On the Unitary Binary Group♣

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Abstract

One consider the variety of the unitary binary group in n variables and it is shown that this algebraic variety is rational and it has n^2 parameters. Then it is given a parametrical rational representation of this variety. We have established some one parameter subgroups of the unitary binary group. Besides we showed that these subgroups are geodesics of the pseudo-Riemannian space V_{n^2}.

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§1. Introduction

It is known that the unitary complex group $U(n)$ is formed by the complex matrices A of n-th order which satisfy the relation $AA^* = E_n$, where A^* is the adjoint of A and E_n is the identity matrix of n-th order. This group is isomorphic with the symplectic orthogonal group in $2n$ variables formed by the real matrices B of $2n$-th order which satisfy the relations:

$$\begin{cases} B\,\tilde{B} = E_{2n} \\ B\,I = I\,B \end{cases}$$

where $\tilde{B}$ is the transpose of B, E_{2n} is the identity matrix of $2n$-th order and I is the matrix

$$\left\| \begin{array}{cccc} I_2 & 0 & \dots & 0 \\ 0 & I_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_2 \end{array} \right\| \qquad \text{where} \quad I_2 = \left\| \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right\|.$$

The purpose of this speach is to show that the unitary binary group $\beta(n)$ formed by the matrices A of n-th order which elements belong to the algebra of binary numbers

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which satisfy the relation $A\,A^* = E_n$, with A^* the adjoint of A, is isomorphic with the group of real matrices B of $2n$-th order which fulfill the relations $B\,I\tilde{B} = I$, $BJ = JB$, where

$$I = \left\| \begin{matrix} I_2 & 0 & 0 & \dots & 0 \\ 0 & I_2 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_2 \end{matrix} \right\| \qquad J = \left\| \begin{matrix} J_2 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & J_2 \end{matrix} \right\|$$

with

$$I_2 = \left\| \begin{matrix} 0 & 1 \\ -1 & 0. \end{matrix} \right\| \quad J_2 = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|.$$

Then we prove that this algebraic variety is rational, finding effectively a rational parametrical representation and proving that it depends on n^2 parameters.

Finally we shall give some geometrical applications.

§2. The variety of the unitary binary group

The algebra of binary numbers $\mathbf{A}$ is a commutative algebra of order two over the field $\mathbf{R}$ of real numbers generated by two elements $1, e$ that satisfy the relations:

$$1^2 = e^2 = 1 \quad 1 \cdot e = e \cdot 1 = e.$$

An element $x \in \mathbf{A}$ of the algebra $\mathbf{A}$ is like $x = a + e\,b$, with $a, b \in \mathbf{R}$ and the conjugate of x is by definition $\bar{x} = a - eb$.

The set U_2 of real matrices of order two of the form $\left\| \begin{matrix} a & b \\ b & a \end{matrix} \right\|$ is an algebra over the field $\mathbf{R}$ of real numbers.

It is known that the function $f : \mathbf{A} \to U_2$ defined by $f(a + eb) = \left\| \begin{matrix} a & b \\ b & a \end{matrix} \right\|$ is an isomorphism of algebras (Rosenfeld).

The unitary binary group $\beta(n)$ is formed by the matrices A of n-th order with elements form the algebra $\mathbf{A}$, which fulfills the relations $AA^* = E_n$, and the multiplication of matrices is a composition law. (A^* is the adjoint of A and E_n is the identity matrix of n-th order).

Let $SO(2n)$ be the group formed by the real matrices B of $2n$-th order, which satisfy the relations $BI\tilde{B} = I$, $BJ = JB$, where $\tilde{B}$ is the transpose of B and I, J are the matrices of $2n$-th order that satisfy the relation $IJ = -JI$.

$$I = \begin{Vmatrix} I_2 & 0 & 0 & \dots & 0 \\ 0 & I_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_2 \end{Vmatrix} \qquad J = \begin{Vmatrix} J_2 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & J_2 \end{Vmatrix}$$

with

$$I_2 = \begin{Vmatrix} 0 & 1 \\ -1 & 0. \end{Vmatrix} \qquad J_2 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}.$$

Theorem 1. *The groups $\beta(n)$ and $SO(2n)$ are isomorphic.*

We have shown by the computation of the number of independent relations that the number of parameters of the variety of the unitary binary group is n^2. The next result will give a parametrical rational representation of the unitary binary group.

Theorem 2. *If X is a symmetric matrix and $XJ = -JX$, then the variety of the unitary binary group V_{n^2} formed by the matrices A with*

$$\begin{cases} AI\tilde{A} = I \\ AJ = JA \end{cases}$$

admits a rational parametrical representation of the form $A = (I + X)^{-1}(I - X)$ in the neighbourhood determined by the condition $|I + X| \neq 0$.

§3. Some geometrical applications

The vectorial space $\mathbf{R}_{2n\times 2n}$ of real matrices of $2n$-th order endowed with the pseudo-scalar product (indefinite bilinear symmetric form),

$$< P, Q >= -\frac{1}{2}Tr(IPI\tilde{Q})$$

where $Tr(A)$ represents the trace of the matrix A, is a pseudo-Euclidian space.

The metric of this space is

$$ds^2 = -\frac{1}{2}Tr(IdPId\tilde{P})$$

where dP is the differential of the matrix P.

The path $B : \mathbf{R} \to V_{n^2}$ of pseudo-Riemannian space V_{n^2} defined by $B(t) = e^{tX}$, where X is an antisymmetric matrix of $2n$-th order ($\tilde{X} = -X$) which satisfies the relations

$$\begin{cases} XI = IX \\ XJ = JX \end{cases}$$

where I and J are matrices introduced above, is an one parameter subgroup of the unitary binary group $\beta(n)$.

For proving this result we showed that the equalities

$$\begin{cases} B(t)I\tilde{B}(t) = I \\ B(t)J = JB(t) \end{cases}$$

are fulfilled.

Theorem 3. *The one parameter subgroups of the unitary binary group are geodesics of the pseudo-Riemannian space* V_{n^2}.

For showing that the path $B(t) = e^{tX}$, $t \in \mathbf{R}$, is geodesic in the space $\beta(n)$, we proved that the vector $B''(t)$ is orthogonal to every vector $B'(t)$ which satisfies the equality $B'(t)I\tilde{B}(t) + B(t)I\tilde{B}'(t) = 0$.

Some one parameter subgroups of the orthogonal group in n variables $O(n)$ were established by K. Teleman showing that these subgroups are geodesics of the Riemannian space $O(n)$.

E. Grecu has established some one parameter subgroups of the semi-pseudo-orthogonal group $O^{r,t}(p)$ in p variables and of the semi-orthogonal group $O^r(p)$, proving that these subgroups are geodesics of the semi-Riemannian spaces $O^{r,t}(p)$ respectively $O^r(p)$.

A more detailed work will be found in [2].

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