Special Properties of Second-Order Dynamical Systems♣

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Abstract

Given a second-order differential system, a basic problem is to establish conserved quantities. Therefore we will associate to the given system the some of variational forms ([1]) for which, it is known, there is a unique adjoint system ([4]). We also study cases of self-adjoint degenerated systems. In the last part, there are some illustrative examples. Related works were developed by F. Mimura and T. Nôno ([2],[3]).

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§1. Introduction

Let M be a smooth manifold of dimension m. We consider a second-order differential system defined by a section in the bundle $\delta = (J^2M \times_M T^*M, p_\delta, J^2M)$ expressed locally by

$$F_i(t, x, \dot{x}, \ddot{x}) = 0 \tag{1}$$

and, respectively, a section in the bundle $\tau = (J^1M \times_M TM, p_\tau, J^1M)$ with the local expression $X = \xi^i(t, x, \dot{x})\frac{\partial}{\partial x^i}$.

We associate to the system (1) and to the section X a *variational form*:

$$\begin{aligned} M_F^X &= \left[\frac{\partial F_i}{\partial x^j}\xi^j + \frac{\partial F_i}{\partial \dot{x}^j}\frac{d\xi^j}{dt} + \frac{\partial F_i}{\partial \ddot{x}^j}\frac{d^2\xi^j}{dt^2}\right] dx^i = \\ &= \left[a_{ij}\xi^j + b_{ij}\frac{d\xi^j}{dt} + c_{ij}\frac{d^2\xi^j}{dt^2}\right] dx^i \end{aligned} \tag{2}$$

For another section Y in the bundle τ, with the local representation $Y = \eta^i(t, x, \dot{x})\frac{\partial}{\partial x^i}$, we will associate the form

$$\widetilde{M}^Y = \left(\tilde{a}_{ij}\eta^j + \tilde{b}_{ij}\frac{d\eta^j}{dt} + \tilde{c}_{ij}\frac{d^2\eta^j}{dt^2}\right) dx^i, \tag{3}$$

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where its coefficients are, for the moment, arbitrary.

Definition. Two forms M_F^X and $\widetilde{M}^Y$ are called *adjoint* one to the other if there is a bilinear function $Q(X,Y)$, such that, on solutions of the given system, the relation:

$$M^X(Y) - \widetilde{M}^Y(X) = \frac{d}{dt}Q(X,Y)\ ,\ \forall X,Y \in \mathcal{S}(\tau) \tag{4}$$

holds.

We have the following

Theorem 1. *Given a system of variational forms M_F^X, there is a unique adjoint system of forms $\widetilde{M}^Y$* ([4]).

By direct computation, we obtain

$$\widetilde{M}_i^Y = \left(a_{ji} - \frac{db_{ji}}{dt} + \frac{d^2c_{ji}}{dt^2}\right)\eta^j - \left(b_{ji} - 2\frac{dc_{ji}}{dt}\right)\frac{d\eta^j}{dt} + c_{ji}\frac{d^2\eta^j}{dt^2} \tag{5}$$

and, respectively,

$$Q(X,Y) = c_{ij}\eta^i\frac{d\xi^j}{dt} - \left[\frac{d}{dt}(c_{ij}\eta^i) - b_{ij}\eta^i\right]\xi^j. \tag{6}$$

§2. Conservation Laws

If we equalize to zero each term from the left hand side of the adjointness relation (4), we obtain the system

$$\begin{cases} a_{ij}\xi^j + b_{ij}\dfrac{d\xi^j}{dt} + c_{ij}\dfrac{d^2\xi^j}{dt^2} = 0 \\ \left(a_{ji} - \dfrac{db_{ji}}{dt} + \dfrac{d^2c_{ji}}{dt^2}\right)\eta^j - \left(b_{ji} - 2\dfrac{dc_{ji}}{dt}\right)\dfrac{d\eta^j}{dt} + c_{ji}\dfrac{d^2\eta^j}{dt^2} = 0. \end{cases} \tag{7}$$

Let ξ^i and η^i be a set of functions satisfying respectively the equations (7_1) and (7_2) on solutions of (1). Then the conserved quantity $Q = const.$, given by (6), can be constructed ([3]).

A solution of the first set of equations (7_1), on solutions of (1), is $\xi^i = \dot{x}^i$. For showing this, we remark that the total differentiation with respect to t of F_i is

$$\frac{dF_i}{dt} = \frac{\partial F_i}{\partial t} + \dot{x}^j\frac{\partial F_i}{\partial x^j} + \ddot{x}^j\frac{\partial F_i}{\partial \dot{x}^j} + \dddot{x}^j\frac{\partial F_i}{\partial \ddot{x}^j}.$$

On solutions of (1), we find $\frac{dF_i}{dt} = 0$, $\frac{\partial F_i}{\partial t} = 0$, which imply

$$\dot{x}^j a_{ij} + \ddot{x}^j b_{ij} + \dddot{x}^j c_{ij} = 0.$$

So $\xi^i = \dot{x}^i$ is a solution of (7_1), on solutions of (1).

It holds:

Theorem 2. *If η^i is a set of functions which satisfy the equations (7_2) on solutions of (1), then*

$$Q = -c_{ij}\dot{x}^j \frac{d\eta^i}{dt} + \left[c_{ij}\ddot{x}^j - \left(\frac{dc_{ij}}{dt} - b_{ij}\right)\dot{x}^j\right]\eta^i = const. \tag{8}$$

is a conserved quantity.

If the system (1) is *self-adjoint*, that is $M^X_{F,i} = \widetilde{M}^X_i$, for any X, then there exists a Lagrange function $L(t, x, \dot{x})$, such that the Euler-Lagrange equations can be identified with the given system (1).

In this case, the two sets of equations (7_1) and (7_2) become equivalent and are rewritten as

$$\begin{aligned} &\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\frac{d^2\xi^j}{dt^2} + \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\right) + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j}\right]\frac{d\xi^j}{dt} + \\ &+ \left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\right) - \frac{\partial^2 L}{\partial x^i \partial x^j}\right]\xi^j = 0. \end{aligned} \tag{9}$$

The conserved quantity (6) is constructed from each pair $\xi_1(t, x, \dot{x})$, $\xi_2(t, x, \dot{x})$ satisfying the equations (9), on solutions of (1), as

$$Q = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\left(\xi_1^i \frac{d\xi_2^j}{dt} - \frac{d\xi_1^i}{dt}\xi_2^j\right) + \left(\frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} - \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j}\right)\xi_1^i \xi_2^j = const. \tag{10}$$

A solution is $\xi_1^i = \dot{x}^i$ and considering another solution $\xi_2^i = \xi^i$ of the equations (9), on solutions of (1), we obtain the following conserved quantity ([1])

$$Q = \dot{x}^i \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\frac{d\xi^j}{dt} + \left(\dot{x}^i \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} + \frac{\partial^2 L}{\partial \dot{x}^j \partial t} - \frac{\partial L}{\partial x^j}\right)\xi^j = const. \tag{11}$$

In particular, for a time-independent Lagrangian $L(x, \dot{x})$, we obtain the case considered by Mimura and Nôno in [3, pp6-7].

§3. Degenerated systems written in main form

We consider a dynamical system written in main form:

$$F_i \equiv A_{ij}(t, x, \dot{x})\ddot{x}^j + B_i(t, x, \dot{x}) = 0, \tag{12}$$

with $rank\left(\frac{\partial F_i}{\partial \ddot{x}^j}\right) = rank(A_{ij}) = r$, $0 < r < m$, ($det(A_{ab}) \neq 0$, $a, b = \overline{1, r}$).

In all what follows we will use $i, j, h = \overline{1, m}$; $a, b, c = \overline{1, r}$; $\alpha, \beta, \gamma = \overline{r+1, m}$.

There is a matrix $(C^a_\alpha)_{\substack{a=\overline{1,r} \\ \alpha=\overline{r+1,m}}}$ such that we have the relations

$$A_{\alpha b} = C^a_\alpha A_{ab}\ , \quad A_{\alpha\beta} = C^a_\alpha A_{a\beta}$$

and the system (12) can be rewritten as

$$\begin{cases} A_{ab}\ddot{x}^b + A_{a\beta}\ddot{x}^\beta + B_a = 0 \\ C^a_\alpha A_{ab}\ddot{x}^b + C^a_\alpha A_{a\beta}\ddot{x}^\beta + B_\alpha = 0. \end{cases} \tag{13}$$

The system (13) is equivalent (it describe the same dynamics) with the system which is formed just from r second-order differential equations, and $m-r$ first-order equations.

$$(14)\qquad \begin{cases} A_{ab}\ddot{x}^b + A_{a\beta}\ddot{x}^\beta + B_a = 0 \\ B_\alpha - C^a_\alpha B_a = 0. \end{cases}$$

§4. Self-adjoint degenerated dynamical systems

It is known ([1]) that a second-order dynamical system is self-adjoint, only if it is written in main form (12).

In what follows we consider a system of the form

$$(15)\qquad \begin{cases} F_a(t,x^h,\dot{x}^h,\ddot{x}^h) = 0 \\ F_\alpha(t,x^h,\dot{x}^h) = 0 \end{cases} \quad \left(\begin{matrix} a=\overline{1,r}\,, \quad \alpha = \overline{r+1,m} \\ h=\overline{1,m} \end{matrix} \right).$$

Then we have the following result

Theorem 3. *A differential system of the form (15) is self-adjoint if and only if it is of the form*

$$(16)\qquad \begin{cases} A_{ab}(t,x^h,\dot{x}^c)\ddot{x}^b + B_{a\beta}(t,x^h,\dot{x}^c)\ddot{x}^\beta + B_a(t,x^h,\dot{x}^c) = 0 \\ C_{\alpha\beta}(t,x^h)\dot{x}^\beta + D_\alpha(t,x^h,\dot{x}^c) = 0 \end{cases}$$

and satisfies the conditions

$$(17)\qquad \begin{cases} A_{ab} = A_{ba}\,, \quad \dfrac{\partial A_{ac}}{\partial \dot{x}^b} = \dfrac{\partial A_{bc}}{\partial \dot{x}^a} \\ \dfrac{\partial B_a}{\partial \dot{x}^b} + \dfrac{\partial B_b}{\partial \dot{x}^a} = 2\left(\dfrac{\partial}{\partial t} + \dot{x}^c\dfrac{\partial}{\partial x^c}\right) A_{ab} \\ \dfrac{\partial B_a}{\partial x^b} - \dfrac{\partial B_b}{\partial x^a} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial t} + \dot{x}^c\dfrac{\partial}{\partial x^c}\right)\left(\dfrac{\partial B_a}{\partial \dot{x}^b} - \dfrac{\partial B_b}{\partial \dot{x}^a}\right) \\ \dfrac{\partial B_{a\alpha}}{\partial \dot{x}^b} = \dfrac{\partial A_{ab}}{\partial \dot{x}^\alpha}\,, \quad B_{a\alpha} = -\dfrac{\partial D_\alpha}{\partial \dot{x}^a} \\ \dfrac{\partial B_a}{\partial x^\alpha} - \dfrac{\partial D_\alpha}{\partial x^a} = \left(\dfrac{\partial}{\partial t} + \dot{x}^c\dfrac{\partial}{\partial x^c}\right) B_{a\alpha} \\ C_{\alpha\beta} + C_{\beta\alpha} = 0\,, \quad \dfrac{\partial C_{\alpha\beta}}{\partial x^\gamma} + \dfrac{\partial C_{\beta\gamma}}{\partial x^\alpha} + \dfrac{\partial C_{\gamma\alpha}}{\partial x^\beta} = 0 \\ \dfrac{\partial C_{\alpha\beta}}{\partial t} + \dot{x}^a\dfrac{\partial C_{\alpha\beta}}{\partial x^a} = \dfrac{\partial D_\alpha}{\partial x^\beta} - \dfrac{\partial D_\beta}{\partial x^\alpha} \end{cases}$$

§5. Example

An illustrative example of self-adjoint degenerated system of the form (16) is

$$\left\{\begin{aligned} \ddot{x}^1 + k\dot{x}^2 - x^1 = 0 \\ -k\dot{x}^1 + x^2 = 0, \end{aligned}\right. \tag{18}$$

with the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^1)^2 + \tfrac{1}{2}kx^2\dot{x}^1 - \tfrac{1}{2}kx^1\dot{x}^2 + \tfrac{1}{2}(x^1)^2 - \tfrac{1}{2}(x^2)^2. \tag{19}$$

To obtain conserved quantities, we write the equations (9)

$$\left\{\begin{aligned} \tfrac{d^2\xi^1}{dt^2} + k\tfrac{d\xi^2}{dt} - \xi^1 = 0 \\ -k\tfrac{d\xi^1}{dt} + \xi^2 = 0, \end{aligned}\right. \tag{20}$$

and with a pair of solutions $\xi_u = (\xi_u^1, \xi_u^2)$ and $\xi_v = (\xi_v^1, \xi_v^2)$, on solutions to (18), we construct the conserved quantity of the form (10).

As shown in the first part of this paper, a solution is $\xi_1 = (\dot{x}^1, \dot{x}^2)$. Other solutions, on solutions of (18), are

$$\begin{aligned} \xi_2 &= (x^1, x^2) \\ \xi_3 &= \left(k_1 e^{\frac{1}{1+k^2}t}, \tfrac{k_1 k}{1+k^2} e^{\frac{1}{1+k^2}t}\right) \\ \xi_4 &= \left(k_2 e^{-\frac{1}{1+k^2}t}, -\tfrac{k_2 k}{1+k^2} e^{-\frac{1}{1+k^2}t}\right), \end{aligned}$$

where k_1 and k_2 are arbitrary constants.

For the following couples of solutions, we obtain respectively the conserved quantities:

$$\begin{aligned} &\xi_1 \text{ and } \xi_2 : && (\dot{x}^1)^2 - (x^1)^2 + (x^2)^2 = const. \\ &\xi_1 \text{ and } \xi_3 : && e^{\frac{1}{1+k^2}t}(\dot{x}^1 + kx^2 - x^1) = const. \\ &\xi_1 \text{ and } \xi_4 : && e^{-\frac{1}{1+k^2}t}(\dot{x}^1 + kx^2 + x^1) = const. \\ &\xi_2 \text{ and } \xi_3 : && e^{\frac{1}{1+k^2}t}(\dot{x}^1 + kx^2 - \sqrt{1+k^2}\, x^1) = const. \\ &\xi_2 \text{ and } \xi_4 : && e^{-\frac{1}{1+k^2}t}(\dot{x}^1 + kx^2 + \sqrt{1+k^2}\, x^1) = const. \end{aligned}$$

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