INFINITESIMAL SYMMETRIES OF CAMASSA-HOLM EQUATION

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Abstract

§1 recalls known results about symmetry group of a third order PDE. One determines the Lie algebra **g** of the infinitesimal transformations (theorem 2) of the Camassa-Holm equation (7), and finds the family of third order PDEs which admits as symmetry group, the Lie group associated to the Lie algebra **g** (theorem 3). This class contains the Camassa-Holm equation and the Rosenau-Hyman equation.

Key-words: symmetry group, infinitesimal symmetries, criterion of infinitesimal invariance, Camassa-Holm equation.

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Camassa and Holm derived a new completely integrable dispersive shallow watter equation that is bi-Hamiltonian and thus possesses an infinite member of conservation laws in involution. The Camassa-Holm (CH) PDE is obtained by using an asymptotic expansion directly in the Hamiltonian for Euler's equations in the shallow watter regime. Holm, Marsden and Ratiu [10] have shown that Camassa-Holm equation [6] in n dimensions describes geodesic motion on the diffeomorphism group of $\mathbf{R}^n$ with respect to metric given by the H^1 norm of Eulerian fluid velocity. Misiolek [13] has shown that the CH equation represents a geodesic flow on the Bott-Virasoro group. Kouranbaeva [11] has shown that the CH equation (for the case $k = 0$) is a geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle obtained by the right translation of the H^1 inner product over the entire group.

In the first part of this article one makes a short presentation of the theory of the infinitesimal symmetries associated with a third order PDE ([5],[14],[16]), and in the second part one applies this theory to the case of the Camassa-Holm PDE. It should be notice that compared to the paper [9], the present work provides the whole family of PDEs of order three which are invariant under the same group which preserves the Camassa-Holm PDE.

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1 Symmetry group of a third order PDE

Let $\pi : \mathbf{R}^{n+1} \to \mathbf{R}^n$, $\pi(x,u) = x$, $x = (x^1, ..., x^n)$, be the projection map. Let $U \subset \mathbf{R}^{n+1}$ be an open set and $U_0 = \pi(U)$.

Definition 1. A smooth map $s : U_0 \to U$, $s(x) = (x, u(x))$ is called *local section of π (on U_0).*

For the function u, we note

$$u_{i_1...i_p}(x) = \frac{\partial^p u}{\partial x^{i_1} ... \partial x^{i_p}}(x),\ x \in U_0,\ 1 \leq i_1 \leq ... \leq i_p \leq n,\ p \geq 1.$$

Let consider

$$J^k(U) = \{(x^i, u, u_{i_1}, ..., u_{i_1...i_k}) | (x^i, u) \in U\},$$

$J^k(U) \subset \mathbf{R}^{n+1} \times \mathbf{R}^{N_1} \times ... \times \mathbf{R}^{N_k}$, where $N_k = \frac{(n+k-1)!}{k!(n-1)!}$, $k > 0$, and the projection

$$\pi^k : J^k(U) \to U_0,\ \pi^k(x, u, u_{i_1}, ..., u_{i_1...i_k}) = x.$$

Convention: for $k = 0$, $J^0(U) = U$ and $\pi^0 = \pi$.

Definition 2. Let $s : U_0 \to U$ be a local section of π (over U_0). The section $j^k(s)$ of π^k over U_0, called *the k-jet of s*, is defined as follows:

$$j^k(s)(x) = (x, u, u_{i_1}, ..., u_{i_1...i_k}),\ x \in U_0,$$

$$u_{i_1...i_r} = \frac{\partial^r u}{\partial x^{i_1} ... \partial x^{i_r}}(p),\ 0 \leq r \leq k.$$

The space $J^k(U)$ is called *the kth order jet space.*

Let $\Omega^q_k(U)$ be the vector space of q-forms on $J^k(U)$ with exterior differential d. In particular $\Omega^q_0(U) = \Omega^q(U)$ is the exterior algebra of q-forms on U and $\Omega^0_k = C^\infty(J^k(U))$ is the algebra of real-functions

$$f = f(x^i, u, u_{i_1}, ..., u_{i_1...i_k})$$

on $J^k(U)$. A basis for $\Omega^1_k(U)$ (as a module $C^\infty(J^k(U))$) consists of the 1-forms dx^i, du, $du_{i_1}, ..., du_{i_1...i_k}$. For $f \in C^\infty(J^k(U))$ we have

$$df = \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial u_{i_1}} du_{i_1} + ... + \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} \frac{\partial f}{\partial u_{i_1...i_k}} du_{i_1...i_k}.$$

Definition 3. A form $\omega \in \Omega^q_k(U)$ is called *basic* if

$$\omega = A_{j_1...j_q} dx^{j_1} \wedge ... \wedge dx^{j_q},$$

where $A_{j_1...j_q}$ are differentiable functions on $J^k(U)$. We note by $\mathcal{B}^q_k(U)$, the space of the basic q-forms.

Definition 4. The operator $D : \mathcal{B}^q_k(U) \to \mathcal{B}^{q+1}_{k+1}(U)$,

$$Df = \left(\frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial u} u_j + \frac{\partial f}{\partial u_{i_1}} u_{i_1 j} + ... + \sum_{1 \leq i_1 \leq ... \leq i_k \leq n} \frac{\partial f}{\partial u_{i_1...i_k}} u_{i_1...i_k j} \right) dx^j,$$

for $f \in C^\infty(J^k(U))$, and

$$D\omega = DA_{j_1 \dots j_q} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

for

$$\omega = A_{j_1 \dots j_q} dx^{j_1} \wedge \dots \wedge dx^{j_q},$$

is called *the total exterior derivative on* $\mathcal{B}_k^q(U)$.

Convention: $Df = D_i f \ dx^i$, where $D_i f$ is the total derivative of f with respect to x^i, and

$$D_{ij} = D_i D_j, \quad D_{ijk} = D_i D_{jk}.$$

Let $\Omega_3^{n+1}(U)$ be the space of $(n+1)$-forms, and $\theta \in \Omega_3^{n+1}(U)$,

$$\theta = F(x, u, u_l, u_{lk}, u_{ijk}) du \wedge dx^1 \wedge \dots \wedge dx^n.$$

Definition 5. *A solution of the equation*

$$\theta = 0,$$

on $U_0 \subset \mathbf{R}^n$, is a section $s : U_0 \to U$ such that $\theta \cdot j^3(s) = 0$.

In other words, θ determines the PDE

$$F(x, u, u_l, u_{lk}, u_{ijk}) = 0, \tag{1}$$

a solution of which is a function $u = u(x)$ such that

$$F\left(x, u(x), \frac{\partial u}{\partial x^l}(x), \frac{\partial^2 u}{\partial x^l \partial x^k}(x), \frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}(x)\right) = 0, \ \ \forall x \in U_0.$$

We note $u^{(3)} = (u, u_l, u_{lk}, u_{ijk})$.

Definition 6. The PDE (1) is called of *maximal rang* if the Jacobi matrix

$$J_F(x, u^{(3)}) = (F_{x^i}; F_u; F_{u_l}; F_{u_{lk}}, F_{u_{ijk}})$$

has rank 1 whenever $F(x, u^{(3)}) = 0$.

Then the subset

$$S = \{(x, u^{(3)}) \in J^{(3)}(U) | F(x, u^{(3)}) = 0\}$$

is a hypersurface.

Definition 7. *A symmetry group* of PDE (1) is a local transformations Lie group G acting on an open set U of the associated space of independent and dependent variables, with the properties:

(a) if $u = f(x, y)$ is a solution of the equation and if $g \cdot f$ has sense for $g \in G$, then $v = g \cdot f(x, y)$ is also a solution.

(b) any solution of the equation can be obtain by a DE associated to PDE (hence any solution is G-invariant $g \cdot f = f$, $\forall g \in G$).

For the determination of the symmetry group of PDE (1) is used the following *criterion of infinitesimal invariance* ([14]).

Theorem 1. *Let (1) be a PDE of maximal rank defined on an open set U_0. If G is a local group of transformations acting on U and*

$$pr^{(3)}X[F(x,u^{(3)})] = 0 \quad whenever \quad F(x,u^{(3)}) = 0, \tag{2}$$

for every infinitesimal generator X of G, then G is a symmetry group of the given equation.

Let

$$X = \zeta^i(x,u)\frac{\partial}{\partial x^i} + \phi(x,u)\frac{\partial}{\partial u}$$

be vector field on U. *The first, second and respectively third order prolongations of the vector field X* are

$$\mathrm{pr}^{(1)}X = X + \Phi_i\frac{\partial}{\partial u_i}, \tag{3}$$

$$\mathrm{pr}^{(2)}X = pr^{(1)}X + \Phi_{ij}\frac{\partial}{\partial u_{ij}},$$

$$\mathrm{pr}^{(3)}X = pr^{(2)}X + \Phi_{ijk}\frac{\partial}{\partial u_{ijk}},$$

where

$$\Phi_i = D_i(\phi - \zeta^k u_k) + \zeta^k u_{ik} = D_i\phi - u_k D_i(\zeta^k),$$

$$\Phi_{ij} = D_{ij}(\phi - \zeta^k u_k) + \zeta^k u_{ijk} = D_{ij}(\phi) - u_k D_{ij}(\zeta^k) - u_{ki}D_j(\zeta^k) - u_{kj}D_i(\zeta^k),$$

$$\Phi_{ijk} = D_{ijk}(\phi - \zeta^l u_l) + \zeta^l u_{ijkl} = D_{ijk}(\phi) - u_l D_{ijk}(\zeta^l) - u_{lk}D_{ij}(\zeta^l) -$$

$$-u_{li}D_{jk}(\zeta^l) - u_{lj}D_{ik}(\zeta^l) - u_{lkj}D_i(\zeta^l) - u_{lki}D_j(\zeta^l) - u_{lji}D_k(\zeta^l).$$

Proposition. *Let be a PDE of the maximal rank defined on U_0. The set of all infinitesimal symmetries of the equation forms a Lie algebra on U. Moreover, if this Lie algebra is finite-dimensional, the symmetry group of the equation is a local Lie group of transformations acting on U.*

Algorithm for finding the symmetry group of PDE (1)

-one considers a vector field X on U and one writes the infinitesimal invariance condition (2);

-one eliminates any dependence between partial derivatives of the function u, determined by the PDE (1);

-one writes the condition (2) like a polynomial in the partial derivatives of u;

-one equates with zero the coefficients of partial derivatives of u in (2), written as a polynomial in the derivatives of the function u; it follows a PDEs system with respect to the unknown functions ζ^i, ϕ, and this system defines the Lie symmetry group G of the given PDE.

Every s-parametric subgroup H of the group G determines a family of group-invariant solutions. The problem of classification of group-invariant solutions reverts to the problem of classification of Lie subalgebras of Lie algebra $\mathbf{g}$ of the group G ([14], 186). For 1-dimensional algebras one considers a general element X, and we simplify this as much as possible using the adjoint transformations.

Remark. We will compute the adjoint representation $Ad\, G$ of the underlying Lie group G, by using the Lie series

$$(4) \qquad Ad(exp(\varepsilon X)Y) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(adX)^n(Y) = Y - \varepsilon[X,Y] + \frac{\varepsilon^2}{2}[X,[X,Y]] - ...$$

In the case $n = 2$, we note $x^1 = x$, $x^2 = y$, $\zeta^1 = \zeta$, $\zeta^2 = \eta$, $\Phi_1 = \Phi^x$, $\Phi_2 = \Phi^y$, $\Phi_{11} = \Phi^{xx}$, $\Phi_{12} = \Phi^{xy}$, $\Phi_{22} = \Phi^{yy}$, $\Phi_{111} = \Phi^{xxx}$, $\Phi_{112} = \Phi^{xxy}$, $\Phi_{122} = \Phi^{xyy}$, $\Phi_{222} = \Phi^{yyy}$ and $u^{(3)} = (u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy})$.

If

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u},$$

is the infinitesimal generator of the symmetry group of the PDE

$$(5) \qquad F(x,y,u^{(3)}) = 0,$$

then the infinitesimal condition (2) becomes

$$(6) \quad \zeta\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} + \phi\frac{\partial F}{\partial u} + \Phi^x\frac{\partial F}{\partial u_x} + \Phi^y\frac{\partial F}{\partial u_y} + \Phi^{xx}\frac{\partial F}{\partial u_{xx}} + \Phi^{xy}\frac{\partial F}{\partial u_{xy}} +$$

$$+\Phi^{yy}\frac{\partial F}{\partial u_{yy}} + \Phi^{xxx}\frac{\partial F}{\partial u_{xxx}} + \Phi^{xxy}\frac{\partial F}{\partial u_{xxy}} + \Phi^{xyy}\frac{\partial F}{\partial u_{xyy}} + \Phi^{yyy}\frac{\partial F}{\partial u_{yyy}} = 0.$$

2 Symmetry Lie group of Camassa-Holm equation

Let us consider the Camassa-Holm PDE ([6])

$$(7) \qquad uu_{xxx} + u_{xxy} + 2u_x u_{xx} - 3uu_x - u_y = 0.$$

The Jacobi matrix of the function

$$F(x,y,u^{(3)}) = uu_{xxx} + u_{xxy} + 2u_x u_{xx} - 3uu_x - u_y$$

is

$$J_F = (0,0;u_{xxx} - 3u_x; 2u_{xx} - 3u, -1; 2u_x, 0, 0; u, 1, 0, 0).$$

Because $rankJ_F = 1$, it results that the PDE (7) is of maximal rank.

Let

$$X = \zeta(x,y,u)\frac{\partial}{\partial x} + \eta(x,y,u)\frac{\partial}{\partial y} + \phi(x,y,u)\frac{\partial}{\partial u}$$

be the infinitesimal generator of the symmetry group of the PDE (7). In this case, the condition (6) turns in

$$\phi(u_{xxx} - 3u_x) + \Phi^x(2u_{xx} - 3u) - \Phi^y + 2u_x\Phi^{xx} + u\Phi^{xxx} + \Phi^{xxy} = 0.$$

On the other hand, the relations (3) implies

$$\Phi^x = \phi_x + (\phi_u - \zeta_x)u_x - \eta_x u_y - \zeta_u u_x^2 - \eta_u u_x u_y,$$

$$\Phi^y = \phi_y - \zeta_y u_x + (\phi_u - \eta_y)u_y - \zeta_u u_x u_y - \eta_u u_y^2,$$

$$\begin{aligned}
\Phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \zeta_{xx})u_x - \eta_{xx}u_y + (\phi_{uu} - 2\zeta_{xu})u_x^2 - \\
&- 2\eta_{xu}u_xu_y - \zeta_{uu}u_x^3 - \eta_{uu}u_x^2u_y + (\phi_u - 2\zeta_x)u_{xx} - 2\eta_x u_{xy} - \\
&- 3\zeta_u u_x u_{xx} - \eta_u u_y u_{xx} - 2\eta_u u_x u_{xy},
\end{aligned}$$

$$\begin{aligned}
\Phi^{xxx} &= \phi_{xxx} + u_x(3\phi_{xxu} - \zeta_{xxx}) - u_y\eta_{xxx} + 3u_x^2(\phi_{xuu} - \zeta_{xxu}) - 3u_xu_y\eta_{xxu} + \\
&+ u_x^3(\phi_{uuu} - 3\zeta_{xuu}) - 3u_x^2u_y\eta_{xuu} - u_x^4\zeta_{uuu} - u_x^3u_y\eta_{uuu} + 3u_{xx}(\phi_{xu} - \\
&- \zeta_{xx}) - 3u_{xy}\eta_{xx} + 3u_xu_{xx}(\phi_{uu} - 3\zeta_{xu}) - 3u_yu_{xx}\eta_{xu} - 6u_xu_{xy}\eta_{xu} - \\
&- 6u_x^2u_{xx}\zeta_{uu} - 3u_xu_yu_{xx}\eta_{uu} - 3u_x^2u_{xy}\eta_{uu} - 3u_{xx}^2\zeta_u - 3u_{xx}u_{xy}\eta_u + \\
&+ u_{xxx}(\phi_u - 3\zeta_x) - 3u_{xxy}\eta_x - 4u_xu_{xxx}\zeta_u - u_yu_{xxx}\eta_u - 3u_xu_{xxy}\eta_u,
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{xxy} &= \phi_{xxy} + u_x(2\phi_{xyu} - \zeta_{xxy}) + u_y(\phi_{xxu} - \eta_{xxy}) + u_x^2(\phi_{yuu} - 2\zeta_{xyu}) + \\
&+ u_xu_y(2\phi_{xuu} - \zeta_{xxu} - 2\eta_{xyu}) - u_y^2\eta_{xxu} - u_x^3\zeta_{yuu} + u_x^2u_y(\phi_{uuu} - \eta_{yuu} - \\
&- 2\zeta_{xuu}) - 2u_xu_y^2\eta_{xuu} - u_x^3u_y\zeta_{uuu} - u_x^2u_y^2\eta_{uuu} + u_{xx}(\phi_{yu} - 2\zeta_{xy}) + \\
&+ u_{xy}(2\phi_{xu} - \zeta_{xx} - 2\eta_{xy}) - u_{yy}\eta_{xx} - 3u_xu_{xx}\zeta_{yu} + u_yu_{xx}(\phi_{uu} - \eta_{yu} - \\
&- 2\zeta_{xu}) + 2u_xu_{xy}(\phi_{uu} - 2\zeta_{xu} - \eta_{yu}) - 4u_yu_{xy}\eta_{xu} - 2u_xu_{yy}\eta_{xu} - \\
&- 3u_xu_yu_{xx}\zeta_{uu} - 4u_xu_yu_{xy}\eta_{uu} - 3u_x^2u_{xy}\zeta_{uu} - u_y^2u_{xx}\eta_{uu} - u_x^2u_{yy}\eta_{uu} - \\
&- 3u_{xx}u_{xy}\zeta_u - 2u_{xy}^2\eta_u - u_{xx}u_{yy}\eta_u - u_{xxx}\zeta_y + u_{xxy}(\phi_u - \eta_y - 2\zeta_x) - \\
&- 2u_{xyy}\eta_x - u_yu_{xxx}\zeta_u - 2u_yu_{xxy}\eta_u - 3u_xu_{xxy}\zeta_u - 2u_xu_{xyy}\eta_u.
\end{aligned}$$

One substitutes the functions Φ^x, Φ^y, Φ^{xx}, Φ^{xxx}, Φ^{xxy}, and one eliminates any dependence among the derivatives of the function u, by substituting

$$u_{xxy} = -uu_{xxx} - 2u_xu_{xx} + 3uu_x + u_y$$

in the above relation. Thus, one finds

$$\begin{gathered}
-3u\phi_x - \phi_y + u\phi_{xxx} + \phi_{xxy} + u_x(-3\phi - 3u\zeta_x - 3u\eta_y + \zeta_y - 9u^2\eta_x - u\zeta_{xxx} - \\
-\zeta_{xxy} + 2\phi_{xx} + 3u\phi_{xxu} + 2\phi_{xyu}) + u_y(-2\zeta_x - u\eta_{xxx} - \eta_{xxy} + \phi_{xxu}) + u_x^2(-6u\zeta_u - \\
-9u^2\eta_u - 2\zeta_{xx} - 3u\zeta_{xxu} - 2\zeta_{xyu} + 3u\phi_{xuu} + \phi_{yuu} + 4\phi_{xu}) - u_xu_y(2\zeta_u + 6u\eta_u + \\
+2\eta_{xx} + 3u\eta_{xxu} + \zeta_{xxu} + 2\eta_{xyu} - 2\phi_{xuu}) - u_y^2(\eta_u + \eta_{xxu}) - u_x^3(4\zeta_{xu} + 3u\zeta_{xuu} + \\
+\zeta_{yuu} - 2\phi_{uu} - u\phi_{uuu}) - u_x^2u_y(4\eta_{xu} + 3u\eta_{xuu} + \eta_{yuu} + 2\zeta_{xuu} - \phi_{uuu}) - 2u_xu_y^2\eta_{xuu} - \\
-u_x^4(2\zeta_{uu} + u\zeta_{uuu}) - u_x^3u_y(2\eta_{uu} + u\eta_{uuu} + \zeta_{uuu}) - u_x^2u_y^2\eta_{uuu} + u_{xx}(2\phi_x + 3u\phi_{xu} - \\
-3u\zeta_{xx} + \phi_{yu} - 2\zeta_{xy}) - u_{xy}(\zeta_{xx} + 2\eta_{xy} + 3u\eta_{xx} - 2\phi_{xu}) - u_{yy}\eta_{xx} + u_xu_{xx}(-2\zeta_x + 2\eta_y + \\
+2\phi_u + 6u\eta_x - 9u\zeta_{xu} - 3\zeta_{yu} + 3u\phi_{uu}) - u_yu_{xx}(2\eta_x + 3u\eta_{xu} + 2\zeta_{xu} + \eta_{yu} - \phi_{uu}) - \\
-2u_xu_{xy}(2\eta_x + 3u\eta_{xu} - \phi_{uu} + 2\zeta_{xu} + \eta_{yu}) - 4u_yu_{xy}\eta_{xu} - 2u_xu_{yy}\eta_{xu} - 2u_x^2u_{xx}(\zeta_u - \\
-3u\eta_u + 3u\zeta_{uu}) - 3u_xu_yu_{xx}(\zeta_{uu} + u\eta_{uu}) - u_y^2u_{xx}\eta_{uu} - u_x^2u_{xy}(4\eta_u + 3u\eta_{uu} + 3\zeta_{uu}) -
\end{gathered}$$

$$-4u_xu_yu_{xy}\eta_{uu} - u_x^2u_{yy}\eta_{uu} - 3u_{xx}^2u\zeta_u - 3u_{xx}u_{xy}(u\eta_u + \zeta_u) - u_{xx}u_{yy}\eta_u -$$
$$-2u_{xy}^2\eta_u + u_{xxx}(\phi - u\zeta_x - \zeta_y + u\eta_y + 3u^2\eta_x) - 2u_{xyy}\eta_x +$$
$$+u_xu_{xxx}(-u\zeta_u + 3u^2\eta_u) + u_yu_{xxx}(-\zeta_u + u\eta_u) - 2u_xu_{xyy}\eta_u = 0.$$

Looking at this condition as a polynomial in the partial derivatives of the function u, and identifying with the polynom zero, we obtain the PDEs system

$$\begin{array}{lll} \zeta_x = 0 & \zeta_y = 0 & \zeta_u = 0 \\ \eta_x = 0 & \eta_u = 0 & \eta_{yy} = 0 \\ \phi_x = 0 & \phi_y = 0 & \phi_{uu} = 0 \end{array}$$

$$\phi + u\eta_y = 0,$$

for which

$$\begin{cases} \zeta &= C_1 \\ \eta &= C_3y + C_2 \\ \phi &= -C_3u, \end{cases}$$

is the general solution, where $C_1, C_2, C_3 \in R$. We get

$$X = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_3\left(y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}\right).$$

Theorem 2. *The Lie algebra $\mathbf{g}$ of the infinitesimal transformations associated with the Camassa-Holm PDE is described by the vector fields*

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}. \tag{8}$$

Remarks.

1. The structure constants of the group G (associated with the Lie algebra $\mathbf{g}$) are finding from

$$[X_1, X_2] = 0,\ [X_2, X_3] = X_2,\ [X_3, X_1] = 0.$$

2. If $u = f(x, y)$ is a solution of the PDE (7), then each function

$$\begin{aligned} u^{(1)} &= f(x - \varepsilon, y), \\ u^{(2)} &= f(x, y - \varepsilon), \\ u^{(3)} &= e^{-\varepsilon}f(x, e^{-\varepsilon}y),\ \varepsilon \in R, \end{aligned}$$

is another solution for the equation.

3. By using the relation (4), the adjoint representation AdG of the Lie group G, is described by the table

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$X_3 - \varepsilon X_2$
X_3	X_1	$e^{\varepsilon}X_2$	X_3

Now we shall study the converse of the Theorem 2: given the Lie group G of transformations associated to the Lie algebra $\mathbf{g}$, determine the third order PDE which admits G like group of symmetries. The algorithm of determination of PDE ([5], 303), implies the using of a maximal chain of Lie subalgebras of the algebra $\mathbf{g}$ of the group G, in the case in which $\mathbf{g}$ is solvable.

In the case of the Camassa-Holm equation, the Lie algebra $\mathbf{g}$ (8) is solvable. Thus, we have

$$\{X_1\} \subset \{X_1, X_2\} \subset \{X_1, X_2, X_3\}.$$

Theorem 3. *The PDE of order III*

$$H(x, y, u^{(3)}) = 0$$

whose Lie algebra of the infinitesimal symmetries is described by the vectors fields (8), has the form

$$h\left(\frac{u_x}{u}, \frac{u_y}{u^2}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3}\right) = 0. \tag{9}$$

Proof. We consider the condition (6) for each above subalgebras.

1. $\{X_1\}$: $X_1 = \frac{\partial}{\partial x}$ şi $pr^{(3)}X_1 = \frac{\partial}{\partial x}$. The condition (6) implies $pr^{(3)}X_1(H) = 0$, i.e.,

$$\frac{\partial H}{\partial x} = 0.$$

It results

$$H = f(y, u^{(3)}).$$

2. $\{X_1, X_2\}$: $X_2 = \frac{\partial}{\partial y}$ and $pr^{(3)}X_2 = \frac{\partial}{\partial y}$. Analogously, we get

$$H = g(u^{(3)}).$$

3. $\{X_1, X_2, X_3\}$: $X_3 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$, and

$$\begin{aligned} pr^{(3)}X_3 &= y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - u_x\frac{\partial}{\partial u_x} - 2u_y\frac{\partial}{\partial u_y} - u_{xx}\frac{\partial}{\partial u_{xx}} - 2u_{xy}\frac{\partial}{\partial u_{xy}} - \\ &- 3u_{yy}\frac{\partial}{\partial u_{yy}} - u_{xxx}\frac{\partial}{\partial u_{xxx}} - 2u_{xxy}\frac{\partial}{\partial u_{xxy}} - 3u_{xyy}\frac{\partial}{\partial u_{xyy}} - 3u_{yyy}\frac{\partial}{\partial u_{yyy}}, \end{aligned}$$

which implies

$$H = h\left(\frac{u_x}{u}, \frac{u_y}{u^2}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3}\right).$$

Particular cases.

In the paper [9] P.A.Clarkson, E.L.Mansfield and T.J.Priestley studied the symmetries of a special class of third order PDE

$$u_y - \varepsilon u_{xxy} + 2ku_x = uu_{xxx} + \alpha uu_x + \beta u_x u_{xx}, \tag{10}$$

where ε, k, α and β are arbitrary constants.

If the PDE (9) is written in the form

$$u_y = u^2\varphi\left(\frac{u_x}{u}, \frac{u_{xx}}{u}, \frac{u_{xy}}{u^2}, \frac{u_{yy}}{u^3}, \frac{u_{xxx}}{u}, \frac{u_{xxy}}{u^2}, \frac{u_{xyy}}{u^3}, \frac{u_{yyy}}{u^3}\right),$$

then, for

$$\varphi(C_1, ..., C_8) = C_6 + C_5 - 3C_1 + 2C_1C_2,$$

we obtain the *Camassa-Holm PDE* (the case $\varepsilon = 1$, $\alpha = -3$, $\beta = 2$ and $k = 0$ in (10)).

For

$$\varphi(C_1, ..., C_8) = \varepsilon C_6 + C_5 + \alpha C_1 + \beta C_1C_2,$$

we get a subclass of the class of PDEs (10), for the parameter $k = 0$. In this case, for the parameters $\varepsilon = 0$, $\alpha = 1$, $\beta = 3$ and $k = 0$, it results the *Rosenau-Hyman PDE*, which is considered in [9] also.

Remark. There exists PDEs of order one or two which admit (8) like infinitesimal symmetries. For example: the elementary nonlinear wave PDE [14, 300]

$$u_y = uu_x,$$

and the Liouville-Ţiţeica PDE [3]

$$uu_{xy} - u_xu_y = u^3.$$

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