Recurrent Weyl Spaces Having Decomposable Projective Curvature Tensor♣

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Abstract

In this work, decomposable recurrent Weyl spaces and recurrent Weyl spaces having decomposable projective curvature tensor are examined.

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§1. Introduction.

A differentiable manifold of dimension n having conformal metric tensor g and symmetric connection ∇ satisfying

$$\nabla g = 2T \otimes g, \tag{1.1}$$

where T is a 1-form is called *a Weyl space* which we denote by $W_n(g,T)$. Accordingly, in local coordinates,

$$\nabla_k g_{ij} = 2T_k g_{ij}. \tag*{(1.1)'}$$

Under the renormalization

$$\tilde{g} = \lambda^2 g, \tag{1.2}$$

the vector field T is transformed into ([1]),

$$\tilde{T} = T + d\ln\lambda.$$

This transformation is called *Gauge transformation.* If the vector field T is locally zero or locally gradient, then W_n is locally Riemannian.

The quantity A is called a *satellite* with weight p of the tensor g_{ij} if it admits a transformation of the form $\tilde{A} = \lambda^p A$ under the renormalization (1.1).[2]

The *prolonged covariant derivative* of a satellite A of g with weight p is defined by

$$\dot{\nabla} A = \nabla A - p(T \otimes A). \tag{1.3}$$

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It is easy to see that prolonged covariant derivative preserves weight.

Let $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ denote the curvature tensor associated with the connection ∇.

The first and the second Bianchi identities for Weyl spaces are ([3]),

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \tag{1.4}$$

$$(\dot{\nabla}_X R)(Y,Z) + (\dot{\nabla}_Y R)(Z,X) + (\dot{\nabla}_Z R)(X,Y) = 0 \tag{1.5}$$

A non-flat Weyl space is called *recurrent* if its curvature tensor R of type (1,3) satisfies the condition

$$(\dot{\nabla}_U R)(X,Y,Z) = \phi(U) R(X,Y,Z) \tag{1.6}$$

where ϕ is a nonzero 1-form with weight zero and such a space is denoted by $(RW)_n$, [3].

Then by expressing (1.6) in local coordinates, we have

$$\dot{\nabla}_r R^i_{jkl} = \phi_r R^i_{jkl}. \tag{1.7}$$

It is easy to see that a recurrent manifold is Ricci-Recurrent, that is,

$$\dot{\nabla}_l R_{jk} = \phi_l R_{jk}, \tag{1.8}$$

where R_{jk} is the Ricci tensor.

A Weyl space W_n is called ***projectively recurrent*** if its projective curvature tensor W ([4]), given in local coordinates by

$$W^i_{jkl} = R^i_{jkl} + \frac{2}{n+1}\delta^i_j R_{[kl]} + \frac{1}{n^2-1}\left[\delta^i_k \left(nR_{jl} + R_{lj}\right) - \delta^i_l \left(nR_{jk} + R_{kj}\right)\right], \tag{1.9}$$

satisfies the condition

$$(\dot{\nabla}_U W)(X,Y,Z) = A(U) W(X,Y,Z), \tag{1.10}$$

where A is a nonzero 1-form and the bracket denotes antisymmetrization.

The projective curvature tensor satisfies the following identities ([4]),

$$W^i_{jkl} + W^i_{jlk} = 0 \tag{1.11}$$

$$W^i_{jkl} + W^i_{ljk} + W^i_{klj} = 0 \tag{1.12}$$

$$W^i_{ikl} = W^i_{jki} = 0 \tag{1.13}$$

$$\dot{\nabla}_i W^i_{jkl} = \frac{(n-2)}{(n-1)}\left[\dot{\nabla}_l R_{jk} - \dot{\nabla}_k R_{jl} + \frac{1}{(n+1)}\left(\dot{\nabla}_k R_{[jl]} - \dot{\nabla}_l R_{[jk]}\right)\right]. \tag{1.14}$$

Theorem 1.1. *A recurrent Weyl space is projectively recurrent.*

Proof. By taking prolonged covariant derivative of (1.9) we have

$$\begin{aligned}\dot{\nabla}_p W^i_{jkl} = {} & \tfrac{1}{n^2-1}\left[\delta^i_k \left(n\dot{\nabla}_p R_{jl} + \dot{\nabla}_p R_{lj}\right) \delta^i_l \left(n\dot{\nabla}_p R_{jk} + \dot{\nabla}_p R_{kj}\right)\right] + \\ & + \dot{\nabla}_p R^i_{jkl} + \dot{\nabla}_p \tfrac{2}{n+1}\delta^i_j R_{[kl]}.\end{aligned}$$

Since W_n is recurrent, by using (1.7) and (1.8), we obtain

$$\dot{\nabla}_p W^i_{jkl} = \phi_p W^i_{jkl}. \tag{1.15}$$

□

§2. Weyl spaces having decomposable curvature tensor

Assume W_n locally admits tensor fields a^i_l and φ_{jk} with weights -1 and 1, respectively, satisfying the condition

$$R^i_{jkl} = a^i_l \varphi_{jk}. \tag{2.1}$$

Then we say that the curvature tensor of W_n is decomposable in the form (2.1).

Theorem 2.1. *Let the curvature tensor of W_n be decomposable in the form (2.1). Then, the tensor fields a^i_l and φ_{jk} satisfy the following equation*

$$a^i_l \varphi_{jk} + a^i_k \varphi_{jl} = 0.$$

Proof. By contracting i and l in (2.1) and remembering the definition of the Ricci tensor R_{ij} we get

$$R_{jk} = \rho \varphi_{jk} \qquad where \quad \rho = a^l_l = tr(a^j_i). \tag{2.2}$$

Then by using (1.11) and (1.12), we obtain

$$a^i_l \varphi_{jk} + a^i_k \varphi_{jl} = 0.$$

□

Theorem 2.2. *Let a Weyl space have decomposable curvature tensor in the form (2.1).If the tensor field φ_{jk} is symmetric, then the Weyl space is flat.*

Proof. By using the properties of the projective curvature tensor and (2.2) we have

$$a^i_l \varphi_{jk} + a^i_j \varphi_{kl} + a^i_k \varphi_{lj} = 0. \tag{2.3}$$

In virtue of the Theorem 2.1, we obtain

$$a^i_k (\varphi_{lj} - \varphi_{jl}) + a^i_j \varphi_{kl} = 0. \tag{2.3}$$

Hence the result follows. □

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Let v^i be a vector field which is normalized by the condition

$$v^i v^j g_{ij} = 1, \tag{3.1}$$

and A_{jkl} be a non-zero 3-form with weight 1.

Assume a Weyl space W_n locally admits a vector field v^i and a 3-form A_{jkl} satisfying the condition

$$W^i_{jkl} = v^i A_{jkl}. \tag{3.2}$$

Then we say that the projective curvature tensor of W_n is decomposable in the form (3.2).

Theorem 3.1. *The tensor field A_{ijk} satisfies the following identities*
i. $A_{jkl} + A_{jlk} = 0$
ii. $A_{jkl} + A_{ljk} + A_{klj} = 0$
iii. $v^i A_{jki} = 0, \quad v^i A_{ikl} = 0.$

Proof. i) By putting (3.2) in (1.11) we get

$$v^i\ A_{jkl} + v^i A_{jlk} = 0.$$

Hence we obtain,

$$A_{jkl} + A_{jlk} = 0.$$

ii) By using (3.2) in (1.12) we infer

$$v^i(A_{jkl} + A_{ljk} + A_{klj}) = 0.$$

Therefore we get

$$A_{jkl} + A_{ljk} + A_{klj} = 0.$$

iii) By using (1.13) and (3.2) we have

$$v^i A_{jki} = 0 \quad , \quad v^i A_{ikl} = 0.$$

□

Theorem 3.2. *Let W_n be a recurrent Weyl space. Then the tensor field A_{ijk} is recurrent with recurrence vector ϕ_r and $\dot{\nabla}_r v^i = 0$.*

Proof. By taking prolonged covariant derivative of (3.2) with respect to r, we have

$$\dot{\nabla}_r W^i_{jkl} = (\dot{\nabla}_r v^i) A_{jkl} + v^i(\dot{\nabla}_r A_{jkl}). \tag{3.3}$$

From Theorem 3.1 and (3.2) we get

$$\phi_r v^i A_{jkl} = (\dot{\nabla}_r v^i) A_{jkl} + v^i(\dot{\nabla}_r A_{jkl}). \tag{3.4}$$

Multiplying both sides of (3.4) by $v^h g_{hi}$, we obtain

$$v^h g_{hi} \phi_r v^i A_{jkl} = v^h g_{hi}(\dot{\nabla}_r v^i) A_{jkl} + v^i v^h g_{hi}(\dot{\nabla}_r A_{jkl}) \tag{3.5}$$

Using (3.1) and the fact that $v^h g_{hi}(\dot{\nabla}_r v^i) = 0$ we infer

$$\dot{\nabla}_r A_{jkl} = \phi_r A_{jkl}. \tag{3.5$'$}$$

Since $A_{jkl} \neq 0$ from Theorem 1.1 , (3.2) and (3.5)$'$ we find

$$\dot{\nabla}_r v^i = 0.$$

□

Theorem 3.3. *Let W_n be Weyl space having decomposable projective curvature tensor in the form $W^i_{jkl} = v^i A_{jkl}$. If the tensor field A_{ijk} is recurrent with recurrence vector ϕ_r and $\dot{\nabla}_r v^i = 0$, or $\dot{\nabla}_r v^i = \phi_r v^i$, then W_n is projectively recurrent with recurrence vector ϕ_r.*

Proof. By taking prolonged covariant derivative of (3.2) and using the assumptions, we find

$$\dot{\nabla}_r W^i_{jkl} = \phi_r W^i_{jkl}.$$

□

Theorem 3.4. *If a recurrent Weyl space has decomposable projective curvature tensor in the form $W^i_{jkl} = v^i A_{jkl}$, then A_{jkl} is determined.*

Proof. From (1.11), (1.12), (1.14) and Theorem 1.1 we have

$$\phi_i v^i A_{jkl} = \frac{(n-2)}{(n-1)} \left[\dot{\nabla}_l R_{jk} - \dot{\nabla}_k R_{jl} + \frac{1}{(n+1)} (\dot{\nabla}_k R_{[jl]} - \dot{\nabla}_l R_{[jk]}) \right] \tag{3.6}$$

Since recurrent Weyl space is Ricci recurrent, then

$$\phi_i v^i A_{jkl} = \frac{(n-2)}{(n-1)} \left[\phi_l R_{jk} - \phi_k R_{jl} + \frac{1}{(n+1)} (\phi_k R_{[jl]} - \phi_l R_{[jk]}) \right] \tag{3.7}$$

From (3.7) we get

$$A_{jkl} = \mu \left[\phi_l R_{jk} - \phi_k R_{jl} + \frac{1}{(n+1)} (\phi_k R_{[jl]} - \phi_l R_{[jk]}) \right]$$

where $\mu = \frac{(n-2)}{(n-1)} (\phi_i v^i)^{-1}$. □

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