^2}{n_{p+1}} = \ldots = \frac{a_r^2}{n_r} = \frac{1}{2(n_{p+1} + \ldots + n_r)} \neq \frac{1}{m}.$$

Moreover, we study the geometric properties of two classes of biharmonic submanifolds of $\mathbb{S}^5$.

Ever since the characterization result obtained by E. Ruh and J. Vilms in [101] and the remarkable link with constant mean curvature hypersurfaces the study of submanifolds with associated harmonic Gauss map in Euclidean spaces has been a classical problem in harmonic maps theory. The sixth chapter proposes a generalization of this problem: the characterization of submanifolds with proper biharmonic Gauss map in Euclidean spaces. The new results of this chapter are gathered in [18].

In the first section we recall the definition and the differential structure of the Grassmannian (see [62]). We use the identification of the Grassmannian as a symmetric space in order to compute its curvature (see [99]).

We then recall the definition of the Gauss map associated to a submanifold in an Euclidean space and compute its differential and its tension field (see [50, 51, 101]).

In the third section we characterize the biharmonicity of the Gauss map associated to a submanifold in an Euclidean space,

Theorem 7.3 ([18]). *The Gauss map associated to an m-dimensional orientable submanifold M of $\mathbb{R}^{m+n}$ is proper biharmonic if and only if $\nabla^\perp H \neq 0$ and*

$$\begin{aligned}\nabla^\perp_X \Delta^\perp H - m\nabla^\perp_{A_H(X)} H + \operatorname{trace} B\big(2A_{\nabla^\perp_{(\cdot)}H}(X) - A_{\nabla^\perp_X H}(\cdot\,),\cdot\big)& \\ -2\operatorname{trace} R^\perp(\cdot\,, X)\nabla^\perp_\cdot H - \operatorname{trace}(\nabla^\perp_\cdot R^\perp)(\cdot\,, X)H &= 0,\end{aligned}$$

for all $X \in C(TM)$, where A denotes the Weingarten operator and H the mean curvature vector field of M in $\mathbb{R}^{m+n}$.

The biharmonic equation for the Gauss map simplifies for the case of hypersurfaces,

Theorem 7.4 ([18]). *The Gauss map of a nowhere zero mean curvature hypersurface M^m in $\mathbb{R}^{m+1}$ is proper biharmonic if and only if* $\operatorname{grad} f \neq 0$ *and*

$$\Delta \operatorname{grad} f + A^2(\operatorname{grad} f) - |A|^2 \operatorname{grad} f = 0,$$

where Δ denotes the rough Laplacian on $C(TM)$ and f and A denote the mean curvature function and, respectively, the shape operator of M in $\mathbb{R}^{m+1}$.

The last section is devoted to the examples of hypercones with biharmonic Gauss map in the Euclidean space. We study the hypercones generated by constant mean curvature hypersurfaces in spheres and, in particular, by the hyperspheres,

Theorem 7.7 ([18]). *Let $\overline{M}$ be an orientable constant non-zero mean curvature hypersurface of $\mathbb{S}^{m+1}$. The Gauss map associated to the hypercone $(0,\infty)\times_{t^2}\overline{M}$ is proper biharmonic if and only if $m>2$ and $|\overline{A}|^2=3(m-2)$, where $\overline{A}$ is the shape operator of $\overline{M}$ in $\mathbb{S}^{m+1}$.*

and

Proposition 7.11 ([18]). *Consider the hypercone $(0,\infty)\times_{t^2}\mathbb{S}^m(a)$ in $\mathbb{R}^{m+2}$, $a\in(0,1)$. Its associated Gauss map is proper biharmonic if and only if $m>2$ and $a=\sqrt{\frac{m}{4m-6}}$.*

Finally, we obtain a non-existence result for cones with biharmonic Gauss map in $\mathbb{R}^3$,

Theorem 7.8 ([18]). *There exist no cones with proper biharmonic Gauss map in $\mathbb{R}^3$.*

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Chapter 2

Preliminaries

This chapter is intended to establish the notations and to recall some basic results which will be used throughout the entire thesis. We focus on calculus on vector bundles and harmonic maps and we closely follow the monographs [8, 50, 115].

The fundamental notions of differentiable and Riemannian geometry, as differentiable manifold, vector bundle, linear connection, Riemannian manifold, Riemannian immersion and submersion, etc, are considered to be known (see, for example, [36, 73, 99, 102]).

Throughout this work we shall place ourselves in the C^∞ category, i.e. all manifolds, connections, metrics, maps are assumed to be C^∞, unless otherwise stated. By (M^m, g) we shall denote an m-dimensional connected manifold, endowed with a Riemannian metric g and by v_g we shall denote its volume form. We shall use the Einstein summation convention for repeated indices.

2.1 Operators on vector bundles

Let $\xi : V \to M$ be a smooth vector bundle of finite rank over M.

We shall denote by $C(\xi)$ or $C(V)$ the vector space of smooth sections of V, i.e. maps $\sigma : M \to V$ satisfying $\xi \circ \sigma = \mathbf{1}_M$, the identity map of M.

Remark 2.1.

(i) The real vector space $C(V)$ is infinite-dimensional.

(ii) If TM is the tangent bundle of M, then the space of smooth sections $C(TM)$ is the Lie algebra of vector fields on M, where the Lie bracket is defined by $[X,Y]f = XY(f) - YX(f)$, for all $X, Y \in C(TM)$ and $f \in C^\infty(M)$, where, as usual, we denote by $C^\infty(M)$ the space of smooth functions on M.

If $\xi : V \to M$ and $\eta : W \to N$ are two vector bundles and $\varphi : M \to N$ is a smooth map, we shall denote by

(i) V^* the dual bundle of V,

(ii) $V \oplus W$ the direct (Whitney) sum of V and W,

(iii) $V \otimes W$ the tensor product of V and W,

(iv) $\otimes^k V$ the tensor power of V,

(v) $\wedge^k V$ the exterior power of V, $\odot^k V$ the symmetric power of V,

(vi) $\varphi^{-1}W$ the *pull-back bundle of W over M induced by φ*, the bundle whose fiber over $p \in M$ is $W_{\varphi(p)}$, the fibre of W over $\varphi(p)$. We have the natural commutative diagram

$$\begin{array}{ccc} \varphi^{-1}W & \longrightarrow & W \\ \downarrow & & \downarrow \eta \\ M & \xrightarrow{\varphi} & N \end{array}$$

Definition 2.2.

(i) A *Riemannian metric* on a vector bundle V is a section $a \in C(\odot^2 V^*)$ which induces on each fiber a positive definite inner product. We shall use the notation $a(\sigma, \rho) = \langle \sigma, \rho \rangle$, for $\sigma, \rho \in C(V)$.

(ii) A *linear connection* on a vector bundle $\xi : V \to M$ is a bilinear map ∇ on spaces of sections,

$$\nabla : C(TM) \times C(V) \to C(V), \qquad (X, \sigma) \to \nabla_X \sigma,$$

such that, for all $f \in C^\infty(M)$,

$$\nabla_{fX}\sigma = f\nabla_X \sigma, \qquad \nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X \sigma.$$

Definition 2.3. A *Riemannian structure* on a vector bundle V is a pair (∇, a), where a is a Riemannian metric, ∇ is a connection, and $\nabla a = 0$, i.e.

$$\begin{aligned} (\nabla a)(X, \sigma, \rho) &= (\nabla_X a)(\sigma, \rho) = X\langle \sigma, \rho \rangle - \langle \nabla_X \sigma, \rho \rangle - \langle \sigma, \nabla_X \rho \rangle \\ &= 0, \end{aligned}$$

for all $X \in C(TM)$, $\sigma, \rho \in C(V)$.

Definition 2.4. The *curvature operator of a connection* ∇ on a vector bundle $\xi : V \to M$ is the map

$$R : \wedge^2 C(TM) \otimes C(V) \to C(V),$$

defined by

$$R(X,Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

Given two metrics a, b and two connection ∇^V, ∇^W on V and W, respectively, we can define the induced metrics and connections in the following cases:

(i) For V^*, the metric is defined by

$$\langle \alpha, \beta \rangle_{V^*} = \langle \alpha^\sharp, \beta^\sharp \rangle_V,$$

where the musical isomorphisms between the fibres V_p and V_p^* are used. Indeed, we define $\flat : V_p \to V_p^*$ by

$$\sigma^\flat(\rho) = \langle \sigma, \rho \rangle, \qquad \sigma, \rho \in V_p$$

and $\sharp : V_p^* \to V_p$ by $\sharp = \flat^{-1}$. The connection on V^* is defined by

$$(\nabla_X^* \theta)(\sigma) = X(\theta(\sigma)) - \theta(\nabla_X \sigma),$$

and its curvature operator is

$$(R^*(X,Y)\theta)(\sigma) = -\theta(R(X,Y)\sigma),$$

for all $X, Y \in C(TM)$, $\theta \in C(V^*)$, $\sigma \in C(V)$.

(ii) The metric on $V \oplus W$ is defined by

$$\langle \sigma \oplus \lambda, \rho \oplus \mu \rangle = \langle \sigma, \rho \rangle + \langle \lambda, \mu \rangle,$$

and the connection by

$$\nabla_X(\sigma \oplus \lambda) = \nabla_X^V \sigma \oplus \nabla_X^W \lambda,$$

for $\sigma, \rho \in C(V)$ and $\lambda, \mu \in C(W)$. The curvature operator is in this case given by

$$R(X,Y)(\sigma \oplus \lambda) = R^V(X,Y)\sigma \oplus R^W(X,Y)\lambda.$$

(iii) For $V \otimes W$, the metric has the expression

$$\langle \sigma \otimes \lambda, \rho \otimes \mu \rangle = \langle \sigma, \rho \rangle \langle \lambda, \mu \rangle,$$

the connection is given by

$$\nabla_X(\sigma \otimes \lambda) = (\nabla_X^V \sigma) \otimes \lambda + \sigma \otimes (\nabla_X^W \lambda),$$

and its curvature operator is

$$R(X,Y)(\sigma\otimes\lambda) = (R^V(X,Y)\sigma)\otimes\lambda + \sigma\otimes R^W(X,Y)\lambda.$$

The above product and connection induce products and connections on $\otimes^k V$, $\wedge^k V$ and $\odot^k V$.

(iv) For a given map $\varphi : M \to N$ and a vector bundle $\eta : W \to N$ with metric b, we can identify $\sigma, \rho \in (\varphi^{-1}W)_p$ with $\sigma, \rho \in W_{\varphi(p)}$ and define

$$\langle\sigma,\rho\rangle_{\varphi^{-1}W} = \langle\sigma,\rho\rangle_W.$$

Moreover, it can be proved that there exists a unique linear connection on $\varphi^{-1}W$, called the *pull-back connection*, such that for all $p \in M$, $X \in T_pM$ and $\lambda \in C(W)$,

$$\nabla^{\varphi^{-1}W}_X(\varphi^*\lambda) = \nabla^W_{d\varphi_p(X)}\lambda,$$

where $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is the differential of φ, and $\varphi^*\lambda = \lambda\circ\varphi \in C(\varphi^{-1}W)$. The curvature operator of the induced connection has the expression

$$R_p(X,Y)\rho = R^W_{\varphi(p)}(d\varphi_p(X), d\varphi_p(Y))\rho(p),$$

where $\rho \in C(\varphi^{-1}W)$.

Note that if (∇^V, a) and (∇^W, b) are Riemannian structures on V and W, respectively, then the induced metrics and connections on the vector bundles under consideration give Riemannian structures.

Example 2.5 (The tangent bundle TM)**.** If g is a Riemannian metric on TM, the fundamental theorem of Riemannian geometry asserts that there exists a unique connection on TM, the *Levi-Civita connection*, satisfying

$$\nabla g = 0 \quad \text{and} \quad T = 0,$$

where, in general, the *torsion T of a connection* ∇ on TM is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \forall X, Y \in C(TM).$$

We recall that the Levi-Civita connection is characterized by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) \;=\; & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \end{aligned}$$

for all $X, Y, Z \in C(TM)$.

Throughout this thesis we shall always endow the tangent bundle of a Riemannian manifold (M, g) with the Levi-Civita connection.

Note that, in general, for a given vector bundle, the torsion of a connection could not be defined and a metric does not determine an unique Riemannian structure.

The *sectional curvature* of a 2-plane P of T_pM is defined by

$$\mathrm{Riem}_p(P) = \langle R(Y,X)X, Y\rangle_p,$$

where $\{X, Y\}$ is an arbitrary orthonormal basis of P.

The *Ricci tensor* at p is the map defined by

$$\mathrm{Ricci}_p : T_pM \times T_pM \to \mathbb{R}, \quad \mathrm{Ricci}_p(X,Y) = \mathrm{trace}(Z \to R(Z,X)Y),$$

and the Ricci operator has the expression

$$\mathrm{Ricci}_p : T_pM \to T_pM, \quad \langle \mathrm{Ricci}_p(X), Y\rangle = \mathrm{Ricci}_p(X,Y).$$

For a smooth map $\varphi : M \to N$ we shall consider on $\varphi^{-1}TN$ the pull-back connection ∇^φ induced by the Levi-Civita connection on N. The following formula holds,

$$\nabla^\varphi_X d\varphi(Y) - \nabla^\varphi_Y d\varphi(X) = d\varphi([X,Y]), \quad \forall X, Y \in C(TM), \tag{2.1}$$

where $d\varphi(X) \in C(\varphi^{-1}TN)$ is defined by $d\varphi(X)(p) = d\varphi_p X_p$, $p \in M$. From (2.1) we deduce the symmetry of the section $\nabla d\varphi \in C(\otimes^2 T^*M \otimes \varphi^{-1}TN)$ defined by

$$\nabla d\varphi(X,Y) = \nabla^\varphi_X d\varphi(Y) - d\varphi(\nabla^M_X Y), \tag{2.2}$$

which is called *the second fundamental form* of φ. A map $\varphi : (M,g) \to (N,h)$ with vanishing second fundamental form is said to be *totally geodesic*.

Denote by $\mathcal{A}^k(\xi) = C(\wedge^k T^*M \otimes V)$ the space of smooth k-forms on M which take values in a vector bundle $\xi : V \to M$. Clearly, $\mathcal{A}^0(\xi) = C(V)$.

Definition 2.6. The *exterior differential operator* $d : \mathcal{A}^k(\xi) \to \mathcal{A}^{k+1}(\xi)$ *relative to the connection* ∇^V is given by

$$\begin{aligned} d\sigma(X_1,...,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\nabla^V_{X_i}(\sigma(X_1,...,\widehat{X_i},...,X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j],X_1,...,\widehat{X_i},...,\widehat{X_j},...,X_{k+1}), \end{aligned}$$

where $X_i \in C(TM)$, $i = 1,\dots,k+1$, and the symbols covered by $\widehat{\ }$ are omitted.

If the tangent bundle TM is equipped with any torsion-free connection, in particular with the Levi-Civita connection, then d can also be defined as the antisymmetrization of ∇ on $C(\wedge^k T^*M \otimes V)$,

$$(d\sigma)(X_1,...,X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}(\nabla_{X_i}\sigma)(X_1,...,\widehat{X_i},...,X_{k+1}),$$

where

$$(\nabla_X \rho)(X_1, ..., X_k) = \nabla^V_X(\rho(X_1, ..., X_k)) - \sum_{i=1}^{k} \rho(X_1, ..., \nabla^M_X X_i, ..., X_k).$$

Definition 2.7. The *codifferential operator* relative to the Riemannian structures on V and TM,

$$d^* : \mathcal{A}^k(\xi) \to \mathcal{A}^{k-1}(\xi),$$

is defined by

$$\begin{aligned}(d^*\rho)(X_1, ..., X_{k-1}) &= -\operatorname{trace}(\nabla_\cdot \rho)(\cdot, X_1, ..., X_{k-1}) \\ &= -\sum_{i,j=1}^{m} g^{ij}(\nabla_{e_i}\rho)(e_j, X_1, ..., X_{k-1}), \quad \rho \in \mathcal{A}^k(\xi),\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a basis of T_pM, $X_1, \ldots, X_{k-1} \in T_pM$ and (g^{ij}) is the inverse of the matrix $(g(e_i, e_j))$.

In particular, if $\rho \in \mathcal{A}^1(\xi)$, then $d^*\rho = -\operatorname{trace} \nabla\rho$.

Proposition 2.8. *The codifferential operator is the adjoint of d via the formula*

$$\int_M \langle d\sigma, \rho\rangle \, v_g = \int_M \langle \sigma, d^*\rho\rangle \, v_g, \tag{2.3}$$

where $\sigma \in \mathcal{A}^{k-1}(\xi)$ and $\rho \in \mathcal{A}^k(\xi)$ are of compact support.

The result used for proving Proposition 2.8 is the following

Proposition 2.9 (Divergence theorem)**.** *Let Ω be a compact domain with smooth boundary of a Riemannian manifold (M, g). Let ρ be a 1-form defined on a neighborhood of Ω. Then*

$$-\int_\Omega d^*\rho \, v_g = \int_{\partial\Omega} \rho(\nu) v_{i^*g},$$

where ν denotes the outward pointing unit normal of $\partial\Omega$ in M and $i : \partial\Omega \to M$ the canonical inclusion.

Definition 2.10. The *Laplacian* on V-valued differential forms is defined by

$$\Delta = dd^* + d^*d : \mathcal{A}^k(V) \to \mathcal{A}^k(V).$$

Proposition 2.11 ([50])**.** *The operator Δ is strongly elliptic, selfadjoint and positive.*

Remark 2.12. Let $f \in C^\infty(M)$. Then

$$\Delta f = d^*df = -\operatorname{trace} \nabla df.$$

Notice that this sign convention is different from the one usually encountered in classical papers on harmonic functions.

Definition 2.13. A k-form $\sigma \in \mathcal{A}^k(V)$ is called *harmonic* if $\Delta\sigma = 0$.

Remark 2.14. By using (2.3) it follows that, when M is compact, $\Delta\sigma = 0$ if and only if $d\sigma = 0$ and $d^*\sigma = 0$.

2.2 Harmonic maps

Let (M, g) and (N, h) be two smooth, connected Riemannian manifolds of dimension m and n, respectively.

The differential $d\varphi$ of a smooth map $\varphi : (M, g) \to (N, h)$ can be viewed as a section of the bundle $T^*M \otimes \varphi^{-1}TN$. By endowing $T^*M \otimes \varphi^{-1}TN$ with the Hilbert-Schmidt norm, we have

$$|d\varphi|^2 = \operatorname{trace}_g \varphi^* h.$$

Definition 2.15.

(i) *The energy density* of φ is the smooth function $e(\varphi) : M \to [0, \infty)$, given by

$$e(\varphi) = \frac{1}{2}|d\varphi|^2.$$

(ii) For a compact domain $\Omega \subset M$, *the energy of* φ *over* Ω is defined by

$$E_\Omega(\varphi) = \int_\Omega e(\varphi)\, v_g. \tag{2.4}$$

Note that $E_\Omega(\varphi) \geq 0$, and $E_\Omega(\varphi) = 0$ if and only if φ is constant on Ω.

If $\varphi : (M, g) \to (N, h)$ is a smooth map, by a *smooth variation of* φ we mean a smooth map $\Phi : I \times M \to N$, $I = (-\varepsilon, \varepsilon) \subset \mathbb{R}$, satisfying

$$\Phi(0, p) = \varphi(p), \quad \forall p \in M.$$

We denote a smooth variation of φ by $\{\varphi_t\}_{t\in I}$, where $\varphi_t(p) = \Phi(t, p)$, $(t, p) \in I \times M$, and say that the variation is *supported in* $\Omega \subset M$ if $\varphi_t = \varphi$ on $M \setminus \overset{\circ}{\Omega}$, for all $t \in I$, where $\overset{\circ}{\Omega}$ denotes the interior of Ω.

Definition 2.16. A smooth map $\varphi : (M, g) \to (N, h)$ is said to be *harmonic* if it is a critical point of the *energy functional* $E_\Omega : C^\infty(M, N) \to \mathbb{R}$, for any compact domain $\Omega \subset M$, i.e. for any $\Omega \subset M$, compact, and for any smooth variation $\{\varphi_t\}_{t\in I}$ of φ supported in Ω, we have

$$D_V E_\Omega(\varphi) = \frac{d}{dt}\Big|_{t=0} E_\Omega(\varphi_t) = 0.$$

Here $V \in \varphi^{-1}TN$ denotes the *variation vector field of* $\{\varphi_t\}_{t\in I}$,

$$V(p) = \frac{d}{dt}\Big|_{t=0} \varphi_t(p), \quad \forall p \in M.$$

Remark 2.17. By construction, every variation supported in a compact subset $\Omega \subset M$ has an associated variation vector field supported in Ω.

Conversely, given a section $V \in C(\varphi^{-1}TN)$ supported in Ω there exists a variation (not necessarily unique) supported in Ω with variation vector field V. A method to construct such a variation is provided by the exponential map on N. Since $\varphi(\Omega)$ is compact there exists $\varepsilon \in \mathbb{R}$ such that $\exp_{\varphi(p)} tV(p)$ is defined for all $t \in (-\varepsilon, \varepsilon)$ and $p \in \Omega$. This allows us to define the smooth map $\Phi : I \times M \to N$, $I = (-\varepsilon, \varepsilon)$,

$$\Phi(t, p) = \exp_{\varphi(p)} tV(p),$$

which is the required variation.

2.2.1 The first variation of the energy

Proposition 2.18 (First variation of the energy, [53]). *Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t\in I}$ be a smooth variation of φ supported in Ω. Then*

$$\frac{d}{dt}\Big|_{t=0} E_\Omega(\varphi_t) = -\int_\Omega \langle \tau(\varphi), V\rangle \, v_g, \tag{2.5}$$

where

$$\tau(\varphi) = -d^* d\varphi = \operatorname{trace} \nabla d\varphi \in C(\varphi^{-1}TN) \tag{2.6}$$

is the tension field of φ.

Theorem 2.19 ([53]). *A map $\varphi : (M, g) \to (N, h)$ is harmonic if and only if*

$$\tau(\varphi) = 0. \tag{2.7}$$

Remark 2.20.

(i) Equation (2.7) is the *Euler-Lagrange* equation associated to the energy functional.

(ii) Equation (2.7) is called the *tension field equation* or the *harmonic equation* and, in local coordinates, yields a semilinear elliptic system of PDE's which are quadratic in the first derivatives. Indeed, if we consider (U, x^i) and (W, y^α) two local coordinate charts at p on M and in $\varphi(p)$ on N, respectively, and denote by ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\beta\delta}^\alpha$ the Christoffel symbols of the Levi-Civita connection of M and N, respectively, then

$$\begin{aligned} \tau(\varphi) &= g^{ij}\left(\frac{\partial^2\varphi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x^k} + {}^N\Gamma_{\beta\delta}^\alpha \frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\delta}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha}\circ\varphi \\ &= \left(-\Delta\varphi^\alpha + g^{ij}\,{}^N\Gamma_{\beta\delta}^\alpha \frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\delta}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha}\circ\varphi. \end{aligned} \tag{2.8}$$

Proposition 2.21. *The second fundamental form and the tension field of the composition of two maps $\varphi : M \to N$ and $\psi : N \to P$ are given by*

$$\nabla d(\psi \circ \varphi) = d\psi(\nabla d\varphi) + \nabla d\psi(d\varphi, d\varphi) \tag{2.9}$$

$$\tau(\psi \circ \varphi) = d\psi(\tau(\varphi)) + \operatorname{trace} \nabla d\psi(d\varphi, d\varphi). \tag{2.10}$$

2.2.2 Examples of harmonic maps

Example 2.22 (Constant maps)**.** Any constant map $\varphi : (M, g) \to (N, h)$ is harmonic. In fact it is an absolute minimum for the energy functional.

Example 2.23 (The identity map)**.** The second fundamental form of the identity map $\mathbf{1}_M : (M, g) \to (M, g)$ vanishes identically, thus $\mathbf{1}_M$ is a harmonic map.

Example 2.24 (Harmonic maps to Euclidean spaces)**.** Let $N = \mathbb{R}^n$ be the Euclidean space endowed with the canonical metric and let $\varphi : M \to \mathbb{R}^n$ be a smooth map, $\varphi(p) = (\varphi^1(p), \ldots, \varphi^n(p))$. Since ${}^N\Gamma^\alpha_{\beta\delta} = 0$, the map φ is harmonic if and only if $\Delta^M \varphi^\alpha = 0$, $\alpha = 1, \ldots, n$, i.e. the φ^α are harmonic functions in the classical sense.

Example 2.25 (Geodesics)**.** Let $M = (a, b)$ be an open interval of $\mathbb{R}$. Then a map $\gamma : (a, b) \to N$ is a curve on N and $\tau(\gamma) = 0$ if and only if

$$\frac{d^2\gamma^\alpha}{dt^2} + {}^N\Gamma^\alpha_{\beta\delta} \frac{d\gamma^\beta}{dt} \frac{d\gamma^\delta}{dt} = 0.$$

Thus, γ is harmonic if and only if it is a geodesic.

Example 2.26 (Harmonic maps to spheres)**.**

(i) Let $\varphi : M \to \mathbb{S}^n$ be a smooth map and $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ the canonical inclusion. Denote by $\phi : M \to \mathbb{R}^{n+1}$ the composition $\phi = \mathbf{i} \circ \varphi$. Then φ is harmonic if and only if

$$\Delta\phi = |d\phi|^2 \phi.$$

(ii) The restriction of a homogeneous harmonic polynomial function of degree k, $F : \mathbb{R}^{m+1} \to \mathbb{R}$, to the sphere $\mathbb{S}^m$ is an eigenfunction of the Laplacian on the sphere with eigenvalue $\lambda_k = k(m + k - 1)$. The function $f = F|_{\mathbb{S}^m} : \mathbb{S}^m \to \mathbb{R}$ is called *spherical harmonic* of order k and it is known that the spherical harmonics give all the eigenfunctions of the Laplacian.

(iii) A smooth map $\varphi : M \to \mathbb{S}^n$ is called an *eigenmap* if the components of $\phi : M \xrightarrow{\varphi} \mathbb{S}^n \xrightarrow{\mathbf{i}} \mathbb{R}^{n+1}$ are all eigenfunctions of the Laplacian on M with the same eigenvalue. It was proved (see, for example, [8]) that a smooth map $\varphi : M \to \mathbb{S}^n$ is an eigenmap if and only if it is harmonic with constant energy density.

Moreover, a map between spheres $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ is an eigenmap if and only if the components of $\phi : \mathbb{S}^m \to \mathbb{R}^{n+1}$ are all spherical harmonics of the same order.

Example 2.27 (Minimal immersions)**.** Let $\varphi : (M, g) \to (N, h)$ be a Riemannian immersion, i.e. $\varphi^* h = g$. By identifying $X \in C(TM)$ with $d\varphi(X) \in C(\varphi^{-1}TN)$, we find that the second fundamental form (2.2) can be written as

$$\nabla d\varphi(X, Y) = \nabla^N_X Y - \nabla^M_X Y,$$

which coincides with the second fundamental form of φ used in the theory of Riemannian immersions. Thus, from (2.6),

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = mH,$$

where H is the mean curvature vector field of φ, and φ is harmonic if and only if it is minimal.

2.2.3 The second variation of the energy

Let (M, g) and (N, h) be two Riemannian manifolds. Suppose, for simplicity, that M is compact and let $\varphi : M \to N$ be a harmonic map.

We consider a smooth two parameter variation $\{\varphi_{s,t}\}_{s,t\in I}$, i.e. a smooth map Φ given by

$$\Phi : I \times I \times M \to N, \quad \Phi(s, t, p) = \varphi_{s,t}(p),$$

where $\Phi(0, 0, p) = \varphi_{0,0}(p) = \varphi(p)$, for all $p \in M$.

The variation vector fields corresponding to this variation, V and W, are given by

$$V(p) = \frac{d}{ds}\Big|_{s=0} \varphi_{s,0}(p) = d\Phi_{(0,0,p)}(\partial/\partial s) \in T_{\varphi(p)}N,$$

and

$$W(p) = \frac{d}{dt}\Big|_{t=0} \varphi_{0,t}(p) = d\Phi_{(0,0,p)}(\partial/\partial t) \in T_{\varphi(p)}N.$$

By a procedure similar to that in Remark 2.17, given two sections $V, W \in C(\varphi^{-1}TN)$, one can determine a smooth two parameter variation $\{\varphi_{s,t}\}_{s,t\in I}$ of variation vector fields V and W. The *Hessian of the energy* E in its critical point φ is defined by

$$H(E)_\varphi(V, W) = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} E(\varphi_{s,t}).$$

Proposition 2.28. *The Hessian of a harmonic map φ has the expression*

$$\begin{aligned} H(E)_\varphi(V, W) &= \int_M (\langle \nabla^\varphi V, \nabla^\varphi W\rangle + \langle \operatorname{trace} R^N(d\varphi, V)d\varphi, W\rangle) v_g \\ &= \int_M \langle \Delta^\varphi V + \operatorname{trace} R^N(d\varphi, V)d\varphi, W\rangle v_g \\ &= \int_M \langle J^\varphi(V), W\rangle v_g, \end{aligned}$$

where Δ^φ denotes the rough Laplacian acting on $C(\varphi^{-1}TN)$, defined by

$$\Delta^\varphi = -\operatorname{trace}(\nabla^\varphi)^2 = -\sum_{i=1}^{m}\{\nabla^\varphi_{E_i}\nabla^\varphi_{E_i} - \nabla^\varphi_{\nabla_{E_i}E_i}\}, \tag{2.11}$$

with respect to a local orthonormal frame field $\{E_i\}_{i=1}^m$ on M.

Remark 2.29.

(i) The Hessian $H(E)_\varphi : C(\varphi^{-1}TN)\times C(\varphi^{-1}TN) \to \mathbb{R}$ is a symmetric bilinear form, thus completely determined by $\{H(E)_\varphi(V,V) : V \in C(\varphi^{-1}TN)\}$.

(ii) The operator J^φ is a linear elliptic self-adjoint differential operator. Moreover, since Δ^φ is positive the Hilbert space of sections

$$L^2(\varphi^{-1}TN) = \left\{V \in C(\varphi^{-1}TN) : (V,V) = \int_M |V|^2\, v_g < \infty\right\}$$

splits as the orthogonal sum of the eigenspaces of J^φ, each being finite dimensional, and the spectrum of J^φ consists of a discrete sequence of real numbers

$$\lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots \nearrow \infty.$$

In analogy with the Jacobi operator for vector fields along geodesics, J^φ is called *the Jacobi operator associated to* φ and the elements of its kernel are called *Jacobi fields along* φ.

Definition 2.30. Let $\varphi : M \to N$ be a harmonic map. The *index* of φ is the dimension of the largest subspace of $C(\varphi^{-1}TN)$ on which $H(E)_\varphi$ is negative definite. The *nullity* of φ is the dimension of the kernel of J^φ.

Definition 2.31. A harmonic map $\varphi : M \to N$ is said to be *stable* if $\operatorname{index}(\varphi) = 0$ and otherwise is said to be *unstable*.

Proposition 2.32 ([53]). *If (N,h) has non-positive sectional curvature, then any harmonic map $\varphi : (M,g) \to (N,h)$ is stable.*

Example 2.33 (Geodesics)**.** If $\gamma : \mathbb{S}^1 \to N$ is a geodesic, then

$$J^\gamma(V) = -\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + R^N(\dot\gamma, V)\dot\gamma,$$

for all $V \in C(\gamma^{-1}TN)$, thus $J^\gamma(V) = 0$ if and only if V is a Jacobi vector field along γ in the classical sense.

Example 2.34 (The identity map)**.** Let $\mathbf{1}_M : M \to M$ be the identity map of M. Then the Jacobi operator, $J^{\mathbf{1}_M} : C(TM) \to C(TM)$, assumes the following form

$$J^{\mathbf{1}_M}(X) = \Delta X - \mathrm{Ricci}(X) = -\operatorname{trace}\nabla^2 X - \mathrm{Ricci}(X).$$

We underline the fact that not even the identity map is always stable as a harmonic map. For example, it was proved in [108] that when $M = \mathbb{S}^m$, $m \geq 3$, the identity map is unstable with $\mathrm{index}(\mathbf{1}_{\mathbb{S}^m}) = m + 1$.

2.2.4 Harmonic morphisms

Definition 2.35. Let $\varphi : (M, g) \to (N, h)$ be a smooth map. Then φ is said to be a *harmonic morphism* if, for every harmonic function f defined on an open subset $V \subset N$ with $\varphi^{-1}(V)$ non-empty, $f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$.

Definition 2.36. A map $\varphi : (M, g) \to (N, h)$ is said to be *horizontally weakly conformal* if for any $p \in M$ with $d\varphi_p \neq 0$, the restriction of $d\varphi_p$ to the orthogonal complement of $\ker d\varphi_p$ is conformal and surjective.

We shall use the following notations: $\ker d\varphi_p = T_p^V M$ (the vertical space) and $(\ker d\varphi_p)^\perp = T_p^H M$ (the horizontal space).

If φ is horizontally weakly conformal, then, away from critical points, there exists a function $\lambda : M \to (0, \infty)$ such that

$$h_{\varphi(p)}(d\varphi_p(X), d\varphi_p(Y)) = \lambda^2(p) g_p(X, Y)$$

for all $X, Y \in T_p^H M$. Setting $\lambda(p) = 0$ at critical points, we continuously extend λ to the whole M. The extended function is called the *dilation function* of the horizontally weakly conformal map φ. Since $\lambda^2 = \frac{2}{n} e(\varphi)$, the *square dilation*, $\lambda^2 : M \to \mathbb{R}$, is smooth even at critical points.

Definition 2.37. A horizontally weakly conformal map $\varphi : (M, g) \to (N, h)$ is said to be *horizontally homothetic* if the gradient of its dilation is vertical at regular points.

The fundamental result in the theory of harmonic morphisms is the following characterization theorem proved by B. Fuglede and T. Ishihara.

Theorem 2.38 ([59, 68])**.** *Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. Then φ is a harmonic morphism if and only if φ is harmonic and horizontally weakly conformal. If φ is non-constant, then φ is a submersion on an open dense subset of M. Moreover, if at a point $p \in M$, $\operatorname{rank} d\varphi_p < n$, then $d\varphi_p = 0$.*

For a detailed account on harmonic morphisms see [8].

2.2.5 Harmonic Riemannian submersions

As a consequence of Theorem 2.38, harmonic Riemannian submersions provide the simplest example of harmonic morphisms. In the following we shall present the basic results needed in order to characterize the harmonicity of a Riemannian submersion.

Let $\varphi : (M, g) \to (N, h)$ be a Riemannian submersion. Then

$$T_pM = T_p^H M \oplus T_p^V M,$$

where $p \in M$, $T_p^V M = \ker d\varphi_p$, and $T_p^H M = (T_p^V M)^\perp$, i.e. the orthogonal complement of $T_p^V M$ in T_pM with respect to g.

Lemma 2.39. *Let $\varphi : (M, g) \to (N, h)$ be a Riemannian submersion. We have*

(i) $\nabla d\varphi\big|_{T^H M \times T^H M} = 0$,

(ii) $\nabla d\varphi\big|_{T^V M \times T^V M} = 0$ *if and only if the fibers of φ are totally geodesic submanifolds in M,*

(iii) $\nabla d\varphi\big|_{T^H M \times T^V M} = 0$ *if and only if the horizontal distribution is integrable.*

Corollary 2.40. *If $\varphi : (M, g) \to (N, h)$ is a Riemannian submersion, then the following statements are equivalent:*

(i) *φ is a totally geodesic map,*

(ii) *the fibers of φ are totally geodesic submanifolds and the horizontal distribution is integrable.*

Let $i_p : \varphi^{-1}(\varphi(p)) \to M$ denote the inclusion of the fiber over p of φ in M. We have

$$0 = \tau(\varphi \circ i_p) = d\varphi(\tau(i_p)) + \operatorname{trace} \nabla d\varphi(di_p, di_p) = d\varphi(\tau(i_p)) + \tau(\varphi),$$

and, since $\tau(i_p)$ is horizontal, we have

Theorem 2.41. *A Riemannian submersion $\varphi : (M, g) \to (N, h)$ has constant energy density $e(\varphi) = \frac{n}{2}$, and is harmonic if and only if its fibers are minimal submanifolds.*

Corollary 2.42. *If $\varphi : (M, g) \to (N, h)$ is a Riemannian submersion, then the following statements are equivalent:*

(i) *φ is a harmonic map,*

(ii) *φ is a harmonic morphism,*

(iii) *the fibres of φ are minimal submanifolds.*

Chapter 3

The biharmonic problem

In this chapter we introduce the notion of biharmonic map as it was suggested by J. Eells and J.H. Sampson. Before doing this we shall recall the classical biharmonic problem and some of its applications in continuous media mechanics.

3.1 Physical problems formulated in terms of the biharmonic equation

The classical biharmonic problem consists in finding a continuous function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, with continuous derivatives up to the fourth order, which is a solution of the homogeneous biharmonic equation

$$(\Delta^2 f)(x) = 0, \qquad x \in \Omega,$$

and satisfies the boundary conditions

$$f\big|_{\partial\Omega} = \alpha \quad \text{and} \quad \frac{\partial f}{\partial n} = \beta,$$

i.e. f has prescribed values on the boundary and prescribed values of the outward normal derivatives.

There are several independent mechanical phenomena modeled by the homogeneous or the inhomogeneous biharmonic equation. We present here three of them, following [58, 76, 85].

Example 3.1 (Thin plates)**.**

In the classical theory of thin plates the equilibrium for a thin plate can be characterized in terms of the biharmonic equation.

Consider an elastic isotropic flat plate of uniform thickness h. When it is bent the plate contains a neutral surface, that lies midway through the plate, on which there is

no extension or compression. We consider a coordinate system with the origin on the neutral surface, the z-axis normal to the surface and the xy-plane as the plane of the undeformed plate and denote by ξ the vertical displacement of a point on the neutral surface, i.e. its z-coordinate. Then the total free energy of the deformed plate can be expressed as

$$F(\xi) = 2D \int \left\{ (\Delta\xi)^2 - 2(1-\sigma) \det(\operatorname{Hess} \xi) \right\} dxdy, \tag{3.1}$$

where the constant $D = Eh^3/12(1-\sigma^2)$ is called the flexural rigidity of the plate and E and σ denote the Young's modulus and Poisson's ratio, respectively (see [76, p.13]).

The equations of equilibrium can be derived from the minimum condition for the total free energy of the bent plate. By computing the first variation of (3.1) and by considering P to be the load per unit area of the plate, we get to the inhomogeneous biharmonic equation

$$D\Delta^2\xi = P,$$

which, consequently, is the equation of equilibrium for a thin plate bent by external forces acting on it.

Example 3.2 (The equilibrium in the plane deformation of a homogeneous isotropic elastic medium)**.**

The equations of equilibrium in the absence of external body forces, in the plane deformation of a homogeneous isotropic elastic medium reduce to

$$\sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i = 1, 2 \tag{3.2}$$

where $\{\sigma_{ij}\}_{i,j=1}^{2}$ denote the non-zero components of the stress tensor. The most general functions satisfying equations (3.2) are of the form

$$\sigma_{11} = \frac{\partial^2 A}{\partial x_2^2}, \quad \sigma_{12} = -\frac{\partial^2 A}{\partial x_1 \partial x_2}, \quad \sigma_{22} = \frac{\partial^2 A}{\partial x_1^2},$$

where $A : D \subset \mathbb{R}^2 \to \mathbb{R}$

From the compatibility condition satisfied by the non-zero components of the strain tensor,

$$\frac{\partial^2 u_{11}}{\partial x_2^2} + \frac{\partial^2 u_{22}}{\partial x_1^2} = 2\frac{\partial^2 u_{12}}{\partial x_1 \partial x_2},$$

$u_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, where u denotes the displacement vector, there follows that A is a solution of

$$\Delta^2 A = 0,$$

i.e. it is a biharmonic function.

The function A is called the *stress function* or *the Airy stress function*. The reduction of plane elastic problems to the biharmonic equation is usually attributed to G.B. Airy [2], but it was J.C. Maxwell (see [84]) the first to present the equilibrium equation in the explicit form of the biharmonic equation.

Example 3.3 (The 2-dimensional Stokes flow)**.**

The 2-dimensional Stokes flow can also be described in terms of the biharmonic equation.

Consider a Newtonian incompressible fluid and a set of axes moving with the local fluid velocity $\mathbf{u}$ and another set fixed in space. If the origin of the moving set of axes is at $\mathbf{x} = (x(t), y(t), z(t))$ at time t, we can write its velocity as

$$\mathbf{u}(\mathbf{x}, t) = \frac{d\mathbf{x}}{dt}.$$

The steady flow of a Newtonian incompressible fluid satisfies the Navier-Stokes equations

$$\begin{cases} \operatorname{div} \mathbf{u} = 0, \\ \\ \langle \mathbf{u}, \operatorname{grad} \mathbf{u} \rangle = -\frac{1}{\rho} \operatorname{grad} p - \nu \Delta \mathbf{u}, \end{cases} \tag{3.3}$$

where p denotes the pressure, ρ the density, ν the kinematic viscosity and no body forces are considered to be acting.

The *Stokes flow*, sometimes referred to as *creeping flow*, describes the motion of an extremely viscous fluid. In this case, the terms in (3.3) involving the squares of the velocities may be omitted compared with the viscous terms, and the fluid motion may be approximated by using the Stokes equations

$$\begin{cases} \operatorname{div} \mathbf{u} = 0, \\ \\ 0 = -\operatorname{grad} p - \mu \Delta \mathbf{u}, \end{cases} \tag{3.4}$$

where μ is the viscosity of the fluid.

If the flow in question is 2-dimensional with velocity field $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$, the first equation of (3.4) insures the existence of a function ψ satisfying

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

Since $\psi(x, y) = k$, $k \in \mathbb{R}$, provides a streamline for the motion, the function ψ is called the *stream function*.

Taking the curl of the second equation in (3.4) and using $\operatorname{div} \mathbf{u} = 0$ we conclude that the stream function ψ satisfies the 2-dimensional biharmonic equation

$$\Delta^2 \psi = 0.$$

3.2 Biharmonic maps

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. As we have seen in (2.6), the tension field of φ is given by $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$.

Definition 3.4. For a compact domain $\Omega \subset M$, the *bienergy of φ over Ω* is defined by

$$E_{2,\Omega}(\varphi) = \frac{1}{2} \int_\Omega |\tau(\varphi)|^2 \, v_g.$$

Definition 3.5. A smooth map φ is said to be *biharmonic* if it is a critical point of the bienergy functional $E_{2,\Omega} : C^\infty(M, N) \to \mathbb{R}$, for any compact domain $\Omega \subset M$.

3.2.1 The first variation formula of the bienergy

The first variation formula and, thus, the Euler-Lagrange equation associated to the bienergy was obtained by R. Caddeo and V. Oproiu (unpublished) and by G.Y. Jiang in [69, 70]. We shall present a proof (see [97]) based on techniques introduced by J. Eells.

Proposition 3.6 (First variation of the bienergy)**.** *Let $\varphi : (M, g) \to (N, h)$ be a smooth map and let $\{\varphi_t\}_{t \in I}$, $I = (-\varepsilon, \varepsilon)$, be a smooth variation of φ supported in Ω, where $\Omega \subset M$ is compact. Then*

$$\frac{d}{dt}\Big|_{t=0} E_{2,\Omega}(\varphi_t) = \int_\Omega \langle \tau_2(\varphi), V \rangle \, v_g,$$

where V denotes the variation vector field of $\{\varphi_t\}_{t \in I}$ and

$$\tau_2(\varphi) = -\Delta^\varphi \tau(\varphi) - \operatorname{trace} R^N(d\varphi, \tau(\varphi)) d\varphi \tag{3.5}$$

is the bitension field of φ.

Proof. Let $\{\varphi_t\}_{t \in I}$, $I = (-\varepsilon, \varepsilon)$, be a smooth variation of φ, i.e. a smooth map $\Phi : I \times M \to N$, satisfying

$$\begin{cases} \Phi(t, p) = \varphi_t(p), & \forall (t, p) \in I \times M, \\ \Phi(0, p) = \varphi_0(p) = \varphi(p), & \forall p \in M. \end{cases}$$

The variation vector field $V \in C(\varphi^{-1} TN)$ associated to the variation $\{\varphi_t\}_{t \in I}$ is given by

$$V(p) = \frac{d}{dt}\Big|_{t=0} \varphi_t(p) = d\Phi_{(0,p)}(\partial/\partial t) \in T_{\varphi(p)} N, \ \forall p \in M.$$

We have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} E_{2,\Omega}(\varphi_t) &= \frac{1}{2} \int_\Omega \Big\{ \frac{\partial}{\partial t} \langle \tau(\varphi_t), \tau(\varphi_t) \rangle \Big\}\Big|_{t=0} v_g \\ &= \int_\Omega \langle \nabla^\Phi_{\partial/\partial t} \tau(\varphi_t), \tau(\varphi_t) \rangle \Big|_{t=0} v_g. \end{aligned}$$

Let now $\{E_i\}_{i=1}^m$ be a local orthonormal frame field geodesic at $p \in \Omega$, i.e. $\{E_i\}_{i=1}^m$ is a local orthonormal frame field with $(\nabla_{E_i} E_j)_p = 0$, for all $i, j = 1, \ldots, m$. With respect to $\{E_i\}_{i=1}^m$ we have

$$\begin{aligned}
\nabla^{\Phi}_{\partial/\partial t}\tau(\varphi_t) &= \nabla^{\Phi}_{\partial/\partial t}\sum_{i=1}^{m}\nabla d\Phi(E_i, E_i) \\
&= \sum_{i=1}^{m}\Big\{\nabla^{\Phi}_{\partial/\partial t}[(\nabla^{\Phi}_{E_i} d\Phi)(E_i)]\Big\} \\
&= \sum_{i=1}^{m}\Big\{\nabla^{\Phi}_{\partial/\partial t}\nabla^{\Phi}_{E_i} d\Phi(E_i) - \nabla^{\Phi}_{\partial/\partial t} d\Phi(\nabla_{E_i} E_i)\Big\}.
\end{aligned}$$

For a given $Z \in C(TM)$, since $[\partial/\partial t, Z] = 0$, we get

$$\begin{aligned}
\nabla^{\Phi}_{\partial/\partial t} d\Phi(Z) &= \nabla^{\Phi}_{Z} d\Phi(\partial/\partial t) + d\Phi[\partial/\partial t, Z] \\
&= \nabla^{\Phi}_{Z} d\Phi(\partial/\partial t),
\end{aligned}$$

thus, at p, we obtain

$$\begin{aligned}
\nabla^{\Phi}_{\partial/\partial t}\tau(\varphi_t) &= \sum_{i=1}^{m}\Big\{\nabla^{\Phi}_{\partial/\partial t}\nabla^{\Phi}_{E_i} d\Phi(E_i) - \nabla^{\Phi}_{\nabla_{E_i} E_i} d\Phi(\partial/\partial t)\Big\} \\
&= \sum_{i=1}^{m}\nabla^{\Phi}_{\partial/\partial t}\nabla^{\Phi}_{E_i} d\Phi(E_i) \\
&= \sum_{i=1}^{m}\Big\{\nabla^{\Phi}_{E_i}\nabla^{\Phi}_{\partial/\partial t} d\Phi(E_i) + \nabla^{\Phi}_{[\partial/\partial t, E_i]} d\Phi(E_i) \\
&\quad + R^{\Phi}(\partial/\partial t, E_i) d\Phi(E_i)\Big\} \\
&= \sum_{i=1}^{m}\Big\{\nabla^{\Phi}_{E_i}\nabla^{\Phi}_{E_i} d\Phi(\partial/\partial t) + R^{N}(d\Phi(\partial/\partial t), d\Phi(E_i)) d\Phi(E_i)\Big\}.
\end{aligned}$$

Denote by ν the outward pointing unit normal of $\partial\Omega$ in M and by $i : \partial\Omega \to M$ the canonical inclusion. Now, by using the symmetries of the Riemann-Christoffel tensor field and the Divergence Theorem (see Proposition 2.9, Chapter 2) for $\rho = \langle \tau(\varphi), \nabla V\rangle$ and $\theta = \langle \nabla\tau(\varphi), V\rangle$,

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E_{2,\Omega}(\varphi_t) &= \int_{\Omega}\langle \operatorname{trace}\nabla^2 V - \operatorname{trace} R^N(d\varphi, V)d\varphi, \tau(\varphi)\rangle v_g \\
&= \int_{\Omega}\langle \nabla V, d\tau(\varphi)\rangle v_g - \int_{\partial\Omega}\langle \nabla_\nu V, \tau(\varphi)\rangle v_{i^*g} - \int_{\Omega}\langle V, \operatorname{trace} R^N(d\varphi, \tau(\varphi))d\varphi\rangle v_g \\
&= \int_{\Omega}\langle V, \operatorname{trace}\nabla^2\tau(\varphi)\rangle v_g + \int_{\partial\Omega}\langle V, \nabla_\nu\tau(\varphi)\rangle v_{i^*g} \\
&\quad - \int_{\Omega}\langle V, \operatorname{trace} R^N(d\varphi, \tau(\varphi))d\varphi\rangle v_g \\
&= \int_{\Omega}\langle V, -\Delta^{\varphi}\tau(\varphi) - \operatorname{trace} R^N(d\varphi, \tau(\varphi))d\varphi\rangle v_g.
\end{aligned}$$

□

Theorem 3.7. *A map $\varphi : (M, g) \to (N, h)$ is biharmonic if and only if*

$$\tau_2(\varphi) = 0. \tag{3.6}$$

Remark 3.8.

(i) Equation (3.6) is called the *bitension field equation* or the *biharmonic equation* and, in local coordinates, it becomes a 4-th order non-linear system of PDE's.

If we consider (U, x^i) and (W, y^α) two local coordinate charts at p on M and in $\varphi(p)$ on N, respectively, and denote by ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\beta\delta}^\alpha$ the Christoffel symbols of the Levi-Civita connection of M and N, respectively, then

$$\begin{aligned}\tau_2(\varphi) \;=\; & g^{ij}\Big(\frac{\partial^2\tau^\sigma}{\partial x^i\partial x^j} + 2\frac{\partial\tau^\alpha}{\partial x^j}\frac{\partial\varphi^\beta}{\partial x^j}\,{}^N\Gamma_{\alpha\beta}^\sigma + \tau^\alpha\frac{\partial^2\varphi^\beta}{\partial x^i\partial x^j}\,{}^N\Gamma_{\alpha\beta}^\sigma \\ & + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\,{}^N\Gamma_{\alpha\beta}^\sigma}{\partial x^j} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\rho}{\partial x^j}\,{}^N\Gamma_{\alpha\beta}^\nu\,{}^N\Gamma_{\nu\rho}^\sigma \\ & - {}^M\Gamma_{ij}^k\Big(\frac{\partial\tau^\sigma}{\partial x^k} + \tau^\alpha\frac{\partial\varphi^\beta}{\partial x^k}\,{}^N\Gamma_{\alpha\beta}^\sigma\Big) - \tau^\nu\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\,{}^N R_{\beta\alpha\nu}^\sigma\Big)\frac{\partial}{\partial y^\sigma}\circ\varphi,\end{aligned}$$

where $\tau(\varphi) = \tau^\sigma\frac{\partial}{\partial y^\sigma}\circ\varphi$ has the local expression given by (2.8).

(ii) A map φ is biharmonic if and only if $\tau(\varphi) \in \ker J^\varphi$, where

$$J^\varphi : C(\varphi^{-1}TN) \to C(\varphi^{-1}TN),$$

$$J^\varphi(V) = \Delta^\varphi V + \operatorname{trace} R^N(d\varphi, V)d\varphi,$$

is, formally, the Jacobi operator corresponding to φ.

(iii) Since J^φ is linear, any harmonic map is biharmonic.

(iv) Any harmonic map is an absolute minimum for the bienergy functional.

Definition 3.9. A map between two Riemannian manifolds is said to be *proper biharmonic* if it is a non-harmonic biharmonic map.

In the following we shall give some examples of biharmonic maps and submanifolds.

3.2.2 Examples of biharmonic maps

Example 3.10 (Biharmonic maps to Euclidean spaces)**.** Consider a smooth map $\varphi : (M, g) \to \mathbb{R}^n$, $\varphi(p) = (\varphi^1(p), \ldots, \varphi^n(p))$. In this situation the tension field reduces to $\tau(\varphi) = -\Delta\varphi = -(\Delta\varphi^1, \ldots, \Delta\varphi^n)$ and the bienergy has the expression

$$E_{2,\Omega}(\varphi) = \frac{1}{2}\int_\Omega |\Delta\varphi|^2\, v_g.$$

The biharmonic equation is given by

$$\tau_2(\varphi) = \Delta^2\varphi = (\Delta^2\varphi^1, \ldots, \Delta^2\varphi^n) = 0,$$

thus a biharmonic map to $\mathbb{R}^n$ is equivalent to an n-tuple of biharmonic functions on M.

We now list some examples.

(i) The simplest examples of biharmonic maps are provided by the polynomial maps of degree 2 and 3 between Euclidean spaces.

(ii) The Almansi Property (see [3]) provides a method for constructing proper biharmonic maps by using harmonic ones.
The Almansi property states that if $f : \mathbb{R}^m \to \mathbb{R}$ is harmonic, then the product function $r^2 f : \mathbb{R}^m \to \mathbb{R}$ is proper biharmonic, i.e.

$$\Delta f = 0 \quad \Longrightarrow \quad \Delta^2(r^2 f) = 0.$$

Here $r : \mathbb{R}^m \to \mathbb{R}$ denotes the distance function from the origin defined by

$$r(x^1, \ldots, x^m) = \sqrt{(x^1)^2 + \ldots + (x^m)^2}.$$

The map

$$\varphi : \mathbb{R}^4 \to \mathbb{R}^3, \quad \varphi(z, w) = (|z|^2 + |w|^2)\big(|z|^2 - |w|^2, 2z\overline{w}\big),$$

is an example of proper biharmonic polynomial map of degree 4. It derives from applying the Almansi property to the Hopf map

$$\Psi : \mathbb{R}^4 \to \mathbb{R}^3, \quad \Psi(z, w) = \big(|z|^2 - |w|^2, 2z\overline{w}\big).$$

(iii) The inversion in the unit sphere

$$\varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \varphi(x) = x/|x|^2,$$

is proper biharmonic if and only if $m = 4$. The conclusion follows from a series of straightforward computations, but this example was obtained as a special case in the study of the behavior of the bitension field under conformal changes of the domain metric for the identity map (see [7] and Chapter 4).

Example 3.11 (Biharmonic maps from geodesics)**.** A simple way of constructing examples of biharmonic maps to an arbitrary Riemannian manifold (N, h) is provided by using the geodesics. Consider $\gamma : I \to N$ to be a geodesic and let $t : J \to I$, $t = t(s)$, $I, J \subset \mathbb{R}$, be a change of parameter. It is not difficult to verify that $\widetilde{\gamma} = \gamma \circ t : J \to N$ is a proper biharmonic map if and only if $d^4t/ds^4 = 0$ and $d^2t/ds^2 \neq 0$.

Example 3.12 (A family of biharmonic non-isometric curves on the Enneper surface)**.** Any straight line passing through the origin in $(\mathbb{R}^2, dx^2 + dy^2)$ transforms into a non-isometric biharmonic curve after applying the conformal change of metric $\left(\mathbb{R}^2, \overline{g} = (r^2+1)^2(dx^2+dy^2)\right)$, where $r = \sqrt{x^2+y^2}$ is the distance function from the origin (see [79]). The metric $\overline{g}$ is the metric, in local isothermal coordinates, of the Enneper minimal surface. Figure 3.1 is a plot of the Enneper surface in polar coordinates, so that radial curves on the picture are biharmonic.

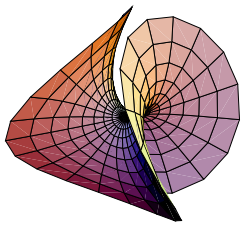

Figure 3.1: The radial curves of the Enneper surface through the origin are non-isometric proper biharmonic curves.

Example 3.13 (Biharmonic projections from the tangent bundle)**.** Two examples of proper biharmonic Riemannian submersions were constructed in [94] by using special "Sasaki type" metrics on the tangent bundle of a Riemannian manifold (M, g). Denote by $V(TM)$ the vertical distribution on TM defined by $V_v(TM) = \ker d\pi_v$, $v \in TM$, where $\pi : TM \to M$ is the canonical projection. We consider a nonlinear connection on TM defined by the distribution $H(TM)$ on TM, complementary to $V(TM)$, i.e. $H_v(TM) \oplus V_v(TM) = T_v(TM)$, $v \in TM$. For any induced local chart $(\pi^{-1}(U), x^i, y^j)$ on TM we have a local adapted frame in $H(TM)$ defined by the local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial x^j}, \qquad i = 1, \ldots, m,$$

where the local functions $N_i^j(x,y)$ are the connection coefficients of the nonlinear connection defined by $H(TM)$.

If TM is endowed with a Riemannian metric defined by

$$S(X^V, Y^V) = S(X^H, Y^H) = g(X,Y), \qquad S(X^V, Y^H) = 0,$$

then the canonical projection $\pi : TM \to M$ is a Riemannian submersion and its biharmonicity can be expressed in the terms of the connection coefficients.

In the case the horizontal distribution is determined by the Levi-Civita connection on the base manifold, i.e. $N_i^j = \Gamma_{ik}^j y^k$, the metric S is the usual Sasaki metric and the projection $\pi : TM \to M$ is harmonic and thus biharmonic.

Other non-trivial examples were generated:

(i) Recall that two torsion-free affine connections ∇ and $\overline{\nabla}$ on a manifold M are said to be *projectively equivalent* if there exists a 1-form α on M such that

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(Y)X + \alpha(X)Y, \qquad X, Y \in C(TM).$$

In local coordinates this condition is expressed as $\overline{\Gamma}^j_{ik} = \Gamma^j_{ik} + \delta^j_i \alpha_k + \delta^j_k \alpha_i$ and we also say that $\nabla \to \overline{\nabla}$ is a *projective change of connection* or that the identity map $1 : (M, \nabla) \to (M, \overline{\nabla})$ is *projective*. Moreover, two torsion-free connections are projectively equivalent if and only they have the same pregeodesics (see [91, p.236]).

If ξ is a unit Killing vector field on (M, g) and the horizontal distribution is determined by the projective change of the Levi-Civita connection on (M, g) induced by ξ, i.e. $N^j_i = \overline{\Gamma}^j_{ik} y^k = (\Gamma^j_{ik} + \delta^j_i \xi_k + \delta^j_k \xi_i) y^k$, then the projection map $\pi : (TM, S) \to (M, g)$ is proper biharmonic.

(ii) A conformal change of metric on a Riemannian manifold $\widetilde{g} = e^{2\rho} g$ induces a change of the Levi-Civita connection $\widetilde{\nabla}_X Y = \nabla_X Y + X(\rho)Y + Y(\rho)X - g(X, Y) \operatorname{grad} \rho$, for all $X, Y \in C(TM)$ (see [25]). In local coordinates $\widetilde{\Gamma}^j_{ik} = \Gamma^j_{ik} + \delta^j_i \alpha_k + \delta^j_k \alpha_i - g_{ik} \alpha^j$, where $\alpha_k = \partial \rho / \partial x^k$.

If $\rho \in C^\infty(M)$ in a non-constant affine function and the horizontal distribution is determined by the conformal change of metric $\widetilde{g} = e^{2\rho} g$, i.e. $N^j_i = \widetilde{\Gamma}^j_{ik} y^k = (\Gamma^j_{ik} + \delta^j_i \alpha_k + \delta^j_k \alpha_i - g_{ik} \alpha^j) y^k$, then the projection map $\pi : (TM, S) \to (M, g)$ is proper biharmonic.

Example 3.14 (Biharmonic maps to spheres)**.** By using the expression for the curvature tensor field for the unit Euclidean sphere, we conclude that $\varphi : (M, g) \to \mathbb{S}^n$ is biharmonic if and only if

$$\Delta^\varphi \tau(\varphi) + 2e(\varphi)\tau(\varphi) - \operatorname{trace}\langle d\varphi, \tau(\varphi)\rangle d\varphi = 0.$$

We now describe a method for constructing biharmonic maps to spheres from harmonic maps of constant energy density (see [81]). Let M be a compact manifold, let $\psi : M \to \mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ be a non-constant map and denote by $\mathbf{i} : \mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \to \mathbb{S}^n$ the canonical inclusion. The composition map $\varphi = \mathbf{i} \circ \psi : M \to \mathbb{S}^n$ is proper biharmonic if and only if ψ is harmonic and $e(\psi)$ is constant. We underline that the converse holds even when M is not compact.

The previous construction can be applied to a series of remarkable examples of harmonic maps of constant energy density:

(i) the generalized Veronese map $\psi : \mathbb{S}^m(\sqrt{(m+1)/m}) \to \mathbb{S}^{m+p}(\frac{1}{\sqrt{2}})$, where $p = \frac{(m+1)(m-2)}{2}$;

(ii) the generalized Clifford torus $\psi : \mathbb{S}^\ell(\frac{1}{2}) \times \mathbb{S}^\ell(\frac{1}{2}) \to \mathbb{S}^{2\ell+1}(\frac{1}{\sqrt{2}})$;

(iii) the Hopf map $\psi : \mathbb{S}^3(\sqrt{2}) \to \mathbb{S}^2(\frac{1}{\sqrt{2}})$, $\psi(z, w) = \frac{1}{2\sqrt{2}}(2z\overline{w}, |z|^2 - |w|^2)$.

3.2.3 Examples of biharmonic submanifolds

Example 3.15 (Biharmonic submanifolds in Euclidean spaces)**.**
In [44], B-Y. Chen classified the Riemannian immersions $\varphi : M \to \mathbb{R}^n$ which satisfy the condition that their mean curvature vector field in $\mathbb{R}^n$ is an eigenvector of the Laplacian, i.e.

$$\Delta H = \lambda H, \quad \text{for some } \lambda \in \mathbb{R}.$$

In particular, when $\lambda = 0$, that is when φ has harmonic mean curvature, from the formula of Beltrami, $\Delta\varphi = -mH$, the condition $\Delta H = 0$ is equivalent to

$$\Delta^2 \varphi = 0.$$

For this reason, the Riemannian submanifolds with harmonic mean curvature vector field in $\mathbb{R}^n$ were called *biharmonic submanifolds* by B-Y. Chen.

As we have already seen, the tension field of a Riemannian immersion has the expression

$$\tau(\varphi) = mH, \tag{3.7}$$

and, replacing (3.7) in (3.5), we get that the bitension field of a Riemannian immersion in $\mathbb{R}^n$ is

$$\tau_2(\varphi) = -m\Delta H.$$

We thus conclude that a Riemannian immersion $\varphi : M \to \mathbb{R}^n$ is biharmonic if and only if it is biharmonic in the sense of B-Y. Chen.

Example 3.16 (Biharmonic curves, [31])**.** Consider $\gamma : I \to (N, h)$ to be a curve parametrized by arc length from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. In this case, as we have seen, by denoting with $T = \dot{\gamma}$, we get that $\tau(\gamma) = \nabla_T T$ and the biharmonic equation (3.6) becomes

$$\nabla_T^3 T - R(T, \nabla_T T)T = 0. \tag{3.8}$$

A curve $\gamma : I \subset \mathbb{R} \to (N^n, h)$, parametrized by arc length, is called a *Frenet curve of osculating order* r, $1 \leq r \leq m$, if there exist orthonormal vector fields $\{F_i\}_{i=1}^r$ along γ such that

$$\begin{aligned}
F_1 &= d\gamma(\partial/\partial s), \\
\nabla^\gamma_{\partial/\partial s} F_1 &= k_1 F_2, \\
\nabla^\gamma_{\partial/\partial s} F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, \ldots, r-1, \\
\nabla^\gamma_{\partial/\partial s} F_r &= -k_{r-1} F_{r-1},
\end{aligned}$$

where the functions $\{k_1, k_2, \ldots, k_{r-1}\}$ are positive functions on I, called the *curvatures* of γ.

A characterization of proper biharmonic curves of osculating order r in terms of the Frenet frame $\{F_i\}_{i=1}^r$ and of the curvatures $\{k_i\}_{i=1}^{r-1}$ was given in [31]. The curve $\gamma : I \to (N^n, h)$, $n \geq 2$, of osculating order r, is proper biharmonic if and only if

$$\begin{cases} k_1 = \text{constant} \neq 0 \\ k_1^2 + k_2^2 = R(F_1, F_2, F_1, F_2) \\ \dot{k}_2 = -R(F_1, F_2, F_1, F_3) \\ k_2 k_3 = -R(F_1, F_2, F_1, F_4) \\ R(F_1, F_2, F_1, F_i) = 0 \qquad i = 5, \ldots, r \\ R(F_1, F_2, F_1, \zeta) = 0, \qquad \forall \zeta \in (\text{span}\{F_i\}_{i=1}^r)^\perp \subset \gamma^{-1}TN. \end{cases} \tag{3.9}$$

More details on this problem can be found in Chapter 5.

Example 3.17 (Biharmonic curves on a surface of revolution, [32])**.**

(i) On a surface of revolution with nonconstant Gauss curvature, obtained by rotating around the z-axis the curve $\alpha(u) = (f(u), 0, g(u))$, parametrized by arc length, the biharmonic curves are the parallels $u = u_0 =$constant, where

$$\dot{f}^2(u_0) + \ddot{f}(u_0) f(u_0) = 0.$$

For example, the torus of revolution with its standard parametrization by arc length,

$$X(u, v) = \big((a + r\cos(u/r))\cos v, (a + r\cos(u/r))\sin v, r\sin(u/r)\big), \quad a > r,$$

has two proper biharmonic curves which are the parallels given by

$$u_1 = r \arccos \frac{-a + \sqrt{a^2 + 8r^2}}{4r}, \qquad u_2 = 2\pi r - r \arccos \frac{-a + \sqrt{a^2 + 8r^2}}{4r}.$$

The proper biharmonic curves on the torus are plotted in Figure 3.2

Figure 3.2: The proper biharmonic curves on the torus.

(ii) The right circular cylinder, obtained when f is constant, and the surface of revolution generated by

$$f(u) = \pm c\sqrt{u},$$
$$g(u) = u\sqrt{\frac{4u - c^2}{4u}} - \frac{c^2}{8} \log\left(8u + 8u\sqrt{\frac{4u - c^2}{4u}} - c^2\right) + c_1,$$

where c and c_1 are positive constants, are the only surfaces of revolution with all parallels biharmonic. Such a surface is plotted in Figure 3.3.

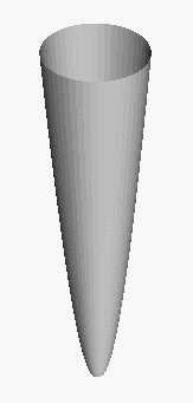

Figure 3.3: Surface of revolution with all parallels proper biharmonic.

(iii) The surfaces of revolution with positive constant Gauss curvature, $K = 1/a^2$, can be parametrized by

$$X(u,v) = \left(b\cos(u/a)\cos v, b\cos(u/a)\sin v, \int_0^u \sqrt{1 - \frac{b^2}{a^2}\sin^2(s/a)}\, ds \right), \qquad b > 0,$$

and the biharmonic curves which are not parallels are obtained for

$$u(t) = -a \arcsin\Big(\frac{c \pm \sqrt{2b^2 - c^2}\sin(A + \frac{\sqrt{2}}{a}t)}{2b}\Big)$$
$$v(t) = \int_0^t \Big(\frac{-\sin\frac{u(s)}{a}}{b\cos^2\frac{u(s)}{a}} + \frac{c}{b^2\cos^2\frac{u(s)}{a}}\Big)ds,$$

where $A, c \in \mathbb{R}$ and $2b^2 - c^2 > 0$.

The biharmonic parallels are obtained for $u = \pm a\pi/4$.

Notice that a biharmonic curve on a sphere of radius r is a circle of radius $r/\sqrt{2}$ (see Figure 3.4).

Example 3.18 (Biharmonic curves of Cartan-Vranceanu spaces)**.**

Let m be a real parameter. Denote by N the whole $\mathbb{R}^3$ if $m \geq 0$, and by $N = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 < -\frac{1}{m}\}$ otherwise. Consider on N the following two-parameter family of Riemannian metrics

$$ds^2_{\ell,m} = \frac{dx^2 + dy^2}{[1 + m(x^2+y^2)]^2} + \left(dz + \frac{\ell}{2}\frac{ydx - xdy}{[1 + m(x^2+y^2)]}\right)^2, \tag{3.10}$$

where $\ell, m \in \mathbb{R}$. The family of metrics (3.10) includes all 3-dimensional homogeneous metrics whose group of isometries has dimension 4 or 6, except for those of constant negative sectional curvature.

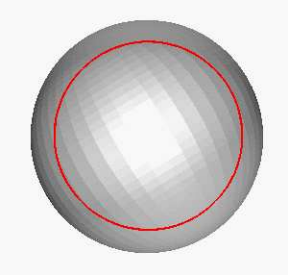

Figure 3.4: Biharmonic curve on $\mathbb{S}^2$.

The proper biharmonic curves in Cartan-Vranceanu spaces were studied in [31]. It was proved that any proper biharmonic curve parametrized by arc length is a helix, i.e. it has constant geodesic curvature and geodesic torsion, and the explicit parametrization of proper biharmonic curves was given.

If $m = 0$ and $\ell \neq 0$, $(N, ds^2_{\ell,m})$ is the Heisenberg space $\mathbb{H}_3$ endowed with a left invariant metric and its biharmonic curves were studied in [33]. The parametric equations of all proper biharmonic curves of $\mathbb{H}_3$ are

$$\begin{cases} x(t) = \frac{1}{A}\sin\alpha_0 \sin(At+a) + b, \\ y(t) = -\frac{1}{A}\sin\alpha_0 \cos(At+a) + c, \\ z(t) = \big(\cos\alpha_0 + \frac{\sin^2\alpha_0}{2A}\big)t - \frac{b}{2A}\sin\alpha_0\cos(At+a) \\ \qquad\quad -\frac{c}{2A}\sin\alpha_0\sin(At+a) + d, \end{cases}$$

where a, b, c, d are real constants $\alpha_0 \in (0, \arccos(2\sqrt{5}/5)] \cup [\arccos(-2\sqrt{5}/5), \pi)$ and $2A = \cos\alpha_0 \pm \sqrt{5\cos^2\alpha_0 - 4}$.

Geometrically, proper biharmonic curves in $\mathbb{H}_3$ can be obtained as the intersection of a minimal helicoid with a round cylinder. Moreover, they are geodesics of this round cylinder (see Figure 3.5).

Example 3.19 (Biharmonic submanifolds in spheres)**.**

(i) The biharmonic submanifolds of $\mathbb{S}^3$ were completely classified in [29].

A curve parametrized by arc length is proper biharmonic on $\mathbb{S}^3$ if and only if it is either the circle of radius $\frac{1}{\sqrt{2}}$, or a geodesic of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ with slope different from ± 1.

The only proper biharmonic surfaces in $\mathbb{S}^3$ are open parts of the sphere $\mathbb{S}^2(\frac{1}{\sqrt{2}})$.

(ii) In the following we shall present some basic examples of proper biharmonic submanifolds of $\mathbb{S}^n$ (see [30]).

The hyperspheres $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ and the generalized Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$, are the only known examples of proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$.

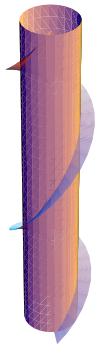

Figure 3.5: Biharmonic curve in the Heisenberg space as the intersection of a minimal helicoid with a round cylinder.

A submanifold M of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ is proper biharmonic as a submanifold of $\mathbb{S}^n$ if and only if it is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.

If $M_1^{m_1}$ is a minimal submanifold in $\mathbb{S}^{n_1}(r_1)$, $0 < m_1 < n_1$, and $M_2^{m_2}$ is a minimal submanifold in $\mathbb{S}^{n_2}(r_2)$, $0 < m_2 < n_2$, $n_1 + n_2 = n - 1$, then the product manifold $M_1 \times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if $r_1 = r_2 = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$.

The proper biharmonic curves of $\mathbb{S}^n$ were studied in [30] and it was proved that, up to isometries, there are only two families of biharmonic curves in spheres:

The circles $\gamma(t) = \cos(\sqrt{2}t)c_1 + \sin(\sqrt{2}t)c_2 + c_4$, where $c_1, c_2, c_4 \in \mathbb{R}^{n+1}$ are constant orthogonal vectors with $|c_1| = |c_2| = |c_4| = \frac{1}{\sqrt{2}}$.

The helices $\gamma(t) = \cos(at)c_1 + \sin(at)c_2 + \cos(bt)c_3 + \sin(bt)c_4$, where $c_1, \ldots, c_4 \in \mathbb{R}^{n+1}$ are constant orthogonal vectors with $|c_1| = |c_2| = |c_3| = |c_4| = \frac{1}{\sqrt{2}}$ and $a^2 + b^2 = 2$, $a^2 \neq b^2$.

Example 3.20 (New examples of biharmonic submanifolds in spheres via conformal changes)**.** Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and denote by $(x^1, \ldots, x^{n+1})$ the canonical coordinates on $\mathbb{R}^{n+1}$. Consider the metric $g = e^{2x^{n+1}} g_0$, conformally equivalent to the canonical metric g_0 on $\mathbb{S}^n$. It was proved in [96] that if M is a minimal submanifold of $(\mathbb{S}^{n-1} \times \{0\}, g_0)$, then M is a proper biharmonic submanifold of $(\mathbb{S}^n, g)$.

Recall that if we consider a Sasakian manifold $(N^{2m+1}, \Psi, \xi, \eta, g)$ and a submanifold $M \subset N$,

(i) M is called *a Legendre submanifold* if it is an m-dimensional integral submanifold of the contact distribution $\mathcal{D} = \{X \in C(TN) : \eta(X) = 0\}$. A one-dimensional integral submanifold of $\mathcal{D}$ is called a *Legendre curve*.

(ii) M is called *anti-invariant* if it is tangent to ξ and Ψ maps any tangent vector to M which is normal to ξ to a vector which is normal to M.

Example 3.21 (Biharmonic submanifolds in $\mathbb{S}^5$ as a Sasakian space form, [4, 106]).

(i) The proper biharmonic Legendre surfaces in $\mathbb{S}^5$ are parametrized by

$$x_0(u,v) = \tfrac{1}{\sqrt{2}}(e^{iu}, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v).$$

(ii) The 3-dimensional anti-invariant proper biharmonic submanifolds in $\mathbb{S}^5$ are parametrized by

$$y_0(u,v,w) = \tfrac{1}{\sqrt{2}}e^{iw}(e^{iu}, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v).$$

For details see also Chapter 6.

Example 3.22 (Biharmonic submanifolds in 3-dimensional Sasakian space forms, [67]). Denote by $N^3(\epsilon)$ a Sasakian space form of sectional curvature ϵ.

If $\epsilon \leq 1$, then there exist no proper biharmonic Legendre curves or Hopf cylinders in $N^3(\epsilon)$.

For $\epsilon > 1$, the Legendre helices of curvature $\sqrt{\epsilon - 1}$ are the only proper biharmonic Legendre curves of $N^3(\epsilon)$ and the Hopf cylinders $S_{\overline{\gamma}}$, where the generating curve $\overline{\gamma} : I \to N/G$ is a Riemannian circle of curvature $\overline{k} = \sqrt{\epsilon - 1}$, are the only proper biharmonic Hopf cylinders of $N^3(\epsilon)$.

3.2.4 Obstructions to the existence of proper biharmonic maps and submanifolds

In the following we shall present some results concerning the non-existence of proper biharmonic maps and submanifolds.

We begin with results of unique continuation, analogous to the ones which hold for harmonic maps.

Theorem 3.23 ([30]). *Let $\varphi : (M,g) \to (N,h)$ be a biharmonic map. If φ is harmonic on an open subset U of M, then φ is harmonic.*

Proof. The proof is based on the following general result concerning the strongly elliptic operators

Theorem 3.24 ([107]). *Let $V(M)$ be a Riemannian vector bundle and let L be a symmetric section in $Hom(V(M), V(M))$. Then $-\Delta + L$ is a strongly elliptic operator with uniqueness for the Cauchy problem in any $p \in M$, i.e. if σ satisfies $-\Delta(\sigma) + L(\sigma) = 0$ and $\sigma_{/U} = 0$, where U is an open subset, then $\sigma = 0$.*

We can apply Theorem 3.24 for $L(\rho) = -\operatorname{trace} R^N(d\varphi, \rho)d\varphi$ and $\sigma = \tau(\varphi)$. Indeed, by considering a local orthonormal frame field $\{E_i\}_{i=1}^m$ on M we can immediately verify that L is a symmetric operator,

$$\begin{aligned}\langle L(\rho_1), \rho_2\rangle &= -\langle \operatorname{trace} R^N(d\varphi, \rho_1)d\varphi, \rho_2\rangle = \sum_{i=1}^m \langle R^N(\rho_1, d\varphi(E_i))d\varphi(E_i), \rho_2\rangle \\ &= \sum_{i=1}^m R^N(\rho_1, d\varphi(E_i), \rho_2, d\varphi(E_i)) = \sum_{i=1}^m R^N(\rho_2, d\varphi(E_i), \rho_1, d\varphi(E_i)) \\ &= \sum_{i=1}^m \langle R^N(\rho_2, d\varphi(E_i))d\varphi(E_i)), \rho_1\rangle \\ &= \langle L(\rho_2), \rho_1\rangle.\end{aligned}$$

□

Corollary 3.25 ([30]). *Let $\varphi : (M, g) \to (N, h)$ be a biharmonic map. If φ is constant on an open subset U of M, then φ is constant.*

The next result, obtained by using the Weitzenböck formula, reveals the sectional curvature of the codomain as an obstruction element to the existence of proper biharmonic maps.

Theorem 3.26 ([69, 70]). *Let M be compact and orientable and suppose that $\operatorname{Riem}^N \leq 0$. Then a smooth map $\varphi : (M, g) \to (N, h)$ is biharmonic if and only if it is harmonic.*

Proof. Suppose that φ is biharmonic. Since $\tau_2(\varphi) = 0$, we get

$$\langle \Delta\tau(\varphi), \tau(\varphi)\rangle = \sum_{i=1}^m \langle R^N(\tau(\varphi), d\varphi(E_i))d\varphi(E_i), \tau(\varphi)\rangle \leq 0, \tag{3.11}$$

where $\{E_i\}_{i=1}^m$ is a local orthonormal frame field on M. By replacing (3.11) in the Weitzenböck formula (see, for example, [50, p.13]),

$$\frac{1}{2}\Delta|\tau(\varphi)|^2 = \langle \Delta\tau(\varphi), \tau(\varphi)\rangle - |\nabla\tau(\varphi)|^2,$$

we get $\frac{1}{2}\Delta|\tau(\varphi)|^2 \leq 0$. As M is compact, by applying the maximum principle, we deduce that $|\tau(\varphi)|^2$ is constant and thus $\nabla\tau(\varphi) = 0$. We integrate and obtain

$$\begin{aligned}0 &= \int_M \langle \nabla\tau(\varphi), d\varphi\rangle v_g = \int_M \langle \tau(\varphi), d^*d\varphi\rangle v_g \\ &= -\int_M |\tau(\varphi)|^2 v_g.\end{aligned}$$

Consequently, φ is harmonic.

□

One can remove the condition that the domain is compact and obtain a similar result with the additional assumption that φ is a Riemannian immersion of constant mean curvature.

Theorem 3.27 ([94]). *Let $\varphi : (M, g) \to (N, h)$ be a Riemannian immersion of constant mean curvature and suppose that* $\mathrm{Riem}^N \leq 0$. *Then φ is biharmonic if and only if it is minimal.*

Proof. Since φ is a Riemannian immersion we obtain

$$\langle d\varphi(X), \tau(\varphi)\rangle = m\langle d\varphi(X), H\rangle = 0, \quad \forall X \in C(TM). \tag{3.12}$$

and the condition that its mean curvature $|H|$ is constant is equivalent to $|\tau(\varphi)|^2 = \text{constant}$.

Suppose now that $\tau_2(\varphi) = 0$. The Weitzenböck formula and the condition $\mathrm{Riem}^N \leq 0$ lead again to $\nabla\tau(\varphi) = 0$.

Let $\{E_i\}_{i=1}^m$ be a local orthonormal frame field on M. By using (3.12) and since $\nabla\tau(\varphi) = 0$, we get

$$\begin{aligned} |\tau(\varphi)|^2 &= \langle \tau(\varphi), \sum_{i=1}^m \nabla d\varphi(E_i, E_i)\rangle = \sum_{i=1}^m \langle \tau(\varphi), \nabla_{E_i} d\varphi(E_i)\rangle \\ &= \sum_{i=1}^m \left\{ E_i\langle \tau(\varphi), d\varphi(E_i)\rangle - \langle d\varphi(E_i), \nabla_{E_i}\tau(\varphi)\rangle \right\} \\ &= 0 \end{aligned}$$

and we conclude.

□

The condition on the sectional curvature can be weakened for codimension 1, i.e. in the case of hypersurfaces.

Theorem 3.28 ([94]). *Let $\varphi : (M, g) \to (N, h)$ be a Riemannian immersion of codimension 1 and suppose that* $\mathrm{Ricci}^N \leq 0$.

(i) *If M is compact and orientable, then φ is biharmonic if and only if it is minimal.*

(ii) *If φ has constant mean curvature, then φ is biharmonic if and only if it is minimal.*

We end this subsection by presenting a list of non-existence results for proper biharmonic submanifolds of Euclidean and hyperbolic spaces.

Theorem 3.29 ([45]). *A spherical submanifold $M \subset \mathbb{S}^{n-1}$ cannot be biharmonic in $\mathbb{R}^n$.*

Theorem 3.30 ([48, 49, 63]). *For any of the following classes of submanifolds $M^m \subset \mathbb{R}^n$ biharmonicity is equivalent to minimality:*

(i) *curves,*

(ii) *submanifolds of finite type,*

(iii) *submanifolds of constant mean curvature,*

(iv) *pseudo-umbilical submanifolds of dimension* $m \neq 4$,

(v) *hypersurfaces with at most two distinct principal curvatures,*

(vi) *conformally flat hypersurfaces,*

(vii) *hypersurfaces of* $\mathbb{R}^3$ *and* $\mathbb{R}^4$.

Theorem 3.31 ([30]). *For any of the following classes of submanifolds* $M^m \subset \mathbb{H}^n$ *biharmonicity is equivalent to minimality:*

(i) *curves,*

(ii) *surfaces in* $\mathbb{H}^3$,

(iii) *submanifolds of constant mean curvature,*

(iv) *pseudo-umbilical submanifolds of dimension* $m \neq 4$.

The non-existence theorems previously presented suggested the following

Generalized Chen Conjecture: *The biharmonic submanifolds of a manifold* N *with* $\mathrm{Riem}^N \leq 0$ *are minimal.*

Chapter 4

New methods of construction for biharmonic maps

The present chapter contains several new methods for constructing proper biharmonic maps by using harmonic ones. We present the Baird-Kamissoko method (see [7]) and new results that are gathered in [11] and [14].

A natural way for constructing proper biharmonic maps is the following: given a harmonic map $\varphi : (M, g) \to (N, h)$ one can conformally change the metric g, or h, in order to render φ proper biharmonic. This problem represents the object of Section 4.1 in the present chapter. The behavior of the bitension field, under the conformal change of the domain metric, was studied by P. Baird and D. Kamissoko in [7] and we shall present here the main results. A metric $\widetilde{g}$, which renders the identity map $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ biharmonic, was called a *biharmonic metric with respect to* g and it was proved that on Einstein manifolds the conformally equivalent biharmonic metrics are defined only by means of isoparametric functions.

In [96], the author studied a particular case of the conformal change of the codomain metric, providing a new method to construct proper biharmonic submanifolds in spheres. For example, the equatorial hypersphere $\mathbb{S}^{n-1}$ is totally geodesic, thus minimal, in $\mathbb{S}^n$ and under a certain conformal change of the Euclidean metric of $\mathbb{S}^n$, it becomes proper biharmonic.

In Section 4.1 we study the biharmonicity of the identity after a conformal change of the codomain metric. The results are similar to those obtained in [7], but we underline the fact that, in general, the biharmonicity of $\overline{\mathbf{1}} : (M, g) \to (M, e^{2\rho}g)$ is not equivalent to the biharmonicity of $\widetilde{\mathbf{1}} : (M, e^{2\rho}g) \to (M, g)$ (see Remark 4.22).

The results gathered in Section 4.2 extend the idea of studying the effect of conformal changes of metric to the biharmonic equation by considering the situation of warped products. The main problems consist in the study of the biharmonicity of the inclusion

of a Riemannian manifold N into the warped product $M \times_{f^2} N$ and of the projection $\overline{\pi} : M \times_{f^2} N \to M$ (Corollary 4.29 and Proposition 4.47). With this setting we obtain new examples of proper biharmonic maps and recover some of the examples first obtained in [29] and [32].

We then consider the product of two harmonic maps $\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$. By warping the metric on the domain or codomain we lose the harmonicity; nevertheless, under certain conditions on the warping function the product map remains biharmonic. When the product map is the identity map $\overline{\mathbf{1}} : M \times_{f^2} N \to M \times N$ we shall call the warping function, which is a solution of the above problem, a *biharmonic warping function*. In the case that M is an Einstein manifold we show that the isoparametric functions provide examples of biharmonic warping ones. We also give the complete classification of the biharmonic warping functions when $M = \mathbb{R}$ (see Example 4.46).

In the last section we use the warped product setting to study axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to an n-dimensional space form. We discuss certain examples and, when the target manifold is $\mathbb{R}^n \setminus \{0\}$, we give the complete classification of biharmonic axially symmetric maps. From the former classification we also deduce that the generalised Kelvin transformation $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(y) = y/|y|^\ell$ is a proper biharmonic map if and only if $m = \ell + 2$.

4.1 Biharmonic maps and conformal changes

4.1.1 Conformal changes of metric

In order to be able to develop the computations we need to recall some well known results on the conformal changes of metric on a Riemannian manifold (M, g). Denote by ∇ the Levi-Civita connection of the metric g and by $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{g} = e^{2\rho} g$, $\rho \in C^\infty(M)$. The relation between the two connections is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + P(X, Y),$$

where the $(1,2)$-type tensor field P has the expression

$$P(X, Y) = X(\rho)Y + Y(\rho)X - g(X, Y)\operatorname{grad}\rho, \tag{4.1}$$

for all $X, Y \in C(TM)$.

The curvature tensors of $\widetilde{\nabla}$ and ∇ satisfy

$$\begin{aligned} \widetilde{R}(X, Y)Z &= R(X, Y)Z + (\nabla_X P)(Y, Z) - (\nabla_Y P)(X, Z) \\ &\quad + P(X, P(Y, Z)) - P(Y, P(X, Z)), \end{aligned} \tag{4.2}$$

for all $X, Y, Z \in C(TM)$.

4.1.2 Isoparametric functions

As we have already announced there is a remarkable connection between conformally equivalent biharmonic metrics and isoparametric functions on Einstein spaces. In this subsection we recall the definition and the fundamental properties of isoparametric functions on Riemannian manifolds.

Definition 4.1 ([6]). A smooth function $f : (M, g) \to \mathbb{R}$ is called *isoparametric* if locally there are smooth real functions γ and σ such that

$$\begin{cases} \text{(i)} \ |df|^2 = \gamma \circ f, \\ \\ \text{(ii)} \ \Delta f = \sigma \circ f, \end{cases} \tag{4.3}$$

at all points where $\operatorname{grad} f \neq 0$.

Proposition 4.2 ([6]). *A function f is isoparametric if and only if for all $x \in M$ with* $\operatorname{grad}_x f \neq 0$,

$$\begin{cases} \text{(i)} \ \ \operatorname{grad}(|\operatorname{grad} f|) \ \textit{is parallel to} \operatorname{grad} f, \\ \\ \text{(ii)} \ \ \operatorname{grad}(\Delta f) \ \textit{is parallel to} \operatorname{grad} f. \end{cases} \tag{4.4}$$

Remark 4.3 ([6]). Condition (4.3)(i) is called the *eiconal equation* for f and its solutions are called *transnormal functions*. It is equivalent to the fact that the regular level hypersurfaces of f are parallel, i.e. the integral curves of $\operatorname{grad} f/|\operatorname{grad} f|$ are geodesics. Condition (4.3)(ii) is equivalent to the assertion that the regular levels of f have constant mean curvature in M.

Remark 4.4. Notice that the isoparametric condition is not conformally invariant. If $f \in C^\infty(M)$ is isoparametric with respect to g, then f is isoparametric with respect to $\widetilde{g} = e^{2\rho} g$ if and only if ρ is locally a reparametrization of f, i.e. locally $\rho = \rho \circ f$ for some smooth function $\rho : \mathbb{R} \to \mathbb{R}$. Indeed, denoting by $\widetilde{\operatorname{grad}}$ and $\widetilde{\Delta}$ the gradient and the Laplacian with respect to $\widetilde{g}$, we have

$$\begin{cases} |\widetilde{\operatorname{grad}} f|^2_{\widetilde{g}} = e^{-2\rho} |\operatorname{grad} f|^2 \\ \\ \widetilde{\Delta} f = e^{-2\rho} \big\{ \Delta f - (m-2) g(\operatorname{grad} \rho, \operatorname{grad} f) \big\} \end{cases} \tag{4.5}$$

and we conclude.

Obviously, from (4.5), if ρ is isoparametric with respect to g, then ρ is isoparametric with respect to $\widetilde{g} = e^{2\rho} g$.

4.1.3 The Baird-Kamissoko method

Since the direct integration of the general biharmonic equation often proves to be a difficult task, in [7] the authors adopted a different approach in order to construct proper biharmonic maps.

Let $\varphi : (M, g) \to (N, h)$ be a smooth map. In the following we shall present the expression for the tension field after a conformal change of metric $\widetilde{g} = e^{2\rho} g$ on the domain. In order to avoid confusion we shall denote by $\widetilde{\varphi} : (M, \widetilde{g}) \to (N, h)$ the map defined by $\widetilde{\varphi}(p) = \varphi(p)$, for all $p \in M$.

By using (4.1), we obtain

$$\tau(\widetilde{\varphi}) = e^{-2\rho}\big(\tau(\varphi) + (m-2)d\varphi(\operatorname{grad}\rho)\big).$$

The assumption that φ is harmonic leads to

$$\tau(\widetilde{\varphi}) = (m-2)e^{-2\rho}d\varphi(\operatorname{grad}\rho). \tag{4.6}$$

Notice that if $m = 2$, then $\tau(\widetilde{\varphi}) = 0$ and this case is of no interest.

If $m \neq 2$, then $\tau(\widetilde{\varphi})$ does not, generally, vanish, so $\widetilde{\varphi} : (M, \widetilde{g}) \to (N, h)$ will not be harmonic. This suggested the idea of searching for functions $\rho : M \to \mathbb{R}$ which render φ biharmonic. The advantage of the method consists in the fact that the biharmonic equation for $\widetilde{\varphi}$ reduces to a second-order equation in the 1-form $d\rho$.

For simplicity, only the case when φ is the identity map of M will be presented. It is useful to introduce the following

Definition 4.5. Let (M, g) be a Riemannian manifold. A Riemannian metric $\widetilde{g}$ on M is said to be *biharmonic with respect to* g if the identity $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ is a biharmonic map.

From the expression for the bitension field,

$$\begin{aligned}\tau_2(\widetilde{\mathbf{1}}) &= e^{-4\rho}(m-2)\big(\operatorname{trace}\nabla^2\operatorname{grad}\rho + \frac{m-6}{2}\operatorname{grad}(|\operatorname{grad}\rho|^2)\\ &\quad +2(\Delta\rho - (m-4)|\operatorname{grad}\rho|^2)\operatorname{grad}\rho + \operatorname{Ricci}(\operatorname{grad}\rho)\big),\end{aligned} \tag{4.7}$$

there follows

Theorem 4.6 ([7]). *Let $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$, $m \neq 2$, be the identity map with $\widetilde{g} = e^{2\rho} g$ a metric conformally equivalent to g. Then $\widetilde{\mathbf{1}}$ is biharmonic, i.e. $\widetilde{g}$ is biharmonic with respect to g, if and only if*

$$\begin{aligned}0 &= \operatorname{trace}\nabla^2\operatorname{grad}\rho + \frac{m-6}{2}\operatorname{grad}(|\operatorname{grad}\rho|^2)\\ &\quad +2(\Delta\rho - (m-4)|\operatorname{grad}\rho|^2)\operatorname{grad}\rho + \operatorname{Ricci}(\operatorname{grad}\rho).\end{aligned} \tag{4.8}$$

Then, by using the musical isomorphisms, the authors obtained

Proposition 4.7 ([7]). *Let $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ be the identity map with $\widetilde{g} = e^{2\rho} g$ conformally equivalent to g. Then $\widetilde{\mathbf{1}}$ is biharmonic if and only if*

$$\Delta\alpha - \frac{m-6}{2} d|\alpha|^2 - (2d^*\alpha - (m-4)|\alpha|^2)\alpha - 2(\text{Ricci } \alpha^\sharp)^\flat = 0, \tag{4.9}$$

where $d\rho = \alpha$ (or $\operatorname{grad}\rho = \alpha^\sharp$).

and

Corollary 4.8 ([7]). *Let $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ be the identity map, $\widetilde{g} = e^{2\rho} g$ conformally equivalent to g and put $d\rho = \alpha$. If $\widetilde{\mathbf{1}}$ is biharmonic, then*

$$-(m-2)d|\alpha|^2 \wedge \alpha + 4(\text{Ricci } \alpha^\sharp)^\flat \wedge \alpha + 2d(\text{Ricci } \alpha^\sharp)^\flat = 0. \tag{4.10}$$

As an application of the maximum principle, the following non-existence result for compact manifolds of strictly negative Ricci curvature was obtained.

Proposition 4.9 ([7]). *Let (M, g), $m \neq 2$, be a compact Riemannian manifold of strictly negative Ricci curvature. Then there exists no biharmonic metric conformally equivalent to g other than a constant multiple of g.*

Furthermore, the structure of (4.9) suggested the study of the relation between conformal changes which render the identity map biharmonic and isoparametric functions in the case of Einstein spaces.

The following

Lemma 4.10 ([7]). *Let (M,g) be an Einstein space. Then* (4.9) *has the form*

$$\Delta\alpha - F(p, \alpha)\,\alpha = 0,$$

where F is a real valued function depending on $p \in M$ and α, if and only if ρ satisfies (4.3)(i)*, i.e locally there exists a smooth real function γ such that $|d\rho|^2 = \gamma \circ \rho$.*

and

Corollary 4.11 ([7]). *Let (M, g) be an Einstein space and let $\rho : M \to \mathbb{R}$ be a function satisfying* (4.4)(i)*. If ρ satisfies* (4.9)*, then ρ is isoparametric.*

are used in order to prove the main result. For the sake of completeness, we shall present it together with its proof.

Theorem 4.12 ([7]). *Let (M^m, g), $m \neq 2$, be an Einstein space and $\widetilde{g} = e^{2\rho} g$ a metric conformally equivalent to g such that $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ is biharmonic. Then $\rho : M \to \mathbb{R}$ is isoparametric.*
Conversely, if $f : M \to \mathbb{R}$ is isoparametric, then, away from critical points of f, there exists a reparametrisation $\rho = \rho \circ f$ such that $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ is biharmonic.

Proof. We shall prove that ρ satisfies (4.4)(i) and so, the conclusion follows from Corollary 4.11. Equation (4.10) reduces to

$$(m-2)d|\alpha|^2 \wedge \alpha = 0. \tag{4.11}$$

Since $m \neq 2$, evaluating (4.11) on X, we find that $X(|\operatorname{grad}\rho|^2) = 0$, for all X with $X(\rho) = 0$. Thus the part of $\operatorname{grad}(|\operatorname{grad}\rho|^2)$ tangent to the regular levels of ρ is zero and (4.4)(i) follows.

Conversely, if $f : M \to \mathbb{R}$ is an isoparametric non-constant function, we shall determine a reparametrisation $\rho = \rho \circ f$ of f, with $\alpha = d\rho$ that satisfies (4.9). We work on an open connected set $U \subset M$ with no critical points for f. If $|df|^2 = \gamma \circ f$, by choosing s to be a solution of the differential equation $s' = \gamma^{-\frac{1}{2}}$, we can reparametrize f such that $s = s \circ f$, and $|ds|^2 = 1$. The function s is isoparametric with $\Delta s = \sigma \circ s$, for a smooth function σ. We have

$$d\rho = (\rho' \circ s)ds,$$

$$\Delta\rho = (-\rho'' + \rho'\sigma) \circ s.$$

From here we get

$$\Delta\alpha = dd^*d\rho = d(\Delta\rho) = (-\rho''' + \rho''\sigma + \rho'\sigma')ds,$$

$$d|\alpha|^2 = 2\rho'\rho''ds,$$

$$(\text{Ricci } \alpha^\sharp)^\flat = c\,\alpha = c\rho' ds.$$

and, by substituting into (4.9), we obtain a second order ordinary differential equation in $y(s) = \rho'(s)$

$$y'' - \sigma y' + (m-8)y'y + (2c - \sigma')y + 2\sigma y^2 - 2(m-4)y^3 = 0. \tag{4.12}$$

The general theory of ODE' s offers local solutions $y = y(s)$ for (4.12).
Finally, $\rho(s) = \int y(s)ds$. □

In [7] the authors gave the complete description of the biharmonic metrics on $\mathbb{R}^m$ with the regular levels of ρ being parallel hyperplanes and found some examples of biharmonic metrics on $\mathbb{R}^m$ with the regular levels of ρ being hyperspheres. These metrics depend on the dimension of M and dimension 4 seems to be somehow special. We shall present here only the following

Example 4.13 ([7])**.** Let $M = \mathbb{R}^m \setminus \{0\}$ and consider the function $s : M \to \mathbb{R}$, $s(x^1, x^2, ..., x^m) = \sqrt{(x^1)^2 + ... + (x^m)^2}$ to be the distance function from the origin. The standard metric on M has the expression $g = ds^2 + s^2 g_{\mathbb{S}^{m-1}}$. With respect to g, the

function s is isoparametric with $|ds|^2 = 1$ and $\Delta s = -(m-1)/s$, so $\sigma(s) = -(m-1)/s$. In this case (4.12) becomes

$$y'' + \frac{m-1}{s}y' + (m-8)yy' - \frac{m-1}{s^2}y - 2\frac{m-1}{s}y^2 - 2(m-4)y^3 = 0. \tag{4.13}$$

We shall search for special solutions satisfying $y = a/s$, with $a \in \mathbb{R}$. For $y = a/s$, (4.13) becomes, excluding $y = 0$,

$$2(m-4)a^2 + (3m-10)a + 2(m-2) = 0. \tag{4.14}$$

There are two cases to be considered:

(i) $m = 4$. Then $a = -2$ and we obtain the metric (up to a constant multiple)

$$\widetilde{g} = \frac{1}{s^4}(ds^2 + s^2 g_{\mathbb{S}^3}). \tag{4.15}$$

(ii) $m \neq 4$. The discriminant of equation (4.14) is positive only for $m = 1, 3$. For $m = 1$, $a = -1, -4/3$. For $m = 3$, $a = -(1 \pm \sqrt{17})/4$. In all cases the deformed metric is given (up to a constant multiple) by $\widetilde{g} = s^{2a} g$.

Remark 4.14. The inversion in the unit sphere

$$\begin{aligned} \varphi : \mathbb{R}^4 \setminus \{0\} &\to \mathbb{R}^4 \setminus \{0\} \\ x &\mapsto x/|x|^2 \end{aligned}$$

corresponds to the substitution $u = 1/s$ with respect to which the metric $\widetilde{g}$ in (4.15) assumes the expression $\widetilde{g} = du^2 + u^2 g_{\mathbb{S}^3}$. Since this is the Euclidean metric, the biharmonicity of the inversion is equivalent to the biharmonicity of the identity map $\widetilde{\mathbf{1}} : (\mathbb{R}^4 \setminus \{0\}, \widetilde{g}) \to (\mathbb{R}^4 \setminus \{0\}, g)$. From the case (i) in Example 4.13 we conclude that the inversion $\varphi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^4 \setminus \{0\}$ is a biharmonic map.

Notice that the only other dimension for which the inversion is biharmonic is dimension 2, but in this case it is harmonic.

4.1.4 Conformal changes of metric on the codomain

We first show that for any arbitrary metrics g and $\widetilde{g}$ on a manifold M, the tension and the bitension fields of the identity map $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ are unchanged when $\overline{\mathbf{1}}$ is precomposed with a harmonic Riemannian submersion.

Proposition 4.15 ([11]). *Let $\varphi : (N, h) \to (M, g)$ be a harmonic Riemannian submersion and $\widetilde{g}$ a Riemannian metric on M. Denote by $\overline{\varphi} = \overline{\mathbf{1}} \circ \varphi : (N, h) \to (M, \widetilde{g})$, where $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ is the identity map. Then*

$$\tau(\overline{\varphi}) = \tau(\overline{\mathbf{1}}) \qquad \textit{and} \qquad \tau_2(\overline{\varphi}) = \tau_2(\overline{\mathbf{1}}).$$

Proof. Let $\{E_i\}_{i=1}^m$ be a local orthonormal frame field on M. Then $\overline{E}_i$ generate a local orthonormal base in T^HN, where $\overline{E}_i$ denotes the basic lift of E_i with respect to φ. We can complete $\{\overline{E}_i\}_{i=1}^m$ with $\{\overline{E}_\alpha\}_{\alpha=m+1}^n \subset T^VN$ to a local orthonormal frame field $\{\overline{E}_a\}_{a=1}^n$ on N.

Using the chain rule (2.10) for the tension field we obtain

$$\begin{aligned}\tau(\overline{\varphi}) &= d\overline{\mathbf{1}}(\tau(\varphi)) + \operatorname{trace}_h \nabla d\overline{\mathbf{1}}(d\varphi, d\varphi) = \operatorname{trace}_h \nabla d\overline{\mathbf{1}}(d\varphi, d\varphi)\\ &= \sum_{a=1}^n \nabla d\overline{\mathbf{1}}(d\varphi(\overline{E}_a), d\varphi(\overline{E}_a)) = \sum_{i=1}^m \nabla d\overline{\mathbf{1}}(E_i, E_i) = \operatorname{trace}_g \nabla d\overline{\mathbf{1}}\\ &= \tau(\overline{\mathbf{1}}).\end{aligned}$$

In order to study the biharmonicity of $\overline{\varphi}$ we compute

$$\begin{aligned}-\Delta\tau(\overline{\varphi}) &= \operatorname{trace}_h \nabla^2\tau(\overline{\varphi})\\ &= \sum_{a=1}^n \{\nabla^{\overline{\varphi}}_{\overline{E}_a}\nabla^{\overline{\varphi}}_{\overline{E}_a}\tau(\overline{\varphi}) - \nabla^{\overline{\varphi}}_{\nabla^M_{\overline{E}_a}\overline{E}_a}\tau(\overline{\varphi})\}\\ &= \sum_{i=1}^m \{\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\tau(\overline{\varphi}) - \widetilde{\nabla}_{\nabla^M_{E_i}E_i}\tau(\overline{\varphi})\} - \sum_{\alpha=m+1}^n \widetilde{\nabla}_{d\varphi(\nabla^N_{\overline{E}_\alpha}\overline{E}_\alpha)}\tau(\overline{\varphi})\\ &= \sum_{i=1}^m \{\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\tau(\overline{\varphi}) - \widetilde{\nabla}_{\nabla^M_{E_i}E_i}\tau(\overline{\varphi})\} + \widetilde{\nabla}_{d\overline{\mathbf{1}}(\tau(\varphi))}\tau(\overline{\varphi})\\ &= \sum_{i=1}^m \{\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\tau(\overline{\mathbf{1}}) - \widetilde{\nabla}_{\nabla^M_{E_i}E_i}\tau(\overline{\mathbf{1}})\}\\ &= -\Delta\tau(\overline{\mathbf{1}})\end{aligned}$$

and

$$\begin{aligned}\operatorname{trace}_h \widetilde{R}(d\overline{\varphi}, \tau(\overline{\varphi}))d\overline{\varphi} &= \sum_{i=1}^m \widetilde{R}(E_i, \tau(\overline{\varphi}))E_i\\ &= \operatorname{trace}_g \widetilde{R}(d\overline{\mathbf{1}}, \tau(\overline{\mathbf{1}}))d\overline{\mathbf{1}}.\end{aligned}$$

Thus,

$$\begin{aligned}\tau_2(\overline{\varphi}) &= -\Delta\tau(\overline{\varphi}) - \operatorname{trace}_h \widetilde{R}(d\overline{\varphi}, \tau(\overline{\varphi}))d\overline{\varphi}\\ &= -\Delta\tau(\overline{\mathbf{1}}) - \operatorname{trace}_g \widetilde{R}(d\overline{\mathbf{1}}, \tau(\overline{\mathbf{1}}))d\overline{\mathbf{1}}\\ &= \tau_2(\overline{\mathbf{1}}).\end{aligned}$$

□

By using Proposition 4.15 we conclude,

Corollary 4.16 ([11]). *Let $\varphi : (N,h) \to (M,g)$ be a harmonic Riemannian submersion and $\widetilde{g} = e^{2\rho}g$, $\rho \in C^\infty(M)$, a conformal change of metric on (M,g). Denote by $\overline{\varphi} = \overline{\mathbf{1}} \circ \varphi : (N,h) \to (M,\widetilde{g})$, where $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$ is the identity map. Then $\overline{\varphi}$ is proper biharmonic if and only if $\overline{\mathbf{1}}$ is proper biharmonic.*

Remark 4.17. We recall that a harmonic Riemannian submersion is a morphism both of harmonic and of biharmonic functions (see [8, 83]). A more general result was obtained in [98] in the study of p-harmonic morphisms.

Theorem 4.18 ([98]). *Let $\varphi : (N,h) \to (M,g)$ be a non-constant horizontally homothetic harmonic morphism with harmonic energy density, and let $\psi : (M,g) \to (Q,k)$ be a map. Then, the composition $\psi \circ \varphi : (N,h) \to (Q,k)$ is a proper biharmonic map if and only if ψ is proper biharmonic on the open subset $\varphi(N) \subset M$.*

Noting that a harmonic Riemannian submersion is a horizontally homothetic harmonic morphism with constant, hence harmonic, energy density the conclusion in Proposition 4.15 follows directly by applying Theorem 4.18 with ψ being the identity map $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$. Thus the study of the biharmonicity for a harmonic Riemannian submersion $\varphi : (N,h) \to (M,g)$ after a change of metric $\widetilde{g}$ on its codomain is reduced to the study of the biharmonicity of the identity map $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$.

Consider now the case when $\widetilde{g} = e^{2\rho}g$, i.e. $\widetilde{g}$ is conformally equivalent to g. We can apply the Baird-Kamissoko method for determining the expressions of the tension and the bitension fields for $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$. In order to do so the identity map $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$ is thought of as the map obtained by performing the conformal change of metric $g = e^{-2\rho}\widetilde{g}$ on the domain of the identity map of $(M,\widetilde{g})$. Thus, by using (4.6) and (4.8), we obtain

$$\tau(\overline{\mathbf{1}}) = -(m-2)e^{2\rho}\widetilde{\operatorname{grad}}\rho = -(m-2)\operatorname{grad}\rho \tag{4.16}$$

and

$$\begin{aligned}\tau_2(\overline{\mathbf{1}}) \;=\; & (m-2)e^{4\rho}\big\{ -\operatorname{trace}_{\widetilde{g}}\widetilde{\nabla}^2\widetilde{\operatorname{grad}}\rho + \frac{m-6}{2}\widetilde{\operatorname{grad}}(|\widetilde{\operatorname{grad}}\rho|^2_{\widetilde{g}}) \\ & +2(\widetilde{\Delta}\rho + (m-4)|\widetilde{\operatorname{grad}}\rho|^2_{\widetilde{g}})\widetilde{\operatorname{grad}}\rho - \widetilde{\operatorname{Ricci}}(\widetilde{\operatorname{grad}}\rho)\big\},\end{aligned} \tag{4.17}$$

where $\widetilde{\operatorname{grad}}$ and $\widetilde{\Delta}$ denote the gradient and the Laplacian on M with respect to the metric $\widetilde{g}$.

Remark 4.19. We underline the fact that $\overline{\mathbf{1}} : (M,g) \to (M,\widetilde{g})$ is harmonic if and only if $\widetilde{\mathbf{1}} : (M,\widetilde{g}) \to (M,g)$ is harmonic and this holds if and only if $m = 2$ or $m > 2$ and ρ is constant. The expression for the bitension field (4.17) does not allow us to draw immediately the same conclusion for what concerns the biharmonicity. As we shall see,

the equivalence is no longer valid in this case (see Remark 4.22). Nevertheless, by using Remark 4.4 it is obvious that an analogous version of Theorem 4.12 holds in the case of conformal changes of metric on the codomain.

In order to study the connection between the biharmonicity of $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ and that of $\widetilde{\mathbf{1}} : (M, \widetilde{g}) \to (M, g)$ we choose not to use (4.17), but to directly compute the bitension field of $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ in terms of the metric g.

Theorem 4.20 ([11]). *Let $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ be the identity map and $\widetilde{g} = e^{2\rho} g$ a metric conformally equivalent to g. Then $\overline{\mathbf{1}} : (M, g) \to (M, \widetilde{g})$ is proper biharmonic if and only if ρ is a non-constant solution of*

$$\begin{aligned} 0 &= \operatorname{trace} \nabla^2 \operatorname{grad} \rho + (2\Delta\rho - (m-2)|\operatorname{grad}\rho|^2) \operatorname{grad}\rho \\ &\quad - \frac{m-6}{2} \operatorname{grad}(|\operatorname{grad}\rho|^2) + \operatorname{Ricci}(\operatorname{grad}\rho). \end{aligned} \tag{4.18}$$

Proof. Let $\{E_i\}_{i=1}^m$ be a geodesic frame at $p \in M$. As $\tau(\overline{\mathbf{1}}) = -(m-2)\operatorname{grad}\rho$ we get

$$\begin{aligned} -\Delta\tau(\overline{\mathbf{1}}) &= \operatorname{trace} \nabla^2 \tau(\overline{\mathbf{1}}) \\ &= -(m-2) \sum_{i=1}^m \big(\widetilde{\nabla}_{E_i} \widetilde{\nabla}_{E_i} \operatorname{grad}\rho - \widetilde{\nabla}_{\nabla_{E_i} E_i} \operatorname{grad}\rho \big) \\ &= -(m-2) \sum_{i=1}^m \widetilde{\nabla}_{E_i} \widetilde{\nabla}_{E_i} \operatorname{grad}\rho. \end{aligned}$$

But, using (4.1),

$$\begin{aligned} \widetilde{\nabla}_{E_i} \operatorname{grad}\rho &= \nabla_{E_i} \operatorname{grad}\rho + P(E_i, \operatorname{grad}\rho) \\ &= \nabla_{E_i} \operatorname{grad}\rho + |\operatorname{grad}\rho|^2 E_i \end{aligned}$$

and we obtain

$$\begin{aligned} \widetilde{\nabla}_{E_i} \widetilde{\nabla}_{E_i} \operatorname{grad}\rho &= \nabla_{E_i} \nabla_{E_i} \operatorname{grad}\rho + \frac{3}{2} E_i(|\operatorname{grad}\rho|^2) E_i \\ &\quad + \nabla_{E_i(\rho) E_i} \operatorname{grad}\rho - g(E_i, \nabla_{E_i} \operatorname{grad}\rho) \operatorname{grad}\rho \\ &\quad + |\operatorname{grad}\rho|^2 \big(2E_i(\rho) E_i - \operatorname{grad}\rho \big). \end{aligned}$$

Thus,

$$\begin{aligned} -\Delta\tau(\overline{\mathbf{1}}) &= -(m-2) \big\{ \operatorname{trace} \nabla^2 \operatorname{grad}\rho + \frac{3}{2} \operatorname{grad}(|\operatorname{grad}\rho|^2) \\ &\quad + \nabla_{\operatorname{grad}\rho} \operatorname{grad}\rho + (\Delta\rho - (m-2)|\operatorname{grad}\rho|^2) \operatorname{grad}\rho \big\}. \end{aligned}$$

In order to determine $\operatorname{trace} \widetilde{R}(d\overline{\mathbf{1}}, \tau(\overline{\mathbf{1}})) d\overline{\mathbf{1}}$, by using (4.2), we first have

$$\sum_{i=1}^m P(E_i, P(\operatorname{grad}\rho, E_i)) = |\operatorname{grad}\rho|^2 \sum_{i=1}^m P(E_i, E_i) = -(m-2)|\operatorname{grad}\rho|^2 \operatorname{grad}\rho,$$

and

$$\begin{aligned}\sum_{i=1}^{m} P(\operatorname{grad}\rho, P(E_i,E_i)) &= -(m-2)P(\operatorname{grad}\rho,\operatorname{grad}\rho)\\ &= -(m-2)|\operatorname{grad}\rho|^2\operatorname{grad}\rho.\end{aligned}$$

Also, since

$$P = d\rho\otimes I + I\otimes d\rho - g\otimes\operatorname{grad}\rho,$$

where $I : C(TM)\to C(TM)$, $I(X)=X$, $\forall X\in C(TM)$,

$$\begin{aligned}(\nabla_X P)(Y,Z)-(\nabla_Y P)(X,Z) &= -g(Z,\nabla_Y\operatorname{grad}\rho)X + g(Z,\nabla_X\operatorname{grad}\rho)Y\\ &\quad -g(Y,Z)\nabla_X\operatorname{grad}\rho + g(X,Z)\nabla_Y\operatorname{grad}\rho.\end{aligned}$$

We shall use the last relation for $X=Z=E_i$ and $Y=\operatorname{grad}\rho$ and sum up. We get

$$\begin{aligned}\sum_{i=1}^{m}\{(\nabla_{E_i}P)(\operatorname{grad}\rho,E_i)-(\nabla_{\operatorname{grad}\rho}P)(E_i,E_i)\} &= -(\Delta\rho)\operatorname{grad}\rho\\ &\quad +(m-2)\nabla_{\operatorname{grad}\rho}\operatorname{grad}\rho.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{trace}\widetilde{R}(d\overline{\mathbf{1}},\tau(\overline{\mathbf{1}}))d\overline{\mathbf{1}} &= \operatorname{trace}\widetilde{R}(\cdot,\tau(\overline{\mathbf{1}}))\cdot\\ &= -(m-2)\big\{-\operatorname{Ricci}(\operatorname{grad}\rho)-(\Delta\rho)\operatorname{grad}\rho\\ &\quad +(m-2)\nabla_{\operatorname{grad}\rho}\operatorname{grad}\rho\big\},\end{aligned}$$

and by summing up all of the above we obtain

$$\begin{aligned}\tau_2(\overline{\mathbf{1}}) &= -(m-2)\big(\operatorname{trace}\nabla^2\operatorname{grad}\rho+(2\Delta\rho-(m-2)|\operatorname{grad}\rho|^2)\operatorname{grad}\rho\\ &\quad -\frac{m-6}{2}\operatorname{grad}(|\operatorname{grad}\rho|^2)\big)+\operatorname{Ricci}(\operatorname{grad}\rho).\end{aligned}\tag{4.19}$$

□

Now, by using (4.7) and (4.19), we get

$$\tau_2(\widetilde{\mathbf{1}})+e^{-4\rho}\tau_2(\overline{\mathbf{1}}) = e^{-4\rho}(m-2)(m-6)\big\{\operatorname{grad}(|\operatorname{grad}\rho|^2)-|\operatorname{grad}\rho|^2\operatorname{grad}\rho\big\}.$$

From here we have

Proposition 4.21 ([11]). *If $m=6$ or $\operatorname{grad}(|\operatorname{grad}\rho|^2)=|\operatorname{grad}\rho|^2\operatorname{grad}\rho$, then $\overline{\mathbf{1}}:(M,g)\to(M,\widetilde{g})$ is biharmonic if and only if $\widetilde{\mathbf{1}}:(M,\widetilde{g})\to(M,g)$ is biharmonic, i.e. $\tau_2(\overline{\mathbf{1}})=0$ if and only if $\tau_2(\widetilde{\mathbf{1}})=0$.*

Remark 4.22 ([11])**.** In general, the condition

$$\operatorname{grad}(|\operatorname{grad}\rho|^2) = |\operatorname{grad}\rho|^2 \operatorname{grad}\rho$$

is not satisfied.

For example, on $M = \mathbb{R}^m$, $m \neq 6$, with the canonical metric, the isoparametric function $\rho(x^1, \ldots, x^m) = \ln x^1$ satisfies (4.18), but

$$\operatorname{grad}(|\operatorname{grad}\rho|^2) = \operatorname{grad}\frac{1}{(x^1)^2} = -\frac{2}{(x^1)^3}e_1,$$

while

$$|\operatorname{grad}\rho|^2 \operatorname{grad}\rho = \frac{1}{(x^1)^3}e_1,$$

thus the canonical metric g is biharmonic with respect to $\widetilde{g}$, but not conversely.

4.2 Biharmonic maps and warped product manifolds

The concept of warped metric was first introduced by R.L. Bishop and B. O'Neill in [26] and it plays an important role in differential geometry as well as in theoretical physics. In Riemannian geometry, warped products have offered new examples of Riemannian manifolds with special curvature properties like Einstein spaces (see [25, 72]), or (locally) symmetric spaces (see [24]). In Lorentzian geometry some important concepts can be expressed in terms of warped products and the geometric properties of Lorentzian warped products are of great interest due to their relativistic applications (see [20, 25, 92]).

4.2.1 Riemannian structure of warped products

Let M and N be two Riemannian manifolds equipped with Riemannian metrics g and h, respectively, and let $f \in C^\infty(M)$ be a positive function. Consider the product manifold $M \times N$ and denote by $\pi : M \times N \to M$ and $\eta : M \times N \to N$ its projections.

Definition 4.23. The *warped product* $M \times_{f^2} N$ is the product manifold $M \times N$ endowed with the Riemannian metric G_{f^2} defined, for $X, Y \in T_{(x,y)}(M \times N)$, by

$$G_{f^2}(X, Y) = g(d\pi(X), d\pi(Y)) + (f \circ \pi)^2 h(d\eta(X), d\eta(Y)).$$

The function f is called the *warping function* of the warped product.

In the following we shall need the expressions of the Levi-Civita connection and of the curvature tensor field for a warped product manifold (see, for example, [92, p.204] and [24]). Let $X, Y \in C(T(M \times N))$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, where $X_1, Y_1 \in C(TM)$

and $X_2, Y_2 \in C(TN)$. Denote by ∇ the Levi-Civita connection on the Riemannian product $M \times N$ with respect to the product metric G and by R its curvature tensor field. The Levi-Civita connection $\widetilde{\nabla}$ of $M \times_{f^2} N$ is given by

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + \frac{1}{2f^2} X_1(f^2)(0, Y_2) + \frac{1}{2f^2} Y_1(f^2)(0, X_2) \\ &\quad -\frac{1}{2} h(X_2, Y_2)(\operatorname{grad} f^2, 0),\end{aligned} \tag{4.20}$$

and the relation between the curvature tensor fields of G_{f^2} and G is

$$\begin{aligned}\widetilde{R}(X,Y) &= R(X,Y) + \frac{1}{2f^2}\Big\{ \big(\nabla^M_{Y_1} \operatorname{grad} f^2 - \frac{1}{2f^2} Y_1(f^2) \operatorname{grad} f^2, 0\big) \wedge_{G_{f^2}} (0, X_2) \\ &\quad -\big(\nabla^M_{X_1} \operatorname{grad} f^2 - \frac{1}{2f^2} X_1(f^2) \operatorname{grad} f^2, 0\big) \wedge_{G_{f^2}} (0, Y_2) \\ &\quad -\frac{1}{2f^2} |\operatorname{grad} f^2|^2 (0, X_2) \wedge_{G_{f^2}} (0, Y_2)\Big\},\end{aligned} \tag{4.21}$$

where

$$(X \wedge_{G_{f^2}} Y)Z = G_{f^2}(Z, Y)X - G_{f^2}(Z, X)Y,$$

for all $X, Y, Z \in C(T(M \times N))$.

4.2.2 Affine functions

Definition 4.24 ([65, 104]). A function $\alpha \in C^\infty(M)$ on a Riemannian manifold (M, g) is called an *affine function* if $\alpha \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}$ is an affine function for any geodesic $\gamma : I \to M$.

We recall the following

Lemma 4.25 ([104]). *Let (M, g) be a Riemannian manifold and $\alpha \in C^\infty(M)$. Then the following statements are equivalent:*

(i) *α is an affine function;*

(ii) $\operatorname{grad} \alpha$ *is a parallel vector field;*

(iii) *the Hessian of α vanishes identically;*

(iv) $\operatorname{grad} \alpha$ *is a Killing vector field.*

Remark 4.26. From the characterization of affine functions we deduce that they are totally geodesic and thus harmonic. In particular, when M is compact any affine function $\alpha \in C^\infty(M)$ is constant.

The classification result for complete, simply connected manifolds admitting a non-trivial affine function was given in [65], and then treated as a particular case in [104]. We present here another proof of this classification, based on a result of M. Svensson in [111].

Proposition 4.27 ([14]).

(i) *Let c be a positive real constant and take (P, q) to be a Riemannian manifold. Then the projection map $\overline{\eta} : P \times (\mathbb{R}, c^2 g_{can}) \to \mathbb{R}$ is an affine function on the Riemannian product $P \times (\mathbb{R}, c^2 g_{can})$.*

(ii) *Any complete, simply connected Riemannian manifold (M, g) admitting a non-trivial affine function is isometric to a Riemannian product $P \times (\mathbb{R}, c^2 g_{can})$, with $c \in \mathbb{R}_+$. Moreover, the affine function is in this case the projection map $\overline{\eta} : P \times (\mathbb{R}, c^2 g_{can}) \to \mathbb{R}$.*

Proof. In order to prove (i), consider $\{\partial/\partial t\}$ to be the orthonormal basis on $\mathbb{R}$ with the Euclidean metric. Then $\operatorname{grad} \overline{\eta} = \frac{1}{c^2} \partial/\partial t$ and this obviously is a parallel vector field on $P \times (\mathbb{R}, c^2 g_{can})$.

To prove (ii), we first use the fact that any horizontally homothetic submersion $\phi : (M, g) \to (N, h)$ with totally geodesic fibres and integrable horizontal distribution is locally the projection of a warped product onto its second factor [111]. In addition to this, if (M, g) is complete and M and N are simply connected, it is globally such a projection.

In the following we shall prove that any non-trivial affine function is a horizontally homothetic submersion with totally geodesic fibres and integrable horizontal distribution.

Denote by α the nontrivial affine function on M. As $|\operatorname{grad} \alpha| = \text{const} > 0$, α is a submersion with vertical distribution

$$T^V M = \ker d\alpha = (\operatorname{grad} \alpha)^{\perp},$$

and integrable horizontal distribution

$$T^H M = \operatorname{span}\{\operatorname{grad} \alpha\}.$$

For $v, w \in T^H_x M$, with $v = c_1 \operatorname{grad} \alpha$ and $w = c_2 \operatorname{grad} \alpha$, we have

$$\begin{aligned} h_{\alpha(x)}(d\alpha(v), d\alpha(w)) &= c_1 c_2 |\operatorname{grad} \alpha|_x^4 \\ &= |\operatorname{grad} \alpha|_x^2 g(v, w), \end{aligned}$$

and α is a horizontally conformal submersion of constant dilation, and so a horizontally homothetic submersion.

As α is an affine function, $\nabla_X \operatorname{grad}\alpha = 0$ and the second fundamental form of any fibre is identically zero. So α has totally geodesic fibres.

By summing up all of the above we conclude that M is isometric to a warped product $P \times_{f^2} \mathbb{R}$, with $f \in C^\infty(P)$, and α is the projection $P \times_{f^2} \mathbb{R} \to \mathbb{R}$ with dilation $1/f = |\operatorname{grad}\alpha|$. Since $|\operatorname{grad}\alpha|$ is constant we conclude. □

4.2.3 The biharmonicity of the inclusion

Let x_0 be an arbitrary point of M, and denote by

$$\mathbf{i}_{x_0} : (N, h) \to (M \times_{f^2} N, G_{f^2})$$

$$y \to (x_0, y)$$

the inclusion map of N at the x_0 level in $M \times_{f^2} N$. The goal of this section is to characterize the biharmonicity of $\mathbf{i}_{x_0}$ in terms of f.

We note that the inclusion $\mathbf{j}_{y_0} : M \to M \times_{f^2} N$, defined by $\mathbf{j}_{y_0}(x) = (x, y_0)$, $y_0 \in N$, is always a totally geodesic map, thus harmonic for any warping function $f \in C^\infty(M)$.

We have

Theorem 4.28 ([14]). *The bitension field of the inclusion $\mathbf{i}_{x_0} : (N, h) \to M \times_{f^2} N$ is given by*

$$\tau_2(\mathbf{i}_{x_0}) = \frac{n^2}{8}(\operatorname{grad}(|\operatorname{grad} f^2|^2), 0) \circ \mathbf{i}_{x_0}.$$

Proof. Let $\{F_a\}_{a=1}^n$ be a local orthonormal frame field on (N, h). By making use of (4.20) we obtain the following expression for the tension field of $\mathbf{i}_{x_0}$,

$$\begin{aligned}
\tau(\mathbf{i}_{x_0}) &= \operatorname{trace}_h \nabla d\mathbf{i}_{x_0} \\
&= \sum_{a=1}^{n} \left\{ \nabla_{F_a} d\mathbf{i}_{x_0}(F_a) - d\mathbf{i}_{x_0}(\nabla^N_{F_a} F_a) \right\} \\
&= \sum_{a=1}^{n} \left\{ \widetilde{\nabla}_{(0,F_a)}(0, F_a) - (0, \nabla^N_{F_a} F_a) \right\} \circ \mathbf{i}_{x_0} \\
&= \sum_{a=1}^{n} \left\{ (0, \nabla^N_{F_a} F_a) - \frac{1}{2} h(F_a, F_a)(\operatorname{grad} f^2, 0) - (0, \nabla^N_{F_a} F_a) \right\} \circ \mathbf{i}_{x_0} \\
&= -\frac{n}{2}(\operatorname{grad} f^2, 0) \circ \mathbf{i}_{x_0}.
\end{aligned}$$

Note that $\mathbf{i}_{x_0}$ is harmonic if and only if $(\operatorname{grad} f^2)_{x_0} = 0$.

Let us now compute the bitension field of the inclusion. First we write down the rough Laplacian. We have

$$\begin{aligned}\nabla_{F_a}\tau(\mathbf{i}_{x_0}) &= -\frac{n}{2}\nabla_{F_a}(\operatorname{grad} f^2,0)\circ\mathbf{i}_{x_0}\\ &= -\frac{n}{2}\big(\widetilde{\nabla}_{(0,F_a)}(\operatorname{grad} f^2,0)\big)\circ\mathbf{i}_{x_0}\\ &= -\frac{n}{2}\Big(\frac{1}{2f^2}\ \operatorname{grad} f^2(f^2)(0,F_a)\Big)\circ\mathbf{i}_{x_0}\\ &= \Big(-\frac{n}{4f^2}\,|\operatorname{grad} f^2|^2(0,F_a)\Big)\circ\mathbf{i}_{x_0}.\end{aligned}$$

Thus

$$\begin{aligned}\nabla_{F_a}\nabla_{F_a}\tau(\mathbf{i}_{x_0}) &= -\frac{n}{2}\Big\{\frac{1}{2f^2}|\operatorname{grad} f^2|^2\,\widetilde{\nabla}_{(0,F_a)}(0,F_a)\Big\}\circ\mathbf{i}_{x_0}\\ &= -\frac{n}{2}\Big\{\frac{1}{2f^2}|\operatorname{grad} f^2|^2\big((0,\nabla^N_{F_a}F_a)-\frac{1}{2}(\operatorname{grad} f^2,0)\big)\Big\}\circ\mathbf{i}_{x_0}.\end{aligned}$$

Moreover,

$$\begin{aligned}\nabla_{\nabla^N_{F_a}F_a}\tau(\mathbf{i}_{x_0}) &= -\frac{n}{2}\Big(\widetilde{\nabla}_{(0,\nabla^N_{F_a}F_a)}(\operatorname{grad} f^2,0)\Big)\circ\mathbf{i}_{x_0}\\ &= -\frac{n}{2}\Big(\frac{1}{2f^2}|\operatorname{grad} f^2|^2(0,\nabla^N_{F_a}F_a)\Big)\circ\mathbf{i}_{x_0}\end{aligned}$$

and by summing all terms we find

$$-\Delta\tau(\mathbf{i}_{x_0}) = \Big(\frac{n^2}{8f^2}|\operatorname{grad} f^2|^2(\operatorname{grad} f^2,0)\Big)\circ\mathbf{i}_{x_0}.$$

Now, since $\nabla^M_{\operatorname{grad}\rho}\operatorname{grad}\rho=\frac{1}{2}\operatorname{grad}(|\operatorname{grad}\rho|^2)$, taking into account (4.21) we obtain

$$\begin{aligned}\widetilde{R}((0,F_a),(\operatorname{grad} f^2,0))(0,F_a) &= \frac{1}{2}\big(\nabla^M_{\operatorname{grad} f^2}\operatorname{grad} f^2-\frac{1}{2f^2}|\operatorname{grad} f^2|^2\operatorname{grad} f^2,0\big)\\ &= \frac{1}{4}\big(\operatorname{grad}(|\operatorname{grad} f^2|^2)-\frac{1}{f^2}|\operatorname{grad} f^2|^2\operatorname{grad} f^2,0\big).\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{trace}_h\widetilde{R}(d\mathbf{i}_{x_0},\tau(\mathbf{i}_{x_0}))d\mathbf{i}_{x_0} &= \frac{n^2}{8}\Big\{-\big(\operatorname{grad}(|\operatorname{grad} f^2|^2),0\big)\\ &\quad+\frac{1}{f^2}|\operatorname{grad} f^2|^2(\operatorname{grad} f^2,0)\Big\}\circ\mathbf{i}_{x_0},\end{aligned}$$

and we conclude. □

Corollary 4.29 ([14]). *The inclusion map $\mathbf{i}_{x_0}:N\to M\times_{f^2}N$ is a proper biharmonic map if and only if x_0 is not a critical point for f^2, but is a critical point for $|\operatorname{grad} f^2|^2$.*

Corollary 4.30 ([14]). *Each and every inclusion $\mathbf{i}_x:N\to M\times_{f^2}N$, $x\in M$, is a proper biharmonic map if and only if $\operatorname{grad} f^2$ is a non-zero constant norm vector field.*

Corollary 4.31 ([14]). *Let (M,g) be a Riemannian manifold with a positive nontrivial affine function f^2 and let (N,h) be an arbitrary Riemannian manifold. Then any inclusion $\mathbf{i}_x : N \to M \times_{f^2} N$, $x \in M$, is a proper biharmonic map.*

Remark 4.32. The existence of a function with constant norm gradient on a given Riemannian manifold (M,g) was studied in [104], where the author underlined that such existence is considerably controlled by the Ricci curvature of M. Of course, any affine function has constant norm gradient and, conversely, if M is a complete connected Riemannian manifold of non-negative Ricci curvature, then any smooth function with constant norm gradient is an affine function. We point out that functions with constant norm gradient form a subclass of transnormal functions.

An important example of Riemannian manifold admitting a non-affine function of constant norm gradient is provided by the warped products: the projection $\overline{\pi} : \mathbb{R} \times_{f^2} N \to \mathbb{R}$ has constant norm gradient. Another such example is given by Busemann functions on Hadamard manifolds. Also, in [104], the author offers a wide range of examples of functions with $|\operatorname{grad}\alpha| = 1$, other than affine functions, on the m-dimensional hyperbolic space and gives a characterization of the Busemann functions from this point of view.

In the following, by using the above results, we shall obtain some new examples of proper biharmonic maps and we shall recover existing ones, previously studied in different settings.

Example 4.33 (The compact case)**.** Take M to be compact and let $f \in C^\infty(M)$ be a non-constant positive function. Then there exists $x_0 \in M$ a maximum point for $|\operatorname{grad} f^2|^2$, so the inclusion $\mathbf{i}_{x_0} : N \to M \times_{f^2} N$ is a proper biharmonic map.

Note that the equivalent conditions of Corollary 4.30 imply the non-existence of critical points for f^2 on M. As a consequence, if M is a compact manifold the inclusions $\mathbf{i}_x : N \to M \times_{f^2} N$, $x \in M$, are all biharmonic if and only if they are harmonic, and this holds only when f is constant.

Example 4.34. We shall now apply Theorem 4.28 in order to obtain new examples of biharmonic maps.

Take M to be

$$\mathbb{R}^m_+ = \{(x^1, ..., x^m) \in \mathbb{R}^m : x^i > 0, \forall i = 1, ..., m\}$$

and let (N,h) be arbitrary. By Corollary 4.30, all the inclusions $\mathbf{i}_x : N \to \mathbb{R}^m_+ \times_{f^2} N$ are proper biharmonic maps if $f^2(x) = \langle x_0, x\rangle + a$, $\forall x \in \mathbb{R}^m_+$, where $x_0 \in \mathbb{R}^m_+$ and $a \in \mathbb{R}_+$. When $m = 1$, all the inclusions $\mathbf{i}_x : N \to \mathbb{R}_+ \times_{f^2} N$ are proper biharmonic maps if and only if $f^2(x) = x_0 x + a$, $\forall x \in \mathbb{R}^+$, where $x_0, a \in \mathbb{R}_+$. We thus obtain a generalization of rotational surfaces with all parallels proper biharmonic (see Chapter 3 and [32]).

Another example is obtained by taking M as $\mathbb{R}^m \setminus \{0\}$ and $f(x) = \sqrt{|x|}$, for $x \in M$. In this case $\operatorname{grad} f^2(x) = \frac{x}{|x|}$, and, as before, $|\operatorname{grad} f^2|$ is constant and the hypothesis of Corollary 4.30 are satisfied. This can be also verified by noting that f^2 is in fact the projection onto the first factor of the warped product $(0,\infty) \times_{t^2} \mathbb{S}^{m-1} = \mathbb{R}^m \setminus \{0\}$ and using Remark 4.32. We underline the fact that f^2 is no longer, as in the previous example, an affine function on M.

Example 4.35 (Biharmonic inclusions in spheres as warped products)**.** By using Theorem 4.28 we find a well known example of biharmonic map. Let $\mathbb{S}^m$ be the $m-$dimensional unit Euclidean sphere. Then, for a given point $p \in \mathbb{S}^m$, the space $\mathbb{S}^m \setminus \{\pm p\}$ is the warped product

$$(0,\pi) \times_{\sin^2 t} \mathbb{S}^{m-1}.$$

Consider now the inclusion map

$$\mathbf{i}_{t_0} : \mathbb{S}^{m-1} \to (0,\pi) \times_{\sin^2 t} \mathbb{S}^{m-1}.$$

By Theorem 4.28, $\mathbf{i}_{t_0}$ is a proper biharmonic map if and only if $t_0 \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$. This example was first obtained in a different manner in the study of proper biharmonic submanifolds of $\mathbb{S}^m$ (see [29, 30]).

Example 4.36 (Biharmonic inclusions in rotational surfaces in $\mathbb{R}^3$ as warped products)**.** Let $S \subset \mathbb{R}^3$ be a rotational surface obtained by rotating the curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$, parametrized by arc length,

$$\alpha : \begin{cases} x = x(t) \\ y = 0 \\ z = z(t) \end{cases}$$

in the xz-plane, around the z-axis.

It is immediate to see that the rotational surface endowed with the induced metric can be expressed as the warped product

$$S = I \times_{x^2} \mathbb{S}^1.$$

By Corollary 4.29, the inclusion $\mathbf{i}_{t_0} : \mathbb{S}^1 \to S$ is proper biharmonic if and only if t_0 satisfies

$$\dot{x}^2(t_0) + x(t_0)\ddot{x}(t_0) = 0 \quad \text{and} \quad \dot{x}(t_0) \neq 0.$$

We recover in this way a result obtained in a different context in [32].

4.2.4 The case of product maps

In this section we give a method to construct proper biharmonic maps of product type.

Let $\mathbf{1}_M : M \to M$ be the identity map of M and $\psi : N \to N$ be an arbitrary harmonic map. The product map

$$\phi = \mathbf{1}_M \times \psi : M \times N \to M \times N$$

is clearly a harmonic map. If we modify the product metric on $M \times N$ (either as the domain or the codomain), using a warping function $f \in C^\infty(M)$, then the product map may fail to remain harmonic; nonetheless we can find the condition on f so that the product map preserves its biharmonicity.

We first examine the case when the domain is endowed with a warped metric, thus we are looking at the product map

$$\overline{\phi} = \overline{\mathbf{1}_M \times \psi} : M \times_{f^2} N \to M \times N. \tag{4.22}$$

We have

Theorem 4.37 ([14]). *Let $\psi : N \to N$ be a harmonic map and let $f \in C^\infty(M)$ be a positive function. Then $\overline{\phi} = \overline{\mathbf{1}_M \times \psi} : M \times_{f^2} N \to M \times N$ is a proper biharmonic map if and only if f is a non-constant solution of*

$$\operatorname{trace}_g \nabla^2 \operatorname{grad} \ln f + \operatorname{Ricci}^M(\operatorname{grad} \ln f) + \frac{n}{2} \operatorname{grad}(|\operatorname{grad} \ln f|^2) = 0. \tag{4.23}$$

Proof. Let $\{E_i\}_{i=1}^m$ and $\{F_a\}_{a=1}^n$ be local orthonormal frame fields on (M, g) and (N, h), respectively. Then $\big\{(E_i, 0), \frac{1}{f}(0, F_a)\big\}$ is a local orthonormal frame field on the warped product $(M \times_{f^2} N, G_{f^2})$. Using the harmonicity of ψ, the tension field of $\overline{\phi}$ is

$$\begin{aligned}
\tau(\overline{\phi}) &= \operatorname{trace}_{G_{f^2}} \nabla d\overline{\phi} \\
&= \sum_{i=1}^m \Big\{ \nabla_{(E_i,0)} d\overline{\phi}(E_i, 0) - d\overline{\phi}(\widetilde{\nabla}_{(E_i,0)}(E_i, 0)) \Big\} \\
&\quad + \frac{1}{f^2} \sum_{a=1}^n \big\{ \nabla_{(0,F_a)} d\overline{\phi}(0, F_a) - d\overline{\phi}(\widetilde{\nabla}_{(0,F_a)}(0, F_a)) \big\} \\
&= \frac{1}{f^2}(0, \tau(\psi)) + \frac{1}{f^2} \sum_{a=1}^n d\overline{\phi}\big(\frac{1}{2} h(F_a, F_a)(\operatorname{grad} f^2, 0)\big) \\
&= n\big(\operatorname{grad} \ln f, 0\big).
\end{aligned}$$

From the expression for the curvature tensor field for the Riemannian product $M \times N$

we get

$$\begin{aligned}\operatorname{trace}_{G_{f^2}} R\big(d\overline{\phi},\tau(\overline{\phi})\big)d\overline{\phi} &= n\Big\{\sum_{i=1}^{m} R\big((E_i,0),(\operatorname{grad}\ln f,0)\big)(E_i,0)\\ &\quad +\frac{1}{f^2}\sum_{a=1}^{m} R\big((0,d\psi(F_a)),(\operatorname{grad}\ln f,0)\big)(0,d\psi(F_a))\Big\}\\ &= -n\big(\operatorname{Ricci}^M(\operatorname{grad}\ln f),0\big).\end{aligned}$$

Also, as $\nabla_{(0,X_2)}(Y_1,0)=0$, for all $X_1\in C(TM)$ and $Y_2\in C(TN)$, it follows that $\nabla_{(0,X_2)}\tau(\overline{\phi})=0$, thus

$$\begin{aligned}-\Delta\tau(\overline{\phi}) &= \operatorname{trace}_{G_{f^2}}\nabla^2\tau(\overline{\phi})\\ &= \sum_{i=1}^{m}\Big\{\nabla_{(E_i,0)}\nabla_{(E_i,0)}\tau(\overline{\phi})-\nabla_{\tilde{\nabla}_{(E_i,0)}(E_i,0)}\tau(\overline{\phi})\Big\}\\ &\quad +\frac{1}{f^2}\sum_{a=1}^{n}\Big\{\nabla_{(0,F_a)}\nabla_{(0,F_a)}\tau(\overline{\phi})-\nabla_{\tilde{\nabla}_{(0,F_a)}(0,F_a)}\tau(\overline{\phi})\Big\}\\ &= n\big(\operatorname{trace}_g\nabla^2\operatorname{grad}\ln f+n\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln f,0\big).\end{aligned}$$

Finally, summing the two terms up, we find

$$\tau_2(\overline{\phi}) = n\big(\operatorname{trace}_g\nabla^2\operatorname{grad}\ln f+\operatorname{Ricci}^M(\operatorname{grad}\ln f)+\frac{n}{2}\operatorname{grad}(|\operatorname{grad}\ln f|^2),0\big),$$

and we conclude. □

Remark 4.38. We note that, besides its harmonicity, the map ψ gives no other contribution to the bitension field of $\overline{\phi}$. This enables us to construct a wide range of examples of proper biharmonic maps of product type (4.22).

Definition 4.39. A function $f\in C^\infty(M)$ is called a *biharmonic warping function* if the identity $\overline{\mathbf{1}}:M\times_{f^2}N\to M\times N$ is a proper biarmonic map for some N of dimension greater than zero.

From Theorem 4.37 we immediately deduce

Proposition 4.40 ([14]). *Let $\psi:N\to N$ be a harmonic map, and $f\in C^\infty(M)$ a positive non-constant function. Then $\overline{\mathbf{1}_M\times\psi}:M\times_{f^2}N\to M\times N$ is a proper biharmonic map if and only if f is a biharmonic warping function.*

Corollary 4.41 ([14]). *Let M be non-compact and $f\in C^\infty(M)$ a positive non-constant function such that $\ln f$ is an affine function on M. Then f is a biharmonic warping function.*

Remark 4.42. Equation (4.23) can be written in the form

$$-\Delta_H \operatorname{grad} \ln f + 2\operatorname{Ricci}^M(\operatorname{grad} \ln f) + \frac{n}{2}\operatorname{grad}|\operatorname{grad} \ln f|^2 = 0, \tag{4.24}$$

where $\Delta_H : C(TM) \to C(TM)$ is defined by $\Delta_H X = -\operatorname{trace}_g \nabla^2 X + \operatorname{Ricci}^M(X)$.

Using the musical isomorphisms $\sharp : T_x^*M \to T_xM$, $\flat : T_xN \to T_x^*M$, $x \in M$, the condition (4.24) can be rewritten as

$$-\Delta\omega + \frac{n}{2}d|\omega|^2 + 2(\operatorname{Ricci}^M \omega^\sharp)^\flat = 0, \tag{4.25}$$

where $\omega = d(\ln f)$.

Proposition 4.43 ([14]). *Let M be a compact Riemannian manifold of negative Ricci curvature. Then there exist no biharmonic warping functions on M.*

Proof. Equation (4.25) together with the Weitzenböck formula for one forms on M (see, for example, [50, p.13]),

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega\rangle - |\nabla\omega|^2 - \operatorname{Ricci}^M(\omega^\sharp, \omega^\sharp),$$

implies

$$\frac{1}{2}\Delta|\omega|^2 = \frac{n}{2}\langle d|\omega|^2, \omega\rangle - |\nabla\omega|^2 + \operatorname{Ricci}^M(\omega^\sharp, \omega^\sharp).$$

Consider now x_0 to be a maximum point of $|\omega|^2$. Then $(d|\omega|^2)_{x_0} = 0$ and $(\Delta|\omega|^2)_{x_0} \geq 0$, and, as Ricci^M is negative definite, we conclude. □

Equation (4.25) also assumes a simpler form when M is an Einstein manifold and we have

Proposition 4.44 ([14]). *Let M be an Einstein space. If $\rho \in C^\infty(M)$ is an isoparametric function, then it admits away from its critical points a local reparameterization f which is a biharmonic warping function.*

Proof. Consider c to be the Einstein constant of M. By the hypothesis, $|d\rho|^2 = \gamma\circ\rho$ and, by choosing s to be a solution of the differential equation $s' = \frac{1}{\gamma^{\frac{1}{2}}}$, we can reparametrize ρ such that $s = s \circ \rho$ has $|ds|^2 = 1$. The function s is isoparametric with $\Delta s = \sigma \circ s$, for a smooth real function σ. We are looking for $\alpha = \alpha \circ s$, such that $\omega = d\alpha$ satisfies (4.25).

For the desired reparameterization α we have

$$d\alpha = (\alpha' \circ s)ds,$$

$$\Delta\alpha = (-\alpha'' + \alpha'\sigma) \circ s,$$

and from here

$$\Delta\omega = dd^*d\alpha = d(\Delta\alpha) = (-\alpha''' + \alpha''\sigma + \alpha'\sigma') \circ s\, ds,$$

$$d|\omega|^2 = 2(\alpha'\alpha'') \circ s\, ds,$$

$$(\mathrm{Ricci}^M \omega^\sharp)^\flat = c\,\omega = c\alpha' \circ s\, ds.$$

By substituting into (4.25), we obtain a second order ordinary differential equation in $y = \alpha'$

$$y'' - \sigma y' + nyy' + (2c - \sigma')y = 0. \tag{4.26}$$

The general theory of differential equations offers local solutions for this type of problem. Finally, by integrating we obtain $\alpha(t) = \int y(t)dt$, t being the real parameter, and $f = e^{\alpha \circ s}$. □

Example 4.45. Let $M = \mathbb{R}^m \setminus \{0\}$, $m \neq 2$, be endowed with the canonical metric and $s : M \to \mathbb{R}$, $s(x^1, x^2, ..., x^m) = \sqrt{(x^1)^2 + ... + (x^m)^2}$. Then s is isoparametric with $|ds|^2 = 1$ and $\Delta s = -(m-1)/s$, so $\sigma(t) = -(m-1)/t$. In this case (4.26) becomes

$$y'' + \frac{m-1}{t}y' + nyy' - \frac{m-1}{t^2}y = 0. \tag{4.27}$$

Looking for particular solutions of type $y = a/t$, $a \in \mathbb{R}$, for (4.27) we get $a = 0$ or $a = \frac{2(2-m)}{n}$, so $\alpha(t) = \frac{2(2-m)}{n} \ln t$ and $f = s^{\frac{2(2-m)}{n}}$.

Example 4.46. When $M = \mathbb{R}$ we can classify all the biharmonic warping functions. In fact, in this case, (4.23) becomes

$$\alpha''' + n\alpha'\alpha'' = 0, \tag{4.28}$$

where $\alpha = \ln f$. We are only interested in the global solutions of (4.28). Considering the case when $\alpha'' = 0$ we obtain the biharmonic warping functions generated by the affine functions on $\mathbb{R}$,

$$f(t) = e^{a_1 t + a_2}, \quad a_1, a_2 \in \mathbb{R}.$$

Assume now that $\alpha'' \neq 0$. Equation (4.28) becomes

$$\frac{\alpha''}{\frac{2c_1}{n} - (\alpha')^2} = \frac{n}{2}, \qquad c_1 \in \mathbb{R}. \tag{4.29}$$

We have three cases,

(i) If $c_1 > 0$, integrating (4.29), we obtain

$$\left| \frac{\sqrt{\frac{2c_1}{n}} + \alpha'}{\sqrt{\frac{2c_1}{n}} - \alpha'} \right| = e^{\sqrt{2nc_1}(t + c_2)}, \qquad c_2 \in \mathbb{R},$$

with the solutions

$$\alpha' = \sqrt{\frac{2c_1}{n}}\tanh\big(\sqrt{\frac{nc_1}{2}}(t+c_2)\big)$$

and

$$\alpha' = \sqrt{\frac{2c_1}{n}}\frac{1}{\tanh\big(\sqrt{\frac{nc_1}{2}}(t+c_2)\big)}.$$

The first equation provides the general global solution of (4.28),

$$\alpha(t) = \ln\Big(a_1\big(\cosh(a_2t+a_3)\big)^{\frac{2}{n}}\Big),$$

with $a_1, a_2 > 0$, and $a_3 \in \mathbb{R}$. This leads to

$$f(t) = a_1\big(\cosh(a_2t+a_3)\big)^{\frac{2}{n}}.$$

In addition to this, from the second equation, we obtain the local solution

$$f(t) = a_1\big(\sinh(a_2t+a_3)\big)^{\frac{2}{n}}.$$

(ii) If $c_1 = 0$, integrating (4.29), we obtain $-\frac{\alpha''}{(\alpha')^2} = \frac{n}{2}$, thus $\alpha' = \frac{2}{n}\frac{1}{t+c_2}, c_2 \in \mathbb{R}$ and

$$\alpha = \frac{2}{n}\ln|t+c_2| + c_3, \qquad c_3 \in \mathbb{R}.$$

This provides a new local solution,

$$f(t) = a_2(t+a_3)^{\frac{2}{n}},$$

where $a_2 > 0$ and $a_3 \in \mathbb{R}$.

(iii) If $c_1 < 0$, integrating (4.29), we obtain

$$\frac{1}{\sqrt{-\frac{2c_1}{n}}}\arctan\frac{\alpha'}{\sqrt{-\frac{2c_1}{n}}} = -\frac{n}{2}(t+c_2), \qquad c_2 \in \mathbb{R},$$

thus

$$\alpha' = \sqrt{-\frac{2c_1}{n}}\tan\big(\sqrt{-\frac{nc_1}{2}}(t+c_2)\big).$$

By integrating and recalling that $\alpha = \ln f$, we obtain another local solution,

$$f(t) = a_1\big(\cos(a_2t+a_3)\big)^{\frac{2}{n}},$$

where $a_1, a_2 > 0$ and $a_3 \in \mathbb{R}$.

We shall now analyze the case of the projection $\overline{\pi} : M \times_{f^2} N \to M$. Recall that the projection onto the second factor, $\overline{\eta} : M \times_{f^2} N \to N$, is a horizontally homothetic submersion with totally geodesic fibres, and thus harmonic (see, for example [8, pag.53]).

By computing the second fundamental form of $\overline{\pi}$ we get $\tau(\overline{\pi}) = n\,\mathrm{grad}(\ln f)\circ\overline{\pi}$, and the bitension field of $\overline{\pi}$ has the expression

$$\tau_2(\overline{\pi}) = n\Big\{\,\mathrm{trace}_g\,\nabla^2\,\mathrm{grad}\ln f + \mathrm{Ricci}^M(\mathrm{grad}\ln f) + \frac{n}{2}\,\mathrm{grad}(|\,\mathrm{grad}\ln f|^2)\Big\}\circ\overline{\pi}.$$

As a consequence,

Proposition 4.47 ([14]). *The projection $\overline{\pi} : M \times_{f^2} N \to M$ is proper biharmonic if and only if f is a biharmonic warping function.*

Consider now the case of the product map

$$\widetilde{\phi} = \widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N, \tag{4.30}$$

that is, when the product metric on the codomain is warped. In this case, as we shall see, the role of ψ increases, involving its energy density. The difficulties are due to the contribution of the curvature tensor field of $M \times_{f^2} N$ in the expression for the bitension field. We have

Theorem 4.48 ([14]). *Let $\mathbf{1}_M : M \to M$ be the identity map, $\psi : N \to N$ be a non-constant harmonic map and let $f \in C^\infty(M)$ be a non-constant positive function. Then $\widetilde{\phi} = \widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ is a proper biharmonic map if and only if*

$$\begin{aligned} e(\psi) \quad \Big(- \operatorname{trace}_g \nabla^2 \operatorname{grad} f^2 - \operatorname{Ricci}^M(\operatorname{grad} f^2) + \tfrac{e(\psi)}{2} \operatorname{grad} |\operatorname{grad} f^2|^2 \Big) \\ + (\Delta e(\psi)) \operatorname{grad} f^2 = 0 \end{aligned} \tag{4.31}$$

and

$$d\psi(\operatorname{grad} e(\psi)) = 0. \tag{4.32}$$

Proof. Let $\{E_i\}_{i=1}^m$ and $\{F_a\}_{a=1}^n$ be local orthonormal frame fields on (M, g) and (N, h), respectively. The tension field of $\widetilde{\phi}$ is given by

$$\begin{aligned} \tau(\widetilde{\phi}) &= \operatorname{trace}_G \nabla d\widetilde{\phi} \\ &= \sum_{i=1}^m \{\nabla^{\widetilde{\phi}}_{(E_i,0)} d\widetilde{\phi}(E_i, 0) - d\widetilde{\phi}(\nabla_{(E_i,0)}(E_i, 0))\} \\ &\quad + \sum_{a=1}^n \{\nabla^{\widetilde{\phi}}_{(0,F_a)} d\widetilde{\phi}(0, F_a) - d\widetilde{\phi}(\nabla_{(0,F_a)}(0, F_a))\} \\ &= \sum_{a=1}^n \Big\{ \big(0, \nabla^{\psi}_{F_a} d\psi(F_a) - d\psi(\nabla^N_{F_a} F_a)\big) - \frac{1}{2} h(d\psi(F_a), d\psi(F_a))(\operatorname{grad} f^2, 0) \circ \widetilde{\phi} \Big\} \\ &= (0, \tau(\psi)) - e(\psi)(\operatorname{grad} f^2, 0) \circ \widetilde{\phi} \\ &= -e(\psi)(\operatorname{grad} f^2, 0) \circ \widetilde{\phi}. \end{aligned}$$

Next, taking into account (4.21), the curvature term becomes

$$\begin{aligned}
\operatorname{trace}_G \widetilde{R}(d\widetilde{\phi}, \tau(\widetilde{\phi}))d\widetilde{\phi} &= \sum_{i=1}^{m} \widetilde{R}((E_i,0),\tau(\widetilde{\phi}))(E_i,0) \\
&\quad + \sum_{a=1}^{n} \widetilde{R}((0,d\psi(F_a)),\tau(\widetilde{\phi}))(0,d\psi(F_a)) \\
&= -e(\psi)\Big\{ \sum_{i=1}^{m} \big(R^M(E_i,\operatorname{grad} f^2)E_i,0\big) \\
&\quad + \sum_{a=1}^{n} \frac{1}{2} h(d\psi(F_a),d\psi(F_a))\Big(\nabla^M_{\operatorname{grad} f^2}\operatorname{grad} f^2 - \frac{1}{2f^2}|\operatorname{grad} f^2|^2 \operatorname{grad} f^2, 0\Big)\circ\widetilde{\phi}\Big\} \\
&= -e(\psi)\big(\operatorname{trace}_g R^M(,\operatorname{grad} f^2),0\big) \\
&\quad - \frac{e(\psi)^2}{2}\Big(\operatorname{grad}|\operatorname{grad} f^2|^2 - \frac{1}{f^2}|\operatorname{grad} f^2|^2\operatorname{grad} f^2, 0\Big)\circ\widetilde{\phi}.
\end{aligned}$$

For the rough Laplacian we have

$$\begin{aligned}
-\Delta^{\widetilde{\phi}}\tau(\widetilde{\phi}) &= -e(\psi)(\operatorname{trace}_g \nabla^2 \operatorname{grad} f^2, 0) + (\Delta e(\psi))(\operatorname{grad} f^2,0)\circ\widetilde{\phi} \\
&\quad + e(\psi)^2\Big(\frac{1}{2f^2}|\operatorname{grad} f^2|^2\operatorname{grad} f^2,0\Big)\circ\widetilde{\phi} \\
&\quad - \Big(\frac{1}{f^2}|\operatorname{grad} f^2|^2\Big)\ (0,d\psi(\operatorname{grad} e(\psi))).
\end{aligned}$$

Finally, by putting together all terms, we determine the bitension field of $\widetilde{\phi}$ and we conclude. □

Remark 4.49.

(i) Using the same argument of Corollary 4.41, if M is non-compact, $\psi : N \to N$ is a harmonic map with constant energy density, and $f \in C^\infty(M)$ is a non-constant function such that f^2 is an affine function on M, then $\widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ is a proper biharmonic map. Indeed, since f^2 is an affine function it follows that $\operatorname{trace}_g \nabla^2 \operatorname{grad} f^2 + \operatorname{Ricci}^M(\operatorname{grad} f^2) = 0$, $\operatorname{grad}|\operatorname{grad} f^2|^2 = 0$ and, since $\psi : N \to N$ has constant energy density, $\Delta e(\psi) = 0$ and $\operatorname{grad} e(\psi) = 0$, thus equations (4.31) and (4.32) are identically satisfied.

(ii) As in Proposition 4.43, if M is compact of negative Ricci curvature and $\psi : N \to N$ is a harmonic map of constant energy density, there exist no non-constant functions f that could render $\widetilde{\mathbf{1}_M \times \psi} : M \times N \to M \times_{f^2} N$ proper biharmonic.

(iii) Let M be an Einstein space and $\psi : N \to N$ a harmonic map of constant energy density λ. The analogy of (4.25) is

$$-\Delta\omega - \frac{\lambda}{2} d|\omega|^2 + 2(\operatorname{Ricci}^M\ \omega^\sharp)^\flat = 0, \tag{4.33}$$

where $\omega = d(f^2)$. Using the same argument as in Proposition 4.44 we conclude that if $\rho \in C^\infty(M)$ is an isoparametric function, then it admits, away from critical points, a local reparameterization $f : U \subset M \to (0, \infty)$ such that $\widetilde{1_U \times \psi} : U \times N \to U \times_{f^2} N$ is a proper biharmonic map.

Example 4.50. If $M = \mathbb{R}$, then (4.31) becomes

$$\alpha''' - \frac{n}{2}\alpha'\alpha'' = 0,$$

where $\alpha = f^2$. In this case we have no global solutions.

Indeed, when $\alpha'' = 0$ we get the local solutions, provided by affine functions on $\mathbb{R}_+$,

$$f(t) = \sqrt{a_1 t + a_2}, \quad a_1 > 0, a_2 \in \mathbb{R}_+.$$

For $\alpha'' \neq 0$ we obtain the local solutions:

$$f(t) = \sqrt{a_3 - \ln\big(\sinh(a_1 t + a_2)\big)^{\frac{4}{n}}},$$

$$f(t) = \sqrt{a_3 - \ln\big(\cosh(a_1 t + a_2)\big)^{\frac{4}{n}}},$$

$$f(t) = \sqrt{a_3 - \ln\big(\cos(a_1 t + a_2)\big)^{\frac{4}{n}}},$$

$$f(t) = \sqrt{a_3 - \ln(t + a_2)^{\frac{4}{n}}},$$

where $a_1 > 0$.

Thus, we have the full classification of warping functions that render the identity $\widetilde{1} : M \times N \to M \times_{f^2} N$ a proper biharmonic map.

Remark 4.51. The following generalizations of Theorems 4.37 and 4.48 can be obtained, but the equations characterizing the biharmonicity of the product maps prove to be quite complicated.

Theorem 4.52. *Let $\varphi : M \to M$ and $\psi : N \to N$ be two harmonic maps and $f \in C^\infty(M)$ a non-constant function with $d\varphi(\operatorname{grad} \ln f) \neq 0$. Then $\overline{\phi} = \overline{\varphi \times \psi} : M \times_{f^2} N \to M \times N$ is a proper biharmonic map if and only if*

$$-J^\varphi(d\varphi(\operatorname{grad} \ln f)) + n\nabla_{\operatorname{grad} \ln f} d\varphi(\operatorname{grad} \ln f) = 0, \tag{4.34}$$

where J^φ denotes the Jacobi operator associated to φ.

Theorem 4.53. *Let $\varphi : M \to M$ and $\psi : N \to N$ be two harmonic maps and let $f \in C^\infty(M)$ be a non-constant function. Then $\widetilde{\phi} = \widetilde{\varphi \times \psi} : M \times N \to M \times_{f^2} N$ is a proper biharmonic map if and only if*

$$e(\psi)\Big(J^{\varphi}(\operatorname{grad} f^2\circ\varphi)+\frac{e(\psi)}{2}\operatorname{grad}|\operatorname{grad} f^2|^2\circ\varphi\Big)+(\Delta e(\psi))\operatorname{grad} f^2\circ\varphi=0 \quad (4.35)$$

and

$$\Big(\frac{1}{f^2}|\operatorname{grad} f^2|^2\Big)\circ\varphi\; d\psi(\operatorname{grad} e(\psi))=0. \quad (4.36)$$

4.3 Axially symmetric biharmonic maps

This section is devoted to the study of the biharmonicity of axially symmetric maps, by using the warped products setting.

Definition 4.54. A map $\phi:\mathbb{R}^m\setminus\{0\}\to\mathbb{R}\times_{f^2}\mathbb{S}^{n-1}$ is called *axially symmetric* if there exist a map $\varphi:\mathbb{S}^{m-1}\to\mathbb{S}^{n-1}$ and a function $\rho:(0,\infty)\to\mathbb{R}$ such that

$$\phi=\rho\times\varphi:(0,\infty)\times_{t^2}\mathbb{S}^{m-1}\to\mathbb{R}\times_{f^2}\mathbb{S}^{n-1}.$$

We first determine the tension field of an axially symmetric map ϕ. Take $\partial/\partial t\in C(T(0,\infty))$, $\partial/\partial s\in C(T(0,\infty))$ and let $\{E_i\}_{i=1}^{m-1}$ be a local orthonormal frame field on the unit Euclidean sphere $\mathbb{S}^{m-1}$. Then

$$\begin{aligned}\nabla d\phi\big((\partial/\partial t,0),(\partial/\partial t,0)\big) &= \Big(\nabla_{\partial/\partial t}d\rho(\partial/\partial t)-d\rho\big(\nabla_{\partial/\partial t}\partial/\partial t\big),0\Big)\\ &= \rho''\big(\partial/\partial s,0\big)\circ\phi\end{aligned}$$

and

$$\begin{aligned}\nabla d\phi\big((0,E_i),(0,E_i)\big) &= \nabla_{(0,E_i)}(0,d\varphi(E_i))-d\phi\big(\widetilde{\nabla}_{(0,E_i)}(0,E_i)\big)\\ &= \big(0,\nabla_{E_i}d\varphi(E_i)\big)-\frac{1}{2}\langle d\varphi(E_i),d\varphi(E_i)\rangle(\operatorname{grad} f^2,0)\circ\phi\\ &\quad -d\phi\big((0,\nabla_{E_i}E_i)-\frac{1}{2}(\operatorname{grad} t^2,0)\big)\\ &= \Big(t\rho'-\langle d\varphi(E_i),d\varphi(E_i)\rangle\,(ff')\circ\rho\Big)\big(\partial/\partial s,0\big)\circ\phi\\ &\quad +\big(0,\nabla d\varphi(E_i,E_i)\big)\end{aligned}$$

As $\{(\partial/\partial t,0),\frac{1}{t}(0,E_i)\}_{i=1}^{m-1}$ is a local orthonormal frame field on $(0,\infty)\times_{t^2}\mathbb{S}^{m-1}$, by summing all of the above, we get

$$\tau(\phi)_{(t,x)}=F(t,x)\Big(\frac{\partial}{\partial s},0\Big)_{\phi(t,x)}+\frac{1}{t^2}(0,\tau(\varphi)_x), \quad (4.37)$$

where $F:\mathbb{R}^m\setminus\{0\}\to\mathbb{R}$ is defined by

$$F(t,x)=\rho''(t)+\frac{m-1}{t}\rho'(t)-\frac{2e(\varphi)(x)}{t^2}f(\rho(t))f'(\rho(t)). \quad (4.38)$$

Analyzing expression (4.37) of the tension field of ϕ we have

Proposition 4.55 ([14]). *An axially symmetric map $\phi = \rho\times\varphi : \mathbb{R}^m\setminus\{0\} \to \mathbb{R}\times_{f^2}\mathbb{S}^{n-1}$ is harmonic if and only if either φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a non-constant solution of*

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2k}{t^2}f(\rho(t))f'(\rho(t)) = 0. \tag{4.39}$$

or φ is harmonic, $\rho = \rho_0$ is constant and $f(\rho_0)f'(\rho_0) = 0$.

Remark 4.56. A general equation which includes (4.39) as a special case was studied in [6].

Consider now φ to be a harmonic map and let us compute the rough Laplacian. We have

$$\nabla_{(\partial/\partial t,0)}\tau(\phi) = \frac{\partial F}{\partial t}\big(\partial/\partial s, 0\big)\circ\phi,$$

so

$$\nabla^2\tau(\phi)\big((\partial/\partial t,0),(\partial/\partial t,0)\big) = \frac{\partial^2 F}{\partial t^2}\big(\partial/\partial s,0\big)\circ\phi.$$

Also,

$$\begin{aligned}\nabla_{(0,E_i)}\tau(\phi) &= E_i(F)\big(\partial/\partial s,0\big)\circ\phi + F\nabla_{(0,E_i)}\big(\partial/\partial s,0\big)\circ\phi\\ &= E_i(F)\big(\partial/\partial s,0\big)\circ\phi + \frac{f'(\rho)}{f(\rho)}F\big(0,d\varphi(E_i)\big),\end{aligned}$$

so

$$\begin{aligned}\nabla_{(0,E_i)}\nabla_{(0,E_i)}\tau(\phi) &= \Big(E_i(E_i(F)) - Ff'^2(\rho)\langle d\varphi(E_i),d\varphi(E_i)\rangle\Big)\big(\partial/\partial s,0\big)\circ\phi\\ &\quad +2\frac{f'(\rho)}{f(\rho)}E_i(F)\big(0,d\varphi(E_i)\big) + \frac{f'(\rho)}{f(\rho)}F\big(0,\nabla_{E_i}d\varphi(E_i)\big)\end{aligned}$$

and

$$\begin{aligned}\nabla_{\widetilde{\nabla}_{(0,E_i)}(0,E_i)}\tau(\phi) &= \Big(\big(\nabla_{E_i}E_i\big)(F) - t\frac{\partial F}{\partial t}\Big)\big(\partial/\partial s,0\big)\circ\phi\\ &\quad +F\frac{f'(\rho)}{f(\rho)}\big(0,d\varphi(\nabla_{E_i}E_i)\big).\end{aligned}$$

From here, as $\tau(\varphi) = 0$ and

$$\operatorname{trace}_{G_{t^2}}\widetilde{R}(d\phi,\tau(\phi))d\phi = 2F\frac{f(\rho)f''(\rho)}{t^2}e(\varphi)\big(\partial/\partial s,0\big)\circ\phi,$$

the expression for the bitension field of ϕ can be written as

$$\begin{aligned}\tau_2(\phi) &= \Big\{\frac{\partial^2 F}{\partial t^2} + \frac{m-1}{t}\frac{\partial F}{\partial t} - \frac{2e(\varphi)}{t^2}\big(f'^2(\rho)+f(\rho)f''(\rho)\big)F\\ &\quad +\frac{2f(\rho)f'(\rho)}{t^4}\Delta e(\varphi)\Big\}\big(\partial/\partial s,0\big)\circ\phi - \frac{4}{t^4}f'^2(\rho)\big(0,d\varphi(\operatorname{grad}e(\varphi))\big)\end{aligned}$$

The calculations above lead to the following

Theorem 4.57 ([14]). *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k \geq 0$. Then $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is biharmonic if and only if ρ is a solution of*

$$F'' + \frac{m-1}{t}F' - \frac{2k}{t^2}\big(f'^2(\rho) + f(\rho)f''(\rho)\big)F = 0,$$

where

$$F = \rho'' + \frac{m-1}{t}\rho' - \frac{2k}{t^2}f(\rho)f'(\rho).$$

In the following we shall treat the case when $k = 0$, i.e. φ is constant.

Proposition 4.58 ([14]). *The axially symmetric map $\phi = \rho \times \varphi_0 : \mathbb{R}^m \setminus \{0\} \to \mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is:*

(i) *harmonic if and only if*

$$\rho(t) = \begin{cases} c_1 \ln t + c_2, & \text{when} \quad m = 2 \\ c_1 t^{2-m} + c_2, & \text{when} \quad m \geq 3, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$;

(ii) *proper biharmonic if and only if*

$$\rho(t) = \begin{cases} c_1 t^2 \ln t + c_2 t^2 + c_3 \ln t + c_4, & \text{when} \quad m = 2 \\ c_1 \ln t + c_2 t^2 + \frac{c_3}{t^2} + c_4, & \text{when} \quad m = 4 \\ c_1 t^{4-m} + c_2 t^2 + c_3 t^{2-m} + c_4, & \text{otherwise}, \end{cases}$$

where $c_1^2 + c_2^2 \neq 0$.

Remark 4.59. Since the inclusion $\mathbb{R} \times \{\varphi_0\}$ in $\mathbb{R} \times_{f^2} \mathbb{S}^{n-1}$ is totally geodesic, the harmonicity of ϕ is equivalent to the harmonicity of ϕ thought of as a real function. In fact, in Proposition 4.58 (i) we obtain the well known formulae of harmonic radial functions on $\mathbb{R}^n \setminus \{0\}$. In the same way, using Proposition 4.4 in [98], for the biharmonic case, we get all the biharmonic radial functions on $\mathbb{R}^n \setminus \{0\}$.

4.3.1 Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{R}^n \setminus \{0\}$

As before, expressing the Euclidian space $\mathbb{R}^m \setminus \{0\}$ as the warped product

$$\mathbb{R}^m \setminus \{0\} = (0, \infty) \times_{t^2} \mathbb{S}^{m-1},$$

a smooth map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \infty)$ such that, for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \rho(|y|)\varphi\Big(\frac{y}{|y|}\Big).$$

In this case the function $F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ of Theorem 4.57 is given by

$$F(t,x) = \rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2e(\varphi)(x)}{t^2}\rho(t),$$

so,

Proposition 4.60 ([14]). *The axially symmetric map $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, with φ non-constant, is harmonic if and only if φ is an eigenmap of eigenvalue $2k > 0$ and*

$$\rho(t) = c_1 t^{A_1} + c_2 t^{A_2}, \tag{4.40}$$

where

$$A_{1,2} = \frac{-(m-2) \pm \sqrt{(m-2)^2 + 8k}}{2},$$

and $c_1, c_2 \in \geq 0$ with $c_1^2 + c_2^2 \neq 0$.

Proof. From Proposition 4.55, φ is an eigenmap of eigenvalue $2k > 0$ and ρ is a solution of

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{2k}{t^2}\rho(t) = 0. \tag{4.41}$$

By making the change of variable $t = e^x$, (4.41) becomes the following homogeneous linear equation with constant coefficients

$$\frac{d^2\rho}{dx^2} + (m-2)\frac{d\rho}{dx} - 2k\rho = 0,$$

which has the desired solution. □

Assume that φ is an eigenmap of eigenvalue $2k > 0$. Then the bitension field given in (4.40) becomes

$$\tau_2(\phi) = \big(F'' + \frac{m-1}{t}F' - \frac{2k}{t^2}F\big)\frac{\partial}{\partial s},$$

where

$$F(t) = \rho'' + \frac{m-1}{t}\rho' - \frac{2k}{t^2}\rho.$$

We shall now study the equation $\tau_2(\phi) = 0$.

Case I. $[\rho' = 0]$. Then $\tau_2(\phi) = 0$ is equivalent to $k = -m + 4$.

Recall that (see [52] or Chapter 2) a map between spheres $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is an eigenmap if and only if its components $\Phi^\alpha : \mathbb{S}^{m-1} \to \mathbb{R}$ are spherical harmonics of the same order, i.e. restrictions of harmonic homogeneous polynomial of the same degree on $\mathbb{R}^m$. Also, for any eigenmap $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$, there exists $h \in \mathbb{N}$ (the homogeneity degree) such that its eigenvalue is

$$2e(\varphi) = 2k = h(m+h-2).$$

Consequently,

(i) for $m \geq 4$, ϕ cannot be biharmonic.

(ii) for $m = 3$, ϕ is a proper biharmonic map if and only if φ is an eigenmap of homogeneous degree $h = 1$; as an example, consider $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ to be the identity map.

(iii) for $m = 2$, ϕ is a proper biharmonic map if and only if φ is an eigenmap of homogeneous degree $h = 2$; the map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(z) = \overline{z}^2$ provides us such an example.

Case II. $[\rho' \neq 0]$. In this case $\phi = \rho \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a proper biharmonic map if and only if

$$\rho(t) = \begin{cases} c_1 t^3 + c_2 t \ln t + c_3 t + c_4 t^{-1}, & \text{when} \quad m = 2 \,\text{and}\, k = \frac{1}{2} \\ \frac{c_1}{2(m+2A_1)} t^{A_1+2} + \frac{c_2}{2(m+2A_2)} t^{A_2+2} + c_3 t^{A_1} + c_4 t^{A_2}, & \text{otherwise.} \end{cases} \tag{4.42}$$

where $A_{1,2} = \frac{-(m-2) \pm \sqrt{(m-2)^2+8k}}{2}$, $c_1^2 + c_2^2 \neq 0$ and c_1, c_2, c_3, c_4 arbitrary such that ρ takes values in $(0, \infty)$.

Remark 4.61. An important class of axially symmetric maps of $\mathbb{R}^m \setminus \{0\}$ is given by

$$\begin{aligned} \phi : \mathbb{R}^m \setminus \{0\} &\longrightarrow \mathbb{R}^m \setminus \{0\} \\ y &\longmapsto y/|y|^\ell, \end{aligned}$$

which for $\ell = 1$ provides the radial projection which is harmonic, and thus biharmonic, from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{S}^{m-1}$, and for $\ell = 2$ provides the well known Kelvin transformation.

Consider now $\ell \neq 0, 1$. For these maps, $\rho(t) = 1/t^{\ell-1}$ and $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ is the identity map. An easy computation shows that ϕ is harmonic if and only if $m = \ell$.

Using (4.42) we conclude that ϕ is a proper biharmonic map if and only if $m = \ell + 2$, or $\ell = -2$ and m arbitrary; the second case is obvious since ϕ becomes a third degree polynomial map. For $\ell = 2$ this result was first obtained in [7].

Note that the proper biharmonic map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $\phi(y) = y/|y|^{m-2}$, is harmonic with respect to the conformal metric on the domain given by $\widetilde{g} = |y|^{\frac{4}{3-m}} g_{\text{can}}$. This property is similar to that of the Kelvin transformation proved by B. Fuglede in [59, p.143].

Consider now the metric $h = \phi^* g_{\text{can}}$ on $\mathbb{R}^m \setminus \{0\}$,

$$h_y(X, Y) = \frac{\ell(\ell-2)}{|y|^{2\ell+2}} \langle y, X \rangle \langle y, Y \rangle + \frac{1}{|y|^{2\ell}} \langle X, Y \rangle, \qquad y \in \mathbb{R}^m \setminus \{0\}, X, Y \in T_y(\mathbb{R}^m \setminus \{0\}).$$

Obviously, the identity map $\overline{1} : (\mathbb{R}^m \setminus \{0\}, g_{\text{can}}) \to (\mathbb{R}^m \setminus \{0\}, h)$ is proper biharmonic if and only if ϕ is proper biharmonic, i.e. g_{can} is proper biharmonic with respect to h

if and only if $\ell = m-2$ or $\ell = -2$. In the same manner, since $\phi^{-1}(x) = x/|x|^{\frac{\ell}{\ell-1}}$, the identity map $\widetilde{1} : (\mathbb{R}^m \setminus \{0\}, h) \to (\mathbb{R}^m \setminus \{0\}, g_{\text{can}})$ is proper biharmonic if and only if ϕ^{-1} is proper biharmonic, i.e. h is proper biharmonic with respect to g_{can} if and only if $\ell = \frac{m-2}{m-3}$, $m \geq 4$, or $\ell = 2/3$. Thus, the generalized Kelvin transformation allows us to construct examples of flat metrics which are proper biharmonic with respect to the canonical metric on $\mathbb{R}^m \setminus \{0\}$.

We also conclude that $\overline{1}$ and $\widetilde{1}$ are simultaneously proper biharmonic if and only if $m = 4$ and $\ell = 2$. Moreover, h is conformally equivalent to g_{can} if and only if $\ell = 2$. Notice that in this case ϕ and ϕ^{-1} coincide.

4.3.2 Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{S}^n \setminus \{\pm p\}$

The space $\mathbb{S}^n \setminus \{\pm p\}$, where $p \in \mathbb{S}^m$, can be described as a warped product

$$\mathbb{S}^n \setminus \{\pm p\} = (0, \pi) \times_{\sin^2 s} \mathbb{S}^{n-1}.$$

So, a smooth map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \pi)$ such that for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \Big(\cos \rho(|y|), \varphi\Big(\frac{y}{|y|}\Big) \sin \rho(|y|) \Big).$$

The map $F : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ of (4.38) is given in this case by

$$F(t, x) = \rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{1}{t^2} e(\varphi) \sin 2\rho(t).$$

Proposition 4.62 ([14]). *The axially symmetric map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is harmonic if and only if either φ is harmonic and $\rho \equiv \frac{\pi}{2}$, or φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a non-constant solution of*

$$\rho''(t) + \frac{m-1}{t}\rho'(t) - \frac{k}{t^2} \sin 2\rho(t) = 0. \tag{4.43}$$

For the biharmonicity we shall only study the case when φ is an eigenmap of eigenvalue $2k > 0$ and $\rho = \rho_0 \neq \frac{\pi}{2}$ is constant, i.e. the image of ϕ is contained in a hypersphere of $\mathbb{S}^n$. We underline here the fact that $e(\phi)$ is not constant, thus the result in Example 3.14 of Chapter 3 does not apply. The function F becomes $F = F(t) = -\frac{k}{t^2} \sin 2\rho_0$, so the bitension field of ϕ assumes the form

$$\tau_2(\phi) = \frac{2k \sin 2\rho_0}{t^4} \Big(m - 4 + k \cos 2\rho_0 \Big) \frac{\partial}{\partial s},$$

and we get

Proposition 4.63 ([14]). *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k > 0$. The axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is proper biharmonic if and only if $\frac{4-m}{k} \in (-1, 1)$ and $\rho_0 \in \left\{ \frac{1}{2} \arccos \frac{4-m}{k}, \pi - \frac{1}{2} \arccos \frac{4-m}{k} \right\}$.*

Corollary 4.64 ([14]). *Let $\varphi : \mathbb{S}^3 \to \mathbb{S}^{n-1}$ be an arbitrary non-constant eigenmap. Then $\phi = \rho_0 \times \varphi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{S}^n \setminus \{\pm p\}$ is a proper biharmonic map if and only if $\rho_0 \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\}$.*

In the following we shall present some examples of eigenmaps satisfying the condition $\frac{4-m}{k} \in (-1, 1)$.

Example 4.65. The map $\varphi_h : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi_h(z) = \overline{z}^h$, $h > 2$, is an eigenmap with $\frac{4-m}{k} = \frac{4}{h^2} \in (0, 1)$.

Example 4.66. The Veronese map $\widetilde{\varphi}_{m-1} : \mathbb{S}^{m-1}(\sqrt{\frac{2m}{m-1}}) \to \mathbb{S}^{\frac{(m-1)(m+2)}{2}-1}$, $m \geq 3$, induces an eigenmap $\varphi_{m-1} : \mathbb{S}^{m-1} \to \mathbb{S}^{\frac{(m-1)(m+2)}{2}-1}$ with $\frac{4-m}{k} = \frac{4-m}{m} \in (-1, 1), \forall m \geq 3$.

Example 4.67. The Hopf maps $\varphi : \mathbb{S}^3 \to \mathbb{S}^2$, $\varphi : \mathbb{S}^7 \to \mathbb{S}^4$, $\varphi : \mathbb{S}^{15} \to \mathbb{S}^8$ also satisfy the above condition.

4.3.3 Axially symmetric biharmonic maps from $\mathbb{R}^m \setminus \{0\}$ to $\mathbb{H}^n \setminus \{0\}$

Consider $\mathbb{R}^{n+1}$ equipped with the standard Lorenzian inner product

$$\langle v, w \rangle_1 = v^1 w^1 + ... + v^n w^n - v^{n+1} w^{n+1}, \quad v, w \in \mathbb{R}^{n+1}.$$

Then the hyperboloid model of $\mathbb{H}^n$ is (the upper sheet of the hyperboloid)

$$H^n_+ = \{(x^1, ..., x^n, x^{n+1}) \in \mathbb{R}^{n+1} : (x^1)^2 + ... + (x^n)^2 - (x^{n+1})^2 = -1, x^{n+1} > 0\},$$

equipped with the Riemannian metric given by the restriction to H^n_+ of $\langle\ ,\ \rangle_1$

Let $p = (0, \ldots, 0, 1)$, then $H^n_+ \setminus \{(0, ..., 0, 1)\}$ is diffeomorphic to $(0, \infty) \times \mathbb{S}^{n-1}$ by

$$(x^1, ..., x^n, x^{n+1}) \longrightarrow \Big(s, \frac{(x^1, ..., x^n)}{\sinh s} \Big),$$

where $x^{n+1} = \cosh s$. In these new coordinates, the Riemannian metric on $\mathbb{H}^n \setminus \{p\}$ has the expression

$$g = ds^2 + \sinh^2 s \; g_{\mathbb{S}^{n-1}},$$

so $\mathbb{H}^m \setminus \{p\}$ can be thought of as the warped product

$$\mathbb{H}^n \setminus \{p\} = (0, \infty) \times_{\sinh^2 s} \mathbb{S}^{n-1}.$$

Let $\phi : \mathbb{R}^m \to \mathbb{H}^n$ be a smooth map. The map ϕ is *axially symmetric* if there exists a map $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ and a function $\rho : (0, \infty) \to (0, \infty)$ such that for $y \in \mathbb{R}^m \setminus \{0\}$,

$$\phi(y) = \Big(\cosh \rho(|y|), \varphi\Big(\frac{y}{|y|}\Big) \sinh \rho(|y|) \Big).$$

Proposition 4.68 ([14]). *The axially symmetric map $\phi : \mathbb{R}^m \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ is harmonic if and only if φ is an eigenmap of eigenvalue $2e(\varphi) = 2k \geq 0$ and ρ is a solution of*

$$\rho''(t) + \frac{n-1}{t}\rho'(t) - \frac{k}{t^2}\sinh 2\rho(t) = 0. \tag{4.44}$$

For the biharmonicity, as for the case of the sphere, we shall only consider the case when $\rho = \rho_0$ is constant and $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ is an eigenmap of eigenvalue $2k > 0$. We have

$$F = F(t) = -\frac{k}{t^2}\sinh 2\rho_0.$$

The bitension field of ϕ assumes the form

$$\tau_2(\phi) = \frac{2k\sinh 2\rho_0}{t^4}\Big(m - 4 + k\cosh 2\rho_0\Big)\frac{\partial}{\partial s}.$$

Proposition 4.69 ([14]). *Let $\varphi : \mathbb{S}^{m-1} \to \mathbb{S}^{n-1}$ be an eigenmap of eigenvalue $2k > 0$.*

(i) *If $m \geq 3$, then the axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^m \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ cannot be biharmonic.*

(ii) *If $m = 2$, then the axially symmetric map $\phi = \rho_0 \times \varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{H}^n \setminus \{p\}$ is proper biharmonic if and only if φ has the homogeneity degree $h = 1$ and $\rho_0 = \frac{1}{2}\ln(4 + \sqrt{15})$.*

Example 4.70. The map $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(z) = \overline{z}$, is an eigenmap of homogeneity degree $h = 1$ and so gives a proper biharmonic map from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{H}^2 \setminus \{p\}$.

Chapter 5

Biharmonic curves in spheres

Due to the non-existence results for proper biharmonic submanifolds in negatively curved manifolds presented in Chapter 2, efforts have been concentrated on the study of biharmonic submanifolds in spheres. The first ones to be considered were the proper biharmonic curves.

The goal of the present chapter is to give a geometric approach for the study of the proper biharmonic curves of the 3-dimensional unit Euclidean sphere $\mathbb{S}^3$, since it was proved that the proper biharmonic curves of $\mathbb{S}^n$, $n \geq 3$, are essentially the ones of $\mathbb{S}^3$ composed with a totally geodesic embedding of $\mathbb{S}^3$ in $\mathbb{S}^n$. We also study the biharmonic curves on 3-dimensional spheres endowed with distorted metrics called the Berger spheres. The new results of this chapter can be found in [12] and [19].

The first section contains general characterization results for biharmonic curves in Riemannian manifolds which were obtained in [31] by expressing the biharmonic equation in terms of the Frenet frame associated to the curve.

In the next section we present the classification result for proper biharmonic curves of $\mathbb{S}^n$, and by using the stereographic projection we obtain the image in $\mathbb{R}^3$ of the biharmonic curves of $\mathbb{S}^3$.

The last section deals with proper biharmonic curves in Berger spheres. There are several ways to introduce the 3-dimensional Berger metrics; here we choose to define them as distortions of the canonical metric on $\mathbb{S}^3$ constructed by performing a biconformal change of metric with respect to the Hopf fibration. Even if they were originally constructed in order to provide counter-examples to various geometric conjectures (see [61] and [119]), their geometry has proved to be interesting for a wide range of other aspects ([22, 52, 103]). The study of the behavior of biharmonic curves under these distortions comes as a natural question. In this study we use the same techniques used in [33], for the case of the Heisenberg space, which have been afterwards developed for the general case of biharmonic curves in $SO(2)$-isotropic 3-dimensional manifolds (see

[31]). We obtain the explicit parametrization of proper biharmonic curves in Berger spheres (see Theorem 5.19) and analyze the particular case of Legendre curves (see Corollary 5.20).

5.1 Biharmonic curves on Riemannian manifolds

We shall first present some general results obtained in [31] concerning the biharmonic equation for curves parametrized by arc length on Riemannian manifolds.

Let $\gamma : I \to (N, h)$ be a curve parametrized by arc length from an open interval $I \subset \mathbb{R}$ to a Riemannian manifold. By denoting with $T = \dot{\gamma}$ the unit tangent vector field along the curve, the expression for the tension field of γ is $\tau(\gamma) = \nabla_T T$ and the biharmonic equation (3.6) becomes

$$\nabla_T^3 T - R(T, \nabla_T T)T = 0. \tag{5.1}$$

Equation (5.1) can be rewritten in terms of the associated Frenet frame.

Definition 5.1 (see, for example, [27]). Let (N^n, h) be a Riemannian manifold and $\gamma : I \subset \mathbb{R} \to (N^n, h)$ a curve parametrized by arc length. Then γ is called a *Frenet curve of osculating order* r, $1 \leq r \leq m$, if there exist orthonormal vector fields $\{F_i\}_{i=1}^r$ along γ such that

$$\begin{aligned} F_1 &= d\gamma(\partial/\partial s), \\ \nabla^\gamma_{\partial/\partial s} F_1 &= k_1 F_2, \\ \nabla^\gamma_{\partial/\partial s} F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, \ldots, r-1, \\ \nabla^\gamma_{\partial/\partial s} F_r &= -k_{r-1} F_{r-1}, \end{aligned}$$

where the functions $\{k_1, k_2, \ldots, k_{r-1}\}$ are positive functions on I, called the *curvatures* of γ.
Note that $F_1 = T = \dot{\gamma}$.

Remark 5.2. A geodesic is a Frenet curve of osculating order 1; a *circle* is a Frenet curve of osculating order 2 with constant k_1; a *helix of order* r, $r \geq 3$, is a Frenet curve of osculating order r with constant $k_1, \ldots, k_{r-1}$; a helix of order 3 is called, simply, helix.

We also recall that, as in $\mathbb{R}^3$, two helices of the same curvature k_1 and torsion k_2 in $\mathbb{S}^3$ are congruent, i.e. there exists an isometry of $\mathbb{S}^3$ which takes one onto another (see, for example, [60]).

By using the Frenet frame, the biharmonic equation of a curve parametrized by arc length reduces to a differential system in the first three curvatures.

Proposition 5.3 ([31]). *Let $\gamma : I \subset \mathbb{R} \to (N^n, h)$, $n \geq 2$, be a Frenet curve of osculating order r, from an open interval of $\mathbb{R}$ into a Riemannian manifold (N, g). Then γ is biharmonic if and only if*

$$\begin{cases} k_1\dot{k}_1 = 0 \\ \ddot{k}_1 - k_1^3 - k_1k_2^2 + k_1R(F_1,F_2,F_1,F_2) = 0 \\ 2\dot{k}_1k_2 + k_1\dot{k}_2 + k_1R(F_1,F_2,F_1,F_3) = 0 \\ k_1k_2k_3 + k_1R(F_1,F_2,F_1,F_4) = 0 \\ k_1R(F_1,F_2,F_1,F_i) = 0, \qquad i = 5,\ldots,r \\ k_1R(F_1,F_2,F_1,\zeta) = 0, \qquad \forall \zeta \in (\mathrm{span}\{F_i\}_{i=1}^r)^\perp \subset \gamma^{-1}TN. \end{cases}$$

Proof. With respect to its Frenet frame, the biharmonic equation (5.1) for γ becomes

$$\begin{aligned} \nabla^3_{F_1}F_1 - R(F_1,\nabla_{F_1}F_1)F_1 &= -3k_1\dot{k}_1F_1 + (\ddot{k}_1 - k_1^3 - k_1k_2^2)F_2 \\ &\quad +(2\dot{k}_1k_2 + k_1\dot{k}_2)F_3 + k_1k_2k_3F_4 - k_1R(F_1,F_2)F_1 \\ &= 0, \end{aligned}$$

and we conclude. □

We are interested in the proper biharmonic solutions and we have

Corollary 5.4 ([31]). *Let $\gamma : I \subset \mathbb{R} \to (N^n, h)$, $n \geq 2$, be a Frenet curve of osculating order r from an open interval of $\mathbb{R}$ into a Riemannian manifold (N, g). Then γ is proper biharmonic if and only if*

$$\begin{cases} k_1 = \text{constant} \neq 0 \\ k_1^2 + k_2^2 = R(F_1,F_2,F_1,F_2) \\ \dot{k}_2 = -R(F_1,F_2,F_1,F_3) \\ k_2k_3 = -R(F_1,F_2,F_1,F_4) \\ R(F_1,F_2,F_1,F_i) = 0, \qquad i = 5,\ldots,r \\ R(F_1,F_2,F_1,\zeta) = 0, \qquad \forall \zeta \in (\mathrm{span}\{F_i\}_{i=1}^r)^\perp \subset \gamma^{-1}TN. \end{cases} \tag{5.2}$$

5.2 Biharmonic curves on Euclidean spheres

Along with Corollary 5.4, which allows us to conclude that the proper biharmonic curves in spheres are helices, the following result is fundamental for their complete classification.

Theorem 5.5 ([30]). *Let $\gamma : I \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a curve parametrized by arc length. Then it is biharmonic on $\mathbb{S}^n$ if and only if, as a curve in $\mathbb{R}^{n+1}$, it satisfies*

$$\gamma^{iv} + 2\ddot{\gamma} + (1 - k_1^2)\gamma = 0, \tag{5.3}$$

where k_1 is the curvature of γ.

By integrating equation (5.3) we obtain

Theorem 5.6 ([30]). *Let $\gamma : I \to \mathbb{S}^n$, $n \geq 3$, be a curve parametrized by arc length. Then γ is proper-biharmonic if and only if it is either the circle*

$$\gamma(s) = \tfrac{1}{\sqrt{2}}\cos(\sqrt{2}s)e_1 + \tfrac{1}{\sqrt{2}}\sin(\sqrt{2}s)e_2 + \tfrac{1}{\sqrt{2}}e_3,$$

or a helix

$$\gamma(s) = \tfrac{1}{\sqrt{2}}\cos(As)e_1 + \tfrac{1}{\sqrt{2}}\sin(As)e_2 + \tfrac{1}{\sqrt{2}}\cos(Bs)e_3 + \tfrac{1}{\sqrt{2}}\sin(Bs)e_4,$$

where A and B are given by

$$A = \sqrt{1+k_1}, \qquad B = \sqrt{1-k_1}, \qquad k_1 \in (0,1),$$

and $\{e_i\}$ are constant unit vectors orthogonal to each other.

5.2.1 Biharmonic curves on $\mathbb{S}^3$ via the stereographic projection

Since from Theorem 5.6 it follows that, up to a totally geodesic embedding, the proper-biharmonic curves of $\mathbb{S}^n$ are those of $\mathbb{S}^3$, in the following we shall offer a more geometric approach for the study of biharmonic curves of $\mathbb{S}^3$.

Notice first that all the biharmonic helices of $\mathbb{S}^3$ are geodesics of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$. We shall study here the conditions under which a curve on $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ is biharmonic on $\mathbb{S}^3$.

Let

$$N = \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) \times \mathbb{S}^1(\tfrac{1}{\sqrt{2}}) = \Big\{p = (x^1, x^2, x^3, x^4) : (x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2 = \frac{1}{2}\Big\}.$$

The set of all vector fields tangent to N is given by

$$C(TN) = \big\{X = (X^1, X^2, X^3, X^4) : X^1x^1 + X^2x^2 = X^3x^3 + X^4x^4 = 0\big\}.$$

Let $p \in N$ and $\eta(p) = (x^1, x^2, -x^3, -x^4)$. Then η satisfies

$$|\eta|^2 = 1, \qquad \langle \eta, p\rangle = 0 \qquad \text{and} \qquad \langle \eta, X\rangle = 0, \quad \forall X \in C(TN),$$

i.e. η is a unit section in the normal bundle of N in $\mathbb{S}^3$.

For $X = (X^1, X^2, X^3, X^4) \in C(TN)$, we have

$$\nabla^{\perp}_X \eta = 0 \qquad \text{and} \qquad A_\eta X = (-X^1, -X^2, X^3, X^4),$$

where A is the Weingarten operator, and $\nabla^{\perp}$ is the normal connection in the normal bundle of N in $\mathbb{S}^3$. We also recall that N is a flat, parallelized Riemannian manifold.

Indeed,

$$U_1(p) = (-x^2, x^1, 0, 0) \qquad \text{and} \qquad U_2(p) = (0, 0, -x^4, x^3)$$

are globally defined vector fields on N and they satisfy

$$|U_1|^2 = |U_2|^2 = \frac{1}{2}, \qquad \langle U_1, U_2 \rangle = 0$$

and

$$\nabla^N_{U_i} U_j = 0, \quad \forall\, i, j = 1, 2.$$

The second fundamental form of N in $\mathbb{S}^3$ is given by

$$B(U_1, U_1) = -\frac{1}{2}\eta, \qquad B(U_1, U_2) = 0, \qquad B(U_2, U_2) = \frac{1}{2}\eta.$$

Let $\mathbf{i} : N \to \mathbb{S}^3$ be the inclusion map and for $\widetilde{\gamma} : I \to N$, a curve parametrized by arc length on N, denote by $\gamma = \mathbf{i} \circ \widetilde{\gamma} : I \to \mathbb{S}^3$. We get

$$\tau(\gamma) = \tau(\widetilde{\gamma}) + \mu\eta,$$

where

$$\begin{cases} T = \dot{\widetilde{\gamma}} = f_1 U_1 + f_2 U_2 \\ \tau(\widetilde{\gamma}) = \nabla^N_T T = \dot{f}_1 U_1 + \dot{f}_2 U_2 \\ \mu = \frac{(f_2)^2 - (f_1)^2}{2}. \end{cases} \tag{5.4}$$

In order to determine the bitension field of γ, we have

$$\nabla^{\mathbb{S}^3}_T \tau(\gamma) = \nabla^N_T \tau(\widetilde{\gamma}) + \mu(f_1 U_1 - f_2 U_2) + \frac{3}{2}\dot{\mu}\eta,$$

$$\begin{aligned} \nabla^{\mathbb{S}^3}_T \nabla^{\mathbb{S}^3}_T \tau(\gamma) &= \nabla^N_T \nabla^N_T \tau(\widetilde{\gamma}) + ((\mu\dot{f}_1) + \frac{3}{2}\dot{\mu} f_1) U_1 - ((\mu\dot{f}_2) + \frac{3}{2}\dot{\mu} f_2) U_2 \\ &\quad + \frac{1}{2}(-f_1\ddot{f}_1 + f_2\ddot{f}_2 + 3\ddot{\mu} - 2\mu)\eta \end{aligned}$$

and

$$R^{\mathbb{S}^3}(T, \tau(\gamma))T = -\tau(\widetilde{\gamma}) - \mu\eta,$$

so

$$\tau_2(\gamma)^{\perp} = \frac{1}{2}(-f_1\ddot{f}_1 + f_2\ddot{f}_2 + 3\ddot{\mu})\eta.$$

Theorem 5.7 ([19]). *Let $\widetilde{\gamma} : I \to N$ be a curve parametrized by arc length. Then $\gamma = \mathbf{i} \circ \widetilde{\gamma} : I \to \mathbb{S}^3$ is proper biharmonic if and only if $\widetilde{\gamma}$ is a geodesic of slope different from ± 1.*

Proof. If $\gamma : I \to \mathbb{S}^3$ is a biharmonic curve, then $\tau_2(\gamma)^\perp = 0$, i.e.

$$-f_1\ddot{f}_1 + f_2\ddot{f}_2 + 3\ddot{\mu} = 0. \tag{5.5}$$

From $2\mu = -(f_1)^2 + (f_2)^2$ and $(f_1)^2 + (f_2)^2 = 2$ we get $(f_1)^2 = 1-\mu$, and $(f_2)^2 = 1+\mu$.

If $\dot{\mu} = 0$, then f_1 and f_2 are constant, which implies that $\widetilde{\gamma}$ is a geodesic.

Assume now that $\dot{\mu} \neq 0$.

As (5.5) is equivalent to

$$4\ddot{\mu} + \frac{\mu\,\dot{\mu}^2}{2(1-\mu^2)} = 0,$$

dividing by $\dot{\mu}$ and integrating, we obtain

$$\frac{\dot{\mu}^{16}}{1-\mu^2} = \text{constant}\,. \tag{5.6}$$

Since γ is biharmonic, from Theorem 5.6, we have

$$k_1^2 = |\tau(\gamma)|^2 = \frac{(\dot{f}_1)^2 + (\dot{f}_2)^2}{2} + \mu^2 = \frac{\dot{\mu}^2}{4(1-\mu^2)} + \mu^2 = \text{constant}\,. \tag{5.7}$$

Substituting $\dot{\mu}$ from (5.6) in (5.7) we get a polynomial equation in μ, so $\mu =$ constant and we have a contradiction.

Conversely, let $\widetilde{\gamma} : I \to N$ be a geodesic parametrized by arc length on N. Writing

$$\widetilde{\gamma}(t) = \frac{1}{\sqrt{2}}(\cos\alpha(s), \sin\alpha(s), \cos\beta(s), \sin\beta(s)),$$

we get $T = \dot{\widetilde{\gamma}} = \dot{\alpha}U_1 + \dot{\beta}U_2$, and thus $\dot{\alpha}$ and $\dot{\beta}$ are constant, i.e. up to an isometry of $\mathbb{S}^3$,

$$\widetilde{\gamma}(t) = \frac{1}{\sqrt{2}}(\cos(\sqrt{2}a_1 s), \sin(\sqrt{2}a_1 s), \cos(\sqrt{2}a_2 s), \sin(\sqrt{2}a_2 s)),$$

where $(a_1)^2 + (a_2)^2 = 1$.

Further, we obtain

$$\tau(\gamma) = ((a_2)^2 - (a_1)^2)\eta$$

and conclude that a geodesic on N is geodesic on $\mathbb{S}^3$ if and only if its slope is equal to ± 1.

Moreover, for $(a_1)^2 \neq (a_2)^2$, a straightforward computation shows that $\tau_2(\gamma) = 0$.

□

Remark 5.8. If $(a_2)^2 - (a_1)^2 \in \{\pm 1\}$, then γ is the circle of radius $\frac{1}{\sqrt{2}}$.

Remark 5.9. As the curvature of the proper biharmonic curves $\gamma = \mathbf{i} \circ \widetilde{\gamma} : I \to \mathbb{S}^3$, $k_1 = |\tau(\gamma)| = |(a_2)^2 - (a_1)^2|$, can take any value in the interval $(0, 1]$, we conclude that they are, up to an isometry, the unique proper biharmonic curves of $\mathbb{S}^3$.

We can isometrically identify $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}})$ with $\mathcal{T} = \mathbb{R}^2/\Lambda$, $\Lambda = \sqrt{2}\pi\mathbb{Z} \times \sqrt{2}\pi\mathbb{Z}$, and choose as a model for $\mathcal{T}$ the square generated by the vectors $v_1 = (0, \sqrt{2}\pi)$ and $v_2 = (\sqrt{2}\pi, 0)$. The biharmonic curves of $\mathbb{S}^3$ are determined by the geodesics of $\mathcal{T}$ of slope different from ± 1. The geodesics of $\mathcal{T}$ corresponding to v_1 and v_2 determine the biharmonic circles of radius $\frac{1}{\sqrt{2}}$ of $\mathbb{S}^3$.

Let $pr : \mathbb{S}^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$, $pr(x^1, x^2, x^3, x^4) = \left(\frac{x^1}{1-x^4}, \frac{x^2}{1-x^4}, \frac{x^3}{1-x^4}\right)$, be the stereographic projection from $(0,0,0,1) \in \mathbb{S}^3$.

The projection pr transforms the Clifford torus, $r : [0, \sqrt{2}\pi] \times [0, \sqrt{2}\pi] \to \mathbb{S}^3$

$$r(u,v) = \frac{1}{\sqrt{2}}(\cos\sqrt{2}u, \sin\sqrt{2}u, \cos\sqrt{2}v, \sin\sqrt{2}v),$$

onto the torus of revolution in $\mathbb{R}^3$, $pr \circ r : [0, \sqrt{2}\pi] \times [0, \sqrt{2}\pi] \to \mathbb{R}^3$

$$(pr \circ r)(u,v) = \Big(\frac{\cos\sqrt{2}u}{\sqrt{2} - \sin\sqrt{2}v}, \frac{\sin\sqrt{2}u}{\sqrt{2} - \sin\sqrt{2}v}, \frac{\cos\sqrt{2}v}{\sqrt{2} - \sin\sqrt{2}v}\Big).$$

In the following we shall present the geometric images through stereographic projection of the three classes of geodesics on the Clifford torus.

We noted that when the slope of the geodesic on N is ± 1, it is also a geodesic on $\mathbb{S}^3$. These curves are also known as the *Villarceau circles*. A pair of such geodesics is parametrized by

$$\widetilde{\gamma}_0(s) = \frac{1}{\sqrt{2}}(\cos\sqrt{2}s, \sin\sqrt{2}s, \cos\sqrt{2}s, \pm\sin\sqrt{2}s),$$

and through stereographic projection becomes

$$(pr \circ \widetilde{\gamma}_0)(s) = \Big(\frac{\cos\sqrt{2}s}{\sqrt{2} \mp \sin\sqrt{2}s}, \frac{\sin\sqrt{2}s}{\sqrt{2} \mp \sin\sqrt{2}s}, \frac{\cos\sqrt{2}s}{\sqrt{2} \mp \sin\sqrt{2}s}\Big).$$

Their geometric image is plotted in Figure 5.1.

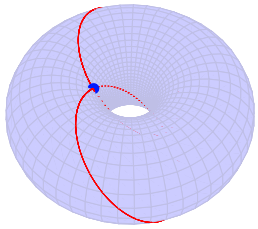

Figure 5.1: Geodesics on $\mathbb{S}^3$ (Villarceau circles on the Clifford torus). Image through stereographic projection.

Considering the geodesic $\widetilde{\gamma}_1 : I \to N$,

$$\widetilde{\gamma}_1(s) = \frac{1}{\sqrt{2}}\Big(\cos 2\sqrt{\frac{2}{5}}s, \sin 2\sqrt{\frac{2}{5}}s, \cos\sqrt{\frac{2}{5}}s, \sin\sqrt{\frac{2}{5}}s\Big)$$

of rational slope, different from ± 1, we get a closed proper biharmonic curve on $\mathbb{S}^3$. We have

$$(pr \circ \widetilde{\gamma}_1)(s) = \Big(\frac{\cos 2\sqrt{\frac{2}{5}}s}{\sqrt{2} - \sin\sqrt{\frac{2}{5}}s}, \frac{\sin 2\sqrt{\frac{2}{5}}s}{\sqrt{2} - \sin\sqrt{\frac{2}{5}}s}, \frac{\cos\sqrt{\frac{2}{5}}s}{\sqrt{2} - \sin\sqrt{\frac{2}{5}}s}\Big)$$

and its image is plotted in Figure 5.2.

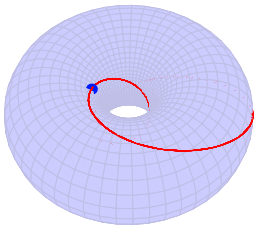

Figure 5.2: Closed proper biharmonic curve on $\mathbb{S}^3$ (geodesic of rational slope on the Clifford torus). Image through stereographic projection.

The curve $\widetilde{\gamma}_2 : I \to N$,

$$\widetilde{\gamma}_2(s) = \frac{1}{\sqrt{2}}\Big(\cos\sqrt{\frac{3}{2}}s, \sin\sqrt{\frac{3}{2}}s, \cos\sqrt{\frac{1}{2}}s, \sin\sqrt{\frac{1}{2}}s\Big),$$

is a geodesic of irrational slope, so dense, on the Clifford torus, and its image through stereographic projection,

$$(pr \circ \widetilde{\gamma}_2)(s) = \Big(\frac{\cos\sqrt{\frac{3}{2}}s}{\sqrt{2} - \sin\sqrt{\frac{1}{2}}s}, \frac{\sin\sqrt{\frac{3}{2}}s}{\sqrt{2} - \sin\sqrt{\frac{1}{2}}s}, \frac{\cos\sqrt{\frac{1}{2}}s}{\sqrt{2} - \sin\sqrt{\frac{1}{2}}s}\Big),$$

is plotted in Figure 5.3.

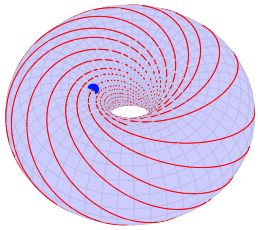

Figure 5.3: Proper biharmonic curve on $\mathbb{S}^3$ (geodesic of irrational slope, so dense, on the Clifford torus). Image through stereographic projection.

5.3 Biharmonic curves on Berger spheres

The present section is devoted to the study of proper biharmonic curves on Berger spheres.

5.3.1 Berger spheres. Definition and Riemannian structure

There are in the literature several ways for defining the Berger spheres. In the following we shall describe them and present some details.

The Berger spheres can be obtained:

(i) As the quotient spaces M^3_α of the Lie group $SU(2)\times\mathbb{R}$ by the subgroup, depending on a parameter $\alpha \in (0, \pi/2]$,

$$H_\alpha = \left\{ \begin{pmatrix} e^{i\ell\cos\alpha} & 0 & 0 \\ 0 & e^{-i\ell\cos\alpha} & 0 \\ 0 & 0 & \ell\sin\alpha \end{pmatrix}, \ell \in \mathbb{R} \right\},$$

where $SU(2)\times\mathbb{R}$ is endowed with the left invariant metric generated by the Killing form $B(X,Y) = -\frac{1}{2}\operatorname{trace} XY$ on $\mathfrak{su}(2)$ and the canonical one on $\mathbb{R}$ (see [23]). Notice that $M^3_{\pi/2}$ is the unit Euclidean sphere.

(ii) As the distance spheres in $\mathbb{CP}^2$ with the Fubini-Study metric (see [119]).

(iii) By performing $\mathcal{D}$-homothetic deformations of the canonical Sasakian structure on $\mathbb{S}^3$.

(iv) By homothetically deforming the canonical metric on $\mathbb{S}^3$ along the fibres of the Hopf fibration (see, for example, [99]).

Sasakian space forms and Berger spheres

Definition 5.10. A *contact metric structure* on a manifold M^{2n+1} is given by (Ψ, ξ, η, g), where $\Psi \in C(T^*M \otimes TM)$, $\xi \in C(TM)$, $\eta \in C(T^*M)$ and g is a Riemannian metric

on M, such that

$$\begin{cases} \Psi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \\ \\ g(\Psi X, \Psi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad \forall X, Y \in C(TM) \\ \\ 2g(X, \Psi Y) = d\eta(X,Y). \end{cases}$$

The vector field ξ is called the *Reeb vector field*. The *contact distribution* $\mathcal{D}$ is defined by $\mathcal{D} = \{X \in C(TM) : \eta(X) = 0\}$ and a one-dimensional integral submanifold of $\mathcal{D}$ is called a *Legendre curve*.

A contact metric structure (Ψ, ξ, η, g) is called *Sasakian* if it is normal, i.e.

$$N_\Psi + d\eta \otimes \xi = 0,$$

where N_Ψ is the Nijenhuis tensor field of Ψ, or equivalently if

$$(\nabla_X \Psi)(Y) = g(X,Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TM).$$

The sectional curvature of a 2-plane generated by X and ΨX, where X is an unit vector field orthogonal to ξ, is called the *holomorphic sectional curvature* determined by X. A Sasakian manifold with constant holomorphic sectional curvature c is called a *Sasakian space form*.

Consider the following structure tensor fields on the 3-dimensional unit Euclidean sphere $\mathbb{S}^3$:

(i) $\xi_0 = -J(z,w)$, $(z,w) \in \mathbb{S}^3 \subset \mathbb{C}^2$, where J is the usual complex structure on $\mathbb{C}^2$, $J(z,w) = i(z,w)$,

(ii) $\Psi_0 = P \circ J$, where $P : T_{(z,w)}\mathbb{C}^2 \to T_{(z,w)}\mathbb{S}^3$ denotes the orthogonal projection.

These tensors induce a Sasakian space structure $(\mathbb{S}^3, \Psi_0, \xi_0, \eta_0, g)$ of constant holomorphic sectional curvature 1, where g denotes the canonical metric on $\mathbb{S}^3$.

Consider now the deformed structure

$$\eta = \alpha\eta_0, \quad \xi = \frac{1}{\alpha}\xi_0, \quad \Psi = \Psi_0,$$

$$g_\alpha = \alpha g + \alpha(\alpha - 1)\eta_0 \otimes \eta_0, \tag{5.8}$$

where α is a positive constant. Such a deformation is called a $\mathcal{D}$-homothetic deformation and the deformed space $(\mathbb{S}^3, \Psi, \eta, \xi, g_\alpha)$ is still a Sasakian space form with constant holomorphic sectional curvature $c = \frac{4}{\alpha} - 3$ (see [113]).

The spheres $(\mathbb{S}^3, g_\alpha)$, with $\alpha \in (0,1)$, are sometimes called the Tanno model for Berger spheres.

The Hopf fibration and Berger spheres

In this section we shall use a slightly different definition obtained by using the Hopf fibration.

Let $\psi : \mathbb{S}^3(1) \to \mathbb{S}^2(\frac{1}{2})$ be the Hopf map given by

$$\psi(x^1, x^2, x^3, x^4) = \frac{1}{2}(2x^1x^3 + 2x^2x^4, 2x^2x^3 - 2x^1x^4, (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2),$$

or

$$\psi(z, w) = \frac{1}{2}\big(2z\overline{w}, |z|^2 - |w|^2\big),$$

where $z = x^1 + ix^2$ and $w = x^3 + ix^4$. The map ψ is a Riemmanian submersion with totally geodesic fibres, $\psi^{-1}(\psi(z, w))$ being the great circle passing through (z, w) and (iz, iw).

A variation of the canonical metric g on $\mathbb{S}^3$ can be obtained by performing the following biconformal change of metric with respect to the Hopf map:

$$g_\varepsilon\big|_{T^H\mathbb{S}^3 \times T^H\mathbb{S}^3} = g\big|_{T^H\mathbb{S}^3 \times T^H\mathbb{S}^3}, \qquad g_\varepsilon\big|_{T^V\mathbb{S}^3 \times T^V\mathbb{S}^3} = \varepsilon^2 g, \qquad g_\varepsilon\big|_{T^H\mathbb{S}^3 \times T^V\mathbb{S}^3} = 0, \tag{5.9}$$

where, as usually, $T^V\mathbb{S}^3$ and $T^H\mathbb{S}^3$ denote respectively the vertical and the horizontal spaces determined by ψ. The sphere endowed with the metric g_ε is called a *Berger sphere* and is denoted by $\mathbb{S}^3_\varepsilon$.

Let $x \in \mathbb{S}^3$. The vector fields

$$\begin{aligned} X_1(x) &= (-x^2, x^1, -x^4, x^3), \\ X_2(x) &= (-x^4, -x^3, x^2, x^1), \\ X_3(x) &= (-x^3, x^4, x^1, -x^2) \end{aligned} \tag{5.10}$$

parallelize $\mathbb{S}^3$ and

(i) X_1 is tangent to the fibres of the Hopf map (i.e. $d\psi(X_1) = 0$),

(ii) X_2 and X_3 are horizontal and $\{d\psi(X_2), d\psi(X_3)\}$ is an orthonormal basis of $T_{\psi(x)}\mathbb{S}^2(\frac{1}{2})$.

From (5.9) we immediately deduce that the global frame field

$$\{E_1 = \varepsilon^{-1}X_1, E_2 = X_2, E_3 = X_3\} \tag{5.11}$$

is an orthonormal frame field on $\mathbb{S}^3_\varepsilon$. In the given frame field, the Levi-Civita connection of the metric g_ε has the expression

$$\begin{cases} {}^\varepsilon\nabla_{E_1}E_1 = 0, & {}^\varepsilon\nabla_{E_1}E_2 = \varepsilon^{-1}(2-\varepsilon^2)E_3, & {}^\varepsilon\nabla_{E_1}E_3 = -\varepsilon^{-1}(2-\varepsilon^2)E_2, \\ {}^\varepsilon\nabla_{E_2}E_1 = -\varepsilon E_3, & {}^\varepsilon\nabla_{E_2}E_2 = 0, & {}^\varepsilon\nabla_{E_2}E_3 = \varepsilon E_1, \\ {}^\varepsilon\nabla_{E_3}E_1 = \varepsilon E_2, & {}^\varepsilon\nabla_{E_3}E_2 = -\varepsilon E_1, & {}^\varepsilon\nabla_{E_3}E_3 = 0, \end{cases} \tag{5.12}$$

and the components of the curvature tensor field ${}^{\varepsilon}R$ are determined by

$$\begin{cases} {}^{\varepsilon}R_{312} = 0, & {}^{\varepsilon}R_{212} = \varepsilon^2 E_1, \\ {}^{\varepsilon}R_{323} = (4-3\varepsilon^2)E_2, & {}^{\varepsilon}R_{131} = \varepsilon^2 E_3, \end{cases} \tag{5.13}$$

where ${}^{\varepsilon}R(E_i, E_j)E_k = {}^{\varepsilon}R_{kij}$.

We notice that, for $\varepsilon \in (0,1)$, the Berger sphere $\mathbb{S}^3_\varepsilon$ is a non-constantly positively curved Riemannian manifold.

Remark 5.11. We underline the fact that, by considering $\alpha = \varepsilon^2$, the metrics in (5.8) and (5.9) are homothetic, thus the Berger sphere $\mathbb{S}^3_\varepsilon$ is homothetic to the 3-dimensional Sasakian space form of constant holomorphic sectional curvature $c = \frac{4}{\varepsilon^2} - 3$, even if it cannot be organized as a contact metric manifold with respect to the corresponding vector field E_1 and 1-form $\varepsilon\eta_0$.

5.3.2 Biharmonic curves on $\mathbb{S}^3_\varepsilon$

In analogy with the case of $\mathbb{S}^3$, we shall determine the parametric equations of all proper biharmonic curves of the Berger sphere $\mathbb{S}^3_\varepsilon$ as curves in $\mathbb{R}^4$.

Consider $\gamma : I \to \mathbb{S}^3_\varepsilon$ to be a curve parametrized by arc length and denote by $\{F_1 = T = \sum T_i E_i, F_2 = N = \sum N_i E_i, F_3 = B = \sum B_i E_i\}$ the Frenet trihedron along γ decomposed with respect to the orthonormal frame field (5.11).

The expression (5.13) for the curvature tensor of $\mathbb{S}^3_\varepsilon$ leads to

$$\begin{aligned} {}^{\varepsilon}R(N,T,T,N) &= \sum T_i N_j T_k N_h g_\varepsilon({}^{\varepsilon}R_{kij}, E_h) \\ &= -k_1\{\varepsilon^2 + 4(1-\varepsilon^2)B_1^2\}. \end{aligned}$$

$$\begin{aligned} {}^{\varepsilon}R(B,T,T,N) &= \sum T_i N_j T_k B_h g_\varepsilon({}^{\varepsilon}R_{kij}, E_h) \\ &= 4k_1(1-\varepsilon^2)B_1 N_1, \end{aligned}$$

and from Corollary 5.4 we obtain

Theorem 5.12 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a curve parametrized by arc length. Then γ is proper biharmonic if and only if*

$$\begin{cases} k_1 = \text{constant} \neq 0 \\ k_1^2 + k_2^2 = \varepsilon^2 + 4(1-\varepsilon^2)B_1^2 \\ \dot{k}_2 = 4(1-\varepsilon^2)N_1 B_1. \end{cases} \tag{5.14}$$

Theorem 5.13 ([12]). *If $\gamma : I \to \mathbb{S}^3_\varepsilon$ is a proper biharmonic curve parametrized by arc length, then it is a helix.*

Proof. From the Frenet equations for γ we have

$$g_\varepsilon({}^\varepsilon\nabla_T B, E_1) = -g_\varepsilon(k_2 N, E_1) = -k_2 N_1,$$

while, using (5.12), we get

$$\begin{aligned} g_\varepsilon({}^\varepsilon\nabla_T B, E_1) &= g_\varepsilon(\dot{B}_1 E_1 + T_2 B_3\, {}^\varepsilon\nabla_{E_2} E_3 + T_3 B_2\, {}^\varepsilon\nabla_{E_3} E_2, E_1) \\ &= \dot{B}_1 + \varepsilon(T_2 B_3 - T_3 B_2) \\ &= \dot{B}_1 - \varepsilon N_1. \end{aligned}$$

Combining the two equations we obtain

$$\dot{B}_1 = (\varepsilon - k_2) N_1. \tag{5.15}$$

Assume now that γ is a proper biharmonic curve. From (5.14) follows that $k_2\dot{k}_2 = 4(1-\varepsilon^2)B_1\dot{B}_1$ and thus

$$B_1(\dot{B}_1 - k_2 N_1) = 0. \tag{5.16}$$

By using (5.15) and (5.16) we obtain $B_1 N_1(\varepsilon - 2k_2) = 0$. The last equation provides two possibilities: $B_1 N_1 = 0$, and from the third equation of (5.14) we get $\dot{k}_2 = 0$; or $k_2 = \frac{\varepsilon}{2}$. In both cases k_2 is constant and we conclude.

□

We shall now show that for a proper biharmonic curve we have $B_1 \neq 0$.

Proposition 5.14 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a curve parametrized by arc length. If $B_1 = 0$, $N_1 \neq 0$, then γ cannot be proper biharmonic.*

Proof. From the hypothesis we have $B_1 = 0$, thus $\dot{B}_1 = 0$. Suppose now that γ is biharmonic. From (5.15), since $N_1 \neq 0$, we get $k_2^2 = \varepsilon^2$. On the other hand, (5.14) implies $k_1^2 + k_2^2 = \varepsilon^2$, thus $k_1 = 0$. This implies that γ is a geodesic and we have a contradiction. □

Summing up, we obtain

Corollary 5.15 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a curve parametrized by arc length. Then γ is proper biharmonic if and only if*

$$\begin{cases} k_1 = \text{constant} \neq 0 \\ B_1 = \text{constant} \neq 0 \\ N_1 = 0 \\ k_1^2 + k_2^2 = \varepsilon^2 + 4(1-\varepsilon^2)B_1^2. \end{cases} \tag{5.17}$$

Lemma 5.16 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a non-geodesic curve, parametrized by arc length. If $N_1 = 0$, then*

$$T(s) = \cos\alpha_0 E_1 + \sin\alpha_0 \sin\beta(s) E_2 + \sin\alpha_0 \cos\beta(s) E_3, \tag{5.18}$$

where $\beta : I \to \mathbb{R}$ is a smooth function and $\alpha_0 \in (0, \pi)$.

Proof. Since, by using the Frenet equations and (5.12),

$$\begin{aligned} k_1 N_1 = g_\varepsilon({}^\varepsilon\nabla_T T, E_1) &= g_\varepsilon(\dot{T}_1 E_1 + T_2 T_3 \, {}^\varepsilon\nabla_{E_2} E_3 + T_2 T_3 \, {}^\varepsilon\nabla_{E_3} E_2, E_1) \\ &= \dot{T}_1 \end{aligned}$$

and γ is non-geodesic it follows that $N_1 = 0$ if and only if $\dot{T}_1 = 0$, i.e. if and only if $T_1 = $ constant. But $T_1 \in (-1, 1)$ and the norm of T is one, thus there exists a constant $\alpha_0 \in (0, \pi)$ and a smooth function β, unique up to an additive constant $2h\pi$, such that

$$T(s) = \cos\alpha_0 E_1 + \sin\alpha_0 \sin\beta(s) E_2 + \sin\alpha_0 \cos\beta(s) E_3.$$

□

By using the expression for T we get the parametric equations of proper biharmonic curves in Berger spheres. We first have

Theorem 5.17 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a curve parametrized by arc length. Then γ is proper biharmonic if and only if, as a curve in $\mathbb{R}^4$, it satisfies*

$$\gamma^{iv} + (b^2 - 2a)\ddot{\gamma} + a^2\gamma = 0, \tag{5.19}$$

where

$$\begin{cases} a = -(\varepsilon^{-2}\cos^2\alpha_0 + \sin^2\alpha_0 - \dot{\beta}\varepsilon^{-1}\cos\alpha_0) \\ b = \dot{\beta} = \varepsilon^{-1}(2 - \varepsilon^2)\cos\alpha_0 \pm \sqrt{(5\varepsilon^2 - 4)\cos^2\alpha_0 + 4(1 - \varepsilon^2)}, \end{cases} \tag{5.20}$$

with $\alpha_0 \in (0, \pi)$.

Proof. Consider the expression for $\gamma : I \to \mathbb{S}^3_\varepsilon$ as a curve in $\mathbb{R}^4$,

$$\gamma(s) = (x^1(s), x^2(s), x^3(s), x^4(s)).$$

Since $T = \dot{\gamma}$, from (5.18), we get that the coordinates of the position vector of γ in $\mathbb{R}^4$ verify

$$\begin{cases} \dot{x}^1 = & & -\varepsilon^{-1}\cos\alpha_0 x^2 & -\sin\alpha_0\cos\beta x^3 & -\sin\alpha_0\sin\beta x^4 \\ \dot{x}^2 = & \varepsilon^{-1}\cos\alpha_0 x^1 & & -\sin\alpha_0\sin\beta x^3 & +\sin\alpha_0\cos\beta x^4 \\ \dot{x}^3 = & \sin\alpha_0\cos\beta x^1 & +\sin\alpha_0\sin\beta x^2 & & -\varepsilon^{-1}\cos\alpha_0 x^4 \\ \dot{x}^4 = & \sin\alpha_0\sin\beta x^1 & -\sin\alpha_0\cos\beta x^2 & +\varepsilon^{-1}\cos\alpha_0 x^3. & \end{cases}$$

From here

$$\begin{cases} \ddot{x}^1 = a\dot{x}^1 - b\dot{x}^2 \\ \ddot{x}^2 = b\dot{x}^1 + a\dot{x}^2 \\ \ddot{x}^3 = a\dot{x}^3 - b\dot{x}^4 \\ \ddot{x}^4 = b\dot{x}^3 + a\dot{x}^4, \end{cases} \tag{5.21}$$

with

$$\begin{cases} a = -(\varepsilon^{-2}\cos^2\alpha_0 + \sin^2\alpha_0 - \dot{\beta}\varepsilon^{-1}\cos\alpha_0) \\ b = \dot{\beta}. \end{cases}$$

We shall first prove that b is constant and we shall determine it. In order to compute the curvature of γ, by using (5.18), we have

$$\begin{aligned} {}^{\varepsilon}\nabla_T T &= \dot{T}_1 E_1 + \dot{T}_2 E_2 + \dot{T}_3 E_3 + \sum_{i\neq j} T_i T_j \, {}^{\varepsilon}\nabla_{E_i} E_j \\ &= \dot{T}_1 E_1 + (\dot{T}_2 - 2\varepsilon^{-1}(1-\varepsilon^2)T_1 T_3)E_2 + (\dot{T}_3 + 2\varepsilon^{-1}(1-\varepsilon^2)T_1 T_2)E_3 \\ &= \big(\dot{\beta} - 2\varepsilon^{-1}(1-\varepsilon^2)\big)\sin\alpha_0(\cos\beta E_1 - \sin\beta E_2), \end{aligned} \tag{5.22}$$

thus

$$k_1 = \pm\big(\dot{\beta} - 2\varepsilon^{-1}(1-\varepsilon^2)\big)\sin\alpha_0 > 0. \tag{5.23}$$

From here

$$N = \pm(\cos\beta E_1 - \sin\beta E_2),$$

and

$$B = T \times N = \pm(-\sin\alpha_0 E_1 + \cos\alpha_0 \sin\beta(s) E_2 + \cos\alpha_0 \cos\beta(s) E_3).$$

Moreover, we can obtain the geodesic torsion,

$$\begin{aligned} k_2 &= g_\varepsilon({}^{\varepsilon}\nabla_T N, B) \\ &= \varepsilon - \big(\dot{\beta} - 2\varepsilon^{-1}(1-\varepsilon^2)\big)\cos\alpha_0. \end{aligned} \tag{5.24}$$

Since γ is biharmonic we have $k_1^2 + k_2^2 = \varepsilon^2 + 4(1-\varepsilon^2)\sin^2\alpha_0$, thus, from (5.23) and (5.24), we conclude that $\dot{\beta} = b$ satisfies

$$\big(b - 2\varepsilon^{-1}(1-\varepsilon^2)\big)^2 - 2\varepsilon\cos\alpha_0\big(b - 2\varepsilon^{-1}(1-\varepsilon^2)\big) - 4(1-\varepsilon^2)\sin^2\alpha_0 = 0,$$

thus

$$b = \varepsilon^{-1}(2-\varepsilon^2)\cos\alpha_0 \pm \sqrt{(5\varepsilon^2-4)\cos^2\alpha_0 + 4(1-\varepsilon^2)}.$$

By differentiating twice the equations in (5.21) and considering the fact that b is a constant we obtain that γ is a solution of (5.19).

The converse is immediate. □

From the proof of Theorem 5.17 we immediately obtain

Corollary 5.18 ([12]). *Let $\gamma : I \to \mathbb{S}^3_\varepsilon$ be a proper biharmonic curve. Then it is a helix.*

By integrating (5.19) we get

Theorem 5.19 ([12]). *Let $\mathbb{S}^3_\varepsilon$ be a Berger sphere, $\varepsilon \in (0,1)$. Then the parametric equations of all proper biharmonic curves, parametrized by arc length, of $\mathbb{S}^3_\varepsilon$ are*

$$\gamma(s) = \cos(\alpha_1 s)\, c_1 + \sin(\alpha_1 s)\, c_2 + \cos(\alpha_2 s)\, c_3 + \sin(\alpha_2 s)\, c_4, \tag{5.25}$$

where

$$\alpha_{1,2} = \sqrt{\frac{(b^2-2a) \pm \sqrt{(b^2-2a)^2 - 4a^2}}{2}},$$

$c_1, \ldots, c_4$ are constant vectors in $\mathbb{R}^4$ determined by $|\gamma|^2 = 1$, $|\dot\gamma|^2_\varepsilon = 1$ and by the initial conditions, and a and b are given by (5.20).

The vectors $c_1, \ldots, c_4$ may be explicitly determined in general, but we choose to present here the special case of Legendre curves.

Corollary 5.20. *Let $\mathbb{S}^3_\varepsilon$ be a Berger sphere, $\varepsilon \in (0,1)$. Then the parametric equations of all proper biharmonic Legendre curves, parametrized by arc length, of $\mathbb{S}^3_\varepsilon$ are*

$$\begin{aligned} \gamma(s) &= \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\cos(\alpha_1 s)e_1 - \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\sin(\alpha_1 s)Je_1 \\ &\quad + \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}}\cos(\alpha_1 s)e_3 + \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}}\sin(\alpha_1 s)Je_3 \\ &= \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\exp(-\mathrm{i}\alpha_1 s)e_1 + \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}\exp(\mathrm{i}\alpha_1 s)e_3, \end{aligned} \tag{5.26}$$

where e_1, e_3 are unit orthogonal vectors of $\mathbb{R}^4$ with e_3 orthogonal to Je_1,

$$\alpha_{1,2} = \sqrt{3 - 2\varepsilon^2 \pm 2\sqrt{(1-\varepsilon^2)(2-\varepsilon^2)}}, \tag{5.27}$$

and J is the canonical complex structure on $\mathbb{R}^4$.

Proof. In order to obtain the parametric equations of Legendre proper biharmonic curves in $\mathbb{S}^3_\varepsilon$, since the unit tangent vector field is given by (5.18), we have to consider (5.25) with $\alpha_0 = \frac{\pi}{2}$. In this case $a = -1$ and $b = \pm 2\sqrt{1-\varepsilon^2}$, thus $\alpha_{1,2}$ are given by (5.27).

The following relations

$$\langle\gamma,\gamma\rangle = 1, \quad \langle\dot\gamma,\dot\gamma\rangle = 1, \quad \langle\gamma,\dot\gamma\rangle = 0, \quad \langle\dot\gamma,\ddot\gamma\rangle = 0, \quad \langle\ddot\gamma,\ddot\gamma\rangle = 5 - 4\varepsilon^2,$$

$$\langle\gamma,\ddot\gamma\rangle = -1, \quad \langle\dot\gamma,\dddot\gamma\rangle = -(5-4\varepsilon^2), \quad \langle\ddot\gamma,\dddot\gamma\rangle = 0, \quad \langle\gamma,\dddot\gamma\rangle = 0,$$

$$\langle\dddot\gamma,\dddot\gamma\rangle = 16\varepsilon^4 - 44\varepsilon^2 + 29,$$

allow us to determine the constant vectors $\{c_i\}$. Indeed, denoting by $c_{ij} = \langle c_i, c_j\rangle|_{s=0}$ we have

$$c_{11} + 2c_{13} + c_{33} = 1 \tag{5.28}$$

$$\alpha_1^2 c_{22} + 2\alpha_1\alpha_2 c_{24} + \alpha_2^2 c_{44} = 1 \tag{5.29}$$

$$\alpha_1 c_{12} + \alpha_1 c_{23} + \alpha_2 c_{14} + \alpha_2 c_{34} = 0 \tag{5.30}$$

$$\alpha_1^3 c_{12} + \alpha_1\alpha_2^2 c_{23} + \alpha_1^2\alpha_2 c_{14} + \alpha_2^3 c_{34} = 0 \tag{5.31}$$

$$\alpha_1^4 c_{11} + 2\alpha_1^2\alpha_2^2 c_{13} + \alpha_2^4 c_{33} = 5 - 4\varepsilon^2 \tag{5.32}$$

$$\alpha_1^2 c_{11} + (\alpha_1^2 + \alpha_2^2) c_{13} + \alpha_2^2 c_{33} = 1 \tag{5.33}$$

$$\alpha_1^4 c_{22} + (\alpha_1\alpha_2^3 + \alpha_1^3\alpha_2) c_{24} + \alpha_2^4 c_{44} = 5 - 4\varepsilon^2 \tag{5.34}$$

$$\alpha_1^5 c_{12} + \alpha_1^3\alpha_2^2 c_{23} + \alpha_1^2\alpha_2^3 c_{14} + \alpha_2^5 c_{34} = 0 \tag{5.35}$$

$$\alpha_1^3 c_{12} + \alpha_1^3 c_{23} + \alpha_2^3 c_{14} + \alpha_2^3 c_{34} = 0 \tag{5.36}$$

$$\alpha_1^6 c_{22} + 2\alpha_1^3\alpha_2^3 c_{24} + \alpha_3^6 c_{44} = 16\varepsilon^4 - 44\varepsilon^2 + 29. \tag{5.37}$$

Since $\alpha_1 \neq \alpha_2$, from (5.30), (5.31), (5.35), (5.36) we obtain

$$c_{12} = c_{23} = c_{34} = c_{41} = 0.$$

Equations (5.28), (5.32) and (5.33) imply

$$c_{11} = \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad c_{13} = 0, \quad c_{33} = \frac{\alpha_1}{\alpha_1 + \alpha_2},$$

and from (5.29), (5.34) and (5.37) we have

$$c_{22} = \frac{\alpha_2}{\alpha_1 + \alpha_2}, \quad c_{24} = 0, \quad c_{44} = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

Thus, $\{c_i\}$ are mutually orthogonal vectors in $\mathbb{R}^4$ with $|c_1| = |c_2| = \sqrt{\frac{\alpha_2}{\alpha_1+\alpha_2}}$ and $|c_3| = |c_4| = \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}}$. Consider $e_i = \frac{c_i}{|c_i|}$. Then

$$\begin{aligned}\gamma(s) =& \sqrt{\frac{\alpha_2}{\alpha_1 + \alpha_2}}\cos(\alpha_1 s)e_1 + \sqrt{\frac{\alpha_2}{\alpha_1 + \alpha_2}}\sin(\alpha_1 s)e_2 \\ &+ \sqrt{\frac{\alpha_1}{\alpha_1 + \alpha_2}}\cos(\alpha_1 s)e_3 + \sqrt{\frac{\alpha_1}{\alpha_1 + \alpha_2}}\sin(\alpha_1 s)e_4.\end{aligned}$$

Since γ is a Legendre curve, by using the canonical complex structure of $\mathbb{R}^4$, we get $\langle e_1, Je_2\rangle = -\langle e_3, Je_s\rangle = \pm 1$ and $\langle e_1, Je_3\rangle = \langle e_1, Je_4\rangle = \langle e_2, Je_3\rangle = \langle e_2, Je_4\rangle = 0$. Therefore $e_2 = \mp Je_1$ and $e_4 = \pm Je_3$ and we conclude.

□

These equations were also obtained using Tanno's model of Berger spheres in [56].

Chapter 6

Biharmonic submanifolds in space forms

The non-existence theorems for the case of non-positive sectional curvature codomains, as well as the

Generalized Chen Conjecture. *Biharmonic submanifolds of a non positive sectional curvature manifold are minimal*,

encouraged the study of proper biharmonic submanifolds in spheres and other curved spaces [29, 30, 55, 67, 86, 98].

Although important results and examples were obtained, the classification of proper biharmonic submanifolds in spheres is still an open problem.

This chapter is fully devoted to the classification of proper biharmonic submanifolds in space forms and the new results presented here are contained in [15], [16], [17], [18].

It is organized as follows. In the preliminary section we include definitions and results on some special classes of submanifolds in space forms.

The second section gathers fundamental characterization results on proper biharmonic submanifolds of space forms and, in particular, of the Euclidean sphere. We present the biharmonic equation for this particular case, non-existence results for non-positively curved space forms and the main examples of biharmonic submanifolds in spheres. The section also contains some basic information on finite type Euclidean submanifolds. Although defined in a different manner, finite type submanifolds are, in a natural way, solutions of a variational problem. They are critical points of the volume functional for a certain class of directional deformations (see [46]). We study the type of compact proper biharmonic submanifolds of constant mean curvature in $\mathbb{S}^n$ and prove that, depending on the value of the mean curvature, they are of 1-type or of 2-type as submanifolds of $\mathbb{R}^{n+1}$ (Theorem 6.25).

The third section is devoted to the complete classification of the proper bihar-

monic hypersurfaces with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. We prove that they are open parts either of the hypersphere $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of the Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$ (Theorem 6.30). A similar classification is obtained for conformally flat biharmonic hypersurfaces in spheres. In contrast, for the hyperbolic space $\mathbb{H}^{m+1}$ we prove the non-existence of such hypersurfaces. The problem of the biharmonic hypersurfaces with at most 3 distinct principal curvatures is also investigated. A non-existence result is obtained for hypersurfaces with constant mean curvature and exactly 3 distinct principal curvatures (Theorem 6.35) and the full classification of proper biharmonic compact hypersurfaces in $\mathbb{S}^4$ is given (Theorem 6.38).

In the fourth section we prove that the pseudo-umbilical biharmonic submanifolds in spheres have constant mean curvature and we give an estimate for their scalar curvature. Then we classify the proper biharmonic pseudo-umbilical submanifolds of codimension 2 (Theorem 6.42). We also prove that the only biharmonic surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ are the minimal surfaces of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$ (Theorem 6.45).

The last section contains new examples of proper biharmonic submanifolds in spheres. We analyze biharmonic products of spheres and study the geometric properties of two classes of biharmonic submanifolds of $\mathbb{S}^5$.

6.1 Generalities on some classes of submanifolds

The present section contains the fundamental definitions and results used in this chapter in order to obtain new geometric properties or classification theorems for biharmonic submanifolds in space forms.

Consider M to be a submanifold of a Riemannian manifold N. We shall denote by A the Weingarten operator, by B the second fundamental form, by H the mean curvature vector field, by $\nabla^\perp$ and by $\Delta^\perp$ the connection and the Laplacian in the normal bundle of M in N.

Due to the nature of the biharmonic equation for Riemannian immersions we shall be interested in submanifolds with special properties of the Weingarten operator and of the mean curvature vector field.

6.1.1 Pseudo-umbilical submanifolds

Definition 6.1. (i) A submanifold M of a Riemannian manifold N is said to be *umbilical with respect to a normal section ξ on M* if the Weingarten operator associated to ξ is proportional to the identity, i.e. there exists a function $\lambda \in C^\infty(M)$, such that $A_\xi = \lambda \,\mathrm{Id}$.

(ii) If M is umbilical with respect to every local normal section on M, then M is said to be *totally umbilical*.

(iii) A submanifold M of a Riemannian manifold N is said to be *pseudo-umbilical* if it is umbilical with respect to its mean curvature vector field.

Remark 6.2. If M is a pseudo-umbilical submanifold of N, one can immediately prove that $\lambda = |H|^2$.

We shall recall here an important geometric property concerning pseudo-umbilical submanifolds in spheres.

Theorem 6.3 ([42]). *Let M^m be a pseudo-umbilical submanifold in $\mathbb{S}^{m+2}$. If M has constant mean curvature, then M is either a minimal submanifold of $\mathbb{S}^{m+2}$ or a minimal hypersurface of a hypersphere of $\mathbb{S}^{m+2}$.*

The following estimate of the scalar curvature for submanifolds in spheres will be used.

Theorem 6.4 ([47]). *Let M be an m-dimensional submanifold of an n-dimensional unit Euclidean sphere $\mathbb{S}^n$. Then the scalar curvature s of M satisfies*

$$s \leq m(m-1)(1+|H|^2).$$

The equality holds if and only if M is contained in an $m-$sphere $\mathbb{S}^m\big(\frac{1}{\sqrt{1+|H|^2}}\big)$ of $\mathbb{S}^n$.

6.1.2 Isoparametric hypersurfaces in spheres

Isoparametric hypersurfaces in spheres play an important role in our study.

Definition 6.5. A hypersurface M^m in $\mathbb{S}^{m+1}$ is said to be *isoparametric* of type ℓ if it has constant principal curvatures $k_1 > \ldots > k_\ell$ with respective constant multiplicities $m_1, \ldots, m_\ell$.

The isoparametric hypersurfaces with $\ell \leq 3$ were classified by E. Cartan (see [38]) and P.J. Ryan (see [100]). We present here a general result.

Theorem 6.6 ([1, 38, 90, 109, 112]). *Let M be an m-dimensional compact isoparametric hypersurface in $\mathbb{S}^{m+1}$. Let $k_1 > k_2 > \ldots > k_\ell$ be the distinct principal curvatures with multiplicities $m_1, \ldots, m_\ell$, $m = m_1 + m_2 + \ldots + m_\ell$. Then*

(i) *ℓ is either $1, 2, 3, 4$ or 6.*

(ii) *If $\ell = 1$, then M is totally umbilical.*

(iii) *If $\ell = 2$, then $M = \mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$, $r_1^2 + r_2^2 = 1$.*

(iv) *If $\ell = 3$, then $m_1 = m_2 = m_3 = 2^q$, $q = 0, \ldots, 3$.*

(v) *If* $\ell = 4$, $m_1 = m_3$ *and* $m_2 = m_4$. *Moreover,* $(m_1, m_2) = (2,2)$ *or* $(4,5)$, *or* $m_1 + m_2 + 1$ *is a multiple of* $2^{\zeta(m^*-1)}$. *Here* $\zeta(n)$ *is the number of integers* a *with* $1 < a < n$, $a \equiv 0, 1, 2, 4 \mod 8$ *and* $m^* = \min\{m_1, m_2\}$.

(vi) *If* $\ell = 6$, $m_1 = m_2 = \ldots = m_6 = 1$ *or* 2.

(vii) *There exists an angle* θ, $0 < \theta < \frac{\pi}{\ell}$, *such that*

$$k_\alpha = \cot\big(\theta + (\alpha - 1)\pi/\ell\big), \quad \alpha = 1, \ldots, \ell.$$

The next result on hypersurfaces with 3 distinct principal curvatures was proved

Theorem 6.7 ([41]). *A compact hypersurface* M^m *of constant scalar curvature* s *and constant mean curvature* $|H|$ *in* $\mathbb{S}^{m+1}$ *is isoparametric provided it has* 3 *distinct principal curvatures everywhere.*

In order to analyze the case of $\mathbb{S}^4$, we need

Theorem 6.8 ([39]). *Any compact hypersurface with constant scalar and mean curvature in* $\mathbb{S}^4$ *is isoparametric.*

6.1.3 Hypersurfaces with at most two distinct principal curvatures

Definition 6.9. Let (M^m, g) be a hypersurface in (N^{m+1}, g_0). We say that M is *quasi-umbilical* in N if the Weingarten operator of M has an eigenvalue of multiplicity at least $m - 1$.

Definition 6.10. A Riemannian manifold is called *conformally flat* if, for every point, it admits an open neighborhood conformally diffeomorphic to an open set of the Euclidean space.

Remark 6.11.

(i) For $m \geq 4$, $M^m \subset \mathbb{E}^{m+1}(c)$ is quasi-umbilical if and only if it is conformally flat (see [37]).

(ii) Quasi-umbilical hypersurfaces have at most two principal curvatures, but there exist hypersurfaces with at most two principal curvatures which are not quasi-umbilical. Take for example the tori

$$\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2) \subset \mathbb{S}^{m_1+m_2+1}, \qquad r_1^2 + r_2^2 = 1, \quad m_1, m_2 \geq 2,$$

which have exactly 2 eigenvalues, one of multiplicity m_1 and the other of multiplicity m_2.

In the following we shall construct examples of hypersurfaces with at most two distinct principal curvatures and non-constant mean curvature in any space form.

We start by considering the rotational hypersurfaces in $\mathbb{R}^{m+1}$. Let γ be a curve parametrized by arc length in the x^1Ox^{m+1} plane in $\mathbb{R}^{m+1}$,

$$\gamma(s) = (f(s), g(s)),$$

$f > 0$, $((f')^2 + (g')^2 = 1)$. The rotational hypersurface M generated by γ and with rotational axis Ox^{m+1} in $\mathbb{R}^{m+1}$ is locally given by

$$x(s, \phi) = f(s)\phi + g(s)e_{m+1},$$

where ϕ is a parametrization of a unit Euclidean sphere $\mathbb{S}^{m-1}$ which lies in the hyperplane orthogonal to Ox^{m+1} in $\mathbb{R}^{m+1}$. A unit normal at M in $\mathbb{R}^{m+1}$ has the expression

$$\eta(s, \phi) = -g'(s)\phi + f'(s)e_{m+1}.$$

Indeed, this follows immediately since

$$dx\big(\partial/\partial s\big) = x_s = f'(s)\phi + g'(s)e_{m+1},$$

$$dx(X) = f(s)X, \qquad \forall X \in C(T\mathbb{S}^{m-1}),$$

where we identify X with $d\phi(X)$.

In order to determine the principal curvatures consider now a local orthonormal frame field $X_2, \ldots, X_m$ on $\mathbb{S}^{m-1}$ in $\mathbb{R}^{m+1}$. This induces a frame field on M and from this one we shall get the principal frame field. Indeed,

$$\langle A(x_s), x_s\rangle = \langle \nabla^{\mathbb{R}^{m+1}}_{x_s} x_s, \eta\rangle = f'g'' - f''g' = \frac{g''}{f'}.$$

Here we use the fact that $f'f'' + g'g'' = 0$, thus $f'g'' - f''g' = \frac{g''}{f'}$. On the other hand,

$$\langle A(x_s), dx(X_i)\rangle = \langle \nabla^{\mathbb{R}^{m+1}}_{x_s} dx(X_i), \eta\rangle = \langle f'(s)X_i, \eta\rangle = 0,$$

thus

$$A(x_s) = (f'g'' - f''g')x_s = \frac{g''}{f'}x_s. \tag{6.1}$$

Since x_s has unit norm, (6.1) implies that x_s is a principal unit vector on M and the first principal curvature is $k_1 = \frac{g''}{f'}$.

Moreover,

$$\langle A(dx(X_i)), dx(X_j)\rangle = -\langle dx(X_i), \nabla^{\mathbb{R}^{m+1}}_{dx(X_j)}\eta\rangle = fg'\delta_{ij},$$

and since $|dx(X_i)| = f$ we obtain that $\frac{1}{f}X_i$ are all principal unit vectors on M with principal curvature $k_2 = \frac{g'}{f}$.

We deduce that the principal curvatures of M in $\mathbb{R}^{m+1}$ are

$$k_1 = \frac{g''}{f'}, \qquad k_\alpha = \frac{g'}{f}, \quad \alpha = 2, \dots, m, \tag{6.2}$$

thus M is quasi-umbilical and with a suitable choice of the generating curve M has non-constant mean curvature.

Remark 6.12. Since the sphere and the hyperbolic space are conformally flat, the problem of the existence of hypersurfaces with at most two principal curvatures and of non-constant mean curvature is solved by using the above construction, once we understand the behavior of quasi-umbilical hypersurfaces under conformal changes of metric on the ambient space.

The following Lemma is known in the literature and for completeness we include it here together with its proof.

Lemma 6.13. *Conformal changes in the ambient space of a hypersurface preserve the principal directions and the multiplicity of the principal curvatures. In particular, quasi-umbilical hypersurfaces transform into quasi-umbilical hypersurfaces.*

Proof. Consider (M^m, g) to be a hypersurface in (N^{m+1}, g_0) and denote by $\varphi : M \to N$ the inclusion map. Once we fix a unit normal vector field η at (M, g) in (N, g_0), we are interested in the expressions of the principal curvatures of the hypersurface $(M, \widetilde{g} = e^{2\mu} g)$ in $(N, \widetilde{g}_0 = e^{2\rho} g_0)$ with respect to $\widetilde{\eta} = e^{-\mu}\eta$, where $\rho \in C^\infty(N)$ and $\mu \in C^\infty(M)$, $\mu = \rho \circ \varphi$.

Consider on (M, g) a local orthonormal basis $\{X_i\}_{i=1}^m$ of principal directions with principal curvatures denoted by $\{k_i\}_{i=1}^m$. We obviously have

$$\nabla^N_{X_i} \eta = -A_\eta(X_i) = -k_i X_i, \qquad i = 1, \dots, m.$$

We denote also by X_i and η the extensions of X_i and η to N and consider $\widetilde{X}_i = e^{-\rho} X_i$, $\widetilde{\eta} = e^{-\rho}\eta$. Notice that the restrictions to $(M, \widetilde{g})$ of $\widetilde{X}_i$ represent a local orthonormal frame field and that of $\widetilde{\eta}$ a unit normal at $(M, \widetilde{g})$ in $(N, \widetilde{g}_0)$.

In order to compute the principal curvatures of $(M, \widetilde{g})$ with respect to $\widetilde{\eta}$ we only have to determine $\widetilde{\nabla}^N_{\widetilde{X}_i} \widetilde{\eta}$. For this

$$\begin{aligned}
\widetilde{\nabla}^N_{\widetilde{X}_i} \widetilde{\eta} &= e^{-\rho} \widetilde{\nabla}^N_{X_i} (e^{-\rho}\eta) \\
&= -e^{-2\rho} X_i(\rho)\eta + e^{-2\rho} \widetilde{\nabla}^N_{X_i} \eta \\
&= -e^{-2\rho} X_i(\rho)\eta + e^{-2\rho} \Big(\nabla^N_{X_i} \eta + X_i(\rho)\eta + \eta(\rho) X_i - g_0(X_i, \eta) \operatorname{grad} \rho \Big) \\
&= e^{-2\rho} \Big(\nabla^N_{X_i} \eta + \eta(\rho) X_i - g_0(X_i, \eta) \operatorname{grad} \rho \Big).
\end{aligned} \tag{6.3}$$

We now restrict (6.3) to the hypersurface and obtain

$$\begin{aligned}\widetilde{\nabla}^N_{\widetilde{X}_i}\widetilde{\eta} &= e^{-2\mu}\Big(-k_i+\eta(\rho)\circ\varphi\Big)X_i \\ &= e^{-\mu}\Big(-k_i+\eta(\rho)\circ\varphi\Big)\widetilde{X}_i.\end{aligned} \tag{6.4}$$

From (6.4) we conclude that $\widetilde{X}_i$ are principal directions on $(M,\widetilde{g})$ in $(N,\widetilde{g}_0)$, with principal curvatures given by

$$\widetilde{k}_i = e^{-\mu}\Big(k_i-\eta(\rho)\circ\varphi\Big) = e^{-\mu}k_i+\eta(e^{-\rho})\circ\varphi. \tag{6.5}$$

□

In the case of the sphere, by using the inverse of the stereographic projection from

$$(\mathbb{R}^{m+1}, e^{2\rho}g_0) \xrightarrow{\Pi} (\mathbb{S}^{m+1}\setminus\{N\}, g_{\text{can}}),$$

$$\rho(p) = \ln\Big(\frac{2}{1+|p|^2}\Big),$$

we can transform quasi-umbilical hypersurfaces in the Euclidean $(m+1)$-dimensional space into quasi-umbilical hypersurfaces of the Euclidean sphere.

We have, $\eta(e^{-\rho}) = \langle\eta, \operatorname{grad} e^{-\rho}\rangle$, and since $e^{-\rho} = \dfrac{1+|x|^2}{2}$, we get $\eta(e^{-\rho}) = \langle\eta, x\rangle$. Along M, this becomes

$$\eta(e^{-\rho}) = \langle -g'(s)\phi + f'(s)e_{m+1}, f(s)\phi + g(s)e_{m+1}\rangle = f'(s)g(s) - f(s)g'(s).$$

From (6.5) we get that the image of M through the stereographic projection has the principal curvatures

$$\widetilde{k}_1(s) = \frac{(1+f^2+g^2)g''}{2f'} + f'g - fg',$$

$$\widetilde{k}_\alpha(s) = \frac{(1+f^2+g^2)g'}{2f} + f'g - fg', \qquad \alpha = 2,\dots,m,$$

and its mean curvature, up to a sign, is given by

$$\widetilde{H} = \frac{1+f^2+g^2}{2m}\Big(\frac{g''}{f'} + (m-1)\frac{g'}{f}\Big) + f'g - fg'.$$

If we are interested in the expression for the Riemannian immersion of $\Pi(M)$ into $\mathbb{S}^{m+1}$ we use the warped product setting. The rotational hypersurface in $\mathbb{R}^{m+1}$ has the local expression

$$I \times_{f^2(s)} \mathbb{S}^{m-1},$$

and its Riemannian immersion in $\mathbb{R}^{m+1}$ is locally expressed by

$$x : I \times_{f^2(s)} \mathbb{S}^{m-1} \to \mathbb{R}^{m+1},$$

$$x(s,\phi) = (t(s),\varphi(s),\psi(\phi)) = \Big(\sqrt{f^2(s)+g^2(s)}, \arccos\frac{g(s)}{\sqrt{f^2(s)+g^2(s)}}, \phi\Big),$$

$(f')^2 + (g')^2 = 1$, where $\mathbb{R}^{m+1} \setminus \{0\}$ is expressed as

$$(0,\infty) \times_{t^2} \Big((0,\pi) \times_{\sin^2\varphi} \mathbb{S}^{m-1}\Big).$$

With this approach the inverse of the stereographic projection from $(0,\dots,0,1)$ is expressed as

$$\Pi : (0,\infty) \times_{t^2} \mathbb{S}^m \to (0,\pi) \times_{\sin^2\mu} \mathbb{S}^m$$

$$\Pi(t,p) = (\mu(t), q(p)) = (\pi - 2\arctan t, p).$$

All these lead to the immersion of M in $\mathbb{S}^{m+1}$,

$$\Pi \circ x : I \times \mathbb{S}^{m-1} \to (0,\pi) \times_{\sin^2\mu} \Big((0,\pi) \times_{\sin^2\varphi} \mathbb{S}^{m-1}\Big)$$

$$\Pi\circ x(s,\phi) = (\mu(s),\varphi(s),q(\phi)) = (\pi - 2\arctan\sqrt{f^2(s)+g^2(s)}, \arccos\frac{g(s)}{\sqrt{f^2(s)+g^2(s)}}, \phi).$$

The Riemannian metric on $M = I \times \mathbb{S}^{m-1}$ in $\mathbb{R}^{m+1}$ is

$$g = ds^2 + f^2(s) g_{\mathbb{S}^{m-1}}.$$

In order to render $\Pi\circ x$ a Riemannian immersion in $\mathbb{S}^{m+1}$ we should consider on $I\times\mathbb{S}^{m-1}$ the conformally equivalent metric

$$\widetilde{g} = \frac{4}{[1+f^2(s)+g^2(s)]^2}(ds^2 + f^2(s) g_{\mathbb{S}^{m-1}}).$$

Example 6.14. If we now consider a curve given by $f(s) = as$ and $g(s) = bs$, with $a^2 + b^2 = 1$, for the image of the rotational hypersurface in $\mathbb{S}^{m+1}$ we get

$$\widetilde{H} = \frac{b}{a}\frac{1+s}{s},$$

which is not constant.

Thus there exist non-umbilical quasi-umbilical hypersurfaces in spheres which do not have constant mean curvature. The same procedure can be applied in order to obtain examples of quasi-umbilical hypersurfaces with non-constant mean curvature in the hyperbolic space.

6.2 Biharmonic submanifolds in space forms

Let $\phi : M \to \mathbb{E}^n(c)$ be the canonical inclusion of a submanifold M in a constant sectional curvature c manifold, $\mathbb{E}^n(c)$. The expressions assumed by the tension and bitension fields are

$$\tau(\phi) = mH, \qquad \tau_2(\phi) = -m(\Delta H - mcH), \tag{6.6}$$

where H denotes the mean curvature vector field of M in $\mathbb{E}^n(c)$.

Indeed, we have seen that, in general, for a Riemannian immersion the tension field is given by $\tau(\phi) = mH$. In order to determine the bitension field we recall that the curvature of the ambient space has the expression

$$R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}, \qquad \forall X,Y,Z \in C(T\mathbb{E}^n(c)),$$

thus

$$\operatorname{trace} R^{\mathbb{E}^n(c)}(d\phi, \tau(\phi))d\phi = -mc\tau(\phi) = -m^2cH$$

and we conclude.

The attempt to classify the biharmonic submanifolds in space forms was initiated in [29] and [48] with the following characterization results, obtained by splitting the bitension field in its normal and tangent components.

Theorem 6.15 ([43, 94])**.**

(i) *The canonical inclusion $\phi : M^m \to \mathbb{E}^n(c)$ of a submanifold M in an n-dimensional space form $\mathbb{E}^n(c)$ is biharmonic if and only if*

$$\begin{cases} -\Delta^{\perp}H - \operatorname{trace} B(\cdot, A_H\cdot) + mcH = 0, \\ \\ 2\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + \frac{m}{2}\operatorname{grad}(|H|^2) = 0. \end{cases} \tag{6.7}$$

(ii) *Moreover, if M is a hypersurface with nowhere zero mean curvature of $\mathbb{E}^{m+1}(c)$, then M is biharmonic if and only if*

$$\begin{cases} \Delta^{\perp}H - (mc - |A|^2)H = 0, \\ \\ 2A\big(\operatorname{grad}(|H|)\big) + m|H|\operatorname{grad}(|H|) = 0, \end{cases} \tag{6.8}$$

where grad *denotes the gradient on M^m.*

Proof. (i) From (6.6), the map ϕ is biharmonic if and only if

$$\Delta H - mcH = 0. \tag{6.9}$$

Consider now $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M, geodesic at $p \in M$. At p we have

$$\begin{aligned}\Delta H &= -\sum_{i=1}^m \nabla^{\mathbb{E}^n(c)}_{E_i}\nabla^{\mathbb{E}^n(c)}_{E_i} H = -\sum_{i=1}^m \{\nabla^{\mathbb{E}^n(c)}_{E_i}[\nabla^\perp_{E_i} H - A_H(E_i)]\}\\ &= -\sum_{i=1}^m \{\nabla^\perp_{E_i}\nabla^\perp_{E_i} H - A_{\nabla^\perp_{E_i}H}(E_i) - \nabla_{E_i}A_H(E_i) - B(E_i, A_H(E_i))\}\\ &= \Delta^\perp H + \operatorname{trace} B(\cdot, A_H\cdot) + \operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \operatorname{trace}\nabla A_H.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{i=1}^m \nabla_{E_i}A_H(E_i) &= \sum_{i,j}\langle\nabla_{E_i}A_H(E_i), E_j\rangle E_j = \sum_{i,j} E_i\langle A_H(E_i), E_j\rangle E_j\\ &= \sum_{i,j} E_i\langle B(E_i,E_j), H\rangle E_j = \sum_{i,j} E_i\langle\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, H\rangle E_j\\ &= \sum_{i,j}\big\{\langle\nabla^{\mathbb{E}^n(c)}_{E_i}\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, H\rangle + \langle\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, \nabla^{\mathbb{E}^n(c)}_{E_i}H\rangle\big\}E_j\\ &= \sum_{i,j}\big\{\langle\nabla^{\mathbb{E}^n(c)}_{E_i}\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, H\rangle + \langle B(E_i,E_j), \nabla^\perp_{E_i}H\rangle\big\}E_j\\ &= \sum_{i,j}\langle\nabla^{\mathbb{E}^n(c)}_{E_i}\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, H\rangle E_j + \sum_i A_{\nabla^\perp_{E_i}H}(E_i)\end{aligned}$$

and, since at p,

$$\begin{aligned}\sum_{i=1}^m\langle\nabla^{\mathbb{E}^n(c)}_{E_i}\nabla^{\mathbb{E}^n(c)}_{E_j}E_i, H\rangle &= \sum_{i=1}^m\langle R^{\mathbb{E}^n(c)}(E_i,E_j)E_i + \nabla^{\mathbb{E}^n(c)}_{E_j}\nabla^{\mathbb{E}^n(c)}_{E_i}E_i + \nabla^{\mathbb{E}^n(c)}_{[E_i,E_j]}E_i, H\rangle\\ &= \langle -mcE_j, H\rangle + \sum_{i=1}^m\langle\nabla^{\mathbb{E}^n(c)}_{E_j}B(E_i,E_i), H\rangle = m\langle\nabla^{\mathbb{E}^n(c)}_{E_j}H, H\rangle\\ &= \frac{m}{2}E_j(|H|^2),\end{aligned}$$

we get

$$\begin{aligned}\sum_{i=1}^m[A_{\nabla^\perp_{E_i}H}(X_i) + \nabla_{E_i}A_H(E_i)] &= 2\sum_{i=1}^m A_{\nabla^\perp_{E_i}H}(E_i) + \frac{m}{2}(d|H|^2)^\sharp\\ &= 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \frac{m}{2}\operatorname{grad}(|H|^2).\end{aligned}$$

Thus, by replacing the expression for ΔH in (6.9) we obtain that ϕ is biharmonic if and only if

$$-\Delta^\perp H - \operatorname{trace} B(\cdot, A_H\cdot) + mcH = 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \frac{m}{2}\operatorname{grad}(|H|^2). \tag{6.10}$$

Since the left hand side of (6.10) is normal, and the right hand side is tangent to M, we conclude.

(ii) Consider $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M and $\eta \in C(NM)$ with $|\eta| = 1$. We have $H = \frac{1}{m}(\operatorname{trace} A)\eta$, $A_H = \frac{1}{m}(\operatorname{trace} A)A$ and

$$\begin{aligned}
\operatorname{trace} B(\cdot, A_H\cdot) &= \frac{1}{m}(\operatorname{trace} A)\operatorname{trace} B(\cdot, A\cdot) \\
&= \frac{1}{m}(\operatorname{trace} A)\sum_{i=1}^m B(E_i, A(E_i)) \\
&= \frac{1}{m}(\operatorname{trace} A)\sum_{i=1}^m \langle A(E_i), A(E_i)\rangle\eta \\
&= \frac{1}{m}(\operatorname{trace} A)|A|^2\eta \\
&= |A|^2 H.
\end{aligned}$$

It is easy to verify that if M is a hypersurface with nowhere zero mean curvature, then

$$2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + \frac{m}{2}\operatorname{grad}(|H|^2) = 2A(\operatorname{grad}(|H|)) + m|H|\operatorname{grad}(|H|).$$

□

Corollary 6.16. (i) *A submanifold M with parallel mean curvature vector field in $\mathbb{E}^n(c)$ is biharmonic if and only if*

$$\operatorname{trace} B(\cdot, A_H\cdot) = mcH, \tag{6.11}$$

(ii) *A hypersurface M of constant non-zero mean curvature in $\mathbb{E}^{m+1}(c)$ is proper biharmonic if and only if*

$$|A|^2 = mc. \tag{6.12}$$

Equation (6.12) immediately leads to a non-existence result on proper biharmonic hypersurfaces in space forms with $c \leq 0$. This can also be deduced as a consequence of Theorem 3.27 in Chapter 3, which proves the non-existence of proper biharmonic submanifolds of constant mean curvature in a negatively curved manifold.

Other non-existence results were given for the Euclidean and hyperbolic spaces, see Theorems 3.30 and 3.31 in Chapter 3. They are all based on the idea that by imposing certain geometric conditions on the biharmonic submanifolds, one can prove that they have constant mean curvature.

6.2.1 The main examples of biharmonic submanifolds in spheres

The case of the Euclidean sphere is different. We present here the basic examples of proper biharmonic hypersurfaces in spheres.

The hypersphere $\mathbb{S}^m(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+1}$

We consider

$$\mathbb{S}^m(a) = \Big\{(x^1, \dots, x^m, x^{m+1}, b) \in \mathbb{R}^{m+2} : \sum_{i=1}^{m+1} (x^i)^2 = a^2\Big\} \subset \mathbb{S}^{m+1},$$

where $a^2 + b^2 = 1$. Then

$$C\big(T\mathbb{S}^m(a)\big) = \Big\{(X^1, \dots, X^{m+1}, 0) \in C(T\mathbb{R}^{m+2}) : \sum_{i=1}^{m+1} x^i X^i = 0\Big\},$$

while

$$\eta = \frac{1}{r}\Big(x^1, \dots, x^{m+1}, -\frac{a^2}{b}\Big),$$

with $r^2 = \frac{a^2}{b^2}$, $r > 0$, is a unit section in the normal bundle of $\mathbb{S}^m(a)$ in $\mathbb{S}^{m+1}$.

We compute

$$\nabla^{\mathbb{S}^{m+1}}_X \eta = \frac{1}{r}\nabla^0_X\Big(x^1, \dots, x^n, -\frac{a^2}{b}\Big) = \frac{1}{r}X, \qquad \forall X \in C(T\mathbb{S}^m(a)),$$

thus

$$\nabla^\perp \eta = 0 \qquad \text{and} \qquad A_\eta = -\frac{1}{r}\,\mathrm{Id}\,.$$

This implies that $H = -\frac{1}{r}\eta$ and so $\mathbb{S}^m(a)$ has constant mean curvature $|H| = \frac{1}{r}$ in $\mathbb{S}^{m+1}$. Now, since $|A_\eta|^2 = \frac{m}{r^2}$, from Corollary 6.16 we conclude that $\mathbb{S}^m(a)$ is proper biharmonic in $\mathbb{S}^{m+1}$ if and only if $r = 1$, i.e. $a = \frac{1}{\sqrt{2}}$.

The generalized Clifford torus $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+1}$

The generalized Clifford torus, $M = \mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$, was the first example of proper biharmonic submanifold in $\mathbb{S}^{m+1}$ (see [69]).

Consider

$$\begin{aligned} M &= \Big\{(x^1, \dots, x^{m_1+1}, y^1, \dots, y^{m_2+1}) \in \mathbb{R}^{m+2} : \sum_{i=1}^{m_1+1} (x^i)^2 = a_1^2, \sum_{j=1}^{m_2+1} (y^j)^2 = a_2^2\Big\} \\ &= \mathbb{S}^{m_1}(a_1) \times \mathbb{S}^{m_2}(a_2) \subset \mathbb{S}^{m+1}, \end{aligned}$$

where $a_1^2 + a_2^2 = 1$.

Then

$$C(TM) = \Big\{(X^1, \dots, X^{m_1+1}, Y^1, \dots, Y^{m_2+1}) \in C(T\mathbb{R}^{m+2}) : \sum_{i=1}^{m_1+1} x^i X^i = \sum_{j=1}^{m_2+1} y^j Y^j = 0\Big\},$$

while

$$\eta = \Big(-\frac{a_2}{a_1}(x^1, \dots, x^{m_1+1}), \frac{a_1}{a_2}(y^1, \dots, y^{m_2+1})\Big)$$

is a unit section in the normal bundle of M in $\mathbb{S}^{m+1}$.

Consider $(X,Y)=(X^1,\dots,X^{m_1+1},Y^1,\dots,Y^{m_2+1})\in C(TM)$. Then

$$\nabla^{\mathbb{S}^{m+1}}_{(X,Y)}\eta=\nabla^0_{(X,Y)}\Big(-\frac{a_2}{a_1}(x^1,\dots,x^{m_1+1}),\frac{a_1}{a_2}(y^1,\dots,y^{m_2+1})\Big)=\Big(-\frac{a_2}{a_1}X,\frac{a_1}{a_2}Y\Big),$$

thus

$$\nabla^\perp\eta=0,\qquad\text{and}\qquad A_\eta(X,Y)=\Big(\frac{a_2}{a_1}X,-\frac{a_1}{a_2}Y\Big).$$

We conclude that

$$H=\frac{1}{ma_1a_2}(a_2^2m_1-a_1^2m_2)\eta,$$

thus M has constant mean curvature $|H|=\frac{1}{ma_1a_2}|a_2^2m_1-a_1^2m_2|$ and

$$|A_\eta|^2=\left(\frac{a_2}{a_1}\right)^2m_1+\left(\frac{a_1}{a_2}\right)^2m_2.$$

Thus, from Corollary 6.16, follows that M is proper biharmonic in $\mathbb{S}^{m+1}$ if and only if $a_1=a_2=\frac{1}{\sqrt{2}}$ and $m_1\neq m_2$.

Inspired by this basic examples, two methods for constructing proper biharmonic submanifolds of codimension higher than 1 in $\mathbb{S}^n$ were given.

Theorem 6.17 ([30]). *Let M be a minimal submanifold of $\mathbb{S}^{n-1}(a)\subset\mathbb{S}^n$. Then M is proper biharmonic in $\mathbb{S}^n$ if and only if $a=\frac{1}{\sqrt{2}}$.*

Remark 6.18.

(i) This result, called the *composition property*, proved to be quite useful for the construction of proper biharmonic submanifolds in spheres. For instance, it implies the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$ (see [30] and [78]).

(ii) All minimal submanifolds of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})\subset\mathbb{S}^n$ are pseudo-umbilical, have parallel mean curvature vector in $\mathbb{S}^n$ and $|H|=1$.

Non pseudo-umbilical examples were also produced by proving the following *product composition property*

Theorem 6.19 ([30]). *Let $M_1^{m_1}$ and $M_2^{m_2}$ be two minimal submanifolds of $\mathbb{S}^{n_1}(r_1)$ and $\mathbb{S}^{n_2}(r_2)$, respectively, where $n_1+n_2=n-1$, $r_1^2+r_2^2=1$. Then $M_1\times M_2$ is proper biharmonic in $\mathbb{S}^n$ if and only if $r_1=r_2=\frac{1}{\sqrt{2}}$ and $m_1\neq m_2$.*

Remark 6.20. The proper biharmonic submanifolds of $\mathbb{S}^n$ constructed as above are not pseudo-umbilical, but they have parallel mean curvature vector field, thus constant mean curvature, and $|H|\in(0,1)$.

We end this subsection with a partial classification result for constant mean curvature biharmonic submanifolds in spheres.

Theorem 6.21 ([97]). *Let M be a proper biharmonic submanifold with constant mean curvature in $\mathbb{S}^n$. Then $|H| \in (0, 1]$.*

Moreover, if $|H| = 1$, then M is a minimal submanifold of a hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^n$.

Proof. Let M be a constant mean curvature biharmonic submanifold of $\mathbb{S}^n$. The first equation of (6.7) implies that

$$\langle \Delta^\perp H, H\rangle = m|H|^2 - |A_H|^2,$$

and by using the Weitzenböck formula,

$$\frac{1}{2}\Delta|H|^2 = \langle \Delta^\perp H, H\rangle - |\nabla^\perp H|^2,$$

we obtain

$$m|H|^2 = |A_H|^2 + |\nabla^\perp H|^2. \tag{6.13}$$

Let $p \in M$ and consider $\{e_i\}$ to be an orthonormal basis of T_pM such that $A_H(e_i) = \lambda_i e_i$. From

$$\lambda_i = \langle A_H(e_i), e_i\rangle = \langle B(e_i, e_i), H\rangle$$

and

$$\sum \lambda_i = m|H|^2, \qquad \sum(\lambda_i)^2 = |A_H|^2,$$

using (6.13), we obtain at p

$$\sum \lambda_i = \sum(\lambda_i)^2 + |\nabla^\perp H|^2 \geq \frac{(\sum \lambda_i)^2}{m} + |\nabla^\perp H|^2. \tag{6.14}$$

Thus

$$m|H|^2 \geq m|H|^4 + |\nabla^\perp H|^2.$$

Consequently, if $|H| > 1$, the last inequality leads to a contradiction.

If $|H| = 1$, then the last inequality implies $\nabla^\perp H = 0$ and from (6.13) we have $\sum(\lambda_i)^2 = \frac{(\sum \lambda_i)^2}{m} = m$, thus we get $\lambda_1 = \ldots = \lambda_m$. Therefore M has parallel mean curvature and is pseudo-umbilical in $\mathbb{S}^n$. This implies that M is a minimal submanifold of a hypersphere $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$ (see, for example, [42]), and from Theorem 6.17 we conclude. □

6.2.2 The type of biharmonic submanifolds in spheres

Using the canonical inclusion of the sphere in the Euclidean space, another useful caracterization result for proper biharmonic submanifolds can be obtained.

Theorem 6.22 ([30]). *If $\phi : (M, g) \to \mathbb{S}^n$ is a Riemannian immersion and $\varphi = \mathbf{i} \circ \phi$, where $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical inclusion, then*

$$\tau_2(\phi) = \tau_2(\varphi) + 2m\tau(\varphi) + \{2m^2 - |\tau(\varphi)|^2\}\varphi.$$

In this section we intend to analyze the Chen type of proper biharmonic submanifolds of $\mathbb{S}^n$, as submanifolds in $\mathbb{R}^{n+1}$. In order to do so we shall mainly use the characterization in Theorem 6.22.

Finite type submanifolds in Euclidean spaces

We first recall some basic notions and results in the finite type submanifolds theory. The concept of submanifold of finite type was introduced by B-Y. Chen in the 1970's in his attempt to extend the notion of degree for the algebraic varieties to the case of arbitrary submanifolds in Euclidean spaces. Although the class of submanifolds of finite type proved to be rather large, it contains important examples of submanifolds of Euclidean spaces. We only enumerate here the minimal submanifolds of the Euclidean space and the minimal submanifolds of hyperspheres.

Consider M to be a compact Riemannian manifold and denote by Δ the Laplacian of M acting on $C^\infty(M)$. The eigenvalues of Δ constitute a discrete infinite sequence

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_t < \ldots \nearrow \infty.$$

Denote by $V_t = \{f \in C^\infty(M), \Delta f = \lambda_t f\}$ the eigenspace corresponding to λ_t. Then V_t is finite dimensional and with respect to the inner product $(f, h) = \int_M fh\, v_g$, the decomposition $\sum_t V_t$ is orthogonal and dense in $C^\infty(M)$. Thus every function $f \in C^\infty(M)$ can be decomposed as

$$f = f_0 + \sum_{t=1}^{\infty} f_t,$$

where f_0 is a constant and f_t is the projection of f on V_t. The function f is said to be *of k-type* if its spectral decomposition contains exactly k non-zero terms, excepting f_0. Although one cannot make spectral decomposition on a non-compact Riemannian manifold it is possible to define a function f of finite type, by requesting it to be a finite sum of eigenfunctions of the Laplacian.

For any smooth map $\varphi : M \to \mathbb{R}^{n+1}$ we have, in a similar manner, the spectral decomposition

$$\varphi = \varphi_0 + \sum_{t=1}^{\infty} \varphi_t,$$

where $\varphi_0 \in \mathbb{R}^{n+1}$ is a constant. If the map φ is an isometric immersion, then φ_0 is exactly the center of mass of M in $\mathbb{R}^{n+1}$, i.e. $\varphi_0 = \frac{1}{\operatorname{Vol} M} \int_M \varphi \, v_g$.

We shall use the following

Definition 6.23. An isometric immersion $\varphi : M \to \mathbb{R}^n$ is called *of finite type* if φ can be expressed as a finite sum of $\mathbb{R}^n$-valued eigenfunctions of the Laplacian Δ of M. When M is compact it is called of *k-type* if the spectral decomposition of φ contains exactly k non-zero terms, excepting the center of mass.

We recall that the type is a well-defined geometric invariant.

The following result constitutes a useful tool in determining whether a compact submanifold of $\mathbb{R}^n$ is of finite type.

Theorem 6.24 (Minimal Polynomial Criterion, [43, 46]). *Let $\varphi : M^m \to \mathbb{R}^n$ be an isometric immersion of a compact Riemannian manifold M into $\mathbb{R}^n$ and denote by H^0 the mean curvature vector field of M in $\mathbb{R}^n$. Then*

(i) *M is of finite type if and only if there exists a non-trivial polynomial $Q(t)$ such that $Q(\Delta)H^0 = 0$.*

(ii) *M is of finite type k if and only if there exists a unique monic (i.e. with leading coefficient equal to 1) polynomial $P(t)$ with exactly k distinct positive roots, such that $P(\Delta)H^0 = 0$.*

Also, a submanifold M^m of a hypersphere of $\mathbb{R}^{n+1}$ is said to be *mass-symmetric* if the center of mass of M coincides with the center of the hypersphere in $\mathbb{R}^{n+1}$. Thus, a submanifold of $\mathbb{S}^n$ is mass-symmetric if its center of mass is the origin.

We prove the following

Theorem 6.25 ([15]). *Let M^m be a compact constant mean curvature, $|H|^2 = k$, submanifold in $\mathbb{S}^n$. Then M is proper biharmonic if and only if*

either

(i) *$|H|^2 = 1$ and M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$,*

or

(ii) *$|H|^2 = k \in (0,1)$ and M is a mass-symmetric 2-type submanifold of $\mathbb{R}^{n+1}$ with the eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.*

Proof. Denote by $\phi : M \to \mathbb{S}^n$ the inclusion of M in $\mathbb{S}^n$ and by $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ the canonical inclusion. Let $\varphi : M \to \mathbb{R}^{n+1}$, $\varphi = \mathbf{i} \circ \phi$, be the inclusion of M in $\mathbb{R}^{n+1}$. Denote by H the mean curvature vector field of M in $\mathbb{S}^n$ and by H^0 the mean curvature vector field of M in $\mathbb{R}^{n+1}$.

The tension fields of the immersions ϕ and φ are related by

$$\tau(\varphi) = \tau(\phi) - m\varphi,$$

thus $H^0 = H - \varphi$.

Also, from Theorem 6.22, we get that $\tau_2(\phi) = 0$ if and only if

$$\Delta H^0 - 2mH^0 + m(|H|^2 - 1)\varphi = 0. \tag{6.15}$$

There are two situations to be analyzed.

If $|H|^2 = 1$, then $\Delta H^0 - 2mH^0 = 0$, and Theorem 6.24 implies that M is a 1-type submanifold of $\mathbb{R}^{n+1}$ with eigenvalue $\lambda = 2m$.

If $|H|^2 = k \in (0,1)$, then equation (6.15) implies

$$\begin{aligned} 0 &= \Delta\Delta H^0 - 2m\Delta H^0 + m(k-1)\Delta\varphi \\ &= \Delta\Delta H^0 - 2m\Delta H^0 - m^2(k-1)H^0. \end{aligned}$$

The monic polynomial with positive distinct roots described in Theorem 6.24, which provides the type of the submanifold M, is

$$P(\Delta) = \Delta^2 - 2m\Delta^1 - m^2(k-1)\Delta^0,$$

so M is a 2-type submanifold with eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.

Moreover, since $|H| \neq 1$ and it is constant, by integrating (6.15) on M we get $\varphi_0 = \frac{1}{\operatorname{Vol} M}\int_M \varphi\, v_g = 0$, i.e. M is mass-symmetric.

For the converse, let first M be a constant mean curvature $|H| = 1$ submanifold of $\mathbb{S}^n$. Suppose it is of 1-type with eigenvalue $\lambda = 2m$ in $\mathbb{R}^{n+1}$. This means that $\varphi = \varphi_0 + \varphi_1$, where $\Delta\varphi_1 = 2m\varphi_1$. By applying Δ, this implies $\Delta H^0 - 2mH^0 = 0$. From here we see that M satisfies equation (6.15), i.e. it is biharmonic in $\mathbb{S}^n$.

When $|H|^2 = k \in (0,1)$ and M is a mass-symmetric 2-type submanifold in $\mathbb{R}^{n+1}$ with eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$ we have

$$\varphi = \varphi_1 + \varphi_2,$$

where $\Delta\varphi_i = \lambda_i\varphi_i$, $i = 1, 2$. Applying the Laplacian we obtain

$$H^0 = -\{\varphi_1 + \varphi_2 + \sqrt{k}(\varphi_1 - \varphi_2)\} = -\varphi - \sqrt{k}(\varphi_1 - \varphi_2)$$

and

$$\Delta H^0 = -m\{(k+1)\varphi + 2(-\varphi - H^0)\} = -m\{(k-1)\varphi - 2H^0\}.$$

Finally, using (6.15), M is biharmonic in $\mathbb{S}^n$. □

Remark 6.26. Note that, using Theorem 6.21, we can conclude that all proper biharmonic submanifolds of $\mathbb{S}^n$ with $|H| = 1$ are 1-type submanifolds in $\mathbb{R}^{n+1}$, independently on whether they are compact or not.

6.3 Biharmonic hypersurfaces in spheres

The aim of our study is to classify the biharmonic submanifolds in space forms. We begin with the biharmonic hypersurfaces and our strategy consists in studying their mean curvature by separately analyzing them with respect to their number of distinct principal curvatures.

6.3.1 Biharmonic hypersurfaces with at most two distinct principal curvatures

If M is a proper biharmonic umbilical hypersurface, i.e. all its principal curvatures are equal, in $\mathbb{S}^{m+1}$, then it is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$. Also, since umbilical hypersurfaces in space forms have constant mean curvature, by using Corollary 6.16 we conclude that there exist no proper biharmonic umbilical hypersurfaces in $\mathbb{R}^{m+1}$ or in the hyperbolic space $\mathbb{H}^{m+1}$.

Similarly to the case of the Euclidean space (see [49]), the study of proper biharmonic hypersurfaces with at most two distinct principal curvatures constitutes the next natural step for the classification of proper biharmonic hypersurfaces in space forms.

As we have seen in the preliminary section there are examples of hypersurfaces with at most two distinct principal curvatures and non-constant mean curvature in any space form. In the following we show that, by adding the hypothesis of biharmonicity, the mean curvature proves to be constant.

Theorem 6.27 ([15]). *Let M be a hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$. If M is proper biharmonic in $\mathbb{E}^{m+1}(c)$, then it has constant mean curvature.*

Proof. Suppose that $|H|$ is not constant on M. Then there exists an open connected subset U of M such that $\operatorname{grad}_p |H|^2 \neq 0$, for all $p \in U$. By restricting U if necessary we can suppose that $|H| > 0$ on U, and $\operatorname{grad}_p |H| \neq 0$, for all $p \in U$. If U is umbilical, then its mean curvature is constant and we have a contradiction. Hence there exists a non-umbilical point in U. This implies the existence of an open neighborhood of non-umbilical points and we can suppose, by restricting U if necessary, that all its points are non-umbilical. On U we can consider the unit section in the normal bundle $\eta = \frac{H}{|H|}$ and denote by $f = |H|$, the mean curvature function of U in $\mathbb{E}^{m+1}(c)$. Denote by k_1, k_2 the principal curvature functions with respect to η, with multiplicity functions m_1 and m_2. The principal curvature functions are continuous on U and since,

$$\begin{cases} m_1 + m_2 = m, \\ \\ m_1 k_1 + m_2 k_2 = mf, \end{cases} \tag{6.16}$$

and

$$k_1(p) \neq k_2(p),$$

for all $p \in U$, it follows that m_1 and m_2 are continuous functions from U to $\mathbb{N}$, and thus they are constant on U.

We deduce that the hypothesis for M to be proper biharmonic with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$ and non-constant mean curvature, implies the existence of an open connected subset U of M, with

$$\begin{cases} \operatorname{grad}_p f \neq 0, \\ f(p) > 0, \qquad\qquad \forall p \in U \\ k_1(p) \neq k_2(p), \\ m_1, m_2 \quad \text{constant}, \end{cases} \tag{6.17}$$

where, denoting by η the unit section in the normal bundle, f is the mean curvature function of U in $\mathbb{E}^{m+1}(c)$, i.e. $H = \frac{1}{m}(\operatorname{trace} A)\eta = f\eta$, and k_1, k_2 are the principal curvature functions with respect to η, with multiplicities m_1, m_2.

Under these hypotheses we shall prove that f is constant on U, contradicting the condition $\operatorname{grad}_p f \neq 0, \forall p \in U$.

Since M is proper biharmonic in $\mathbb{E}^{m+1}(c)$, from (6.8) we have

$$\begin{cases} \Delta f = (mc - |A|^2)f, \\ A(\operatorname{grad} f) = -\frac{m}{2} f \operatorname{grad} f. \end{cases} \tag{6.18}$$

Consider now $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ on U. Then X_1 is a principal direction with principal curvature $k_1 = -\frac{m}{2} f$. Suppose that there are m_1 principal directions of principal curvature k_1 and m_2 principal directions of principal curvature $k_2 \neq k_1$ and recall that $mf = m_1 k_1 + m_2 k_2$.

We shall use the moving frames method and denote by $X_1, \{X_i\}_{i=2}^{m_1}, \{X_\alpha\}_{\alpha=m_1+1}^{m}$ the orthonormal frame field of principal directions and by $\{\omega^a\}_{a=1}^m$ the dual frame field of $\{X_a\}_{a=1}^m$ on U.

Obviously,

$$X_i(f) = \langle X_i, \operatorname{grad} f \rangle = |\operatorname{grad} f| \langle X_i, X_1 \rangle = 0, \qquad i = 2, \ldots, m_1$$

and analogously $X_\alpha(f) = 0$, $\alpha = m_1 + 1, \ldots, m$, thus

$$\operatorname{grad} f = X_1(f) X_1.$$

We write

$$\nabla X_a = \omega_a^b X_b, \qquad \omega_a^b \in C(T^*U).$$

From the Codazzi equations for M we get, for distinct $a, b, d = 1, \ldots, m$,

$$X_a(k_b) = (k_a - k_b)\omega_a^b(X_b) \tag{6.19}$$

and

$$(k_b - k_d)\omega_b^d(X_a) = (k_a - k_d)\omega_a^d(X_b). \tag{6.20}$$

First, we shall show that $m_1 = 1$.
Consider in equation (6.19), $a = 1$ and $b = i$. This leads to $X_1(k_1) = 0$, thus $|\operatorname{grad} f| = 0$ on U and we have a contradiction. Hence $m_1 = 1$, and so

$$k_2 = \frac{3m}{2(m-1)} f.$$

Consider now in (6.19), $a = 1$ and $b = \alpha$. We obtain

$$3X_1(f) = -(m+2) f \omega_1^\alpha(X_\alpha). \tag{6.21}$$

For $a = \alpha$ and $b = 1$, as $0 = X_\alpha(k_1)$, equation (6.19) leads to $\omega_1^\alpha(X_1) = 0$ and we can write

$$\omega_1^a(X_1) = 0, \quad \forall a = 1, \ldots, m. \tag{6.22}$$

From (6.20), for $a = 1$, $b = \alpha$ and $d = \beta$, with $\alpha \neq \beta$, we get

$$\omega_1^\beta(X_\alpha) = 0, \qquad \forall \alpha \neq \beta. \tag{6.23}$$

We now compute

$$\begin{aligned} \Delta f &= -\operatorname{div}(\operatorname{grad} f) = -\langle \nabla_{X_1} \operatorname{grad} f, X_1 \rangle - \sum_{\alpha=2}^{m} \langle \nabla_{X_\alpha} \operatorname{grad} f, X_\alpha \rangle \qquad (6.24) \\ &= -X_1\big(X_1(f)\big) - X_1(f) \sum_{\alpha=2}^{m} \omega_1^\alpha(X_\alpha). \end{aligned}$$

By using (6.21) we get that

$$f\Delta f = -f X_1\big(X_1(f)\big) + \frac{3(m-1)}{m+2} \big(X_1(f)\big)^2. \tag{6.25}$$

As $|A|^2 = k_1^2 + (m-1)k_2^2 = \frac{m^2(m+8)}{4(m-1)} f^2$ and M is biharmonic,

$$\Delta f = (mc - |A|^2) f = \Big(mc - \frac{m^2(m+8)}{4(m-1)} f^2\Big) f,$$

and equation (6.25) becomes

$$fX_1\big(X_1(f)\big) - \frac{3(m-1)}{m+2}\big(X_1(f)\big)^2 - \frac{m^2(m+8)}{4(m-1)}f^4 + mcf^2 = 0. \tag{6.26}$$

We shall now use the Gauss and the Cartan structural equations in order to obtain other information on f. We have

$$d\omega_1^\alpha \;=\; -\sum_{a=1}^{m}\omega_1^a \wedge \omega_\alpha^a - (k_1k_2 + c)\omega^1 \wedge \omega^\alpha,$$

thus, using equations (6.22) and (6.23), we get

$$d\omega_1^\alpha(X_1, X_\alpha) = -k_1k_2 - c = \frac{3m^2}{4(m-1)}f^2 - c. \tag{6.27}$$

On the other hand from (6.22) and (6.23) we obtain $\omega_1^\alpha = \omega_1^\alpha(X_\alpha)\omega^\alpha$, thus (6.21) implies

$$3X_1(f)\omega^\alpha = -(m+2)f\omega_1^\alpha. \tag{6.28}$$

By differentiating (6.28) we obtain

$$3d\big(X_1(f)\big) \wedge \omega^\alpha + 3X_1(f)d\omega^\alpha = -(m+2)(df \wedge \omega_1^\alpha + fd\omega_1^\alpha). \tag{6.29}$$

Now, substituting

$$\big(dX_1(f) \wedge \omega^\alpha\big)(X_1, X_\alpha) = X_1\big(X_1(f)\big),$$

$$d\omega^\alpha(X_1, X_\alpha) = \omega_1^\alpha(X_\alpha),$$

$$(df \wedge \omega_1^\alpha)(X_1, X_\alpha) = X_1(f)\omega_1^\alpha(X_\alpha)$$

in (6.29) and taking into account (6.27), we obtain

$$fX_1\big(X_1(f)\big) - \frac{m+5}{m+2}\big(X_1(f)\big)^2 + \frac{m^2(m+2)}{4(m-1)}f^4 - \frac{m+2}{3}cf^2 = 0. \tag{6.30}$$

Consider now an arbitrary integral curve γ of X_1 and denote by f' and f'' the first and the second derivatives of f along this curve. Equations (6.26) and (6.30) become, respectively,

$$ff'' - \frac{3(m-1)}{m+2}(f')^2 - \frac{m^2(m+8)}{4(m-1)}f^4 + mcf^2 = 0 \tag{6.31}$$

and

$$ff'' - \frac{m+5}{m+2}(f')^2 + \frac{m^2(m+2)}{4(m-1)}f^4 - \frac{m+2}{3}cf^2 = 0, \tag{6.32}$$

along γ.

Multiplying equation (6.31) by $(m+5)$ and equation (6.32) by $-3(m-1)$ and summing up, we get

$$(4-m)ff'' = \frac{m^2(m^2+4m+9)}{2(m-1)}f^4 - (m^2+3m-1)cf^2. \tag{6.33}$$

For $m=4$, equation (6.33) implies immediately that f is constant.
For $m \neq 4$, multiply equation (6.33) by f'/f, integrate the result and obtain

$$(f')^2 = \frac{m^2(m^2+4m+9)}{8(4-m)(m-1)}f^4 - \frac{(m^2+3m-1)}{2(4-m)}cf^2 + C. \tag{6.34}$$

On the other hand, multiplying equation (6.31) by -1 and adding it to equation (6.32) leads to

$$(f')^2 = \frac{m^2(m+5)(m+2)}{4(4-m)(m-1)}f^4 - \frac{(2m+1)(m+2)}{3(4-m)}cf^2 \tag{6.35}$$

From (6.34) and (6.35) we conclude that f is a solution of a polynomial equation with constant coefficients, thus f is constant along γ. Since γ is an arbitrary integral curve for X_1 we have $X_1(f)=0$ on U, thus f is constant. □

Corollary 6.28 ([15]). *Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$. Then M is orientable.*

To strengthen the Generalized Chen Conjecture, as an immediate consequence of Theorem 6.27, we have the following non-existence result.

Theorem 6.29 ([15]). *There exist no proper biharmonic hypersurfaces with at most two distinct principal curvatures in $\mathbb{H}^{m+1}$.*

Proof. Suppose that M is a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{H}^{m+1}$. From Theorem 6.27, the mean curvature of M is constant, and applying Corollary 6.16 we obtain $|A|^2 = -m$ and we conclude. □

The case of the sphere is essentially different. Theorem 6.27 proves to be the main ingredient for the following complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures.

Theorem 6.30 ([15]). *Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}})\times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1+m_2=m$, $m_1 \neq m_2$.*

Proof. By Theorems 6.27 and 6.21, the mean curvature of M in $\mathbb{S}^{m+1}$ is constant, $|H| \in (0,1]$. Thus we have a globally defined unit section in the normal bundle $\eta = \frac{H}{|H|}$ and a globally defined mean curvature function $f = |H|$. By using Corollary 6.16, we also obtain $|A|^2 = m$.

We now have two situations.

(i) If there exists an umbilical point $p_0 \in M$ and we denote by $k(p_0)$ the principal curvature with respect to η at p_0, then

$$mk(p_0) = m|H(p_0)|,$$

and $|A|^2 = m$ implies

$$mk^2(p_0) = m.$$

These two relations lead to $|H(p_0)| = 1$, but $|H|$ is constant, thus $|H| = 1$ on M. From Theorem 6.21 we conclude that M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.

(ii) If M has only non-umbilical points, we have the globally defined continuous principal curvature functions k_1 and k_2 with multiplicity functions m_1 and m_2 with respect to $\eta = H/|H|$, and $k_1(p) \neq k_2(p)$, for all $p \in M$. As in the proof of Theorem 6.27, the relations in (6.16) imply that m_1 and m_2 are constant on M. Since k_1 and k_2 are the solutions of

$$\begin{cases} m_1k_1 + m_2k_2 = mf, \\ \\ m_1k_1^2 + m_2k_2^2 = m, \end{cases} \tag{6.36}$$

where m_1, m_2 and f are constant, we conclude that M has two distinct constant principal curvatures. Proposition 2.5 in [100] implies that M is an open part of the product of two spheres $\mathbb{S}^{m_1}(a) \times \mathbb{S}^{m_2}(b)$, such that $a^2 + b^2 = 1$, $m_1 + m_2 = m$. Since M is biharmonic in $\mathbb{S}^n$, from Theorem 6.19, we get that $a = b = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$. □

Remark 6.31. Notice that, for $m = 2$, we recover the classification result for proper biharmonic surfaces in $\mathbb{S}^3$,

Theorem 6.32 ([29]). *A surface M^2 is proper biharmonic in $\mathbb{S}^3$ if and only if it is on open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$.*

As for quasi-umbilical and conformally flat biharmonic hypersurfaces in spheres we obtain

Theorem 6.33 ([15]). *Let M^m, $m \geq 3$, be a proper biharmonic hypersurface in $\mathbb{S}^{m+1}$. The following statements are equivalent*

(i) *M is quasi-umbilical,*

(ii) *M is conformally flat,*

(iii) *M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m-1}(\frac{1}{\sqrt{2}})$.*

Proof. By Theorem 6.30 we get that (i) is equivalent to (iii). Also, note that (iii) obviously implies (ii).

In order to prove that (ii) implies (i), recall that, for $m \geq 4$, by a well-known result (see, for example, [42]), any conformally flat hypersurface of a space form is quasi-umbilical and we conclude.

For $m = 3$, as the hypersurface is conformally flat, it follows that the $(0,2)$-tensor field
$L = -\operatorname{Ricci} + \frac{s}{4}\langle\ ,\ \rangle$, where s is the scalar curvature of M, is a Codazzi tensor field, i.e.

$$(\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z), \quad \forall X,Y,Z \in C(TM). \tag{6.37}$$

Using the notations from the proof of Theorem 6.27, the Gauss equation implies

$$\operatorname{Ricci}(X,Y) = 2\langle X,Y\rangle + 3f\langle A(X),Y\rangle - \langle A(X),A(Y)\rangle$$

and

$$s = 6 + 9f^2 - |A|^2. \tag{6.38}$$

We use the same techniques as in the proof of Theorem 6.27. Suppose the existence of an open subset U of M with 3 distinct principal curvatures.

If f is constant on U, using the above expressions, we conclude that U is flat and that the product of any of its two principal curvatures is -1, thus we get to a contradiction.

Assume that f is not constant on U. We can suppose that $\operatorname{grad}_p f \neq 0$, $\forall p \in U$. Consider $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$. As M is proper biharmonic, X_1 gives a principal direction with principal curvature $k_1 = -\dfrac{3}{2}f$. From $k_1 + k_2 + k_3 = 3f$, we can write $k_2 = \dfrac{9}{4}f + \varepsilon$ and $k_3 = \dfrac{9}{4}f - \varepsilon$, $\varepsilon \in C^\infty(U)$. Using the Codazzi and Gauss equations and equations (6.37) and (6.38) we show that $f = a\varepsilon^5$, $a \in \mathbb{R}$, and combining all these relations we obtain that ε is a solution of a polynomial equation with constant coefficients. Thus ε and f are constant.

Hence M has at most two distinct principal curvatures and this completes the proof. □

6.3.2 Biharmonic hypersurfaces with at most three distinct principal curvatures

For proper biharmonic hypersurfaces with constant mean curvature in spheres we also have the following geometric property

Proposition 6.34 ([15]). *Let M be a proper biharmonic hypersurface with constant mean curvature $|H|^2 = k$ in $\mathbb{S}^{m+1}$, $m \geq 2$. Then M has positive constant scalar curvature,*

$$s = m^2(1+k) - 2m.$$

Proof. Since M is proper biharmonic of constant mean curvature, the squared norm of its second fundamental form is $|A|^2 = m$. By applying the Gauss equation we obtain

$$\mathrm{Ricci}(X,Y) = (m-1)\langle X,Y\rangle + \langle A(X),Y\rangle \operatorname{trace} A - \langle A(X),A(Y)\rangle,$$

and, by considering the trace, we conclude. □

As a consequence, by using Theorem 6.7, the following non-existence result is obtained.

Theorem 6.35 ([16]). *There exist no compact proper biharmonic hypersurfaces of constant mean curvature and with three distinct principal curvatures everywhere in the unit Euclidean sphere.*

Proof. From the hypothesis, by using Proposition 6.34, it follows that M has constant scalar curvature. Since M has three distinct principal curvatures, we can apply Theorem 6.7 and we get that M is isoparametric with $\ell = 3$ in $\mathbb{S}^{m+1}$.

We now use (vii) in Theorem 6.6 in order to express the principal curvatures of M. There exists $\theta \in (0, \pi/3)$ such that

$$k_1 = \cot\theta, \qquad k_2 = \cot\big(\theta + \frac{\pi}{3}\big) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \qquad k_3 = \cot\big(\theta + \frac{2\pi}{3}\big) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}$$

Thus

$$|A|^2 = 2^q(k_1^2 + k_2^2 + k_3^2) = 2^q\left[\frac{9k_1^6 + 45k_1^2 + 6}{(1-3k_1^2)^2}\right]. \tag{6.39}$$

Moreover, from (iv) in Theorem 6.6 we obtain $m = 3\cdot 2^q$, $q = 0, \ldots, 3$ and since M is biharmonic of constant mean curvature, from (6.8), we get $|A|^2 = m = 3\cdot 2^q$.

The last equation together with (6.39) implies that k_1^2 is a solution of $P(t) = 3t^3 - 9t^2 + 21t + 1 = 0$, which is an equation with no positive roots. Indeed, $P(0) = 1 > 0$ and $P'(t) = 9t^2 - 18t + 21 > 0$, for all $t \in \mathbb{R}$, hence P is an increasing function on $\mathbb{R}$.

□

We shall now concentrate on biharmonic hypersurfaces in $\mathbb{E}^4(c)$. Similar techniques to the ones in [5] are used in order to prove

Theorem 6.36 ([16]). *Let M^3 be a biharmonic hypersurface of the space form $\mathbb{E}^4(c)$. Then M has constant mean curvature.*

Proof. Suppose that $|H|$ is not constant on M. Then there exists an open subset U of M such that $\operatorname{grad}_p |H|^2 \neq 0$, for all $p \in U$. By restricting U if necessary we can suppose that $|H| > 0$ on U, and thus $\operatorname{grad}_p |H| \neq 0$, for all $p \in U$. If U has at most two distinct principal curvatures, then by Theorem 6.27 we conclude that its mean curvature is constant and we have a contradiction. Hence there exists a point in U with three distinct principal curvatures. This implies the existence of an open neighborhood of points with three distinct principal curvatures and we can suppose, by restricting U, that all its points have three distinct principal curvatures. On U we can consider the unit section in the normal bundle $\eta = \frac{H}{|H|}$ and denote by $f = |H|$, the mean curvature function of U in $\mathbb{E}^4(c)$ and by k_i, $i = 1, 2, 3$, its principal curvatures with respect to η.

Thus, the hypothesis for M to be proper biharmonic with at most three distinct principal curvatures in $\mathbb{E}^4(c)$ and non-constant mean curvature, implies the existence of an open connected subset U of M, with

$$\begin{cases} \operatorname{grad}_p f \neq 0, \\ \\ f(p) > 0, \\ \\ k_1(p) \neq k_2(p) \neq k_3(p) \neq k_1(p), \qquad \forall p \in U. \end{cases} \tag{6.40}$$

By adding the hypothesis that M is biharmonic, we shall contradict the condition $\operatorname{grad}_p f \neq 0, \forall p \in U$.

Since M is proper biharmonic in $\mathbb{E}^4(c)$, we have

$$\begin{cases} \Delta f = (3c - |A|^2) f, \\ \\ A(\operatorname{grad} f) = -\frac{3}{2} f \operatorname{grad} f. \end{cases} \tag{6.41}$$

Consider now $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ on U. Then X_1 is a principal direction with principal curvature $k_1 = -\frac{3}{2} f$. Recall that $3f = k_1 + k_2 + k_3$, thus

$$k_2 + k_3 = \frac{9}{2} f. \tag{6.42}$$

We shall use the moving frames method and denote by X_1, X_2, X_3 the orthonormal frame field of principal directions and by $\{\omega^a\}_{a=1}^3$ the dual frame field of $\{X_a\}_{a=1}^3$ on U.

Obviously,

$$X_i(f) = \langle X_i, \operatorname{grad} f \rangle = |\operatorname{grad} f| \langle X_i, X_1 \rangle = 0, \qquad i = 2, 3, \tag{6.43}$$

thus

$$\operatorname{grad} f = X_1(f) X_1. \tag{6.44}$$

We write

$$\nabla X_a = \omega_a^b X_b, \qquad \omega_a^b \in C(T^*U).$$

From the Codazzi equations for M we get

$$X_a(k_b) = (k_a - k_b)\omega_a^b(X_b) \tag{6.45}$$

and

$$(k_b - k_d)\omega_b^d(X_a) = (k_a - k_d)\omega_a^d(X_b), \tag{6.46}$$

for distinct $a, b, d = 1, 2, 3$.

Consider now in (6.45), $a = 1$ and $b = i$ and, respectively, $a = i$ and $b = j$ with $i \neq j$. We obtain

$$\omega_i^1(X_i) = \frac{X_1(k_i)}{k_i - k_1}$$

and

$$\omega_j^i(X_j) = \frac{X_i(k_j)}{k_j - k_i}.$$

For $a = i$ and $b = 1$, as $X_i(k_1) = 0$, equation (6.45) leads to $\omega_i^1(X_1) = 0$ and we can write

$$\omega_a^1(X_1) = 0, \quad a = 1, 2, 3.$$

Notice that, since $X_i(f) = 0$, then $\langle [X_i, X_j], X_1 \rangle = 0$, thus $\omega_1^j(X_i) = \omega_1^i(X_j)$. Now, from (6.46), for $a = 1$, $b = i$ and $d = j$, with $i \neq j$, we get

$$\omega_2^1(X_3) = \omega_3^2(X_1) = \omega_1^3(X_2) = 0.$$

The structure 1-forms are thus determined by the following set of relations

$$\begin{cases} \omega_2^1(X_1) = 0, \quad \omega_2^1(X_2) = \frac{X_1(k_2)}{k_2 + \frac{3}{2}f} = \alpha_2, \quad \omega_2^1(X_3) = 0, \\ \omega_3^1(X_1) = 0, \quad \omega_3^1(X_2) = 0, \qquad\qquad \omega_3^1(X_3) = \frac{X_1(k_3)}{k_3 + \frac{3}{2}f} = \alpha_3, \\ \omega_3^2(X_1) = 0, \quad \omega_3^2(X_2) = \frac{X_3(k_2)}{k_3 - k_2} = \beta_2, \quad \omega_3^2(X_3) = \frac{X_2(k_3)}{k_3 - k_2} = \beta_3, \end{cases} \tag{6.47}$$

In order to express the first condition in (6.41), by using (6.42), we compute

$$\begin{aligned} |A|^2 &= k_1^2 + k_2^2 + k_3^2 \\ &= k_1^2 + (k_2 + k_3)^2 - 2k_2k_3 \\ &= \frac{45}{2}f^2 - 2K, \end{aligned} \tag{6.48}$$

where K denotes the product k_2k_3. From (6.44) we deduce that

$$\begin{aligned}\Delta f &= -\operatorname{div}(\operatorname{grad} f) = -\operatorname{div}(X_1(f)X_1) = -X_1(X_1(f)) - X_1(f)\operatorname{div} X_1\\ &= -X_1(X_1(f)) + X_1(f)(\omega_2^1(X_2) + \omega_3^1(X_3))\\ &= -X_1(X_1(f)) + X_1(f)(\alpha_2 + \alpha_3). \end{aligned} \tag{6.49}$$

Now, by using (6.48) and (6.49), the equation $\Delta f = (3c - |A|^2)f$ becomes

$$X_1(X_1(f)) - X_1(f)(\alpha_2 + \alpha_3) + (2K + 3c - \frac{45}{2}f^2)f = 0. \tag{6.50}$$

We also compute

$$\begin{aligned}[X_1, X_i] &= \nabla_{X_1}X_i - \nabla_{X_i}X_1 = \langle \nabla_{X_1}X_i, X_1\rangle X_1 - \langle \nabla_{X_i}X_1, X_i\rangle X_i\\ &= \omega_i^1(X_i)X_i = \alpha_i X_i. \end{aligned} \tag{6.51}$$

We shall now use the Gauss equation

$$\begin{aligned}\langle R^{\mathbb{E}^4(c)}(X,Y)Z, W\rangle &= \langle R(X,Y)Z, W\rangle\\ &\quad +\langle B(X,Z), B(Y,W)\rangle - \langle B(X,W), B(Y,Z)\rangle. \end{aligned} \tag{6.52}$$

From (6.52), for $X = W = X_1$ and $Y = Z = X_i$ we get

$$\begin{cases} X_1(\alpha_2) = \alpha_2^2 + c - \frac{3}{2}fk_2,\\ \\ X_1(\alpha_3) = \alpha_3^2 + c - \frac{3}{2}fk_3. \end{cases} \tag{6.53}$$

From (6.52), for $X = W = X_2$ and $Y = Z = X_3$ we get

$$K + c = X_2(\beta_3) - X_3(\beta_2) - \alpha_2\alpha_3 - \beta_2^2 - \beta_3^2. \tag{6.54}$$

From (6.52), for $Y = W = X_3$, $X = X_2$ and $Z = X_1$ and, respectively, for $X = W = X_2$, $Y = X_3$ and $Z = X_1$, we get

$$\begin{cases} X_2(\alpha_3) = \beta_3(\alpha_3 - \alpha_2),\\ \\ X_3(\alpha_2) = \beta_2(\alpha_3 - \alpha_2). \end{cases} \tag{6.55}$$

From (6.52), for $X = W = X_2$, $Y = X_1$ and $Z = X_3$, and, respectively, for $X = W = X_3$, $Y = X_2$ and $Z = X_1$ we get

$$\begin{cases} X_1(\beta_2) = \alpha_2\beta_2,\\ \\ X_1(\beta_3) = \alpha_3\beta_3. \end{cases} \tag{6.56}$$

Notice now that, from (6.43) and (6.51),

$$X_i(X_1(f)) = -[X_1, X_i]f + X_1(X_i(f)) = -\alpha_i X_i(f) + X_1(X_i(f)) = 0 \tag{6.57}$$

and

$$X_i(X_1(X_1(f))) = 0. \tag{6.58}$$

Also, since $K = \frac{(k_2+k_3)^2-(k_3-k_2)^2}{4}$ we obtain

$$\begin{cases} X_2(K) = -(k_3-k_2)^2\beta_3, \\ \\ X_3(K) = (k_3-k_2)^2\beta_2. \end{cases} \tag{6.59}$$

We differentiate (6.50) along X_2 and X_3 and use (6.55), (6.57), (6.58) and (6.59). We get

$$\begin{cases} X_2(\alpha_2) = -\beta_3(\alpha_3-\alpha_2) - \frac{2f}{X_1(f)}(k_3-k_2)^2\beta_3, \\ \\ X_3(\alpha_3) = -\beta_2(\alpha_3-\alpha_2) + \frac{2f}{X_1(f)}(k_3-k_2)^2\beta_2. \end{cases} \tag{6.60}$$

We intend to prove that $X_i(k_j) = 0$, $i,j = 2,3$. In order to do this we apply $[X_1,X_2] = \alpha_2 X_2$ to the quantity α_2. On one hand, from (6.60), we get

$$[X_1,X_2]\alpha_2 = \alpha_2 X_2(\alpha_2) = \beta_3\Big\{ -\alpha_2\alpha_3 + \alpha_2^2 - \frac{2f}{X_1(f)}(k_3-k_2)^2\alpha_2\Big\} \tag{6.61}$$

On the other hand, by using (6.53) and (6.60) we obtain

$$\begin{aligned} [X_1,X_2]\alpha_2 &= X_1(X_2(\alpha_2)) - X_2(X_1(\alpha_2)) \\ &= \beta_3\Big\{ -2\alpha_3^2 - \alpha_2^2 + 3\alpha_2\alpha_3 + \frac{2f}{X_1(f)}\big[-2(k_3-k_2)X_1(k_3-k_2) \\ &\quad +(k_3-k_2)^2(2\alpha_2-\alpha_3)\big] - 2X_1\Big(\frac{f}{X_1(f)}\Big)(k_3-k_2)^2\Big\} \end{aligned} \tag{6.62}$$

By putting together (6.61) and (6.62) we either have $\beta_3 = 0$ or

$$X_1\Big(\frac{f}{X_1(f)}\Big) = -\frac{(\alpha_3-\alpha_2)^2}{(k_3-k_2)^2} + \frac{f}{X_1(f)}\Big(3\alpha_2 - \alpha_3 - 2\frac{X_1(k_3-k_2)}{k_3-k_2}\Big). \tag{6.63}$$

Moreover,

$$\begin{aligned} X_2\Big(\frac{X_1(k_3-k_2)}{k_3-k_2}\Big) &= -\frac{1}{k_3-k_2}[X_1,X_2](k_3-k_2) + X_1\Big(\frac{X_2(k_3-k_2)}{k_3-k_2}\Big) \\ &= 2(\alpha_3-\alpha_2)\beta_3. \end{aligned}$$

Suppose that $\beta_3 \neq 0$, derive (6.63) along X_2 and use (6.55) and (6.60). We get

$$2(\alpha_3-\alpha_2) = -\frac{f}{X_1(f)}(k_3-k_2)^2. \tag{6.64}$$

We differentiate again along X_2 the equation (6.64) and obtain

$$\alpha_3 - \alpha_2 = -2\frac{f}{X_1(f)}(k_3-k_2)^2, \tag{6.65}$$

and since $k_2 \neq k_3$ the equations (6.64) and (6.65) lead to a contradiction.

Analogously, by using the symmetry of the equations in X_2 and X_3, we immediately prove that $\beta_2 = 0$.

We rewrite equations (6.53) in the form

$$\begin{cases} X_1(X_1(k_2)) = \frac{21}{2}\alpha_2 X_1(f) + 2(K+c)(k_3 + \frac{3}{2}f) + (c - \frac{3}{2}fk_2)(k_2 + \frac{3}{2}f), \\ \\ X_1(X_1(k_3)) = \frac{21}{2}\alpha_3 X_1(f) + 2(K+c)(k_2 + \frac{3}{2}f) + (c - \frac{3}{2}fk_3)(k_3 + \frac{3}{2}f), \end{cases} \tag{6.66}$$

and by summing up we obtain

$$X_1(X_1(f)) = \frac{7}{3}X_1(f)(\alpha_2 + \alpha_3) + f(4K + 5c - 9f^2). \tag{6.67}$$

Now, by using (6.50) and (6.67) we obtain

$$X_1(f)(\alpha_2 + \alpha_3) = f\big(-\frac{9}{2}K - 6c + \frac{189}{8}f^2\big). \tag{6.68}$$

We replace (6.68) in (6.67) and get

$$X_1(X_1(f)) = f\big(-\frac{13}{2}K - 9c + \frac{369}{8}f^2\big). \tag{6.69}$$

In order to get another relation on f and K we first use (6.54), (6.53), (6.42), (6.47) and determine

$$\begin{aligned} X_1(K) &= -X_1(\alpha_2\alpha_3) \\ &= -(\alpha_2\alpha_3 + c)(\alpha_2 + \alpha_3) + \frac{3}{2}f(\alpha_2 k_3 + \alpha_3 k_2) \\ &= (K + 9f^2)(\alpha_2 + \alpha_3) - \frac{27}{4}fX_1(f). \end{aligned} \tag{6.70}$$

By differentiating now (6.68) along X_1, and by using (6.70), (6.69), (6.53), (6.68) we get

$$X_1(f)\Big(\frac{13}{2}K + 10c - 108f^2\Big) = f(\alpha_2 + \alpha_3)\Big(\frac{13}{2}K + 15c - \frac{441}{4}f^2\Big) \tag{6.71}$$

We multiply (6.71) first by $X_1(f)$ and secondly by $\alpha_2 + \alpha_3$ and, by using (6.68), we get

$$\begin{cases} (X_1(f))^2(\frac{13}{2}K + 10c - 108f^2) = f^2(-\frac{9}{2}K - 6c + \frac{189}{8}f^2)(\frac{13}{2}K + 15c - \frac{441}{4}f^2), \\ \\ (\frac{13}{2}K + 10c - 108f^2)(-\frac{9}{2}K - 6c + \frac{189}{8}f^2) = (\alpha_2 + \alpha_3)^2(\frac{13}{2}K + 15c - \frac{441}{4}f^2). \end{cases} \tag{6.72}$$

Differentiating (6.68) along X_1, and using (6.70), (6.69), (6.53), (6.68), (6.72), we get

$$\begin{aligned} &27f^2(4044800c^3 - 49579440c^2f^2 + 187840944cf^4 - 254205945f^6) \\ &-6(51200c^3 - 19600320c^2f^2 + 119328660cf^4 - 80969301f^6)K \\ &-208(2240c^2 - 108396cf^2 - 285363f^4)K^2 \\ &+2704(16c - 2277f^2)K^3 \\ &+140608K^4 = 0. \end{aligned} \tag{6.73}$$

Consider now $\gamma = \gamma(t)$, $t \in I$, to be an integral curve of X_1 passing through $p = \gamma(t_0)$. Since $X_2(f) = X_3(f) = 0$ and $X_2(K) = X_3(K) = 0$ and $X_1(f) \neq 0$, we can write $t = t(f)$ in a neighborhood of $f_0 = f(t_0)$ and thus consider $K = K(f)$.

Notice that if $\frac{13}{2}K + 15c - \frac{441}{4}f^2 = 0$ or $10c + \frac{13}{2}K - 108f^2 = 0$ then from (6.73) the function f would be the solution of a polynomial equation of eighth degree with constant coefficients and we would get a contradiction. Thus, from (6.72) we have that

$$\begin{cases} (\frac{df}{dt})^2 = \frac{f^2(-\frac{9}{2}K-6c+\frac{189}{8}f^2)(\frac{13}{2}K+15c-\frac{441}{4}f^2)}{\frac{13}{2}K+10c-108f^2}, \\ \\ (\alpha_2+\alpha_3)^2 = \frac{(\frac{13}{2}K+10c-108f^2)(-\frac{9}{2}K-6c+\frac{189}{8}f^2)}{\frac{13}{2}K+15c-\frac{441}{4}f^2}. \end{cases} \tag{6.74}$$

We can now compute $\frac{dK}{df}$ by using (6.74), (6.70) and (6.68):

$$\begin{aligned} \frac{dK}{df} &= \frac{dK}{dt}\frac{dt}{df} \\ &= \frac{(K+9f^2)\frac{df}{dt}(\alpha_2+\alpha_3)}{(\frac{df}{dt})^2} - \frac{27}{4}f \\ &= \frac{(K+9f^2)(\frac{13}{2}K+10c-108f^2)}{f(\frac{13}{2}K+15c-\frac{441}{4}f^2)} - \frac{27}{4}f. \end{aligned} \tag{6.75}$$

The next step consists in differentiating (6.73) with respect to f. By substituting $\frac{dK}{df}$ from (6.75) we get another polynomial equation in f and K, of fifth degree in K. We eliminate K^5 between this new polynomial equation and (6.73). The result constitutes a polynomial equation in f and K, of fourth degree in K. In a similar way, by using (6.73) and its consequences we are able to gradually eliminate K^4, K^3, K^2 and K and we are led to a polynomial equation with constant coefficients in f. Thus f is constant and we conclude. □

By using Theorem 6.36 we reobtain the non-existence result concerning the proper biharmonic hypersurfaces of $\mathbb{R}^4$ presented in Chapter 3 and manage to prove that it also holds for the 4-dimensional hyperbolic space. These consequences constitute positive answers for the Generalized Chen Conjecture.

Theorem 6.37 ([16]). *There exist no proper biharmonic hypersurfaces in $\mathbb{R}^4$ or $\mathbb{H}^4$.*

Proof. Suppose that M^3 is a proper biharmonic hypersurface in $\mathbb{R}^4$ or $\mathbb{H}^4$. From Theorem 6.36, the mean curvature of M is constant, and applying Theorem 6.32 we obtain $|A|^2 = 0$ or $|A|^2 = -3$, respectively, and we conclude. □

Theorem 6.36 also allows the classification of all compact hypersurfaces of the sphere $\mathbb{S}^4$.

Theorem 6.38 ([16]). *The only proper biharmonic compact hypersurfaces of* $\mathbb{S}^4$ *are the hypersphere* $\mathbb{S}^3(\frac{1}{\sqrt{2}})$ *and the torus* $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.

Proof. Suppose that M^3 is a compact proper biharmonic hypersurface of $\mathbb{S}^4$. From Theorem 6.36, M has constant mean curvature and, since it satisfies the hypotheses of Proposition 6.34, we deduce that it also has constant scalar curvature. We can thus apply Theorem 6.8 and it follows that M is isoparametric in $\mathbb{S}^4$. From Theorem 6.35 we get that M cannot be isoparametric with $\ell = 3$, and by using Theorem 6.30 we conclude the proof.

□

In view of the above results we propose the following

Conjecture.

The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ *are the open parts of hyperspheres* $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ *or of generalized Clifford tori* $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

6.4 Codimension 2 biharmonic pseudo-umbilical submanifolds in spheres

We shall first prove two general results concerning the biharmonic submanifolds in spheres.

Theorem 6.39 ([15]). *Let* M^m *be a biharmonic submanifold with constant mean curvature in* $\mathbb{S}^n$. *Then its scalar curvature* s *satisfies*

$$s \leq 2m(m-1).$$

The equality holds if and only if M *is open in* $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.

Proof. From Theorem 6.4, the scalar curvature s of a submanifold M of $\mathbb{S}^n$ satisfies $s \leq m(m-1)(1+|H|^2)$, and equality holds if and only if M is contained in an m-sphere of $\mathbb{S}^n$.

By using Theorem 6.21, as M is biharmonic, we deduce that its constant mean curvature satisfies $|H| \in (0, 1]$, and this completes the proof. □

Theorem 6.40 ([15]). *Let* M^m *be a pseudo-umbilical submanifold of* $\mathbb{S}^n$, $m \neq 4$. *If* M *is biharmonic, then it has constant mean curvature.*

Proof. Consider $p \in M$ and let $\{E_i\}_{i=1,}^{m}$ be a local orthonormal frame field geodesic at p. As M is biharmonic, from (6.7), we get

$$\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) = -\frac{m}{4}\operatorname{grad}(|H|^2). \tag{6.76}$$

On the other hand, at p, by standard computations, we obtain

$$\begin{aligned}
\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) &= \sum_i A_{\nabla^\perp_{E_i}H}(E_i) = \sum_{i,j}\langle A_{\nabla^\perp_{E_i}H}E_i, E_j\rangle E_j \\
&= \sum_{i=1,j}^{m}\langle B(E_i,E_j), \nabla^\perp_{E_i}H\rangle\big\}E_j = \sum_{i,j}\langle \nabla^{\mathbb{S}^n}_{E_j}E_i, \nabla^{\mathbb{S}^n}_{E_i}H\rangle E_j \\
&= \sum_{i,j}\big\{E_i\langle\nabla^{\mathbb{S}^n}_{E_j}E_i, H\rangle - \langle\nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_j}E_i, H\rangle\big\}E_j,
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j}E_i\langle\nabla^{\mathbb{S}^n}_{E_j}E_i, H\rangle E_j &= \sum_{i,j}\big\{E_i\langle B(E_i,E_j), H\rangle E_j \\
&= \sum_{i,j}E_i\langle A_H(E_i), E_j\rangle E_j \\
&= \sum_i \nabla_{E_i}A_H(E_i),
\end{aligned}$$

$$\begin{aligned}
\sum_{i,j}\langle\nabla^{\mathbb{S}^n}_{X_i}\nabla^{\mathbb{S}^n}_{E_j}E_i, H\rangle E_j &= \sum_{i,j}\big\{\langle R^{\mathbb{S}^n}(E_i,E_j)E_i + \nabla^{\mathbb{S}^n}_{[E_i,E_j]}E_i, H\rangle E_j \\
&\quad +\langle\nabla^{\mathbb{S}^n}_{E_j}\nabla^{\mathbb{S}^n}_{E_i}E_i, H\rangle E_j\big\} \\
&= \sum_{i,j}\langle\nabla^{\mathbb{S}^n}_{E_j}\nabla^{\mathbb{S}^n}_{E_i}E_i, H\rangle E_j \\
&= \sum_{i,j}\{E_j\langle\nabla^{\mathbb{S}^n}_{E_i}E_i, H\rangle - \langle\nabla^{\mathbb{S}^n}_{E_i}E_i, \nabla^{\mathbb{S}^n}_{E_j}H\rangle\}E_j \\
&= \sum_j\{E_j\langle mH, H\rangle - \langle mH, \nabla^{\mathbb{S}^n}_{E_j}H\rangle\}E_j \\
&= \frac{m}{2}\operatorname{grad}(|H|^2).
\end{aligned}$$

Now, as M is pseudo-umbilical, $\sum_i \nabla_{X_i}A_H(X_i) = \operatorname{grad}(|H|^2)$, thus

$$\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) = \frac{2-m}{2}\operatorname{grad}(|H|^2). \tag{6.77}$$

By putting together expressions (6.76) and (6.77), we conclude. □

As a consequence, from Theorem 6.39, we obtain

Corollary 6.41 ([15]). *Let M^m be a biharmonic pseudo-umbilical submanifold in $\mathbb{S}^n$, $m \neq 4$. Then its scalar curvature s satisfies*

$$s \leq 2m(m-1).$$

6.4.1 Rigidity results

Concerning biharmonic pseudo-umbilical submanifolds of codimension two we obtain the following rigidity result.

Theorem 6.42 ([15]). *Let M^m be a pseudo-umbilical submanifold of $\mathbb{S}^{m+2}$, $m \neq 4$. Then M is proper biharmonic if and only if it is minimal in $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$.*

Proof. From the hypotheses, using Theorem 6.40, we deduce that M has constant mean curvature. Now, Theorem 6.3 implies that any such submanifold is either minimal in $\mathbb{S}^{m+2}$ or minimal in a hypersphere of $\mathbb{S}^{m+2}$. But M is proper biharmonic in $\mathbb{S}^{m+2}$ and, from Theorem 6.17, we conclude. □

We now replace the condition on M to be pseudo-umbilical with that of being a hypersurface of a hypersphere in $\mathbb{S}^{m+2}$.

Theorem 6.43 ([15]). *Let M^m be a hypersurface of $\mathbb{S}^{m+1}(a) \subset \mathbb{S}^{m+2}$, $a \in (0,1)$. Assume that M is not minimal in $\mathbb{S}^{m+1}(a)$. Then it is biharmonic in $\mathbb{S}^{m+2}$ if and only if $a > \frac{1}{\sqrt{2}}$ and M is open in $\mathbb{S}^m(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+1}(a)$.*

Proof. Note that the converse follows immediately from Theorem 6.17.

In order to prove the other implication, denote by $\mathbf{j}$ and $\mathbf{i}$ the inclusion maps of M in $\mathbb{S}^{m+1}(a)$ and of $\mathbb{S}^{m+1}(a)$ in $\mathbb{S}^{m+2}$, respectively.

We consider

$$\mathbb{S}^{m+1}(a) = \Big\{(x^1, \ldots, x^{m+2}, \sqrt{1-a^2}) \in \mathbb{R}^{m+3} : \sum_{i=1}^{m+2} (x^i)^2 = a^2\Big\} \subset \mathbb{S}^{m+2}.$$

Then

$$C\big(T\mathbb{S}^{m+1}(a)\big) = \Big\{(X^1, \ldots, X^{m+2}, 0) \in C(T\mathbb{R}^{m+3}) : \sum_{i=1}^{m+2} x^i X^i = 0\Big\},$$

while

$$\eta = \frac{1}{c}\Big(x^1, \ldots, x^{m+2}, -\frac{a^2}{\sqrt{1-a^2}}\Big)$$

is a unit section in the normal bundle of $\mathbb{S}^{m+1}(a)$ in $\mathbb{S}^{m+2}$, where $c^2 = \frac{a^2}{1-a^2}, c > 0$.

By computing the tension and bitension fields of $\phi = \mathbf{i} \circ \mathbf{j}$, one gets

$$\tau(\phi) = \tau(\mathbf{j}) - \frac{m}{c}\eta,$$

and

$$\tau_2(\phi) = \tau_2(\mathbf{j}) - \frac{2m}{c^2}\tau(\mathbf{j}) + \frac{1}{c}\{|\tau(\mathbf{j})|^2 - \frac{m^2}{c^2}(c^2-1)\}\eta.$$

By the hypotheses M is biharmonic in $\mathbb{S}^{m+2}$, thus

$$|\tau(\mathbf{j})|^2 = \frac{m^2}{c^2}(c^2-1) = \frac{m^2}{a^2}(2a^2-1)$$

and, since $\tau(\mathbf{j}) \neq 0$, this implies $a > \frac{1}{\sqrt{2}}$.

Also,

$$|\tau(\phi)|^2 = |\tau(\mathbf{j})|^2 + \frac{m^2}{c^2} = m^2.$$

This implies that the mean curvature of M in $\mathbb{S}^{m+2}$ is 1, thus, using Theorem 6.21, M has to be a minimal submanifold of the hypersphere $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+2}$, i.e. it is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^{m+2}$.

Since $M \subset \mathbb{S}^{m+1}(a)$ is pseudo-umbilical in $\mathbb{S}^{m+2}$ it is also pseudo-umbilical, and thus totally umbilical, in $\mathbb{S}^{m+1}(a)$. From here we get that M is an open subset of a hypersphere $\mathbb{S}^m(r)$ in $\mathbb{S}^{m+1}(a)$. But it is proper biharmonic in $\mathbb{S}^{m+2}$, thus $r = \frac{1}{\sqrt{2}}$ and we conclude. □

Corollary 6.44 ([15]). *Let M be a proper biharmonic submanifold in $\mathbb{S}^{m+2}$ such that $M \subset \mathbb{S}^{m+1}(a)$, $a \in (0,1)$. Then $a \geq \frac{1}{\sqrt{2}}$. Moreover,*

(i) *if $a = \frac{1}{\sqrt{2}}$, then M is minimal in $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$*

(ii) *if $a > \frac{1}{\sqrt{2}}$, then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.*

We also use Theorem 6.43 in order to prove

Theorem 6.45 ([15]). *Let M^2 be a proper biharmonic surface with parallel mean curvature vector field in $\mathbb{S}^n$. Then M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.*

Proof. B-Y. Chen and S-T. Yau proved (see [42, p.106]) that the only non-minimal surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ are either minimal surfaces of small hyperspheres $\mathbb{S}^{n-1}(a)$ of $\mathbb{S}^n$ or surfaces with constant mean curvature in 3−spheres of $\mathbb{S}^n$.

If M is a minimal surface of a small hypersphere $\mathbb{S}^{n-1}(a)$, then it is biharmonic in $\mathbb{S}^n$ if and only if $a = \frac{1}{\sqrt{2}}$ (see Theorem 6.17).

If M is a surface in a 3-sphere $\mathbb{S}^3(a)$, $a \in (0,1]$, of $\mathbb{S}^n$ then we can consider the composition

$$M \longrightarrow \mathbb{S}^3(a) \longrightarrow \mathbb{S}^4 \longrightarrow \mathbb{S}^n.$$

Note that M is biharmonic in $\mathbb{S}^n$ if and only if it is biharmonic in $\mathbb{S}^4$. From Theorem 6.43, for $a \in (0,1)$, we conclude that either $a = \frac{1}{\sqrt{2}}$ and M is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$, or $a > \frac{1}{\sqrt{2}}$

and M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$. For $a = 1$, from Theorem 6.32, also follows that M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$.

In all cases M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. □

Remark 6.46. All the results we have proved so far could suggest that the codimension 2 biharmonic submanifolds of $\mathbb{S}^n$ arise from minimal submanifolds of $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.

This is not the case since the product $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times M^m$, where M is a minimal non-totally geodesic hypersurface of $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$, is a full proper biharmonic submanifold of $\mathbb{S}^{m+3}$ of codimension 2 (see also Remark 6.53). Recall that a map $\phi : M \to \mathbb{S}^n$ is said to be *full* if the image $\phi(M)$ is contained in no hypersphere of $\mathbb{S}^n$.

Since all the known examples of proper biharmonic submanifolds in spheres have constant mean curvature we propose the following

Conjecture.

Any biharmonic submanifold in $\mathbb{S}^n$ has constant mean curvature.

6.5 New examples of proper biharmonic submanifolds in spheres

This section is devoted to the study of new examples of proper biharmonic submanifolds of codimension greater than 1 in spheres.

6.5.1 Biharmonic products of spheres

We shall begin with the classification of proper biharmonic products of spheres in the unit Euclidean sphere.

Theorem 6.47 ([17]). *Consider $\mathcal{T} = \mathbb{S}^{n_1}(a_1) \times \mathbb{S}^{n_2}(a_2) \times \ldots \times \mathbb{S}^{n_r}(a_r) \subset \mathbb{S}^{m+r-1} \subset \mathbb{R}^{m+r}$, where $m = \sum_{k=1}^{r} n_k$ and $\sum_{k=1}^{r} a_k^2 = 1$. Then*

(i) *$\mathcal{T}$ is minimal in $\mathbb{S}^{m+r-1}$ if and only if*

$$\frac{a_k^2}{n_k} = \frac{1}{m},$$

for all $k = 1, \ldots, r$.

(ii) *$\mathcal{T}$ is proper biharmonic in $\mathbb{S}^{m+r-1}$ if and only if there exists $p = 1, \ldots, r-1$ such that*

$$\frac{a_1^2}{n_1} = \ldots = \frac{a_p^2}{n_p} = \frac{1}{2(n_1 + \ldots + n_p)} \neq \frac{1}{m}$$

and

$$\frac{a_{p+1}^2}{n_{p+1}} = \ldots = \frac{a_r^2}{n_r} = \frac{1}{2(n_{p+1}+\ldots+n_r)} \neq \frac{1}{m}.$$

Proof. Denote by $(x_1^1,\ldots,x_1^{n_1+1},\ldots,x_r^1,\ldots,x_r^{n_r+1}) = p \in \mathcal{T}$ the position vector field in $\mathbb{R}^{m+r}$. Consider $\eta_k = \frac{1}{a_k}x_k$. This proves to be the unit normal vector field of $\mathbb{S}^{n_k}(a_k)$ in $\mathbb{R}^{n_k+1} \subset \mathbb{R}^{m+r}$. Consider $p = (x_1,\ldots,x_r)$ and denote by $\{E_{k,i}\}_{i=1}^{n_k}$ a local orthonormal frame field on $\mathbb{S}^{n_k}(a_k)$ geodesic at x_k. Then $\{E_{k,i}\}_{\substack{k=\overline{1,r}\\ i=\overline{1,n_k}}}$ constitutes a local orthonormal frame field on $\mathcal{T}$, geodesic at p.

Denote by B the second fundamental form of $\mathcal{T}$ in $\mathbb{S}^{m+r-1}$, by ∇^0, ∇ and $\nabla^{\mathcal{T}}$ the Levi-Civita connections on $\mathbb{R}^{m+r}$, $\mathbb{S}^{m+r-1}$ and $\mathcal{T}$, respectively. Then

$$\begin{aligned} B(E_{k,i},E_{k,i}) &= \nabla_{E_{k,i}}E_{k,i} - \nabla^{\mathcal{T}}_{E_{k,i}}E_{k,i} \\ &= \nabla^0_{E_{k,i}}E_{k,i} + \langle E_{k,i},E_{k,i}\rangle p \\ &= \langle \nabla^0_{E_{k,i}}E_{k,i},\eta_k\rangle\eta_k + p \\ &= -\langle E_{k,i},\nabla^0_{E_{k,i}}\eta_k\rangle\eta_k + p = -\frac{1}{a_k}\eta_k + p \\ &= a_1\eta_1 + \ldots + \Big(-\frac{1}{a_k} + a_k\Big)\eta_k + \ldots + a_r\eta_r. \end{aligned} \tag{6.78}$$

Thus the mean curvature vector field of $\mathcal{T}$ in $\mathbb{S}^{m+r-1}$ is given by

$$\begin{aligned} mH &= \sum_{k=1}^{r}\sum_{i=1}^{n_k} B(E_{k,i},E_{k,i}) = \sum_{k=1}^{r}\Big(-\frac{n_k}{a_k}\eta_k + n_k p\Big) = -\sum_{k=1}^{r}\frac{n_k}{a_k}\eta_k + mp \\ &= \sum_{k=1}^{r}\Big(-\frac{n_k}{a_k} + ma_k\Big)\eta_k. \end{aligned} \tag{6.79}$$

We get

$$H = \frac{1}{m}\sum_{k=1}^{r}\Big(-\frac{n_k}{a_k} + ma_k\Big)\eta_k$$

and $\mathcal{T}$ has constant mean curvature in $\mathbb{S}^{m+r-1}$. We shall prove that $\mathcal{T}$ has parallel mean curvature in $\mathbb{S}^{m+r-1}$. Indeed

$$\begin{aligned} \nabla_{E_{k,i}}H &= \nabla^0_{E_{k,i}}H \\ &= \frac{1}{m}\Big(-\frac{n_k}{a_k^2} + m\Big)E_{k,i}, \end{aligned} \tag{6.80}$$

thus

$$\nabla^{\perp}_{E_{k,i}}H = 0, \qquad k = 1,\ldots,r, \quad i = 1,\ldots,n_k$$

and

$$A_H(E_{k,i}) = \frac{1}{m}\Big(\frac{n_k}{a_k^2} - m\Big)E_{k,i}. \tag{6.81}$$

Since H is parallel, the conditions on $\mathcal{T}$ to be biharmonic in $\mathbb{S}^{m+r-1}$ is equivalent to

$$mH = \operatorname{trace} B(\cdot, A_H\cdot). \tag{6.82}$$

From (6.81) we get

$$\begin{aligned}\operatorname{trace} B(\cdot, A_H\cdot) &= \sum_{k=1}^{r}\sum_{i=1}^{n_k} B(E_{k,i}, A_H(E_{k,i}))\\ &= \sum_{k=1}^{r}\frac{1}{m}\Big(\frac{n_k}{a_k^2}-m\Big)\sum_{i=1}^{n_k} B(E_{k,i},E_{k,i}) \qquad (6.83)\\ &= \frac{1}{m}\sum_{k=1}^{r}\Big(\frac{n_k}{a_k^2}-m\Big)n_k\Big(a_1\eta_1+\ldots+\big(-\frac{1}{a_k}+a_k\big)\eta_k+\ldots+a_r\eta_r\Big),\end{aligned}$$

and equation (6.82) becomes

$$\frac{a_k^2}{n_k}\Big(\sum_{j=1}^{r}\frac{n_j^2}{a_j^2}-2m^2\Big)+2m-\frac{n_k}{a_k^2}=0, \qquad k=1,\ldots,r. \tag{6.84}$$

Denote now by $\alpha_k=\frac{a_k^2}{n_k}$ and by $d=\sum_{j=1}^{r}\frac{n_j}{\alpha_j}$. Then equation (6.84) becomes

$$(2m^2-d)\alpha_k^2-2m\alpha_k+1=0, \qquad k=1,\ldots,r. \tag{6.85}$$

We have two cases:

(i) $\alpha_k=\alpha$, for all k, thus $d=m/\alpha$ and we have $2m^2\alpha^2-3m\alpha+1=0$. The condition $\sum_{k=1}^{r}a_k^2=1$ implies that $\alpha=1/m$ is the only solution of (6.85) and, since in this case $a_k^2=n_k/m$, $\mathcal{T}$ is minimal in $\mathbb{S}^{m+r-1}$,

(ii) there exists $p=1,\ldots,r-1$ such that $\alpha_1=\ldots=\alpha_p=A$ and $\alpha_{p+1}=\ldots=\alpha_r=B$, $A\neq B$. Then, from (6.85) it follows that $A=\frac{1}{2(n_1+\ldots+n_p)}\neq\frac{1}{m}$ and $B=\frac{1}{2(n_{p+1}+\ldots+n_r)}\neq\frac{1}{m}$. □

Remark 6.48.

(i) All the examples constructed as above arise from the product composition property (Theorem 6.19). Indeed, $\mathbb{S}^{n_1}(a_1)\times\mathbb{S}^{n_2}(a_2)\times\ldots\times\mathbb{S}^{n_p}(a_p)$ and $\mathbb{S}^{n_{p+1}}(a_{p+1})\times\mathbb{S}^{n_{p+2}}(a_{p+2})\times\ldots\times\mathbb{S}^{n_r}(a_r)$ with the radii given by Theorem 6.47, are minimal in $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}})$ and $\mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, respectively, where $m_1=n_1+\ldots+n_p+p-1$ and $m_2=n_{p+1}+\ldots+n_r+r-p-1$.

(ii) Theorem 6.47 generalizes a result of W. Zhang (see [124]) which characterizes the biharmonicity of products of circles in spheres.

(iii) For $r=2$ in Theorem 6.47 we obtain the example of Jiang of the biharmonic generalized Clifford torus.

6.5.2 Geometric properties of two classes of biharmonic submanifolds of $\mathbb{S}^5$

In the following we shall analyze the geometric properties of two special classes of proper biharmonic submanifolds in $\mathbb{S}^5$. The computations reveal that, although they have constant mean curvature, their mean curvature vector field is not parallel. Notice that all the examples of proper biharmonic submanifolds in spheres previously presented have parallel mean curvature vector field.

In [4, 106] the authors studied the proper biharmonic Legendre surfaces and 3-dimensional anti-invariant submanifolds in Sasakian space forms. They obtained the explicit representations of such submanifolds in the unit Euclidean 5-dimensional sphere $\mathbb{S}^5$.

Theorem 6.49 ([106]). *Let $\phi : M^2 \to \mathbb{S}^5$ be a proper biharmonic Legendre immersion. Then the position vector field $x_0 = \mathbf{i} \circ \phi = x_0(u, v)$ of M in $\mathbb{R}^6$ is given by*

$$x_0(u,v) = \tfrac{1}{\sqrt{2}}(e^{iu}, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v),$$

where $\mathbf{i} : \mathbb{S}^5 \to \mathbb{R}^6$ is the canonical inclusion.

Remark 6.50. The map ϕ is a proper biharmonic Legendre immersion from a torus into $\mathbb{S}^5$. We shall verify that it is biharmonic by using Theorem 6.22.

We have

$$\nabla dx_0\big(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial u}\big) = \tfrac{1}{\sqrt{2}}(-e^{iu}, -ie^{-iu}\sin\sqrt{2}v, -ie^{-iu}\cos\sqrt{2}v) = -x_0,$$

$$\nabla dx_0\big(\tfrac{\partial}{\partial v}, \tfrac{\partial}{\partial v}\big) = \sqrt{2}(0, -ie^{-iu}\sin\sqrt{2}v, -ie^{-iu}\cos\sqrt{2}v) = -2x_0 + \sqrt{2}(e^{iu}, 0, 0),$$

thus

$$\tau(x_0) = \operatorname{trace}\nabla dx_0 = -3x_0 + \sqrt{2}(e^{iu}, 0, 0). \tag{6.86}$$

A straightforward computation leads to

$$|\tau(x_0)|^2 = 5 \tag{6.87}$$

and

$$\tau_2(x_0) = 9x_0 - 4\sqrt{2}(e^{iu}, 0, 0). \tag{6.88}$$

Now, by using Theorem 6.22, from (6.86), (6.87) and (6.88) it follows that

$$\begin{aligned}\tau_2(\phi) &= \tau_2(x_0) + 4\tau(x_0) + \{8 - |\tau(x_0)|^2\}x_0 \\ &= 0.\end{aligned}$$

and we conclude.

Lemma 6.51. *Let $\phi : M^2 \to \mathbb{S}^5$ be a proper biharmonic Legendre immersion. Then it has constant mean curvature, is not pseudo-umbilical and its mean curvature vector field is not parallel.*

Proof. We shall use the explicit parametrization in $\mathbb{R}^6$ obtained in Theorem 6.49. Recall that, if H and H^0 denote the mean curvature vector of ϕ and x_0, respectively, then

$$\begin{aligned} H &= H^0 + x_0 \\ &= -\frac{3}{2}x_0 + \frac{1}{\sqrt{2}}(e^{iu}, 0, 0) + x_0 \\ &= -\frac{1}{2}x_0 + \frac{1}{\sqrt{2}}(e^{iu}, 0, 0). \end{aligned}$$

From here

$$|H|^2 = \frac{1}{4}|x_0|^2 + \frac{1}{2}|(e^{iu}, 0, 0)|^2 - \frac{1}{\sqrt{2}}\langle x_0, (e^{iu}, 0, 0)\rangle = \frac{1}{4}.$$

Therefore ϕ is a constant mean curvature Riemannian immersion in $\mathbb{S}^5$.

By using the relation between the second fundamental form of x_0 and that of ϕ, we obtain that $A_H(\frac{\partial}{\partial u}) = 0$ and $A_H(\frac{\partial}{\partial v}) = \frac{1}{2}\frac{\partial}{\partial v}$, so ϕ is not pseudo-umbilical in $\mathbb{S}^5$.

Also

$$\nabla^{\perp}_{\frac{\partial}{\partial u}} H^0 = \frac{1}{2\sqrt{2}}(ie^{iu}, -e^{-iu}\sin\sqrt{2}v, -e^{-iu}\cos\sqrt{2}v),$$

and consequently $\nabla^{\perp} H \neq 0$, thus the mean curvature vector of ϕ is not parallel. □

Moreover, since $|H| = 1/2$, from Theorem 6.25 we get that x_0 is a 2-type immersion in $\mathbb{R}^6$ of eigenvalues 1 and 3. Indeed, as T. Sasahara proved, $x_0 = x_{01} + x_{02}$,

$$x_{01} = \tfrac{1}{\sqrt{2}}(e^{iu}, 0, 0),$$

$$x_{02} = \tfrac{1}{\sqrt{2}}(0, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v),$$

and

$$\Delta x_{01} = x_{01},$$

$$\Delta x_{02} = 3x_{02}.$$

Recall that the Willmore equation for a surface M^2 in $\mathbb{S}^n$ is

$$-\Delta^{\perp} H - 2|H|^2 H + \operatorname{trace} B(\cdot, A_H \cdot) = 0,$$

where B denotes the second fundamental form of M in $\mathbb{S}^n$ and A_H the Weingarten operator associated to H (see, for example, [118]). By straightforward computations, for $\phi : M^2 \to \mathbb{S}^5$, we obtain $\Delta^{\perp} H = H$ and, since $|H|^2 = 1/4$ and ϕ is biharmonic, i.e.

$$-\Delta^{\perp} H + 2H - \operatorname{trace} B(\cdot, A_H \cdot) = 0,$$

we conclude that ϕ is not a Willmore immersion.

Theorem 6.52 ([4]). *Let $\psi : M^3 \to \mathbb{S}^5$ be a proper biharmonic anti-invariant immersion. Then the position vector field $y_0 = \mathbf{i} \circ \psi = y_0(u, v, w)$ of M in $\mathbb{R}^6$ is given by*

$$y_0(u, v, w) = \tfrac{1}{\sqrt{2}} e^{iw}(e^{iu}, ie^{-iu} \sin\sqrt{2}v, ie^{-iu} \cos\sqrt{2}v),$$

where $\mathbf{i} : \mathbb{S}^5 \to \mathbb{R}^6$ is the canonical inclusion.

Remark 6.53. Recall that if we consider a Sasakian manifold (N, Ψ, ξ, η, g) and a submanifold M tangent to ξ, M is called *anti-invariant* if Ψ maps any tangent vector to M which is normal to ξ to a vector which is normal to M.

Note that ψ is a full proper biharmonic anti-invariant immersion from a 3-dimensional torus into $\mathbb{S}^5$.

As in the previous case one can verify that ψ is biharmonic by using the expressions

$$\tau(y_0) = -4y_0 + \sqrt{2}e^{iw}(e^{iu}, 0, 0),$$
$$|\tau(y_0)|^2 = 10,$$
$$\tau_2(y_0) = 16y_0 - 6\sqrt{2}e^{iw}(e^{iu}, 0, 0),$$

in Theorem 6.22. Indeed, we obtain

$$\begin{aligned} \tau_2(\psi) &= \tau_2(y_0) + 6\tau(y_0) + \{18 - |\tau(y_0)|^2\}y_0 \\ &= 0, \end{aligned}$$

and we conclude.

Lemma 6.54. *Let $\psi : M^3 \to \mathbb{S}^5$ be a proper biharmonic anti-invariant immersion. Then it has constant mean curvature, is non pseudo-umbilical and its mean curvature vector field is parallel.*

In this case,

$$H = -\frac{1}{3}y_0 + \frac{\sqrt{2}}{3}e^{iw}(e^{iu}, 0, 0),$$

thus ψ is a constant mean curvature Riemannian immersion in $\mathbb{S}^5$, $|H|^2 = 1/9$. One can verify that the mean curvature vector of ψ is parallel and ψ is not pseudo-umbilical in $\mathbb{S}^5$, since $A_H(\frac{\partial}{\partial u}) = 0$, $A_H(\frac{\partial}{\partial v}) = \frac{1}{3}\frac{\partial}{\partial v}$ and $A_H(\frac{\partial}{\partial w}) = 0$.

In addition to these properties, since $|H| = 1/3$, from Theorem 6.25 we conclude that y_0 is a 2-type submanifold of $\mathbb{R}^6$ with eigenvalues 2 and 4. Indeed $y_0 = y_{01} + y_{02}$,

$$y_{01} = \tfrac{1}{\sqrt{2}} e^{iw}(e^{iu}, 0, 0),$$
$$y_{02} = \tfrac{1}{\sqrt{2}} e^{iw}(0, ie^{-iu} \sin\sqrt{2}v, ie^{-iu} \cos\sqrt{2}v),$$

and

$$\Delta y_{01} = 2y_{01},$$
$$\Delta y_{02} = 4y_{02}.$$

Chapter 7

On the biharmonicity of the Gauss map

Ever since the characterization result obtained by E. Ruh and J. Vilms in [101] and the remarkable link with constant mean curvature hypersurfaces the study of submanifolds with associated harmonic Gauss map in Euclidean spaces has been a classical problem in harmonic maps theory. The present chapter proposes a generalization of this problem: the characterization of submanifolds with proper biharmonic Gauss map in Euclidean spaces. The new results of this chapter are gathered in [18].

In the first section we recall the definition and the differential structure of the Grassmannian (see [62]). We use the identification of the Grassmannian as a symmetric space in order to compute its curvature (see [99]).

We then recall the definition of the Gauss map associated to a submanifold in an Euclidean space and compute its differential and its tension field (see [51, 50, 101]).

In the third section we characterize the submanifolds of the Euclidean space with biharmonic Gauss map (Theorem 7.3). The biharmonic equation for the Gauss map simplifies for the case of hypersurfaces (see Theorem 7.4).

The last section is devoted to the examples of hypercones with biharmonic Gauss map in the Euclidean space. We study the hypercones generated by constant mean curvature hypersurfaces in spheres and, in particular, by the hyperspheres (Theorem 7.7 and Proposition 7.11). We obtain some examples and a non-existence result for cones with biharmonic Gauss map in $\mathbb{R}^3$ (Theorem 7.8).

7.1 The Grassmannians. Preliminaries

In this section we shall recall the definition and the differential and metric structure of the Grassmannian (see [62]).

7.1.1 Definition and differential structure

Let $\widetilde{G}(m,n)$ be the set of m-dimensional linear subspaces of $\mathbb{R}^{m+n}$. Take $Q \in \widetilde{G}(m,n)$ with $Q = \text{span}\{v_1, \ldots, v_m\}$. To Q we can associate the multivector

$$\lambda = v_1 \wedge \ldots \wedge v_m \in \wedge^m \mathbb{R}^{m+n}.$$

Note that λ is determined up to a scalar by Q. Indeed, if we chose $Q = \text{span}\{\overline{v}_1, \ldots, \overline{v}_m\}$ and denote by $K = (k_i^j)$ the change of basis matrix (i.e. $\overline{v}_i = k_i^j v_j$), then

$$\overline{\lambda} = \overline{v}_1 \wedge \ldots \wedge \overline{v}_m = \det K v_1 \wedge \ldots \wedge v_m = \det K \lambda.$$

We thus have a well-defined map

$$\Psi : \widetilde{G}(m,n) \to \mathbb{P}(\wedge^m \mathbb{R}^{m+n})$$

$$Q \to [\lambda].$$

This is obviously an inclusion because any $[\lambda]$ in the image of Ψ uniquely determines the corresponding subspace Q,

$$Q = \{v \in \mathbb{R}^{m+n} / v \wedge \lambda = 0\}.$$

The inclusion Ψ is called the *Plücker embedding* of $\widetilde{G}(m,n)$.

The homogeneous coordinates on $\mathbb{P}(\wedge^m \mathbb{R}^{m+n})$ are called the *Plücker coordinates* on $\widetilde{G}(m,n)$. We can identify the m-plane Q by the matrix M_Q whose rows are the vectors v_i, this identification being determined up to a multiplication on the left by an invertible $m \times m$ matrix. The Plücker coordinates are just the maximal minors of the matrix M_Q.

We have described until now the Grassmannian $\widetilde{G}(m,n)$ as a subset of the projective space $\mathbb{P}(\wedge^m \mathbb{R}^{m+n})$. In the following we shall see that it is a submanifold, by characterizing the subset of totally decomposable vectors $\lambda \in \wedge^m \mathbb{R}^{m+n}$, i.e. products of type $\lambda = v_1 \wedge \ldots \wedge v_m$. The vector $v \in \mathbb{R}^{m+n}$ divides λ (i.e. there exists φ such that $\lambda = \varphi \wedge v$) if and only if $\lambda \wedge v = 0$. Moreover, λ is decomposable if and only if

$$\dim K_\lambda = \dim\{v \in \mathbb{R}^{m+n} / \lambda \wedge v = 0\} = m.$$

Thus, for $\lambda \in \wedge^m \mathbb{R}^{m+n}$, $[\lambda] \in \widetilde{G}(m,n)$ if and only if the rank of the linear map described by

$$\phi(\lambda) : \mathbb{R}^{m+n} \to \wedge^{m+1} \mathbb{R}^{m+n}$$

$$v \to \lambda \wedge v$$

is equal to n. As $\dim K_\lambda$ is always greater than or equal to m we can conclude

$$[\lambda] \in \widetilde{G}(m,n) \Leftrightarrow \text{rank}\, \phi(\lambda) \leq n.$$

Now, since the map

$$\wedge^m\mathbb{R}^{m+n} \to \mathrm{Hom}(\mathbb{R}^{m+n}, \wedge^{m+1}\mathbb{R}^{m+n})$$

$$\lambda \to \phi(\lambda)$$

is linear, the entries of the matrix $\phi(\lambda)$ are homogeneous coordinates on $\mathbb{P}(\wedge^m\mathbb{R}^{m+n})$. We can thus characterize $\widetilde{G}(m,n) \subset \mathbb{P}(\wedge^m\mathbb{R}^{m+n})$ as the submanifold defined by the vanishing of the $n+1$ minors of the matrix.

We recall that there is another approach of the Grassmanian, by using certain special affine open subsets. To describe them consider $W \subset \mathbb{R}^{m+n}$ an n-dimensional subspace, corresponding to a multivector $\omega \in \wedge^n\mathbb{R}^{m+n} \equiv \wedge^m(\mathbb{R}^{m+n})^*$. We can think of ω as a homogeneous linear form on $\mathbb{P}(\wedge^m\mathbb{R}^{m+n})$. Consider $U_W \subset \mathbb{P}(\wedge^m\mathbb{R}^{m+n})$ the affine open set where $\omega \neq 0$,

$$U_W = \{\lambda \in \wedge^m\mathbb{R}^{m+n}/\omega(\lambda) \neq 0\}.$$

Then

$$\widetilde{G}(m,n) \cap U_W = \{Q \subset \mathbb{R}^{m+n}/\dim Q = m, Q \cap W = \{0\}\}.$$

Any such subspace Q can be viewed as the graph of a linear map from $\mathbb{R}^{m+n}/W \to W$ and viceversa, so that we have the identification

$$\widetilde{G}(m,n) \cap U_W = \mathrm{Hom}(\mathbb{R}^{m+n}/W, W) = \mathcal{M}_{m\times n}(\mathbb{R}).$$

To see this in coordinates consider, for example, $W = \mathrm{span}\{e_{m+1}, \ldots, e_{m+n}\}$, where $\{e_r\}_{r=1}^{m+n}$ is the canonical basis of $\mathbb{R}^{m+n}$. Then

$$\widetilde{G}(m,n) \cap U_W = \{Q \subset \mathbb{R}^{m+n}/\dim Q = m, \text{the first } m \text{ minor of } M_Q \text{ is non-zero}\}.$$

Thus any $Q \in \widetilde{G}(m,n) \cap U_W$ is represented as the row space of a unique matrix

$$M_Q^* = \begin{pmatrix} 1 & \ldots & 0 & a_{1m+1} & \ldots & a_{1m+n} \\ 0 & \ddots & 0 & a_{2m+1} & \ldots & a_{2m+n} \\ 0 & \ldots & 1 & a_{mm+1} & \ldots & a_{mm+n} \end{pmatrix}.$$

The entries $a_{im+\alpha}$ give the bijection of $\widetilde{G}(m,n) \cap U_W$ with $\mathrm{Hom}(\mathbb{R}^{m+n}/W, W)$. Note that the affine coordinates of the affine open subset of $G(m,n)$ are just the m minors of M_Q^*, i.e. the minors of all sizes of the matrix $(a_{im+\alpha})_{\substack{i=\overline{1,m} \\ \alpha=\overline{1,n}}}$.

7.1.2 Tangent spaces to Grassmannians

The last description of the Grassmanian, by using local affine charts, allows us to describe the tangent space to the Grassmannian at every one of its points. Consider $Q \in \widetilde{G}(m,n) \cap U_W$. As $\mathbb{R}^{m+n}/W$ is isomorphic to Q,

$$\widetilde{G}(m,n) \cap U_W = \mathrm{Hom}(Q, W)$$

so that

$$T_Q\widetilde{G}(m,n) = \mathrm{Hom}(Q,W).$$

As W is naturally isomorphic to $\mathbb{R}^{m+n}/Q$, we deduce that our result does not depend on W.

Moreover, by considering the Euclidean metric on $\mathbb{R}^{m+n}$ we can identify

$$T_Q\widetilde{G}(m,n) = \mathrm{Hom}(Q,Q^\perp) = Q^*\otimes Q^\perp.$$

If $G(m,n)$ denotes the set of m-dimensional oriented linear subspaces of $\mathbb{R}^{m+n}$, then $G(m,n)$ is the orientable double covering of $\widetilde{G}(m,n)$. Thus it is locally diffeomorphic to $\widetilde{G}(m,n)$ through the projection map which maps a point of $G(m,n)$ to a point of $\widetilde{G}(m,n)$ by forgetting the orientation.

Therefore, if $P\in G(m,n)$, then $T_PG(m,n) = P^*\otimes P^\perp$. Moreover, if we fix an orthonormal basis $E_1,\ldots,E_m$ of P and complete it with $E_{m+1},\ldots,E_{m+n}$, which span $P^\perp$, to an orthonormal basis of $\mathbb{R}^{m+n}$, then a basis in $T_PG(m,n)$ will be given by $\{E_i^*\otimes E_{m+\alpha}\}_{\substack{i=\overline{1,m}\\ \alpha=\overline{1,n}}}$ which can be also written as $\{E_1\wedge\ldots\wedge E_{i-1}\wedge E_{m+\alpha}\wedge E_{i+1}\wedge\ldots\wedge E_m\}_{\substack{i=\overline{1,m}\\ \alpha=\overline{1,n}}}$.

The Riemannian metric on the Grassmannian $G(m,n)$ is given by requesting that the basis $\{E_i^*\otimes E_{m+\alpha}\}$ is an orthonormal basis.

7.1.3 The compact Grassmannian as a symmetric space. Curvature

In the following we shall identify the Grassmannian $G(m,n)$ as a symmetric space and determine its curvature (see [99]).

The space $\mathcal{O}(m+n)$ acts on $\mathbb{R}^{m+n}$ and maps m-dimensional subspaces into m-dimensional subspaces changing or preserving the orientation, so it acts transitively on $G(m,n)$. The isotropy group consists of the elements that preserve $\mathbb{R}^m$ together with its canonical orientation, and is $\mathcal{SO}(m)\times\mathcal{O}(n)$. As we have seen, the tangent space to $G(m,n)$ at $P=\mathbb{R}^m$ is

$$T_PG(m,n) = Hom(\mathbb{R}^m,\mathbb{R}^n) = \mathcal{M}_{m\times n}(\mathbb{R}).$$

The isotropy action of $\mathcal{SO}(m)\times\mathcal{O}(n)$ on $T_PG(m,n)$ is

$$\mathcal{SO}(m)\times\mathcal{O}(n)\times\mathcal{M}_{m\times n}(\mathbb{R})\to\mathcal{M}_{m\times n}(\mathbb{R})$$

$$(A,B,X)\to AXB^{-1} = AXB^T$$

and the involution that induces the symmetry is obtained for $A=I_m$ and $B=-I_n$.

We shall describe the Lie algebra and compute the curvature tensor. We have the isometry group modulo a finite covering, so

$$\mathfrak{iso} = \mathfrak{so}(m+n),$$

$$\mathfrak{iso}_P = \mathfrak{so}(m) \times \mathfrak{so}(n).$$

We consider the block decomposition of matrices in $\mathfrak{so}(m+n)$,

$$X = \begin{pmatrix} X_1 & B \\ -B^T & X_2 \end{pmatrix} \in \mathfrak{so}(m+n),$$

$X_1 \in \mathfrak{so}(m), X_2 \in \mathfrak{so}(n), B \in \mathcal{M}_{m\times n}(\mathbb{R})$. Thus,

$$\mathfrak{so}(m+n) = \mathfrak{so}(m) \oplus \mathfrak{so}(n) \oplus \mathfrak{t}_P,$$

where

$$\mathfrak{t}_P = \left\{ \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} / B \in \mathcal{M}_{m\times n}(\mathbb{R}) \right\}$$

can be identified with the tangent space $T_P G(m,n)$ by

$$\begin{pmatrix} 0 & B_{i\alpha} \\ -B_{i\alpha}^T & 0 \end{pmatrix} \leftrightarrow e_i^* \otimes e_{m+\alpha} - e_{n+i}^* \otimes e_\alpha \leftrightarrow e_i^* \otimes e_{m+\alpha} \tag{7.1}$$

$$\leftrightarrow e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{m+\alpha} \wedge e_{i+1} \wedge \ldots \wedge e_m,$$

where $B_{i\alpha} = (\delta_{il}\delta_{\alpha\tau})_{\substack{l=\overline{1,m} \\ \tau=\overline{1,n}}}$ and $\{e_r\}_{r=1}^{m+n}$ is the canonical basis on $\mathbb{R}^{m+n}$.

If we consider the standard Euclidean product on $\mathfrak{t}_P$,

$$\left\langle \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \right\rangle = \operatorname{trace} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}^T$$

$$= 2 \operatorname{trace} AB^T,$$

it is twice the usual Euclidean metric, but for what concerns the curvature tensor we obtain the same expression. We recall that for $A, B, C \in \mathfrak{t}_P$,

$$R(A,B)C = [C,[A,B]].$$

We have

$$[A,B] = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} BA^T - AB^T & 0 \\ 0 & B^TA - A^TB \end{pmatrix} \in \mathfrak{so}(m) \times \mathfrak{so}(n),$$

and consequently,

$$R(A,B)C = \begin{pmatrix} 0 & D \\ -D^T & 0 \end{pmatrix},$$

where $D = C(B^TA - A^TB) - (BA^T - AB^T)C$.

We shall only have to express the curvature by using the identification $T_PG(m,n) = P^* \otimes P^\perp$. Consider $\{e_i\}$ to be an orthonormal basis on P and complete it to an orthonormal basis of $\mathbb{R}^{m+n}$, $\{e_r\}$. As $\{e_i^* \otimes e_{m+\alpha}\}$ provides us an orthonormal basis of $T_PG(m,n)$, we intend to determine

$$R(e_i^* \otimes e_{m+\alpha}, e_j^* \otimes e_{m+\beta})e_k^* \otimes e_{m+\gamma}.$$

In our setting, $e_i^* \otimes e_{m+\alpha}$ identifies with $\begin{pmatrix} 0 & B_{i\alpha} \\ -B_{i\alpha} & 0 \end{pmatrix}$, so

$$R(e_i^* \otimes e_{m+\alpha}, e_j^* \otimes e_{m+\beta})e_k^* \otimes e_{m+\gamma} = R(B_{i\alpha}, B_{j\beta})B_{k\gamma}$$

$$= B_{k\gamma}(B_{j\beta}^T B_{i\alpha} - B_{i\alpha}^T B_{j\beta}) - (B_{j\beta}B_{i\alpha}^T - B_{i\alpha}B_{j\beta}^T)B_{k\gamma}.$$

Since

$$B_{k\gamma}B_{j\beta}^T B_{i\alpha} = \sum_{\tau,l}(\delta_{hk}\delta_{\tau\gamma}\delta_{\tau\beta}\delta_{jl}\delta_{li}\delta_{\alpha\rho})_{\substack{h=\overline{1,m} \\ \rho=\overline{1,n}}} = \delta_{\gamma\beta}\delta_{ij}B_{k\alpha},$$

we conclude that

$$R(e_i^* \otimes e_{m+\beta}, e_i^* \otimes e_{m+\alpha})e_k^* \otimes e_{m+\alpha} = e_k^* \otimes e_{m+\beta},$$

$$R(e_i^* \otimes e_{m+\alpha}, e_j^* \otimes e_{m+\alpha})e_j^* \otimes e_{m+\gamma} = e_i^* \otimes e_{m+\gamma},$$

all the other components being zero.

We can thus express the curvature vector field of the Grassmannian in the invariant form,

$$\begin{aligned} R(X_1^* \otimes \eta_1, X_2^* \otimes \eta_2)X_3^* \otimes \eta_3 &= \langle X_1, X_2\rangle\langle \eta_2, \eta_3\rangle X_3^* \otimes \eta_1 \\ &- \langle X_1, X_2\rangle\langle \eta_1, \eta_3\rangle X_3^* \otimes \eta_2 \\ &+ \langle X_2, X_3\rangle\langle \eta_1, \eta_2\rangle X_1^* \otimes \eta_3 \\ &- \langle X_1, X_3\rangle\langle \eta_1, \eta_2\rangle X_2^* \otimes \eta_3, \end{aligned} \tag{7.2}$$

where $X_i \in P$, $\eta_i \in P^\perp$, $i = 1, 2, 3$.

7.2 The Gauss map associated to an m-dimensional submanifold in $\mathbb{R}^{m+n}$

In this section we recall the definition of the Gauss map and compute its differential and tension field (see [50, 51]).

7.2.1 Definition. The pull-back bundle

Definition 7.1. Consider M to be an m-dimensional oriented submanifold of $\mathbb{R}^{m+n}$. The map which assigns to every point $p \in M$ the tangent space T_pM thought of as a point of the Grassmannian of oriented m-dimensional subspaces of $\mathbb{R}^{m+n}$,

$$\gamma : M \to G(m,n),$$

$$\gamma(p) = T_pM \in G(m,n),$$

is called the *Gauss map associated to* M.

The pull-back bundle of the tangent bundle induced by γ can be thought of as

$$\gamma^{-1}(TG(m,n)) = \bigcup_{p\in M} T_{\gamma(p)}G(m,n) = \bigcup_{p\in M}(T_p^*M \otimes N_pM) = T^*M \otimes NM,$$

where NM denotes the normal bundle of M in $\mathbb{R}^{m+n}$, and this is an isometric identification.

Remark 7.2. A generalization of the Gauss map was defined by M. Obata in [93]. We shall only recall here the definition of Obata's Gauss map for submanifolds in spheres. Consider M to be an m dimensional submanifold of the $m+n$ dimensional Euclidean sphere $\mathbb{S}^{m+n}$. The map that assigns to any $p \in M$ the great m-sphere tangent to M at p, or the $m+1$-dimensional subspace in $\mathbb{R}^{m+n+1}$ spanned by the tangent space of M and the normal to $\mathbb{S}^{m+n}$ at p in $\mathbb{R}^{m+n+1}$ is called the *Obata's Gauss map* associated to M.

7.2.2 The harmonicity of the Gauss map

In the following we intend to determine the differential of the Gauss map (see [50]).

Consider $v \in T_pM$. In order to compute $d\gamma(v)$ consider $\sigma : I \to M$ to be a curve with $\sigma(0) = p$ and $\dot{\sigma}(0) = v$. Let now $\{e_i\}_{i=1}^m$ be a positive oriented orthonormal basis in T_pM. By parallel transporting it along σ we obtain $\{e_i(t)\}_{i=1}^m$, a positive oriented orthonormal basis in $T_{\sigma(t)}M$, for all t.

In this way

$$d\gamma_p(v) = \frac{d}{dt}\Big|_{t=0}(\gamma\circ\sigma)(t) = \frac{d}{dt}\Big|_{t=0}(e_i(t)\wedge\ldots\wedge e_m(t)).$$

As $e_i(t)$ are obtained by parallel transport along σ,

$$\dot{e}_i(t) = \nabla^{\mathbb{R}^{m+n}}_{\dot{\sigma}} e_i = \nabla^M_{\dot{\sigma}} e_i + B(\dot{\sigma}, e_i(t)) = B(\dot{\sigma}, e_i(t)),$$

where B is the second fundamental form of M in $\mathbb{R}^{m+n}$.

Thus,

$$\begin{aligned} d\gamma_p(v) &= \sum_{i=1}^{m} e_1 \wedge \ldots \wedge e_{i-1} \wedge \dot{e}_i(0) \wedge e_{i+1} \wedge \ldots \wedge e_m \\ &= \sum_{i=1}^{m} e_1 \wedge \ldots \wedge e_{i-1} \wedge B(v, e_i) \wedge e_{i+1} \wedge \ldots \wedge e_m \end{aligned}$$

and by using the identification in (7.1), we get

$$d\gamma_p(v) = \sum_{i=1}^{m} e_i^* \otimes B(v, e_i). \tag{7.3}$$

The first results concerning the harmonicity of the Gauss map were obtained in [101]. We shall present here a computation that follows [50].

By using expression (7.3) we can compute the tension field of the Gauss map in terms of the mean curvature of M.

Since the bundles $\gamma^{-1}(TG(m,n))$ and $T^*M \otimes NM$ are isometric, we can write (see Chapter 2)

$$\nabla^{\gamma}\rho = \nabla\theta \otimes \eta + \theta \otimes \nabla^{\perp}\eta,$$

where the section $\rho \in C(\gamma^{-1}(TG(m,n)))$ in the pull-back bundle is such that it can naturally be identified with $\theta \otimes \eta \in C(T^*M \otimes NM)$.

Consider $\{E_i\}_{i=1}^m$ to be a local positive oriented orthonormal frame field, geodesic at $p \in M$. By using (7.3) and the consequence of the Codazzi equation, $\nabla^{\perp}_{E_i} B(E_j, E_k) = \nabla^{\perp}_{E_j} B(E_i, E_k)$, for all i, j, k, we get at p,

$$\begin{aligned} \tau(\gamma) &= \sum_{i=1}^{m} \nabla d\gamma(E_i, E_i) \\ &= \sum_{i=1}^{m} \left\{ \nabla^{\gamma}_{E_i} d\gamma(E_i) - d\gamma(\nabla^{M}_{E_i} E_i) \right\} = \sum_{i=1}^{m} \nabla_{E_i} \Big(\sum_{j=1}^{m} E_j^* \otimes B(E_i, E_j) \Big) \\ &= \sum_{i,j=1}^{m} \left(\nabla^{M}_{E_i} E_j^* \otimes B(E_i, E_j) + E_j^* \otimes \nabla^{\perp}_{E_i} B(E_i, E_j) \right) \\ &= \sum_{i,j=1}^{m} E_j^* \otimes \nabla^{\perp}_{E_i} B(E_i, E_j) = \sum_{i,j=1}^{m} E_j^* \otimes \nabla^{\perp}_{E_j} B(E_i, E_i) \\ &= m \sum_{j=1}^{m} E_j^* \otimes \nabla^{\perp}_{E_j} H, \end{aligned} \tag{7.4}$$

where H is the mean curvature vector field of M in $\mathbb{R}^{m+n}$. We note that E_j^* coincides with $E_j^{\flat}$ obtained by the musical isomorphism $\flat$.

The Ruh-Vilms Theorem is an immediate consequence of (7.4).

7.3 The biharmonicity of the Gauss map

Inspired by the expression for the tension field, we now characterize the biharmonicity of the Gauss map by using the second fundamental form of the submanifold. We obtain

Theorem 7.3 ([18]). *The Gauss map associated to an m-dimensional orientable submanifold M of $\mathbb{R}^{m+n}$ is proper biharmonic if and only if $\nabla^\perp H \neq 0$ and*

$$\begin{aligned}\nabla^\perp_X \Delta^\perp H - m\nabla^\perp_{A_H(X)}H + \operatorname{trace} B\big(2A_{\nabla^\perp_{(\,\cdot\,)}H}(X) - A_{\nabla^\perp_X H}(\,\cdot\,),\,\cdot\,\big)\\ -2\operatorname{trace} R^\perp(\,\cdot\,,X)\nabla^\perp_{\cdot}H - \operatorname{trace}(\nabla^\perp_{\cdot}R^\perp)(\,\cdot\,,X)H &= 0, \qquad (7.5)\end{aligned}$$

for all $X \in C(TM)$, where A denotes the Weingarten operator and H the mean curvature vector field of M in $\mathbb{R}^{m+n}$.

Proof. We fix an orientation on M and consider $\{E_i\}_{i=1}^m$ to be a local positive oriented orthonormal frame field, geodesic at $p \in M$. In order to determine the bitension field of the Gauss map, by using (7.2), (7.3) and (7.4) we obtain

$$\begin{aligned}\operatorname{trace} R(d\gamma,\tau(\gamma))d\gamma &= \sum_{i=1}^m R(d\gamma(E_i),\tau(\gamma))d\gamma(E_i)\\
&= m\sum_{h,i,j,k=1}^m R\big(E_j^*\otimes B(E_i,E_j), E_h^*\otimes\nabla^\perp_{E_h}H\big)E_k^*\otimes B(E_i,E_k)\\
&= m\sum_{h,i,j,k=1}^m \Big\{\delta_{jh}\big\langle\nabla^\perp_{E_h}H, B(E_i,E_k)\big\rangle E_k^*\otimes B(E_i,E_j)\\
&\quad -\delta_{jh}\big\langle B(E_i,E_j), B(E_i,E_k)\big\rangle E_k^*\otimes\nabla^\perp_{E_h}H\\
&\quad +\delta_{hk}\big\langle\nabla^\perp_{E_h}H, B(E_i,E_j)\big\rangle E_j^*\otimes B(E_i,E_k)\\
&\quad -\delta_{jk}\big\langle\nabla^\perp_{E_h}H, B(E_i,E_j)\big\rangle E_h^*\otimes B(E_i,E_k)\Big\}\\
&= m\sum_{i,j,k=1}^m \Big\{\ \big\langle\nabla^\perp_{E_j}H, B(E_i,E_k)\big\rangle E_k^*\otimes B(E_i,E_j)\\
&\quad -\big\langle B(E_i,E_j), B(E_i,E_k)\big\rangle E_k^*\otimes\nabla^\perp_{E_j}H\\
&\quad +\big\langle\nabla^\perp_{E_k}H, B(E_i,E_j)\big\rangle E_j^*\otimes B(E_i,E_k)\\
&\quad -\big\langle\nabla^\perp_{E_k}H, B(E_i,E_j)\big\rangle E_k^*\otimes B(E_i,E_j)\Big\}.\end{aligned}$$

Further computations lead to

$$\begin{aligned}\operatorname{trace} R(d\gamma,\tau(\gamma))d\gamma &= m\sum_{k=1}^{m}\Big\{E_k^*\otimes\sum_{i,j=1}^{m}\Big(\big(2\langle\nabla^\perp_{E_j}H,B(E_i,E_k)\rangle\\ &\quad-\langle B(E_i,E_j),\nabla^\perp_{E_k}H\rangle\big)B(E_i,E_j)\\ &\quad-\langle B(E_i,E_j),B(E_i,E_k)\rangle\nabla^\perp_{E_j}H\Big)\Big\}.\end{aligned}$$

By using the Weingarten operator we can express

$$2\langle\nabla^\perp_{E_j}H,B(E_i,E_k)\rangle-\langle B(E_i,E_j),\nabla^\perp_{E_k}H\rangle=\langle 2A_{\nabla^\perp_{E_j}}(E_k)-A_{\nabla^\perp_{E_k}}(E_j),E_i\rangle,$$

and from the Gauss equation of M in $\mathbb{R}^{m+n}$,

$$\langle B(Y,T),B(X,Z)\rangle=\langle B(X,T),B(Y,Z)\rangle-\langle R^M(X,Y)Z,T\rangle,\ \forall X,Y,Z,T\in C(TM),$$

for $Y=Z=E_i$, $X=E_k$, $T=E_j$, we get

$$\begin{aligned}\sum_{i,j=1}^{m}\langle B(E_i,E_j),B(E_i,E_k)\rangle E_j &= \sum_{i,j=1}^{m}\Big\{\langle B(E_k,E_j),B(E_i,E_i)\rangle-\langle R^M(E_k,E_i)E_i,E_j\rangle\Big\}E_j\\ &= \sum_{i=1}^{m}\Big\{m\langle A_H(E_k),E_i)\rangle E_i-R^M(E_k,E_i)E_i\Big\}\\ &= mA_H(E_k)-\operatorname{Ricci}^M(E_k),\end{aligned}$$

where Ricci^M denotes the Ricci tensor field of M.
By summing up all of the above we obtain

$$\begin{aligned}\operatorname{trace} R(d\gamma,\tau(\gamma))d\gamma &= m\sum_{k=1}^{m}E_k^*\otimes\Big(\sum_{j=1}^{m}B\big(2A_{\nabla^\perp_{E_j}H}(E_k)-A_{\nabla^\perp_{E_k}H}(E_j),E_j\big)\\ &\quad-m\nabla^\perp_{A_H(E_k)}H+\nabla^\perp_{\operatorname{Ricci}^M(E_k)}H\Big).\end{aligned}\tag{7.6}$$

In order to compute $-\Delta^\gamma\tau(\gamma)$ we recall that, since $\{E_i\}_{i=1}^m$ is geodesic at p, $(\nabla_{E_i}E_k)_p=0$ and $(\nabla_{E_i}\nabla_{E_i}E_k)_p=0$, for all $i,k=1,\dots,m$. Thus, at p we have

$$\begin{aligned}\operatorname{trace}\nabla^2\tau(\gamma) &= \sum_{i=1}^{m}\nabla^\gamma_{E_i}\nabla^\gamma_{E_i}\tau(\gamma)=m\sum_{i,k=1}^{m}\nabla^\gamma_{E_i}\nabla^\gamma_{E_i}(E_k^*\otimes\nabla^\perp_{E_k}H)\\ &= m\sum_{i,k=1}^{m}\nabla^\gamma_{E_i}\big(\nabla_{E_i}E_k^*\otimes\nabla^\perp_{E_k}H+E_k^*\otimes\nabla^\perp_{E_i}\nabla^\perp_{E_k}H\big)\\ &= m\sum_{i,k=1}^{m}\big(\nabla_{E_i}\nabla_{E_i}E_k^*\otimes\nabla^\perp_{E_k}H+E_k^*\otimes\nabla^\perp_{E_i}\nabla^\perp_{E_i}\nabla^\perp_{E_k}H\big)\\ &= m\sum_{i,k=1}^{m}\Big((\nabla_{E_i}\nabla_{E_i}E_k)^*\otimes\nabla^\perp_{E_k}H+E_k^*\otimes\nabla^\perp_{E_i}\nabla^\perp_{E_i}\nabla^\perp_{E_k}H\Big)\\ &= m\sum_{i,k=1}^{m}E_k^*\otimes\nabla^\perp_{E_i}\nabla^\perp_{E_i}\nabla^\perp_{E_k}H.\end{aligned}\tag{7.7}$$

Moreover, by using the curvature tensor fields R^M and $R^\perp$, for ∇ and $\nabla^\perp$, respectively, at p we obtain

$$\begin{aligned}\nabla^\perp_{E_i}\nabla^\perp_{E_i}\nabla^\perp_{E_k}H &= \nabla^\perp_{E_i}\Big(R^\perp(E_i,E_k)H+\nabla^\perp_{E_k}\nabla^\perp_{E_i}H+\nabla^\perp_{[E_i,E_k]}H\Big)\\ &= (\nabla^\perp_{E_i}R^\perp)(E_i,E_k)H+R^\perp(E_i,E_k)\nabla^\perp_{E_i}H\\ &\quad+\nabla^\perp_{E_i}\nabla^\perp_{E_k}\nabla^\perp_{E_i}H+\nabla^\perp_{E_i}\nabla^\perp_{[E_i,E_k]}H\\ &= (\nabla^\perp_{E_i}R^\perp)(E_i,E_k)H+2R^\perp(E_i,E_k)\nabla^\perp_{E_i}H\\ &\quad+\nabla^\perp_{E_k}\Big(\nabla^\perp_{E_i}\nabla^\perp_{E_i}H-\nabla^\perp_{\nabla_{E_i}E_i}H\Big)+\nabla^\perp_{R^M(E_k,E_i)E_i}H,\end{aligned}$$

thus

$$\begin{aligned}\operatorname{trace}\nabla^2\tau(\gamma) &= m\sum_{k=1}^m E_k^*\otimes\Big(\operatorname{trace}\big\{(\nabla^\perp_\cdot R^\perp)(\,\cdot\,,E_k)H+2R^\perp(\,\cdot\,,E_k)\nabla^\perp_\cdot H\big\}\\ &\quad-\nabla^\perp_{E_k}\Delta^\perp H+\nabla^\perp_{\operatorname{Ricci}^M(E_k)}H\Big).\end{aligned}\tag{7.8}$$

We finally substitute (7.6) and (7.8) into the biharmonic equation and conclude. □

7.3.1 The case of hypersurfaces

Consider M to be a hypersurface with nowhere zero mean curvature in $\mathbb{R}^{m+1}$. We obtain

Theorem 7.4 ([18]). *The Gauss map of a nowhere zero mean curvature hypersurface M^m in $\mathbb{R}^{m+1}$ is proper biharmonic if and only if* $\operatorname{grad} f\neq 0$ *and*

$$\Delta\operatorname{grad} f+A^2(\operatorname{grad} f)-|A|^2\operatorname{grad} f=0,\tag{7.9}$$

where Δ denotes the rough Laplacian on $C(TM)$ and f and A denote the mean curvature function and, respectively, the shape operator of M in $\mathbb{R}^{m+1}$.

Proof. In this case we can consider the expression of the mean curvature vector field as $H=f\eta$, where $f=|H|$ is the mean curvature function of M in $\mathbb{R}^{m+1}$ and $\eta=\frac{1}{|H|}H$ is a unit section in the normal bundle NM.

In the following we shall use the general biharmonic equation (7.5) and express it for the case of hypersurfaces.

Since $H=f\eta$ and $\nabla^\perp\eta=0$, we get that

$$\operatorname{trace}R^\perp(\,\cdot\,,X)\nabla^\perp_\cdot H=0\tag{7.10}$$

and

$$\operatorname{trace}(\nabla^\perp_\cdot R^\perp)(\,\cdot\,,X)H=0.\tag{7.11}$$

By considering now $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M we obtain

$$\begin{aligned}\operatorname{trace} B\big(2A_{\nabla^\perp_{(\cdot)}H}(X) - A_{\nabla^\perp_{\cdot}H}(\cdot\,),\cdot\,\big) &= \sum_{i=1}^m \Big\{2B\big(A(X),E_i(f)E_i\big) - X(f)B\big(A(E_i),E_i\big)\Big\}\\ &= \langle X, 2A^2(\operatorname{grad} f) - |A|^2 \operatorname{grad} f\rangle\eta \qquad (7.12)\end{aligned}$$

and

$$\nabla^\perp_{A_H(X)}H = \langle X, fA(\operatorname{grad} f)\rangle\eta. \qquad (7.13)$$

For the final term, we recall that $\operatorname{grad}\Delta f = \Delta\operatorname{grad} f + \operatorname{Ricci}^M(\operatorname{grad} f)$. Also, as a consequence of the Gauss equation for M in $\mathbb{R}^{m+1}$, we have

$$\operatorname{Ricci}^M(\operatorname{grad} f) = mfA(\operatorname{grad} f) - A^2(\operatorname{grad} f).$$

Thus, it follows that

$$\begin{aligned}\nabla^\perp_X\Delta^\perp H &= \langle X, \operatorname{grad}\Delta f\rangle\eta\\ &= \langle X, \Delta\operatorname{grad} f + mfA(\operatorname{grad} f) - A^2(\operatorname{grad} f)\rangle\eta. \qquad (7.14)\end{aligned}$$

Finally, by replacing the expressions (7.10), (7.11), (7.12), (7.13), (7.14) in (7.5), we deduce that the Gauss map of the hypersurface is biharmonic if and only if

$$\langle X, \Delta\operatorname{grad} f + A^2(\operatorname{grad} f) - |A|^2\operatorname{grad} f\rangle = 0, \qquad \forall X \in C(TM),$$

and this completes the proof. □

Example 7.5. The first example of hypersurface with proper biharmonic Gauss map is obtained by analyzing the right cylinders in $\mathbb{R}^3$. A right cylinder in $\mathbb{R}^3$ determined by a plane curve parametrized by arc length has proper biharmonic Gauss map if and only if the curve is a *clothoid* or a *Cornu spiral*. Indeed, consider a right cylinder in $\mathbb{R}^3$, determined by a curve $\sigma : I \to \mathbb{R}^2$, parametrized by arc length. Denote by s the parameter along σ and by t the parameter along the generatrix. If k denotes the signed curvature of σ, then the mean curvature function is $f(s,t) = \pm\frac{1}{2}k(s)$. Moreover, since $A^2(\operatorname{grad} f) = |A|^2\operatorname{grad} f = k^2(s)\operatorname{grad} f$, equation (7.9) becomes $\dddot{k} = 0$, hence k is a second degree polynomial in s and, by using the fundamental theorem of plane curves, we deduce that σ is a clothoid.

7.4 Hypercones with biharmonic associated Gauss map

In order to obtain some examples of hypersurfaces with biharmonic Gauss map, in the following we shall study the hypercones generated by hypersurfaces of the unit Euclidean sphere.

7.4.1 Riemannian structure and second fundamental form of the cone generated by a submanifold in $\mathbb{S}^{m+1}$

Let $\overline{M}$ be an n-dimensional orientable submanifold of the unit Euclidean sphere $\mathbb{S}^{m+1}$. The cone generated by $\overline{M}$ is defined by the immersion

$$\phi : (0,\infty) \times \overline{M} \longrightarrow \mathbb{R}^{m+2}$$
$$(t,p) \to t \cdot p.$$

The differential of ϕ is determined by

$$d\phi_{(t,p)}(\partial/\partial t_{(t,p)}) = p = x^{\alpha}(p)e_{\alpha}(\phi(p)),$$
$$d\phi_{(t,p)}(X_{(t,p)}) = tX_p = t\xi^{\alpha}(p)e_{\alpha}(\phi(p)),$$

where $X \in C(T\overline{M})$ with $X(p) = \xi^{\alpha}(p)e_{\alpha}(p) \in \mathbb{R}^{m+2}$, for all $p \in \overline{M}$, and $\{e_{\alpha}\}_{\alpha=1}^{m+2}$ is the canonical frame field on $\mathbb{R}^{m+2}$.

From here, as we want $\phi : (0,\infty) \times \overline{M} \to \mathbb{R}^{m+2}$ to be a Riemannian immersion, we deduce that the cone is completely described as the warped product

$$(0,\infty) \times_{t^2} \overline{M}.$$

Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections on $(0,\infty) \times_{t^2} \overline{M}$ and on $\overline{M}$, respectively, and recall that (see [92, p.206]) ∇ is completely determined by

$$\begin{cases} \nabla_{\partial/\partial t}\partial/\partial t = 0 \\ \nabla_{\partial/\partial t}X = \nabla_X \partial/\partial t = \frac{1}{t}X \\ \nabla_X Y = \overline{\nabla}_X Y - t\langle X, Y\rangle \partial/\partial t, \end{cases} \tag{7.15}$$

where $X, Y \in C(T\overline{M})$. In order to determine the shape operator we have

$$\begin{aligned} B(\partial/\partial t, \partial/\partial t) &= \nabla d\phi(\partial/\partial t, \partial/\partial t) \\ &= \nabla_{\partial/\partial t} d\phi(\partial/\partial t) - d\phi(\nabla_{\partial/\partial t}\partial/\partial t) \\ &= \nabla_{\partial/\partial t}(x^{\alpha}(p)e_{\alpha} \circ \phi) = x^{\alpha}(p)\nabla^{\mathbb{R}^{m+2}}_{d\phi(\partial/\partial t)}e_{\alpha} \\ &= 0, \end{aligned} \tag{7.16}$$

$$\begin{aligned} B(X, \partial/\partial t) &= \nabla d\phi(X, \partial/\partial t) \\ &= \nabla_X d\phi(\partial/\partial t) - d\phi(\nabla_X \partial/\partial t) \\ &= \nabla_X(x^{\alpha}(p)e_{\alpha} \circ \phi) - (1/t)d\phi(X) \\ &= 0, \end{aligned} \tag{7.17}$$

$$\begin{aligned} B(X,Y) &= \nabla d\phi(X,Y) \\ &= \nabla_X d\phi(Y) - d\phi(\nabla_X Y) \\ &= \nabla_X(t\zeta^\alpha(p)e_\alpha \circ \phi) - d\phi(\overline{\nabla}_X Y - t\langle X,Y\rangle \partial/\partial t) \\ &= t\Big(\nabla^{\mathbb{R}^{m+2}}_X Y - \overline{\nabla}_X Y + \langle X,Y\rangle p\Big) \\ &= t\overline{B}(X,Y), \end{aligned} \tag{7.18}$$

for all $X, Y \in C(T\overline{M})$, where $\overline{B}$ denotes the second fundamental form of $\overline{M}$ in $\mathbb{S}^{m+1}$.

Thus, if we denote by $\overline{A}$ the shape operator of $\overline{M}$ in $\mathbb{S}^{m+1}$ with respect to an arbitrary fixed unit section $\overline{\eta}$ in the normal bundle of $\overline{M}$ in $\mathbb{S}^{m+1}$, we obtain the expression for the shape operator A of the cone with respect to the unit section $\eta(t,p) = \overline{\eta}(p)$, for all $(t,p) \in M$, in the normal bundle of M in $\mathbb{R}^{m+2}$,

$$A(\partial/\partial t) = 0 \qquad \text{and} \qquad A(X) = \frac{1}{t}\overline{A}(X), \tag{7.19}$$

for all $X \in C(T\overline{M})$, and, consequently, $|A|^2 = \frac{1}{t^2}|\overline{A}|^2$.

Also, for $f \in C^\infty\big((0,\infty) \times_{t^2} \overline{M}\big)$,

$$\operatorname{grad} f = \frac{\partial f}{\partial t}\partial/\partial t + \frac{1}{t^2}\operatorname{grad} f_t \tag{7.20}$$

and

$$\Delta f = -\frac{\partial^2 f}{\partial t^2} - \frac{n}{t}\frac{\partial f}{\partial t} + \frac{1}{t^2}\overline{\Delta} f_t, \tag{7.21}$$

where $f_t \in C^\infty(\overline{M})$, $f_t(p) = f(t,p)$, for all $p \in \overline{M}$ and $t \in (0,\infty)$.

7.4.2 Hypercones with biharmonic Gauss map

Theorem 7.6. *Let $\overline{M}$ be a nowhere zero mean curvature hypersurface of $\mathbb{S}^{m+1}$. The Gauss map associated to the hypercone $(0,\infty) \times_{t^2} \overline{M}$ is proper biharmonic if and only if*

$$\begin{cases} \overline{\Delta}\operatorname{grad}\overline{f} + \overline{A}^2(\operatorname{grad}\overline{f}) + (2m - 3 - |\overline{A}|^2)\operatorname{grad}\overline{f} = 0 \\ \\ 3\overline{\Delta}\,\overline{f} + (3m - 6 - |\overline{A}|^2)\overline{f} = 0, \end{cases} \tag{7.22}$$

where $\overline{A}$ and $\overline{f} \in C^\infty(\overline{M})$ are the shape operator and the mean curvature function of $\overline{M}$ in $\mathbb{S}^{m+1}$, respectively.

Proof. Consider $\partial/\partial t$ and $\{E_i\}_{i=1}^m$ a local orthonormal frame field on $\overline{M}$, geodesic at p. Then $\Big\{\partial/\partial t, \frac{1}{t}E_i\Big\}_{i=1}^m$ constitutes a local orthonormal frame field on $(0,\infty) \times_{t^2} \overline{M}$. Denoting by $\overline{f}$ is the mean curvature function of $\overline{M}$ in $\mathbb{S}^{m+1}$ and using (7.16), (7.17) and (7.18) we get the mean curvature function f of the hypercone,

$$f = \frac{m}{(m+1)}\frac{1}{t}\overline{f}.$$

Using (7.20), we get

$$\operatorname{grad} f = \frac{m}{(m+1)}\Big(-\frac{1}{t^2}\overline{f}\partial/\partial t + \frac{1}{t^3}\operatorname{grad}\overline{f}\Big),$$

and this, together with (7.19), implies

$$A^2(\operatorname{grad} f) = \frac{m}{(m+1)}\frac{1}{t^5}\overline{A}^2(\operatorname{grad}\overline{f}). \tag{7.23}$$

Also,

$$-|A|^2\operatorname{grad} f = \frac{m}{(m+1)}\frac{1}{t^4}|\overline{A}|^2\Big(\overline{f}\partial/\partial t - \frac{1}{t}\operatorname{grad}\overline{f}\Big). \tag{7.24}$$

In order to compute $\Delta(\operatorname{grad} f) = -\operatorname{trace}\nabla^2(\operatorname{grad} f)$ we shall use (7.15). Thus,

$$\begin{aligned}
\operatorname{trace}\nabla^2\Big(\frac{1}{t^2}\overline{f}\partial/\partial t\Big) &= \overline{f}\nabla_{\partial/\partial t}\nabla_{\partial/\partial t}\Big(\frac{1}{t^2}\partial/\partial t\Big) + \frac{1}{t^2}\sum_{i=1}^{m}\Big\{\frac{1}{t^2}\nabla_{E_i}\nabla_{E_i}(\overline{f}\partial/\partial t)\\
&\quad -\nabla_{\nabla_{E_i}E_i}\Big(\frac{1}{t^2}\overline{f}\partial/\partial t\Big)\Big\}\\
&= \frac{6}{t^4}\overline{f}\partial/\partial t + \frac{1}{t^4}\sum_{i=1}^{m}\Big\{E_i(E_i(\overline{f}))\partial/\partial t + \frac{2}{t}E_i(\overline{f})E_i - 3\overline{f}\partial/\partial t\Big\}\\
&= \frac{1}{t^4}\big((6-3m)\overline{f} - \overline{\Delta}\,\overline{f}\big)\partial/\partial t + \frac{2}{t^5}\operatorname{grad}\overline{f},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{trace}\nabla^2\Big(\frac{1}{t^3}\operatorname{grad}\overline{f}\Big) &= \nabla_{\partial/\partial t}\nabla_{\partial/\partial t}\Big(\frac{1}{t^3}\operatorname{grad}\overline{f}\Big)\\
&\quad +\frac{1}{t^2}\sum_{i=1}^{m}\Big\{\frac{1}{t^3}\nabla_{E_i}\nabla_{E_i}\operatorname{grad}\overline{f} - \nabla_{\nabla_{E_i}E_i}\Big(\frac{1}{t^3}\operatorname{grad}\overline{f}\Big)\Big\}\\
&= \frac{6}{t^5}\operatorname{grad}\overline{f} + \frac{1}{t^2}\sum_{i=1}^{m}\Big\{\frac{1}{t^3}\nabla_{E_i}\big(\overline{\nabla}_{E_i}\operatorname{grad}\overline{f} - t\langle E_i, \operatorname{grad}\overline{f}\rangle\partial/\partial t\big)\\
&\quad -\frac{2}{t^3}\operatorname{grad}\overline{f}\Big\}\\
&= \frac{2}{t^4}\overline{\Delta}\,\overline{f}\partial/\partial t + \frac{1}{t^5}\big((5-2m)\operatorname{grad}\overline{f} - \overline{\Delta}\operatorname{grad}\overline{f}\big).
\end{aligned}$$

Using the two expressions above we obtain

$$\Delta(\operatorname{grad} f) = \frac{m}{m+1}\Big\{\frac{1}{t^4}\big((6-3m)\overline{f} - 3\overline{\Delta}\,\overline{f}\big)\partial/\partial t + \frac{1}{t^5}\big((2m-3)\operatorname{grad}\overline{f} + \overline{\Delta}\operatorname{grad}\overline{f}\big)\Big\}. \tag{7.25}$$

By substituting (7.23), (7.24) and (7.25) in (7.9), we obtain the desired result. □

As a consequence of Theorem 7.6 we obtain,

Theorem 7.7. *Let $\overline{M}$ be a constant non-zero mean curvature hypersurface of $\mathbb{S}^{m+1}$. The Gauss map associated to the hypercone $(0,\infty)\times_{t^2}\overline{M}$ is proper biharmonic if and only if $m>2$ and $|\overline{A}|^2 = 3(m-2)$, where $\overline{A}$ is the shape operator of $\overline{M}$ in $\mathbb{S}^{m+1}$.*

Proof. If $\overline{f}$ is constant, then the first condition of (7.22) is identically satisfied and the second one implies $|\overline{A}|^2 = 3(m-2)$. The converse is immediate. □

For the case of hypercones in $\mathbb{R}^3$ we have the following non-existence result.

Theorem 7.8. *There exist no cones in $\mathbb{R}^3$ with proper biharmonic Gauss map.*

Proof. Consider a cone in $\mathbb{R}^3$ generated by a curve $\sigma : I \to \mathbb{S}^2$, parametrized by arc length. Denote by s the parameter on the curve and by $T = \dot{\sigma}$ the tangent vector field along σ. Since $\nabla^{\mathbb{S}^2}_T T = kN$, with N the unit normal vector field along σ, the mean curvature function is given by $\overline{f} = \pm k$,

$$\overline{A}(\partial/\partial s) = k\partial/\partial s \qquad \text{and} \qquad |\overline{A}|^2 = k^2,$$

$$\operatorname{grad}\overline{f} = \pm\dot{k}\partial/\partial s \qquad \text{and} \qquad \overline{\Delta}\operatorname{grad}\overline{f} = \mp\dddot{k}\partial/\partial s.$$

Thus, condition (7.22) becomes

$$\begin{cases} \dddot{k} + \dot{k} = 0 \\ \\ k(3+k^2) + 3\ddot{k} = 0. \end{cases} \tag{7.26}$$

This implies that $\dot{k}k^2 = 0$, hence $k = 0$, i.e. the Gauss map of the cone is harmonic, and we conclude. □

Recall that if $\overline{M}$ is a hypersurface in $\mathbb{S}^{m+1}$, then the cone over $\overline{M}$ has harmonic Gauss map if and only if $\overline{M}$ is minimal in $\mathbb{S}^{m+1}$ (see [107]). This does not hold in the case of the biharmonicity. Indeed, by considering $\overline{M}$ to be a constant mean curvature proper biharmonic hypersurface of $\mathbb{S}^{m+1}$ and by using the fact that the squared norm of the shape operator of such a submanifold is equal to m we get

Theorem 7.9. *Let $\overline{M}$ be a constant mean curvature proper biharmonic hypersurface of $\mathbb{S}^{m+1}$. Then the hypercone $(0,\infty) \times_{t^2} \overline{M}$ has proper biharmonic associated Gauss map if and only if $m = 3$.*

Remark 7.10. Theorem 7.9 can be deduced, for the particular case of the hypersphere of radius equal to $\frac{1}{\sqrt{2}}$, in a more geometrical manner. The argument is the following. In [82], the authors proved that if $\psi : N \to \mathbb{S}^m(\frac{1}{\sqrt{2}})$ is a harmonic map and $\mathbf{i} : \mathbb{S}^m(\frac{1}{\sqrt{2}}) \to \mathbb{S}^{m+1}$ denotes the inclusion map, then the tension and bitension fields of the composition are given by

$$\tau(\mathbf{i}\circ\psi) = -2e(\psi)\overline{\eta} \quad \text{and} \quad \frac{1}{2}\tau_2(\mathbf{i}\circ\psi) = \big(\Delta e(\psi)\big)\overline{\eta} - 2d\psi(\operatorname{grad} e(\psi)), \tag{7.27}$$

where $e(\psi)$ denotes the energy density of the map ψ and $\overline{\eta}$ the unit section of the normal bundle of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ in $\mathbb{S}^{m+1}$. The Gauss map associated to the hypercone $(0,\infty)\times_{t^2}\mathbb{S}^m(\frac{1}{\sqrt{2}})$ are given by

$$\gamma : (0,\infty)\times_{t^2}\mathbb{S}^m(\tfrac{1}{\sqrt{2}})\to\mathbb{S}^{m+1}$$

$$\gamma(t,p)=\overline{\eta}(p),$$

i.e. $\gamma(t,x^1,\dots,x^{m+1},\frac{1}{\sqrt{2}})=(x^1,\dots,x^{m+1},-\frac{1}{\sqrt{2}})$. We can thus think of γ, up to an isometry, as the composition $\mathbf{i}\circ\psi$, where

$$\psi : (0,\infty)\times_{t^2}\mathbb{S}^m(\tfrac{1}{\sqrt{2}})\to\mathbb{S}^m(\tfrac{1}{\sqrt{2}})$$

$$\psi(t,p)=p.$$

The map ψ is the projection onto the second factor of a warped product, so it is a harmonic map. Now, since $d\psi(\partial/\partial t)=0$ and $d\psi(X)=X$, for all $X\in C(T\mathbb{S}^m(\frac{1}{\sqrt{2}}))$, the energy density of ψ is $e(\psi)=\frac{m}{2t^2}$ and $\operatorname{grad}e(\psi)=-\frac{m}{t^3}\partial/\partial t$. Thus, we deduce that

$$\Delta e(\psi)=\frac{m(m-3)}{t^4}\qquad\text{and}\qquad d\psi(\operatorname{grad}e(\psi))=0. \tag{7.28}$$

Finally, by using (7.27) and (7.28), we conclude that the Gauss map associated to the hypercone $(0,\infty)\times_{t^2}\mathbb{S}^m(\frac{1}{\sqrt{2}})$ in $\mathbb{R}^{m+2}$ is proper biharmonic if and only if $m=3$, in accordance with Theorem 7.9.

In order to obtain other examples we consider the hypersphere $\mathbb{S}^m(a)$, $a\in(0,1)$, in $\mathbb{S}^{m+1}$ and we prove

Proposition 7.11 ([18]). *Consider the hypercone $(0,\infty)\times_{t^2}\mathbb{S}^m(a)$, $a\in(0,1)$, in $\mathbb{R}^{m+2}$. Its associated Gauss map is proper biharmonic if and only if $m>2$ and $a=\sqrt{\frac{m}{4m-6}}$.*

Proof. Since $|\overline{A}|^2=m\frac{1-a^2}{a^2}$, by using Theorem 7.7, we conclude.

Notice that in this case,

$$\eta(x^1,...,x^{m+1},b)=\Big(\frac{b}{a}(x^1,...,x^{m+1}),-a\Big),$$

where $a^2+b^2=1, a,b>0$ and $(x^1,...,x^{m+1},b)\in\mathbb{S}^m(a)$. Also, since $\overline{A}(X)=-\frac{1}{c}X$, for all $X\in C(T\mathbb{S}^m(a))$, $c^2=\frac{a^2}{b^2}$, it follows that

$$|A|^2=\frac{m}{c^2t^2}$$

and

$$\tau_2(\gamma)=-\frac{m}{ct^4}\Big(6-3m+\frac{m}{c^2}\Big)\partial/\partial t.$$

□

Remark 7.12. Proposition 7.11 provides us examples of hypersurfaces with proper biharmonic associated Gauss map in any $(m+2)$-dimensional Euclidean space, with $m > 2$.

Also, for the generalized Clifford torus $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2) \subset \mathbb{S}^{m+1}$, $m_1 + m_2 = m$, $r_1^2 + r_2^2 = 1$, since $|\overline{A}|^2 = \left(\frac{r_2}{r_1}\right)^2 m_1 + \left(\frac{r_1}{r_2}\right)^2 m_2$, we get

Proposition 7.13 ([18]). *Consider the hypercone* $(0,\infty) \times_{t^2} \left(\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)\right)$ *in* $\mathbb{R}^{m+2}$. *Its associated Gauss map is proper biharmonic if and only if* $\frac{m_1}{r_1^2} \neq \frac{m_2}{r_2^2}$ *and*

$$\frac{m_1}{r_1^2} + \frac{m_2}{r_2^2} = 4m - 6. \tag{7.29}$$

Remark 7.14. Notice that the conditions in Proposition 7.13 are not satisfied for $m = 2$ and $m = 3$. In order to obtain an example, consider $m > 3$, $m_1 = 1$, and $m_2 = m - 1$. Then $r_1^2 = \frac{3m-4\pm\sqrt{9m^2-40m+40}}{2(4m-6)}$ and $r_2^2 = \frac{5m-8\mp\sqrt{9m^2-40m+40}}{2(4m-6)}$ are solutions for (7.29).

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