Linear transformation of N-connection in Osc^2M ♣

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Abstract

In the tangent space T(E) and in its dual T^*(E), two different adapted basis are introduced. The connections between connection coefficients are given. The conditions when some connection is $d-$connection in both basis are determined.

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§1. Different adapted bases in $T(Osc^2M)$

We define $E = Osc^2M$ as a $3n-$dimensional C^∞ real manifold, in which the transformations of form (1.1) are allowed. It is formed as a tangent space of order two of the $n-$dimensional base manifold M. In some local chart (U,φ), some point $u \in E$ has the coordinates

$$(x^a, y^{1a}, y^{2a}) = (y^{0a}, y^{1a}, y^{2a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = \overline{1,n}, \quad \alpha, \beta, \gamma, \delta, \ldots = \overline{0,2}.$$

The following abbreviations will be used:

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}}, \quad \alpha = \overline{0,2}.$$

If in some other chart (U', φ') the point $u \in E$ has the coordinates $(x^{a'}, y^{1a'}, y^{2a'})$, then in $U \cap U'$ the allowable coordinate transformations are given by

$$(1.1) \quad \begin{cases} x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n), \\ y^{1a'} = (\partial_a x^{a'}) y^{1a} = (\partial_{0a} y^{0a'}) y^{1a}, \\ y^{2a'} = (\partial_{0a} y^{1a'}) y^{1a} + (\partial_{1a} y^{1a'}) y^{2a}. \end{cases}$$

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Theorem 1.1. *The transformations of type (1.1) form a group.* One nice example of the space E can be obtained if the points $(x^a) \in M$, belong to a curve $x^a = x^a(t),\ t \in I$, and $y^{\alpha a},\ \alpha = 1,2$ are determined by

$$(1.2)\quad y^{\alpha a} = d_t^\alpha x^a, \quad d_t^\alpha = \tfrac{d^\alpha}{dt^\alpha}, \quad d_t = \tfrac{d}{dt}.$$

If in $U \cap U'$ the equation $x^{a'} = x^{a'}(x^1(t), x^2(t), \ldots, x^n(t))$ is valid, then it is easy to see that

$$(1.3)\quad y^{1a'} = d_t^1 x^{a'}, \quad y^{2a'} = d_t^2 x^{a'},$$

satisfy (1.1). From (1.2) and (1.3) it follows

$$y^{1a'} = y^{1a'}(x, y^{1a}), \quad y^{2a'} = y^{2a'}(x, y^{1a}, y^{2a}).$$

Let us introduce the notations:

$$(1.4)\quad A^{(0)a'}_{\ a} = \partial_a x^{a'}, A^{(\alpha)a'}_{\ a} = d_t^\alpha A^{(0)a'}_{\ a} = \tfrac{d^\alpha A^{(0)a'}_{\ a}}{dt^\alpha}, \quad \alpha = \overline{1,2}.$$

The natural base of T(E) and of $T^*(E)$ are respectively

$$(1.5)\quad \bar{\mathcal{B}} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}\}$$

and

$$(1.6)\quad \bar{\mathcal{B}}^* = \{dy^{0a}, dy^{1a}, dy^{2a}\}.$$

The elements of $\bar{\mathcal{B}}$ and $\bar{\mathcal{B}}^*$ are dual to each other, i.e.,

$$(1.7)\quad \langle dy^{\alpha a}, \partial_{\beta b}\rangle = \delta^\alpha_\beta \delta^a_b,$$

but with respect to (1.1) they have not a tensorial character.

The adapted basis $\mathcal{B}^*$ of $T^*(E)$ is given by

$$(1.8)\quad \mathcal{B}^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}\},$$

where

$$(1.9)\quad \begin{cases} \delta y^{0a} = dx^a = dy^{0a}, \\ \delta y^{1a} = dy^{1a} + M^{1\ a}_{0\ b} dy^{0b}, \\ \delta y^{2a} = dy^{2a} + M^{2\ a}_{1\ b} dy^{1b} + M^{2\ a}_{0\ b} dy^{0b}. \end{cases}$$

Theorem 1.2. *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as $d-$tensor fields, i.e.,*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{\alpha a}, \quad \alpha = \overline{0,2},$$

are that the following equations are satisfied

$$(1.10)\quad \begin{cases} M_{0\ b}^{1\ a}(\partial_a x^{b'}) = M_{0\ c'}^{1\ b'}(\partial_b x^{c'}) + \partial_b y^{1b'}, \\ M_{1\ b}^{2\ a}(\partial_a x^{b'}) = M_{1\ c'}^{2\ b'}(\partial_{1b} y^{1c'}) + \partial_{1b} y^{2b'}, \\ M_{0\ b}^{2\ a}(\partial_a x^{b'}) = M_{0\ c'}^{2\ b'}(\partial_b x^{c'}) + M_{1\ c'}^{2\ b'}(\partial_b y^{1c'}) + \partial_b y^{2b'}. \end{cases}$$

Proof. We have

$$\begin{aligned} \delta y^{0a'} &= dy^{0a'} = dx^{a'} = \frac{\partial x^{a'}}{\partial x^a} dx^a. \\ \delta y^{1a'} &= dy^{1a'} + M_{0\ b'}^{1\ a'} dy^{0b'} = (\partial_{0a} y^{1a'}) dy^{0a} + \partial_{1a} y^{1a'} dy^{1a} + M_{0\ b'}^{1\ a'} (\partial_{0a} y^{0b'}) dy^{0a} = \\ &= [M_{0\ b'}^{1\ a'}(\partial_a x^{b'}) + \partial_a y^{1a'}] dy^{0a} + (\partial_{1a} y^{1a'}) dy^{1a} = (\partial_a x^{a'}) \delta y^{1a} = \\ &= (\partial_a x^{a'})[dy^{1a} + M_{0\ b}^{1\ a} dy^{0b}] = (\partial_a x^{a'}) dy^{1a} + (\partial_c x^{a'}) M_{0\ a}^{1\ c} dy^{0a}. \end{aligned}$$

Then we infer $M_{0\ a}^{1\ c}(\partial_c x^{a'}) = M_{0\ b'}^{1\ a'}(\partial_a x^{b'}) + \partial_a y^{1a'}$, since $\partial_{1a} y^{1a'} = \partial_a x^{a'}$. □

Using the notations (1.4) the above formulae can be written in the form

$$(1.11)\quad \begin{cases} M_{0\ b}^{1\ a} A^{(0)b'}_{\ \ a} = M_{0\ c'}^{1\ b'} A^{(0)c'}_{\ \ b} + A^{(1)b'}_{\ \ b}, \\ M_{1\ b}^{2\ a} A^{(0)b'}_{\ \ a} = M_{1\ c'}^{2\ b'} A^{(0)c'}_{\ \ b} + A^{(1)b'}_{\ \ b}, \\ M_{0\ b}^{2\ a} A^{(0)b'}_{\ \ a} = M_{0\ c'}^{2\ b'} A^{(0)c'}_{\ \ b} + M_{1\ c'}^{2\ b'} A^{(1)c'}_{\ \ b} + A^{(2)b'}_{\ \ b}. \end{cases}$$

Let us denote the adapted basis of $T(E)$ by $\mathcal{B}$, where

$$(1.12)\quad \mathcal{B} = \{\delta_{oa}, \delta_{1a}, \delta_{2a}\} = \{\delta_{\alpha a}\}, \quad \alpha = \overline{0,2},$$

and

$$(1.13)\quad \begin{cases} \delta_{0a} = \partial_{oa} - N_{0\ a}^{1\ b}\partial_{1b} - N_{0\ a}^{2\ b}\partial_{2b} \\ \delta_{1a} = \qquad\qquad \partial_{1a} - N_{1\ a}^{2\ b}\partial_{2b} \\ \delta_{2a} = \qquad\qquad\qquad\qquad \partial_{2a}. \end{cases}$$

Theorem 1.3. *$\mathcal{B}$ is dual to $\mathcal{B}^*$, ($\langle\delta^{\beta b}, \delta_{\alpha a}\rangle = \delta^\beta_\alpha \delta^b_a$) if and only if the following relations hold* (1.14)

$$\begin{cases} N_{0\ a}^{1\ b} = M_{0\ a}^{1\ b} \\ N_{0\ a}^{2\ b} = M_{0\ a}^{2\ b} - M_{1\ c}^{2\ b} N_{0\ a}^{1\ c} \\ N_{1\ a}^{2\ b} = M_{1\ a}^{2\ b}. \end{cases}$$

Proof.

$$\begin{aligned} \langle\delta^{0b}, \delta_{0a}\rangle &= \langle dx^b, \partial_{0a} - N_{0a}^{1b}\partial_{1b} - N_{0a}^{2b}\partial_{2b}\rangle = \langle dx^b, \partial_a\rangle = \delta^b_a; \\ \langle\delta^{1b}, \delta_{1a}\rangle &= \langle dy^{1b} + M_{0\ c}^{1\ b} dy^{0c}, \partial_{1a} - N_{1a}^{2b}\partial_{2b}\rangle = \langle dy^{1b}, \partial_{1a}\rangle + \\ &\quad + M_{0\ c}^{1\ b}\langle dy^{0c}, \partial_{1a}\rangle - N_{1a}^{2b}\langle dy^{1b}, \partial_{2b}\rangle - M_{0\ c}^{1\ b} N_{1a}^{2b}\langle dy^{0c}, \partial_{2b}\rangle = \delta^1_1 \delta^b_a; \end{aligned}$$

$$\begin{aligned}\langle\delta^{2b},\delta_{1a}\rangle &= \langle dy^{2b}+M^{2b}_{1c}dy^{1c}+M^{2b}_{0c}dy^{0c},\partial_{1a}-N^{2b}_{1a}\partial_{2b}\rangle=\\ &=\langle dy^{2b},\partial_{1a}\rangle-N^{2b}_{1a}\langle dy^{2b},\partial_{2b}\rangle+M^{2b}_{1c}\langle dy^{1c},\partial_{1a}\rangle-\\ &-M^{2b}_{1c}N^{2b}_{1a}\langle dy^{1c},\partial_{2b}\rangle+M^{2b}_{0c}\langle dy^{0c},\partial_{1a}\rangle-M^{2b}_{0c}N^{2b}_{1a}\langle dy^{0c},\partial_{2b}\rangle=\\ &=-N^{2b}_{1a}+M^{2b}_{1c}\delta^1_1\delta^c_a=-N^{2b}_{1a}+M^{2b}_{1a}=0.\end{aligned}$$

We infer $N^{2b}_{1a}=M^{2b}_{1a}$. □

Theorem 1.4. *$\delta_{\alpha 0}$ transform with respect to (1.1) as d−tensors if and only if the following formulae hold*

$$(1.15)\quad\begin{cases} N^{1}_{0}{}^{b'}_{a'}A^{(0)a'}_{a}=N^{1}_{0}{}^{c}_{a}A^{(0)b'}_{c}-A^{(1)b'}_{a}\\ N^{2}_{1}{}^{b'}_{a'}A^{(0)a'}_{a}=N^{2}_{1}{}^{c}_{a}A^{(0)b'}_{c}-2A^{(1)b'}_{a}\\ N^{2}_{0}{}^{b'}_{a'}A^{(0)a'}_{a}=N^{2}_{0}{}^{c}_{a}A^{(0)b'}_{c}+2N^{1}_{0}{}^{c}_{a}A^{(1)b'}_{c}-A^{(2)b'}_{a}.\end{cases}$$

Theorem 1.5. *The basis vectors of $\bar{\mathcal{B}}$ are connected with the basis vectors of $\mathcal{B}$ by*

$$(1.16)\quad\begin{cases}\partial_{0a}=\delta_{oa}+M^{1}_{0}{}^{b}_{a}\delta_{1b}+M^{2}_{0}{}^{b}_{a}\delta_{2b}\\ \partial_{1a}=\delta_{1a}+M^{2}_{1}{}^{b}_{a}\delta_{2b}\\ \partial_{2a}=\delta_{2a}.\end{cases}$$

Proof. From (1.13) we have

$$\begin{cases}\partial_{0a}=\delta_{oa}+N^{1}_{0}{}^{b}_{a}\partial_{1b}+N^{2}_{0}{}^{b}_{a}\partial_{2b}\\ \partial_{1a}=\delta_{1a}+N^{2}_{1}{}^{b}_{a}\partial_{2b}\\ \partial_{2a}=\delta_{2a},\end{cases}$$

whence we infer

$$\begin{aligned}\partial_{0a}&=\delta_{0a}+N^{1}_{0}{}^{b}_{a}(\delta_{1b}+N^{2}_{1}{}^{c}_{b}\delta_{2c})+N^{2}_{0}{}^{b}_{a}\delta_{2b}=\\ &=\delta_{0a}+N^{1}_{0}{}^{b}_{a}\delta_{1b}+(N^{1}_{0}{}^{c}_{a}N^{2}_{1}{}^{b}_{c}+N^{2}_{0}{}^{b}_{a})\delta_{2b}=\\ &\overset{(1.14)}{=}\delta_{0a}+M^{1}_{0}{}^{b}_{a}\delta_{1b}+M^{2}_{0}{}^{b}_{a}\delta_{2b}.\end{aligned}$$

□

Theorem 1.6. *The basis vectors of $\bar{\mathcal{B}}^*$ are connected with the basis vectors of $\mathcal{B}^*$ by*

$$(1.17)\quad\begin{cases}dy^{0a}=\delta y^{0a}\\ dy^{1a}=\delta y^{1a}-N^{1}_{0}{}^{a}_{e}\delta y^{0e}\\ dy^{2a}=\delta y^{2a}-N^{2}_{1}{}^{a}_{e}\delta y^{1e}-N^{2}_{0}{}^{a}_{e}\delta y^{0e}.\end{cases}$$

Proof. From (1.9), using (1.14) it follows

$dy^{0a} = \delta y^{0a}$.

$$dy^{1a} = \delta y^{1a} - M_{0\ b}^{1\ a} dy^{0b} = \delta y^{1a} - M_{0\ b}^{1\ a}\delta y^{0b} \overset{(1.14)}{=} \delta y^{1a} - N_{0\ b}^{1\ a}\delta y^{0b}.$$

$$dy^{2a} = \delta y^{2a} - M_{1\ b}^{2\ a} dy^{1b} - M_{0\ b}^{2\ a} dy^{0b} \overset{(1.14)}{=} \delta y^{2a} - N_{1\ b}^{2\ a}[\delta y^{1b} - N_{0\ c}^{1\ b}\delta y^{0c}] -$$
$$- M_{0\ b}^{2\ a}\delta y^{0b} = \delta y^{2a} - N_{1\ b}^{2\ a}\delta y^{1b} - [M_{0\ b}^{2\ a} - N_{1\ c}^{2\ a} N_{0\ b}^{1\ c}]\delta y^{0b} =$$
$$= \delta y^{2a} - N_{1\ b}^{2\ a}\delta y^{1b} - N_{0\ b}^{2\ a}\delta y^{0b}.$$

□

§2. The metric tensor and the covariant derivation

Let us denote by T_H, T_{V_1}, T_{V_2} the subspaces of $T(E)$ generated by the sets of vectors $\{\delta_{0a}\}, \{\delta_{1a}\}, \{\delta_{2a}\}$ respectively. Then

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2}, \quad dim\ T(E) = 3n.$$

Any vector field X in $T(E)$ can be represented in the basis $\mathcal{B}$ in the form

$$(2.1)\quad X = X^{0a}\delta_{0a} + X^{1a}\delta_{1a} + X^{2a}\delta_{2a} = X^{\alpha a}\delta_{\alpha a}, \quad \alpha = \overline{0,2}.$$

Any form ω in the basis $\mathcal{B}^*$ decomposes

$$(2.2)\quad \omega = \omega_{0a}\delta y^{0a} + \omega_{1a}\delta y^{1a} + \omega_{2a}\delta y^{2a} = \omega_{\alpha a}\delta y^{\alpha a}, \quad \alpha = \overline{0,2}.$$

With respect to the coordinate transformation (1.1) we get

$$X^{\alpha a'} = X^{\alpha a}(\partial_a x^{a'}), \qquad \omega_{\alpha a} = \omega_{\alpha a'}(\partial_a x^{a'}), \quad \alpha = \overline{0,2}.$$

In the space $T^*(E) \otimes T^*(E)$, the metric tensor G can be given by

$$G \ = [\delta y^{0a}\ \delta y^{1a}\ \delta y^{2a}] \begin{bmatrix} g_{0a0b} & g_{0a1b} & g_{0a2b} \\ g_{1a0b} & g_{1a1b} & g_{1a2b} \\ g_{2a0b} & g_{2a1b} & g_{2a2b} \end{bmatrix} \begin{bmatrix} \delta y^{0b} \\ \delta y^{1b} \\ \delta y^{2b} \end{bmatrix} =$$
$$= g_{\alpha a\beta b}\delta y^{\alpha a} \otimes \delta^{\beta b}, \quad \alpha, \beta = \overline{0,2}.$$

For the components of the metric tensor we have

$$g_{\alpha a\beta b} = g_{\alpha a'\beta b'}(\partial_a x^{a'})(\partial_b x^{b'}), \quad \alpha, \beta = \overline{0,2}.$$

Definition 2.1. The *generalized connection*

$$\nabla : T(E) \otimes T(E) \to T(E), \qquad \nabla : (X, Y) \to \nabla_X Y,$$

or equivalently

$$\nabla_X : T(E) \to T(E), \qquad \nabla_X : Y \to \nabla_X Y$$

is a linear connection determined by

$$(2.4) \qquad \nabla_{\delta_{\beta b}} \delta_{\alpha a} = \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b} \delta_{\gamma c},$$

where the summation is going over γ and c. If in (2.4) we set $\gamma = \alpha$ this provides the so called $d-$connection

$$(2.5) \qquad \nabla_{\delta_{\beta b}} \delta_{\alpha a} = \Gamma_{\alpha a}{}^{\alpha c}{}_{\beta b} \delta_{\alpha c},$$

(with no summation over α). The explicit form of (2.4) is given by

$$(2.6) \qquad \begin{aligned} \nabla_{\delta_{0b}} \delta_{0a} &= \underline{\Gamma_{0a}{}^{0c}{}_{0b} \delta_{0c}} + \Gamma_{0a}{}^{1c}{}_{0b} \delta_{1c} + \Gamma_{0a}{}^{2c}{}_{0b} \delta_{2c}, \\ \nabla_{\delta_{0b}} \delta_{1a} &= \Gamma_{1a}{}^{0c}{}_{0b} \delta_{0c} + \underline{\Gamma_{1a}{}^{1c}{}_{0b} \delta_{1c}} + \Gamma_{1a}{}^{2c}{}_{0b} \delta_{2c}, \\ \nabla_{\delta_{0b}} \delta_{2a} &= \Gamma_{2a}{}^{0c}{}_{0b} \delta_{0c} + \Gamma_{2a}{}^{1c}{}_{0b} \delta_{1c} + \underline{\Gamma_{2a}{}^{2c}{}_{0b} \delta_{2c}}. \end{aligned}$$

If in the above formulae we substitute 0b with 1b, and then 0b with 2b, we obtain the complete list of 9 formulae. The underlined terms are $\Gamma_{0a}{}^{0c}{}_X, \Gamma_{1a}{}^{1c}{}_X, \Gamma_{2a}{}^{2c}{}_X$, where instead of X stays 1b or 2b.

Comparing (2.6) with (2.5), we see that if in (2.6) all terms are zero except the underlined ones, we obtain the explicit form of the so called $d-$connection defined by (2.5).

Proposition 2.1. *For the generalized connection ∇, defined by (2.4), and for arbitrary vector fields $X = X^{\alpha a} \delta_{\alpha a}$, $Y = Y^{\beta b} \delta_{\beta b}$, we have*

$$(2.7) \qquad \nabla_X Y = Y^{\beta b}{}_{|\alpha a} X^{\alpha a} \delta_{\beta b},$$

where

$$(2.8) \qquad Y^{\beta b}{}_{|\alpha a} = \delta_{\alpha a} Y^{\beta b} + \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} Y^{\gamma c},$$

where the summation goes over both types of indices (Latin and Greek).

Proof. By straightforward computation, we get

$$\begin{aligned} \nabla_X Y &= \nabla_{X^{\alpha a}\delta_{\alpha a}} Y^{\beta b} \delta_{\beta b} = X^{\alpha a} \nabla_{\delta_{\alpha a}} Y^{\beta b} \delta_{\beta b} = \\ &= X^{\alpha a} [\delta_{\alpha a} Y^{\beta b} \delta_{\beta b} + Y^{\beta b} \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} \delta_{\gamma c}] = \\ &= X^{\alpha a} [\delta_{\alpha a} Y^{\beta b} + Y^{\gamma c} \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a}] \delta_{\beta b} \overset{(2.8)}{=} X^{\alpha a} Y^{\beta b}{}_{|\alpha a} \delta_{\beta b}. \end{aligned}$$

□

Proposition 2.2. *If (y^{0a}, y^{1a}, y^{2a}) and $(y^{0a'}, y^{1a'}, y^{2a'})$ are two coordinate systems connected by (1.1) then*

(2.9) $\quad \nabla_{X'} Y' = \nabla_X Y$

if and only if

$$Y^{\beta b}{}_{|\alpha a} = Y^{\beta b'}{}_{|\alpha a'}(\partial_{b'} x^b)(\partial_a x^{a'}), \quad \text{for all} \;\; \alpha, \beta = \overline{0,2},$$

or, equivalently, if all $\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a}$ are transformed as $d-$tensors, i.e.,

$$\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a} \frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^{a'}}{\partial x^a},$$

except $\Gamma_{\beta c}{}^{\beta b}{}_{0a}$ (with no summation over β, $\beta = \overline{0,2}$), which have the transformation law:

$$\Gamma_{\beta c}{}^{\beta b}{}_{0a} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\beta c'}{}^{\beta b'}{}_{\alpha a'} \frac{\partial x^{c'}}{\partial x^c} + \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c},$$

for $\beta = \overline{0,2}$ [2].

§3. The torsion tensor of the generalized connection

The torsion tensor $T(X,Y)$ is defined as usual by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Theorem 3.1. *The torsion tensor of the generalized connection has the form*

$$T(X,Y) = T_{\beta b}{}^{\gamma c}{}_{\alpha a} X^{\alpha a} Y^{\beta b} \delta_{\gamma c},$$

where

$$T_{\beta b}{}^{\gamma c}{}_{\alpha a} = \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b} - K_{\alpha a}{}^{\gamma c}{}_{\beta b};$$
$$[\delta_{\alpha a}, \delta_{\beta b}] = K_{\alpha a}{}^{\gamma c}{}_{\beta b} \delta_{\gamma c}.$$

The following relations hold true

$$\begin{aligned}
[\delta_{\alpha a}, \delta_{\beta b}] X^{\alpha a} Y^{\beta b} &= [\delta_{0a}, \delta_{0b}] X^{0a} Y^{0b} + [\delta_{0a}, \delta_{1b}] X^{0a} Y^{1b} + [\delta_{0a}, \delta_{2b}] X^{0a} Y^{2b} + \\
&+ [\delta_{1a}, \delta_{0b}] X^{1a} Y^{0b} + [\delta_{1a}, \delta_{1b}] X^{1a} Y^{1b} + [\delta_{1a}, \delta_{2b}] X^{1a} Y^{2b} + \\
&+ [\delta_{2a}, \delta_{0b}] X^{2a} Y^{0b} + [\delta_{2a}, \delta_{1b}] X^{2a} Y^{1b} + [\delta_{2a}, \delta_{2b}] X^{2a} Y^{2b} = \\
&= (K_{0a}{}^{1c}{}_{0b} \delta_{1c} + K_{0a}{}^{2c}{}_{0b} \delta_{2c}) X^{0a} Y^{0b} + \\
&+ (K_{0a}{}^{1c}{}_{1b} \delta_{1c} + K_{0a}{}^{2c}{}_{1b} \delta_{2c})(X^{0a} Y^{1b} - X^{1b} Y^{0a}) + \\
&+ (K_{0a}{}^{1c}{}_{2b} \delta_{1c} + K_{0a}{}^{2c}{}_{1b} \delta_{2c})(X^{0a} Y^{2b} - X^{2b} Y^{0a}) + \\
&+ K_{1a}{}^{2c}{}_{1b} \delta_{2c} X^{1a} Y^{1b} + \\
&+ K_{1a}{}^{2c}{}_{2b} \delta_{2c} (X^{1a} Y^{2b} - X^{2b} Y^{1a}),
\end{aligned}$$

where

$$K_{0a}{}^{1c}{}_{0b} = \delta_{0b}N^{1}_{0}{}^{c}_{a} - \delta_{0a}N^{1}_{0}{}^{c}_{b}, \qquad K_{0a}{}^{2c}{}_{0b} = (\delta_{0b}N^{2}_{0}{}^{c}_{a} - \delta_{0a}N^{2}_{0}{}^{c}_{b}) + M^{2}_{1}{}^{c}_{e}K_{0a}{}^{1e}{}_{0b},$$
$$K_{0a}{}^{1c}{}_{1b} = \delta_{1b}N^{1}_{0}{}^{c}_{a}, \qquad K_{0a}{}^{2c}{}_{1b} = (\delta_{1b}N^{2}_{0}{}^{c}_{a} - \delta_{0a}N^{2}_{1}{}^{c}_{b}) + M^{2}_{1}{}^{c}_{d}K_{0a}{}^{1d}{}_{1b},$$
$$K_{0a}{}^{1c}{}_{2b} = \delta_{2b}N^{1}_{0}{}^{c}_{a}, \qquad K_{0a}{}^{2c}{}_{2b} = \delta_{2b}N^{2}_{0}{}^{c}_{a} + M^{2}_{1}{}^{c}_{d}K_{0a}{}^{1d}{}_{2b},$$
$$K_{1a}{}^{2c}{}_{1b} = \delta_{1b}N^{2}_{1}{}^{c}_{a} - \delta_{1a}N^{2}_{1}{}^{c}_{b}, \qquad K_{1a}{}^{2c}{}_{2b} = \delta_{2b}N^{2}_{1}{}^{c}_{a}.$$

Proposition 3.1. *$K_{0a}{}^{1c}{}_{0b}, K_{0a}{}^{2c}{}_{0b}, K_{1a}{}^{2c}{}_{1b}, K_{1a}{}^{2c}{}_{2b}$ are $d-$tensor fields, $K_{0a}{}^{1c}{}_{1b}$, $K_{0a}{}^{2c}{}_{1b}, K_{0a}{}^{1c}{}_{2b}, K_{0a}{}^{2c}{}_{2b}$ are not $d-$tensor fields, and they transform in the following way*

$$K_{0a}{}^{1c}{}_{1b} = K_{0a'}{}^{1c'}{}_{1b'}(\partial_a x^{a'})(\partial_b x^{b'})(\partial_{c'}x^c) + (\partial_a\partial_b x^{c'})(\partial_{c'}x^c),$$

and similar for the next two sets of coefficients.

§4. The transformation of the nonlinear connection

Assume that $\hat{\mathcal{B}} = \{\hat\delta_{0a}, \hat\delta_{1a}, \hat\delta_{2a}\}$ is another adapted basis of T(E), which is formed as $\mathcal{B}$ (1.13) but with N replaced with $\hat N$, which satisfies (1.15). Another adapted basis of $T^*(E)$ is $\hat{\mathcal{B}}^* = \{\hat\delta y^{0a}, \hat\delta y^{1a}, \hat\delta y^{2a}\}$ which is formed as $\mathcal{B}^*$ (1.9) but with M replaced with $\hat M$, which satisfy (1.10).

We introduce the notation

$$A^{\beta b}_{\alpha a} = N^{\beta b}_{\alpha a} - \hat N^{\beta b}_{\alpha a},\ B^{\beta b}_{\alpha a} = M^{\beta b}_{\alpha a} - \hat M^{\beta b}_{\alpha a}, \tag{4.1}$$

and remark that

$$\begin{aligned}\hat\delta_{0a} &= \partial_{0a} - \hat N^{1b}_{0a}\partial_{1b} - \hat N^{2b}_{0a}\partial_{2b} = \\ &= \partial_{0a} - (N^{1b}_{0a} - A^{1b}_{0a})\partial_{1b} - (N^{2b}_{0a} - A^{2b}_{0a})\partial_{2b},\end{aligned}$$

whence

$$\begin{cases} \hat\delta_{0a} = \delta_{oa} + A^{1}_{0}{}^{b}_{a}\partial_{1b} + A^{2}_{0}{}^{b}_{a}\partial_{2b} \\ \hat\delta_{1a} = \qquad\qquad \delta_{1a} + A^{2}_{1}{}^{b}_{a}\partial_{2b} \\ \hat\delta_{2a} = \qquad\qquad\qquad \delta_{2a} \end{cases} \tag{4.2}$$

$$\begin{cases} \hat\delta y^{0a} = \delta y^{0a} \\ \hat\delta y^{1a} = \delta y^{1a} - B^{1}_{0}{}^{a}_{b}dy^{0b} \\ \hat\delta y^{2a} = \delta y^{2a} - B^{2}_{1}{}^{a}_{b}dy^{1b} - B^{2}_{0}{}^{a}_{b}dy^{0b}. \end{cases} \tag{4.3}$$

As N and $\hat N$ transform as prescribed in (1.15), we have

$$\begin{cases} A^{1b'}_{0a'}(\partial_a x^{a'}) = A^{1c}_{0a}(\partial_c x^{b'}) \\ A^{2b'}_{1a'}(\partial_a x^{a'}) = A^{2c}_{1a}(\partial_c x^{b'}) \\ A^{2b'}_{0a'}(\partial_a x^{a'}) = A^{2c}_{0a}(\partial_c x^{b'}) + A^{1c}_{0a}(\partial_{0c}y^{1b'}), \end{cases} \tag{4.4}$$

which shows that A^{1b}_{0a} and A^{2c}_{1a} transform as $d-$tensors, but A^{2b}_{0a} is not a $d-$tensor.

In a similar way, using (1.11), we get

$$(4.5)\quad \begin{cases} B^{1a}_{0b}(\partial_a x^{b'}) = B^{1b'}_{0c'}(\partial_b x^{c'}) \\ B^{2a}_{1b}(\partial_a x^{b'}) = B^{2b'}_{1c'}(\partial_b x^{c'}) \\ B^{2a}_{0b}(\partial_a x^{b'}) = B^{2b'}_{0c'}(\partial_b x^{c'}) + B^{2b'}_{1c'}(\partial_{0b} y^{1c'}). \end{cases}$$

Then it follows that B^{1b}_{0a} and B^{2c}_{1a} transform as $d-$tensors, but B^{2b}_{0a} is not a $d-$tensor.
The substitution of (1.16) into (4.2) and (1.17) into (4.3) gives

$$(4.6)\quad \begin{cases} \hat{\delta}_{0a} = \delta_{oa} + A^{1b}_{0a}\delta_{1b} + (A^{1c}_{0a}M^{2b}_{1c} + A^{2b}_{0a})\delta_{2b} \\ \hat{\delta}_{1a} = \qquad \delta_{1a} \qquad + A^{2b}_{1a}\delta_{2b} \\ \hat{\delta}_{2a} = \qquad\qquad \delta_{2a} \end{cases}$$

$$(4.7)\quad \begin{cases} \hat{\delta}y^{0a} = \qquad\qquad \delta y^{0a} \\ \hat{\delta}y^{1a} = \qquad \delta y^{1a} \qquad - B^{1a}_{0b}\delta y^{0b} \\ \hat{\delta}y^{2a} = \delta y^{2a} - B^{2a}_{1b}\delta y^{1b} - (B^{2a}_{0b} - B^{2a}_{1c}N^{1c}_{0b})\delta y^{0b}. \end{cases}$$

Theorem 4.1. *The necessary and sufficient condition for the duality of basis* $\hat{\mathcal{B}} = \{\hat{\delta}_{0a}, \hat{\delta}_{1a}, \hat{\delta}_{2a}\}$ *and* $\hat{\mathcal{B}}^* = \{\hat{\delta}y^{0a}, \hat{\delta}y^{1a}, \hat{\delta}y^{2a}\}$ *of* $T(E)$ *and* $T^*(E)$ *respectively, are the following relations*

$$(4.8)\quad \begin{cases} A^{1a}_{0b} - B^{1a}_{0b} = 0 \\ A^{1c}_{0b}M^{2a}_{1c} + A^{2a}_{0b} - A^{1c}_{0b}B^{2a}_{1c} - (B^{2a}_{0b} - B^{2a}_{1c}N^{1c}_{0b}) = 0 \\ A^{2a}_{1b} - B^{2a}_{1b} = 0. \end{cases}$$

Proof. From $\langle\hat{\delta}_{\alpha a}, \hat{\delta}^{\beta b}\rangle = \delta^{\beta}_{\alpha}\delta^{b}_{a}$, it follows (4.8). □

Using (4.8) and the notation

$$(4.9)\quad \bar{A}^{2a}_{0b} = A^{1c}_{0b}M^{2a}_{1c} + A^{2a}_{0b},$$

(which defines a tensor), the equations (4.6) and (4.7) can be written in the form

$$(4.10)\quad \begin{cases} \hat{\delta}_{0a} = \delta_{oa} + A^{1b}_{0a}\delta_{1b} + \bar{A}^{2b}_{0a}\delta_{2b} \\ \hat{\delta}_{1a} = \qquad \delta_{1a} + A^{2b}_{1a}\delta_{2b} \\ \hat{\delta}_{2a} = \qquad\qquad \delta_{2a} \end{cases}$$

$$(4.11)\quad \begin{cases} \hat{\delta}y^{0a} = \qquad\qquad \delta y^{0a} \\ \hat{\delta}y^{1a} = \qquad \delta y^{1a} \qquad - A^{1a}_{0b}\delta y^{0b} \\ \hat{\delta}y^{2a} = \delta y^{2a} - A^{2a}_{1b}\delta y^{1b} - (\bar{A}^{2a}_{0b} - A^{1c}_{0b}A^{2a}_{1c})\delta y^{0b}. \end{cases}$$

§5. The connection coefficients in the new adapted basis

The generalized connection ∇ defined by (2.4) in the new basis $\hat{B}$ has the form

$$\nabla_{\hat{\delta}_{\beta b}}\hat{\delta}_{\alpha a} = \hat{\Gamma}_{\alpha a}{}^{\gamma c}{}_{\beta b}\hat{\delta}_{\gamma c}. \tag{5.1}$$

Using the linearity of ∇ and (2.6) we get

$$\nabla_{\hat{\delta}_{0b}}\hat{\delta}_{0a} = \hat{\Gamma}_{0a}{}^{0c}{}_{0b}\hat{\delta}_{0c} + \hat{\Gamma}_{0a}{}^{1c}{}_{0b}\hat{\delta}_{1c} + \hat{\Gamma}_{0a}{}^{2c}{}_{0b}\hat{\delta}_{2c}. \tag{5.2}$$

On the other side we have

$$\nabla_{\hat{\delta}_{0b}}\hat{\delta}_{0a} = \nabla_{(\delta_{0b}+A^{1c}_{0b}\delta_{1c}+\bar{A}^{2c}_{0b}\delta_{2c})}(\delta_{0a} + A^{1d}_{0a}\delta_{1d} + \bar{A}^{2d}_{0a}\delta_{2d}) \tag{5.3}$$

whence

$$\begin{aligned}
&\nabla_{\delta_{0b}}\delta_{0a} + A^{1c}_{0b}\nabla_{\delta_{1c}}\delta_{0a} + \bar{A}^{2c}_{0b}\nabla_{\delta_{2c}}\delta_{0a} + A^{1d}_{0a}\nabla_{\delta_{0b}}\delta_{1d} + (\delta_{0b}A^{1d}_{0a})\delta_{1d}+\\
&+A^{1c}_{0b}A^{1d}_{0a}\nabla_{\delta_{1c}}\delta_{1d} + A^{1c}_{0b}(\delta_{1c}A^{1d}_{0a})\delta_{1d} + \bar{A}^{2c}_{0b}A^{1d}_{0a}\nabla_{\delta_{2c}}\delta_{1d} + \bar{A}^{2c}_{0b}(\delta_{2c}A^{1d}_{0a})\delta_{1d}+\\
&+\bar{A}^{2d}_{0a}\nabla_{\delta_{0b}}\delta_{2d} + A^{1c}_{0b}\bar{A}^{2d}_{0a}\nabla_{\delta_{1c}}\delta_{2d} + A^{1c}_{0b}(\delta_{1c}\bar{A}^{2d}_{0a})\delta_{2d} + \bar{A}^{2c}_{0b}\bar{A}^{2d}_{0a}\nabla_{\delta_{2c}}\delta_{2d}+\\
&+\bar{A}^{2c}_{0b}(\delta_{2c}\bar{A}^{2d}_{0a})\delta_{2d} = \Gamma_{0a}{}^{\gamma e}{}_{0b}\delta_{\gamma e} + A^{1c}_{0b}\Gamma_{0a}{}^{\gamma e}{}_{1c}\delta_{\gamma e} + \bar{A}^{2c}_{0b}\Gamma_{0a}{}^{\gamma e}{}_{2c}\delta_{\gamma e}+\\
&+A^{1d}_{0a}\Gamma_{1d}{}^{\gamma e}{}_{0b}\delta_{\gamma e} + \delta_{0b}A^{1e}_{0a}\delta_{1e} + A^{1c}_{0b}A^{1d}_{0a}\Gamma_{1d}{}^{\gamma e}{}_{1c}\delta_{\gamma e} + A^{1c}_{0b}(\delta_{1c}A^{1e}_{0a})\delta_{1e} + \bar{A}^{2c}_{0b}A^{1d}_{0a}\Gamma_{1d}{}^{\gamma e}{}_{2c}\delta_{\gamma e}+\\
&+\bar{A}^{2c}_{0b}(\delta_{2c}A^{1d}_{0a})\delta_{1d} + \bar{A}^{2d}_{0a}\Gamma_{2d}{}^{\gamma e}{}_{0b}\delta_{\gamma e} + A^{1c}_{0b}\bar{A}^{2d}_{0a}\Gamma_{2d}{}^{\gamma e}{}_{1c}\delta_{\gamma e} + A^{1c}_{0b}(\delta_{1c}\bar{A}^{2e}_{0a})\delta_{2e}+\\
&+\bar{A}^{2c}_{0b}\bar{A}^{2d}_{0a}\Gamma_{2d}{}^{\gamma e}{}_{2c}\delta_{\gamma e} + \bar{A}^{2c}_{0b}(\delta_{2c}\bar{A}^{2e}_{0a})\delta_{2e} = \hat{\Gamma}_{0a}{}^{\gamma e}{}_{0b}\hat{\delta}_{\gamma e} = \hat{\Gamma}_{0a}{}^{0e}{}_{0b}\delta_{0e} + A^{1e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{0b}\delta_{1e}+\\
&+\bar{A}^{2e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{0b}\delta_{2e} + \hat{\Gamma}_{0a}{}^{1e}{}_{0b}\delta_{1e} + A^{2e}_{1d}\hat{\Gamma}_{0a}{}^{1d}{}_{0b}\delta_{2e} + \hat{\Gamma}_{0a}{}^{2e}{}_{0b}\delta_{2e} = \hat{\Gamma}_{0a}{}^{0e}{}_{0b}\delta_{0e}+\\
&+(\hat{\Gamma}_{0a}{}^{1e}{}_{0b} + A^{1e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{0b})\delta_{1e} + (\hat{\Gamma}_{0a}{}^{2e}{}_{0b} + A^{2e}_{1d}\hat{\Gamma}_{0a}{}^{1d}{}_{0b} + \bar{A}^{2e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{0b})\delta_{2e}.
\end{aligned}$$

Using the notations

$$\bar{\Gamma}_{\delta d}{}^{\gamma e}{}_{0b} = \Gamma_{\delta d}{}^{\gamma e}{}_{0b} + A^{1c}_{0b}\hat{\Gamma}_{\delta d}{}^{\gamma e}{}_{1c} + \bar{A}^{2c}_{0b}\Gamma_{\delta d}{}^{\gamma e}{}_{2c} \tag{5.4}$$

and

$$\bar{\Gamma}_{\delta d}{}^{\gamma e}{}_{1b} = \Gamma_{\delta d}{}^{\gamma e}{}_{1b} + A^{2c}_{1b}\Gamma_{\delta d}{}^{\gamma e}{}_{2c} \tag{5.5}$$

from (5.2), (5.3), (4.9) we get

$$\begin{cases}
\hat{\Gamma}_{0a}{}^{0e}{}_{0b} = \bar{\Gamma}_{0a}{}^{0e}{}_{0b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{0e}{}_{0b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{0e}{}_{0b}\\
\hat{\Gamma}_{0a}{}^{1e}{}_{0b} + A^{1e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{0b} = \delta_{0b}A^{1e}_{0a} + \bar{\Gamma}_{0a}{}^{1e}{}_{0b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{1e}{}_{0b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{1e}{}_{0b}\\
\hat{\Gamma}_{0a}{}^{2e}{}_{0b} + A^{2e}_{1c}\hat{\Gamma}_{0a}{}^{1c}{}_{0b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{0a}{}^{0c}{}_{0b} =\\
\qquad\qquad \delta_{0b}\bar{A}^{2e}_{0a} + \bar{\Gamma}_{0a}{}^{2e}{}_{0b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{2e}{}_{0b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{2e}{}_{0b}.
\end{cases} \tag{5.6}$$

Using the same process, from

$$\nabla_{\hat{\delta}_{0b}}\hat{\delta}_{1a} = \hat{\Gamma}_{1a}{}^{\alpha e}{}_{0b}\hat{\delta}_{\alpha e} = \nabla_{(\delta_{0b}+A^{1c}_{0b}\delta_{1c}+\bar{A}^{2c}_{0b}\delta_{2c})}(\delta_{1a} + A^{2d}_{1a}\delta_{2d})$$

we obtain

$$(5.7)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{0b} = \bar{\Gamma}_{1a}{}^{0e}{}_{0b} + A^{2d}_{1a}\bar{\Gamma}_{2d}{}^{0e}{}_{0b} \\ \hat{\Gamma}_{1a}{}^{1e}{}_{0b} + A^{1e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{0b} = \bar{\Gamma}_{1a}{}^{1e}{}_{0b} + A^{1e}_{2d}\bar{\Gamma}_{1a}{}^{2d}{}_{0b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{0b} + A^{2e}_{1c}\hat{\Gamma}_{1a}{}^{1c}{}_{0b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{0b} = \delta_{0b}A^{2e}_{1a} + \bar{\Gamma}_{1a}{}^{2e}{}_{0b} + A^{2d}_{1a}\bar{\Gamma}_{2d}{}^{2e}{}_{0b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{1b}}\hat{\delta}_{0a}$, we get

$$(5.8)\quad \begin{cases} \hat{\Gamma}_{0a}{}^{0e}{}_{1b} = \bar{\Gamma}_{0a}{}^{0e}{}_{1b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{0e}{}_{1b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{0e}{}_{1b} \\ \hat{\Gamma}_{0a}{}^{1e}{}_{1b} + A^{1e}_{0c}\hat{\Gamma}_{0a}{}^{0c}{}_{1b} = \delta_{1b}A^{1e}_{0a} + \bar{\Gamma}_{0a}{}^{1e}{}_{1b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{1e}{}_{1b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{1e}{}_{1b} \\ \hat{\Gamma}_{0a}{}^{2e}{}_{1b} + A^{2e}_{1c}\hat{\Gamma}_{0a}{}^{1c}{}_{1b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{0a}{}^{0c}{}_{1b} = \\ \qquad \delta_{1b}\bar{A}^{2e}_{0a} + \bar{\Gamma}_{0a}{}^{2e}{}_{1b} + A^{1d}_{0a}\bar{\Gamma}_{1d}{}^{2e}{}_{1b} + \bar{A}^{2d}_{0a}\bar{\Gamma}_{2d}{}^{2e}{}_{1b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{0b}}\hat{\delta}_{2a}$, we have

$$(5.9)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{0b} = \bar{\Gamma}_{2a}{}^{0e}{}_{0b} \\ \hat{\Gamma}_{2a}{}^{1e}{}_{0b} + A^{1e}_{0d}\hat{\Gamma}_{2a}{}^{0d}{}_{0b} = \bar{\Gamma}_{2a}{}^{1e}{}_{0b} \\ \hat{\Gamma}_{2a}{}^{2e}{}_{0b} + A^{2e}_{1d}\hat{\Gamma}_{2a}{}^{1d}{}_{0b} + \bar{A}^{2e}_{0d}\hat{\Gamma}_{2a}{}^{0d}{}_{0b} = \bar{\Gamma}_{2a}{}^{2e}{}_{0b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{2b}}\hat{\delta}_{0a}$, we obtain

$$(5.10)\quad \begin{cases} \hat{\Gamma}_{0a}{}^{0e}{}_{2b} = \Gamma_{0a}{}^{0e}{}_{2b} + A^{1d}_{0b}\Gamma_{1d}{}^{0e}{}_{2e} + \bar{A}^{2d}_{0e}\Gamma_{2d}{}^{0e}{}_{2b} \\ \hat{\Gamma}_{0a}{}^{1e}{}_{2b} + A^{1e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{2b} = \delta_{2b}A^{1e}_{0a} + \Gamma_{0a}{}^{1e}{}_{2b} + A^{1d}_{0a}\Gamma_{1d}{}^{1e}{}_{2b} + \bar{A}^{2d}_{0a}\Gamma_{2d}{}^{1e}{}_{2b} \\ \hat{\Gamma}_{0a}{}^{2e}{}_{2b} + A^{2e}_{1d}\hat{\Gamma}_{0a}{}^{1d}{}_{2b} + \bar{A}^{2e}_{0d}\hat{\Gamma}_{0a}{}^{0d}{}_{2b} = \\ \qquad \delta_{2b}\bar{A}^{2e}_{0a} + \Gamma_{0a}{}^{2e}{}_{2b} + A^{1d}_{0a}\Gamma_{1d}{}^{2e}{}_{2b} + \bar{A}^{2d}_{0a}\Gamma_{2d}{}^{2e}{}_{2b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{1b}}\hat{\delta}_{1a}$, we get

$$(5.11)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{1b} = \bar{\Gamma}_{1a}{}^{0e}{}_{1b} + A^{2d}_{1a}\bar{\Gamma}_{2d}{}^{0e}{}_{1b} \\ \hat{\Gamma}_{1a}{}^{1e}{}_{1b} + A^{1e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{1b} = \bar{\Gamma}_{1a}{}^{1e}{}_{1b} + A^{2d}_{1a}\bar{\Gamma}_{2d}{}^{1e}{}_{1b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{1b} + A^{2e}_{1c}\hat{\Gamma}_{1a}{}^{1c}{}_{1b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{1b} = \delta_{1b}\bar{A}^{2e}_{1a} + \bar{\Gamma}_{1a}{}^{2e}{}_{1b} + A^{2d}_{1a}\bar{\Gamma}_{2d}{}^{2e}{}_{1b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{2b}}\hat{\delta}_{1a}$, we have

$$(5.12)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{2b} = \Gamma_{1a}{}^{0e}{}_{2b} + A^{2d}_{1a}\Gamma_{2d}{}^{0e}{}_{2b} \\ \hat{\Gamma}_{1a}{}^{1e}{}_{2b} + A^{1e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{2b} = \Gamma_{1a}{}^{1e}{}_{2b} + A^{2d}_{1a}\Gamma_{2d}{}^{1e}{}_{2b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{2b} + A^{2e}_{1c}\hat{\Gamma}_{1a}{}^{1c}{}_{2b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{1a}{}^{0c}{}_{2b} = \delta_{2b}A^{2e}_{1a} + \Gamma_{1a}{}^{2e}{}_{2b} + A^{2d}_{1a}\Gamma_{2d}{}^{2e}{}_{2b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{1b}}\hat{\delta}_{2a}$, we get

$$(5.13)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{1b} = \bar{\Gamma}_{2a}{}^{0e}{}_{1b}, \qquad \hat{\Gamma}_{2a}{}^{1e}{}_{1b} + A^{1e}_{0c}\hat{\Gamma}_{2a}{}^{0c}{}_{1b} = \bar{\Gamma}_{2a}{}^{1e}{}_{1b}, \\ \hat{\Gamma}_{2a}{}^{2e}{}_{1b} + A^{2e}_{1c}\hat{\Gamma}_{2a}{}^{1c}{}_{1b} + \bar{A}^{2e}_{0c}\hat{\Gamma}_{2a}{}^{0c}{}_{1b} = \bar{\Gamma}_{2a}{}^{2e}{}_{1b}. \end{cases}$$

From the explicit form of $\nabla_{\hat{\delta}_{2b}}\hat{\delta}_{2a}$, we have

$$(5.14)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{2b} = \Gamma_{2a}{}^{0e}{}_{2b} \\ \hat{\Gamma}_{2a}{}^{1e}{}_{2b} = \Gamma_{2a}{}^{1e}{}_{2b} \\ \hat{\Gamma}_{2a}{}^{2e}{}_{2b} = \Gamma_{2a}{}^{2e}{}_{2b}. \end{cases}$$

Theorem 5.1. *If $\bigtriangledown$ is a $d-$connection in the adapted basis $\mathcal{B} = \{\delta_{0a}, \delta_{1a}, \delta_{2a}\}$, then the coefficients of the same connection $\bigtriangledown$ in the basis $\hat{\mathcal{B}}$ have the form*

$$(5.15)\quad \begin{cases} \hat{\Gamma}_{0a}{}^{0e}{}_{0b} = \bar{\Gamma}_{0a}{}^{0e}{}_{0b} \\ \hat{\Gamma}_{0a}{}^{1e}{}_{0b} + A_{0d}^{1e}\hat{\Gamma}_{0a}{}^{0d}{}_{0b} = \delta_{0b}A_{0a}^{1e} + A_{0a}^{1d}\bar{\Gamma}_{1d}{}^{1e}{}_{0b} \\ \hat{\Gamma}_{0a}{}^{2e}{}_{0b} + A_{1c}^{2e}\hat{\Gamma}_{0a}{}^{1c}{}_{0b} + \bar{A}_{0c}^{2e}\hat{\Gamma}_{0a}{}^{0c}{}_{0b} = \delta_{0b}\bar{A}_{0a}^{2e} + \bar{A}_{0a}^{2d}\bar{\Gamma}_{2d}{}^{2e}{}_{0b}. \end{cases}$$

$$(5.16)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{0b} = 0, \qquad \hat{\Gamma}_{1a}{}^{1e}{}_{0b} = \bar{\Gamma}_{1a}{}^{1e}{}_{0b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{0b} + A_{1c}^{2e}\hat{\Gamma}_{1a}{}^{1c}{}_{0b} + \bar{A}_{0c}^{2e}\hat{\Gamma}_{1a}{}^{0c}{}_{0b} = \delta_{0b}A_{1a}^{2e} + A_{1a}^{2d}\bar{\Gamma}_{2d}{}^{2e}{}_{0b}. \end{cases}$$

$$(5.17)\quad \begin{cases} \hat{\Gamma}_{0a}{}^{0e}{}_{1b} = \bar{\Gamma}_{0a}{}^{0e}{}_{1b} \\ \hat{\Gamma}_{0a}{}^{1e}{}_{1b} + A_{0c}^{1e}\hat{\Gamma}_{0a}{}^{0c}{}_{1b} = \delta_{1b}A_{0a}^{1e} + A_{0a}^{1d}\bar{\Gamma}_{1d}{}^{1e}{}_{1b} \\ \hat{\Gamma}_{0a}{}^{2e}{}_{1b} + A_{1c}^{2e}\hat{\Gamma}_{0a}{}^{1c}{}_{1b} + \bar{A}_{0c}^{2e}\bar{\Gamma}_{0a}{}^{0c}{}_{1b} = \delta_{1b}\bar{A}_{0a}^{2e} + \bar{A}_{oa}^{2d}\bar{\Gamma}_{2d}{}^{2e}{}_{1b}. \end{cases}$$

$$(5.18)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{0b} = 0, \qquad \hat{\Gamma}_{2a}{}^{1e}{}_{0b} = 0 \\ \hat{\Gamma}_{2a}{}^{2e}{}_{0b} = \bar{\Gamma}_{2a}{}^{2e}{}_{0b}. \end{cases}$$

$$(5.19)\quad \begin{cases} \hat{\Gamma}_{0a}{}^{0e}{}_{2b} = \Gamma_{0a}{}^{0e}{}_{2b} \\ \hat{\Gamma}_{0a}{}^{1e}{}_{2b} + A_{0d}^{1e}\hat{\Gamma}_{0a}{}^{0d}{}_{2b} = \delta_{2b}A_{0a}^{1e} + A_{0a}^{1d}\Gamma_{1d}{}^{1e}{}_{2b} \\ \hat{\Gamma}_{0a}{}^{2e}{}_{2b} + A_{1d}^{2e}\hat{\Gamma}_{0a}{}^{1d}{}_{2b} + \bar{A}_{0d}^{2e}\hat{\Gamma}_{0a}{}^{0d}{}_{2b} = \delta_{2b}\bar{A}_{0a}^{2e} + \bar{A}_{0a}^{2d}\Gamma_{2d}{}^{2e}{}_{2b}. \end{cases}$$

$$(5.20)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{1b} = 0, \qquad \hat{\Gamma}_{1a}{}^{1e}{}_{1b} = \bar{\Gamma}_{1a}{}^{1e}{}_{1b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{1b} + A_{1c}^{2e}\hat{\Gamma}_{1a}{}^{1c}{}_{1b} = \delta_{1b}A_{1a}^{2e} + A_{1a}^{2d}\bar{\Gamma}_{2d}{}^{2e}{}_{1b}. \end{cases}$$

$$(5.21)\quad \begin{cases} \hat{\Gamma}_{1a}{}^{0e}{}_{2b} = 0, \qquad \hat{\Gamma}_{1a}{}^{1e}{}_{2b} = \Gamma_{1a}{}^{1e}{}_{2b} \\ \hat{\Gamma}_{1a}{}^{2e}{}_{2b} + A_{1c}^{2e}\hat{\Gamma}_{1a}{}^{1c}{}_{2b} = \delta_{2b}A_{1a}^{2e} + A_{1a}^{2d}\Gamma_{2d}{}^{2e}{}_{2b}. \end{cases}$$

$$(5.22)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{1b} = 0, \qquad \hat{\Gamma}_{2a}{}^{1e}{}_{1b} = 0 \\ \hat{\Gamma}_{2a}{}^{2e}{}_{1b} = \bar{\Gamma}_{2a}{}^{2e}{}_{1b}. \end{cases}$$

$$(5.23)\quad \begin{cases} \hat{\Gamma}_{2a}{}^{0e}{}_{2b} = 0, \qquad \hat{\Gamma}_{2a}{}^{1e}{}_{2b} = 0 \\ \hat{\Gamma}_{2a}{}^{2e}{}_{2b} = \Gamma_{2a}{}^{2e}{}_{2b}. \end{cases}$$

Up till now the adapted basis $\mathcal{B}$ and $\hat{\mathcal{B}}$ was formed with arbitrary N and $\hat{N}$ which satisfy the prescribed transformation laws (1.15). The relation between their coefficients is given by (4.1). The transformation law of A_{0a}^{1b}, A_{1a}^{2b} and $\bar{A}_{0a}^{2b}$ is given by (4.4) and (4.9) and these are $d-$tensors. The following theorem gives the conditions when the $d-$connection in the basis $\mathcal{B}$ will be also a $d-$connection in the basis $\hat{\mathcal{B}}$.

Theorem 5.2. *The necessary and sufficient conditions that the $d-$connection $\bigtriangledown$ in the basis $\mathcal{B}$ be also a $d-$connection in the basis $\hat{\mathcal{B}}$ are*

(5.24) $\hat{\Gamma}_{0a}{}^{0e}{}_{0b} = \bar{\Gamma}_{0a}{}^{0e}{}_{0b}, \quad A^{1e}_{0a\hat{|}0b} = 0, \quad \bar{A}^{2e}_{0a\hat{|}0b} = 0.$

(5.25) $\hat{\Gamma}_{1a}{}^{1e}{}_{0b} = \bar{\Gamma}_{1a}{}^{1e}{}_{0b}, \quad A^{1e}_{0a\hat{|}1b} = 0, \quad \bar{A}^{2e}_{0a\hat{|}1b} = 0.$

(5.26) $\hat{\Gamma}_{0a}{}^{0e}{}_{1b} = \bar{\Gamma}_{0a}{}^{0e}{}_{1b}, \quad A^{1e}_{0a\hat{|}1b} = 0, \quad \bar{A}^{2e}_{0a\hat{|}1b} = 0.$

(5.27) $\hat{\Gamma}_{2a}{}^{2e}{}_{0b} = \bar{\Gamma}_{2a}{}^{2e}{}_{0b}.$

(5.28) $\hat{\Gamma}_{0a}{}^{0e}{}_{2b} = \bar{\Gamma}_{0a}{}^{0e}{}_{2b}, \quad A^{1e}_{0a\hat{|}2b} = 0, \quad \bar{A}^{2e}_{0a\hat{|}2b} = 0.$

(5.29) $\hat{\Gamma}_{1a}{}^{1e}{}_{1b} = \bar{\Gamma}_{1a}{}^{1e}{}_{1b}, \quad A^{2e}_{1a\hat{|}1b} = 0.$

(5.30) $\hat{\Gamma}_{1a}{}^{1e}{}_{2b} = \bar{\Gamma}_{1a}{}^{1e}{}_{2b}, \quad A^{2e}_{1a\hat{|}2b} = 0.$

(5.31) $\hat{\Gamma}_{2a}{}^{2e}{}_{1b} = \bar{\Gamma}_{2a}{}^{2e}{}_{1b}.$

(5.32) $\hat{\Gamma}_{2a}{}^{2e}{}_{2b} = \Gamma_{2a}{}^{2e}{}_{2b},$

and all other coefficients $\hat{\Gamma}$ are equal to zero.

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