On Generalized Birecurrent Weyl Spaces♣

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Abstract

In this paper, generalized birecurrent Weyl spaces are defined and some results concerning generalized birecurrent hypersurfaces of a generalized birecurrent Weyl space are obtained.

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§1. Introduction.

An n-dimensional manifold W_n is said to be a *Weyl space* if it has a conformal metric tensor g_{ij} and a symmetric connection D satisfying the compatibility condition given by the equation

$$D_k g_{ij} - 2T_k g_{ij} = 0,$$

where T_k denotes a covariant vector field and D_k denotes the usual covariant derivative.

Under the renormalization

$$\tilde{g}_{ij} = \lambda^2 g_{ij}, \tag{1.1}$$

the vector field T_k is transformed into ([1], [2]),

$$\tilde{T}_k = T_k + \partial_k \ln \lambda.$$

If T_k is zero or a gradient vector field, then W_n is locally Riemannian.

The quantity A is called a ***satellite*** with weight p of the tensor g_{ij} if it admits a transformation of the form $\tilde{A} = \lambda^p A$ under the renormalization (1.1). The *prolonged covariant derivative* of a satellite A of g_{ij} with weight p is defined by, [1],

$$\dot{D}_k A = D_k A - pT_k A.$$

Let R^i_{jkl} denote the curvature tensor associated with the connection D_k. The first and the second Bianchi identities for a Weyl space are ([3]),

$$R^h_{ijk} + R^h_{kij} + R^h_{jki} = 0 \tag{1.2}$$

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$$\dot{D}_r R^i_{jkh} + \dot{D}_k R^i_{jhr} + \dot{D}_h R^i_{jrk} = 0. \tag{1.3}$$

§2. Generalized birecurrent Weyl spaces

A Weyl space with nonzero R^i_{jkl} is said to be *birecurrent* if the condition

$$\dot{D}_r \dot{D}_s R^i_{jkl} = \phi_{sr} R^i_{jkl} \tag{2.1}$$

is satisfied for a nonzero tensor field ϕ_{sr} with weight 0. A non-flat Weyl space is said to be *generalized birecurrent* if the condition

$$\dot{D}_r \dot{D}_s R^h_{jkl} = A_{sr} R^h_{jkl} + Q^h_{jklsr} \tag{2.2}$$

is satisfied for nonzero tensor fields A_{sr} and Q^h_{jklsr}. Multiplying (2.2) by g_{hi} we get

$$\dot{D}_r \dot{D}_s R_{ijkl} = A_{sr} R_{ijkl} + Q_{ijklsr}, \tag{2.2)'$$

where $Q_{ijklsr} = Q^h_{jklsr} g_{hi}$.
It is easy to see that a generalized birecurrent Weyl space is birecurrent if $Q^h_{jklsr} = 0$. Birecurrent Weyl spaces are examined in [4].

We remark that the definition of a generalized birecurrent Weyl space [5] agrees with that of a generalized birecurrent space with affine symmetric connection.

Theorem 2.1. *Let W_n be a generalized birecurrent Weyl space with non-zero scalar curvature R. If A_{sr} and B_{sr} are both symmetric tensors, then the space is locally Riemannian, where $B_{sr} = Q^l_{jklsr} g^{jk}$.*

Proof. Assume A_{sr} and B_{sr} are symmetric tensors of order two. Changing the order of indices r and s in (2.2), we get

$$\dot{D}_s \dot{D}_r R^h_{jkl} = A_{rs} R^h_{jkl} + Q^h_{jklrs} \tag{2.3}$$

and subtracting (2.3) from (2.2), we have

$$\dot{D}_{[r} \dot{D}_{s]} R^h_{jkl} = A_{[sr]} R^h_{jkl} + Q^h_{jkl[sr]}, \tag{2.4}$$

where bracket denotes antisymmetrization.

Contracting h and l in (2.4), we get

$$\dot{D}_{[r} \dot{D}_{s]} R^l_{jkl} = A_{[sr]} R^l_{jkl} + Q^l_{jkl[sr]}. \tag{2.4)'$$

Multiplying (2.4)′ by g^{jk} and remembering that the Ricci tensor R_{ij} and the scalar curvature R of the Weyl space are respectively defined by $R_{ij} = R^h_{ijh}$ and $R = R_{ij} g^{ij}$, we get

$$\dot{D}_{[r} \dot{D}_{s]} R = R A_{[sr]} + B_{[sr]}, \tag{2.5}$$

where $B_{sr} = Q^l_{jklsr} g^{jk}$. Since, by assumption, A_{sr} and B_{sr} are symmetric tensors, we get

$$\dot{D}_{[r} \dot{D}_{s]} R = 0. \tag{2.6}$$

Expanding $\dot{D}_{[r}\dot{D}_{s]}R$ and using that R is a satellite of g_{ij} with weight -2, we find that

$$\dot{D}_{[r}\dot{D}_{s]}R = D_{[r}D_{s]}R + 2D_{[r}T_{s]}R = 0 \quad with \quad D_{[r}D_{s]}R = 0.$$

Since $R \neq 0$, we have

$$D_{[s}T_{r]} = 0,$$

which means that W_n is locally Riemannian. From this theorem we conclude the following theorems.

Corollary 2.1. *If* $\dot{D}_r\dot{D}_sR^i_{jkh} = 0$*, then the Weyl space is locally Riemannian.*

Corollary 2.2. *If* $\dot{D}_{[r}\dot{D}_{s]}R^i_{jkh} = 0$*, then the Weyl space is locally Riemannian.*

Theorem 2.2. *If* A_{sr} *is symmetric and* B_{sr} *is not, then*

$$B_{[sr]} = 2D_{[r}T_{s]}R.$$

Proof. Assume A_{sr} is symmetric. Then from (2.5) we have

$$\dot{D}_{[s}\dot{D}_{r]}R = B_{[sr]}.$$

Since $R \neq 0$ and a satellite of g_{ij} with weight -2, we infer

$$2D_{[s}T_{r]}R = B_{[sr]}. \tag{2.7}$$

Corollary 2.3. *If* A_{sr} *is symmetric and* W_n *is locally Riemannian, then* B_{sr} *is symmetric.*

Proof. If W_n is locally Riemannian, then (2.7) yields

$$B_{[sr]} = 0.$$

Theorem 2.3. *If* B_{sr} *is symmetric and* A_{sr} *is not, then*

$$A_{[sr]} = 2D_{[r}T_{s]}.$$

Proof. Assume B_{sr} is symmetric. Then from (2.5) we have

$$\dot{D}_{[r}\dot{D}_{s]}R = RA_{[sr]}. \tag{2.8}$$

Expanding $\dot{D}_{[r}\dot{D}_{s]}R$ and remembering that R is a non-zero satellite of g_{ij} with weight -2, we find that

$$A_{[sr]} = 2D_{[r}T_{s]}.$$

Corollary 2.4. *If* B_{sr} *is symmetric and* W_n *is locally Riemannian, then* A_{sr} *is symmetric.*

Proof. If W_n is locally Riemannian, then from (2.7) obtain

$$A_{[sr]} = 0.$$

§3. Hypersurfaces of generalized birecurrent Weyl spaces.

Let $W_n(g_{ij}, T_k)$ be a hypersurface, with coordinates u^i, $i = \overline{1,n}$ of a Weyl space $W_{n+1}(g_{ab}, T_c)$ with coordinates x^a, $a = \overline{1,n+1}$. The metrics of W_n and W_{n+1} are connected by the relations

$$g_{ij} = g_{ab} x_i^a x_j^b, \qquad i,j = \overline{1,n},\ a,b = \overline{1,n+1}, \tag{3.1}$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i.

It is easy to see that the prolonged covariant derivative of a satellite A, relative to W_n, and W_{n+1}, are related by

$$\dot{D}_k A = x_k^c \dot{D}_c A, \qquad k = \overline{1,n},\ c = \overline{1,n+1}. \tag{3.2}$$

Let n^a be the contravariant components of the vector field of W_{n+1} normal to W_n which is normalized by the condition

$$g_{ab} n^a n^b = 1. \tag{3.3}$$

The moving frame $\{x_a^i, n_a\}$ in W_n, reciprocal to the moving frame $\{x_i^a, n^a\}$ is defined by the relations [3]

$$n_a x_i^a = 0, \qquad n^a x_a^i = 0, \qquad x_i^a x_a^j = \delta_i^j. \tag{3.4}$$

Remembering that the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a with respect to u^k is found as

$$\dot{D}_k x_i^a = D_k x_i^a = \omega_{ik} n^a, \tag{3.5}$$

where ω_{ik} is the second fundamental form. It can be shown that ω_{ik} is a satellite of g_{ij} with weight $\{1\}$.

The generalized Gauss and Mainardi-Codazzi equations are obtained in [3], respectively as

$$R_{pijk} = \Omega_{pijk} + \overline{R}_{dbce} x_p^d x_i^b x_j^c x_k^e \tag{3.6}$$

$$\dot{D}_k \omega_{ij} - \dot{D}_j \omega_{ik} + \overline{R}_{dbce} x_i^b x_j^c x_k^e n^d = 0, \tag{3.7}$$

where $\overline{R}_{dbce}$ is the covariant curvature tensor of W_{n+1} and Ω_{pijk} is the Sylvesterian of ω_{ij} defined by $\Omega_{pijk} = \omega_{pj}\omega_{ik} - \omega_{pk}\omega_{ij}$.

In the following we will use the following notation $H_{ijkl}^{abcd} = x_i^a x_j^b x_k^c x_l^d$.

Theorem 3.1. *For a hypersurface of a generalized birecurrent Weyl space W_{n+1} with A_{ef} and Q_{abcdef} we have the identity*

$$\begin{aligned} \dot{D}_r \dot{D}_s R_{ijkl} - A_{sr} R_{ijkl} - Q_{ijklsr} = {} & \dot{D}_r \dot{D}_s \Omega_{ijkl} - A_{sr}\Omega_{ijkl} + 2S_{ijkl(sr)} + \\ & + D_{ijkl}\omega_{sr} + \overline{R}_{abcd} \dot{D}_r \dot{D}_s H_{ijkl}^{abcd}, \end{aligned}$$

where

$$Q_{abcdef} = g_{at}Q^t_{bcdef}$$
$$S_{ijklsr} = \dot{D}_e\overline{R}_{abcd}x^e_s\dot{D}_rH^{abcd}_{ijkl}$$
$$D_{ijkl} = H^{abcd}_{ijkl}n^e\dot{D}_e\overline{R}_{abcd}$$
$$A_{sr} = A_{ef}x^e_sx^f_r$$
$$Q_{ijklsr} = Q_{abcdef}H^{abcdef}_{ijklsr}$$

and the paranthesis $(\ldots)$ *denotes symmetrization.*

Proof. By taking the prolonged covariant derivative of Gauss equation with respect to u^s and u^r successively, we have

$$\begin{aligned}\dot{D}_r\dot{D}_sR_{ijkl} = \quad & \dot{D}_r\dot{D}_s\Omega_{ijkl} + (\dot{D}_f\dot{D}_e\overline{R}_{abcd})H^{abcdef}_{ijklsr} + (\dot{D}_e\overline{R}_{abcd})x^e_s(\dot{D}_rH^{abcd}_{ijkl})+\\ & +(\dot{D}_f\overline{R}_{abcd})x^f_r(\dot{D}_sH^{abcd}_{ijkl})+\\ & +\overline{R}_{abcd}\dot{D}_r\dot{D}_sH^{abcd}_{ijkl} + (\dot{D}_e\overline{R}_{abcd})H^{abcd}_{ijkl}\dot{D}_rx^e_s.\end{aligned}$$

If W_{n+1} is generalized birecurrent Weyl, then by definition

$$\dot{D}_f\dot{D}_e\overline{R}_{abcd} = A_{ef}\overline{R}_{abcd} + Q_{abcdef}. \tag{3.8}$$

Therefore from (3.7) and (3.8), we have

$$\begin{aligned}\dot{D}_r\dot{D}_sR_{ijkl} = \quad & \dot{D}_r\dot{D}_s\Omega_{ijkl} + A_{ef}\overline{R}_{abcd}H^{abcdef}_{ijklsr} + Q_{abcdef}H^{abcdef}_{ijklsr} + \overline{R}_{abcd}\dot{D}_r\dot{D}_sH^{abcd}_{ijkl}+\\ & +(\dot{D}_e\overline{R}_{abcd})x^e_s(\dot{D}_rH^{abcd}_{ijkl}) + (\dot{D}_f\overline{R}_{abcd})x^f_r(\dot{D}_sH^{abcd}_{ijkl}) + \omega_{sr}n^e(\dot{D}_e\overline{R}_{abcd})H^{abcd}_{ijkl}.\end{aligned}$$

By using (3.1) and the Gauss equation (3.6), the above equation rewrites

$$\begin{aligned}\dot{D}_r\dot{D}_sR_{ijkl} = \quad & A_{sr}R_{ijkl} + \dot{D}_s\dot{D}_r\Omega_{ijkl} - A_{sr}\Omega_{ijkl} + \dot{D}_e\overline{R}_{abcd}x^e_s\dot{D}_rH^{abcd}_{ijkl}+\\ & +x^f_r\dot{D}_f\overline{R}_{abcd}\dot{D}_sH^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{D}_r\dot{D}_sH^{abcd}_{ijkl} + Q_{ijklsr},\end{aligned}$$

where $A_{sr} = A_{ef}H^{ef}_{sr}$ and $Q_{ijklrs} = Q_{abcdef}H^{abcdef}_{ijklsr}$.

Hence we get

$$\begin{aligned}\dot{D}_r\dot{D}_sR_{ijkl} \quad & -A_{sr}R_{ijkl} - Q_{ijklsr} = \dot{D}_r\dot{D}_s\Omega_{ijkl} - A_{sr}\Omega_{ijkl}+\\ & +2S_{ijkl(sr)} + D_{ijkl}\omega_{sr} + \overline{R}_{abcd}\dot{D}_r\dot{D}_sH^{abcd}_{ijkl}\end{aligned} \tag{3.9}$$

where

$$Q_{abcdef} = g_{at}Q^t_{bcdef}, \qquad S_{ijklsr} = \dot{D}_e\overline{R}_{abcd}x^e_s\dot{D}_rH^{abcd}_{ijkl},$$
$$D_{ijkl} = H^{abcd}_{ijkl}n^e\dot{D}_e\overline{R}_{abcd}, \qquad A_{sr} = A_{ef}x^e_sx^f_r,$$
$$Q_{ijklsr} = Q_{abcdef}H^{abcdef}_{ijklsr}$$

Theorem 3.2. *If a hypersurface of a generalized birecurrent Weyl space is generalized birecurrrent then*

$$\dot{D}_r\dot{D}_s\Omega_{ijkl} - A_{sr}\Omega_{ijkl} + 2S_{ijkl(sr)} + \omega_{sr}D_{ijkl} + \overline{R}_{abcd}\dot{D}_r\dot{D}_sH^{abcd}_{ijkl} = 0. \tag{3.10}$$

or, equivalently,

$$\dot{D}_{[r}\dot{D}_{s]}\Omega_{ijkl} - A_{[sr]}\Omega_{ijkl} + \overline{R}_{abcd}\dot{D}_{[r}\dot{D}_{s]}H^{abcd}_{ijkl} = 0. \tag{3.11}$$

Proof. It is clear from (2.2) and Theorem 3.1. □

A hypersurface of a Weyl space is called *totally geodesic* if $\omega_{ij} = 0$.

Theorem 3.3. *Every totally geodesic hypersurface of a generalized birecurrent Weyl space is generalized birecurrent.*

Proof. Since the hypersurface is totally geodesic, by putting $\omega_{ij} = 0$ in (3.9) and using (3.5) we get the result. □

References

[1] V.Hlavaty, *Theorie d'immersion d'une W_m dans W_n*, Ann. Soc. Polon. Math., 21(1949), 196-206.

[2] A.Norden, *Affinely Connected Spaces,* GRFML, Moscow, 1976.

[3] Ö.E.Canfes, A.Özdeğer, *Some Applications of Prolonged Covariant Differentiation in Weyl Geometry*, Journal of Geometry, 60(1997), 7-16.

[4] Ö.E.Canfes, A.Özdeğer, *On Birecurrent Weyl Spaces*, to appear in Studia Universitatatis Napocensis.

[5] P.Enghiş, *Generalized Birecurrency in Spaces of with Affine Symmetric Connection*, Proceedings of 23rd. Conference on Geom. and Topology, Cluj Napoca, 1993, 59-62.

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