

Dynamical Systems with Extreme Integrals and Conservation Laws *

Mihaela Neamțu

Abstract

The purpose of this paper is to give conservation laws for a maximizing problem with a finite number of restrictions. Generalizations of some results due to Mimura and Nôno are obtained. An example to illustrate the obtained results is given.

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§1. Conservation Laws for Euler-Lagrange Systems

We consider a system of ordinary implicit second-order differential equations

$$(1) \quad F_\alpha(t, x^\beta, \dot{x}^\beta, \ddot{x}^\beta) = 0, \quad \alpha = \overline{1, n},$$

with $x^\beta = x^\beta(t)$, $\beta = \overline{1, n}$.

We recall the following results [2]:

Theorem 1. *If $\xi^\alpha(t, x, \dot{x})$ and $\eta^\alpha(t, x, \dot{x})$ are two sets of functions which verify the equations*

$$(2) \quad \frac{\partial F_\alpha}{\partial x^\beta} \xi^\beta + \frac{\partial F_\alpha}{\partial \dot{x}^\beta} \frac{d\xi^\beta}{dt} + \frac{\partial F_\alpha}{\partial \ddot{x}^\beta} \frac{d^2\xi^\beta}{dt^2} = 0$$

$$(3) \quad \left[\frac{\partial F_\beta}{\partial x^\alpha} - \frac{d}{dt} \left(\frac{\partial F_\beta}{\partial \dot{x}^\alpha} \right) + \frac{d^2}{dt^2} \left(\frac{\partial F_\beta}{\partial \ddot{x}^\alpha} \right) \right] \eta^\beta - \left[\frac{\partial F_\beta}{\partial \dot{x}^\alpha} - 2 \frac{d}{dt} \left(\frac{\partial F_\beta}{\partial \ddot{x}^\alpha} \right) \right] \frac{d\eta^\beta}{dt} + \frac{\partial F_\beta}{\partial \ddot{x}^\alpha} \frac{d^2\eta^\beta}{dt^2} = 0,$$

along the solutions of the system (1), then

$$\Omega = \frac{\partial F_\alpha}{\partial \ddot{x}^\beta} \frac{d\xi^\beta}{dt} - \left[\frac{d}{dt} \left(\frac{\partial F_\alpha}{\partial \ddot{x}^\beta} \eta^\alpha \right) - \frac{\partial F_\alpha}{\partial \dot{x}^\beta} \eta^\alpha \right] \xi^\beta$$

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represents a conservation law.

Now, we assume that the system of equations (1) is self-adjoint. In this case the systems of equations (2) and (3) coincide. We have:

Theorem 2. *If the system (1) is self-adjoint and $\xi_1^\alpha(t, x, \dot{x})$ and $\xi_2^\alpha(t, x, \dot{x})$ are two sets of functions which verify the relation (2) along the solutions of the system (1), then*

$$\Omega = \frac{\partial F_\alpha}{\partial \dot{x}^\beta} \frac{d\xi_1^\beta}{dt} - \left[\frac{d}{dt} \left(\frac{\partial F_\alpha}{\partial \dot{x}^\beta} \xi_2^\alpha \right) - \frac{\partial F_\alpha}{\partial \dot{x}^\beta} \xi_2^\alpha \right] \xi_1^\beta$$

represents a conservation law.

In all what follows we assume that the system (1) is Lagrangian, i.e., there is a Lagrange function $L(t, x, \dot{x})$, for which we can write the Euler-Lagrange equations

$$(4) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0.$$

Because the system (1) is assumed to be Lagrangian, it is self-adjoint and a conservation law for it is given by Theorem 2. To rewrite Theorem 2 with the help of the function L we must have

$$F_\alpha \equiv \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0$$

and

$$\begin{aligned} \frac{\partial F_\alpha}{\partial \dot{x}^\beta} &= \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \\ \frac{\partial F_\alpha}{\partial \dot{x}^\beta} &= \frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \right) + \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta} - \frac{\partial^2 L}{\partial x^\alpha \partial \dot{x}^\beta} \\ \frac{\partial F_\alpha}{\partial x^\beta} &= \frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta} \right) - \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta}. \end{aligned}$$

Theorem 3 ([2]) *If $\xi_1^\alpha, \xi_2^\alpha$ are two sets of functions which satisfy the equations*

$$(6) \quad \begin{aligned} &\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \frac{d^2 \xi^\beta}{dt^2} + \left[\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \right) + \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta} - \frac{\partial^2 L}{\partial \dot{x}^\beta \partial x^\alpha} \right] \cdot \frac{d\xi^\beta}{dt} + \\ &+ \left[\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta} \right) - \frac{\partial^2 L}{\partial x^\alpha \partial x^\beta} \right] \xi^\beta = 0, \end{aligned}$$

along the solutions of the system (4), then

$$(7) \quad \Omega = \frac{\partial^2 L}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \left(\xi_1^\alpha \frac{d\xi_2^\beta}{dt} - \xi_2^\beta \frac{d\xi_1^\alpha}{dt} \right) + \left(\frac{\partial^2 L}{\partial \dot{x}^\alpha \partial x^\beta} - \frac{\partial^2 L}{\partial \dot{x}^\beta \partial x^\alpha} \right) \xi_1^\alpha \xi_2^\beta$$

represents a conservation law.

§2. Maximizing Problems

We consider a maximizing problem associated to the integral

$$(8) \quad \int_0^T f(t, x, \dot{x}) dt,$$

the variables being subject to the restrictions:

$$(9) \quad F_a = F_a(x, \dot{x}) = 0, \quad a = \overline{1, n},$$

where $x^i = x^i(t)$, $i = \overline{1, m}$.

We consider a Lagrange function of the form

$$(10) \quad L(t, x, \dot{x}, \lambda^a) = f(t, x, \dot{x}) + \lambda^a F_a(x, \dot{x}),$$

where $\lambda^a = \lambda^a(t)$, $a = \overline{1, n}$, are called the *Lagrange multipliers*.

For this case, the equations (6) are:

$$(11) \quad \frac{d}{dt} \left[\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} \xi^j + \frac{\partial F_a}{\partial \dot{x}^i} \xi^a \right] - \left(\frac{\partial^2 L}{\partial \dot{x}^j \partial x^i} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial x^i \partial x^j} \xi^j + \frac{\partial F_a}{\partial x^i} \xi^a \right) = 0,$$

and

$$(12) \quad \frac{\partial F_a}{\partial \dot{x}^j} \frac{d\xi^j}{dt} + \frac{\partial F_a}{\partial x^j} \xi^j = 0.$$

The conservation law (7) is written in the form:

$$(13) \quad \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \left(\xi_1^i \frac{d\xi_2^j}{dt} - \xi_2^j \frac{d\xi_1^i}{dt} \right) + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} (\xi_1^i \xi_2^j - \xi_2^i \xi_1^j) + \frac{\partial F_a}{\partial \dot{x}^i} (\xi_1^i \xi_2^a - \xi_2^a \xi_1^i) = \text{const.}$$

Thus we obtain the following:

Theorem 4. *If $\xi_1^\alpha(t, x, \dot{x}, \lambda, \dot{\lambda}) = (\xi_1^a, \xi_1^i)$ and $\xi_2^\alpha(t, x, \dot{x}, \lambda, \dot{\lambda}) = (\xi_2^a, \xi_2^i)$ are two sets of solutions of the equations (11) and (12), on the optimal path for the maximizing problem of (8) under the restrictions (9), then (13) is a conservation law.*

§3. Special Cases

¹⁰ First we consider the special case in which $f(t, x, \dot{x}) = e^{-\rho t} U(x, \dot{x})$, where ρ is a constant and U is a function of state parameters and its derivatives. In economics the function U is called "utility function".

Proposition 1. $\xi^\alpha = (\dot{\lambda}^a + \rho \lambda^a, \dot{x}^i)$ is a solution for (11) and (12).

For this case the Theorem 1 comes to:

Theorem 5. If the functions $\xi^\alpha = (\xi^a, \xi^i)$ satisfy the equations (11) and (12), then

$$(14) \quad \Omega = \dot{x}^i \left[\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \frac{d\xi^j}{dt} + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j} \xi^j + \frac{\partial F_a}{\partial \dot{x}^i} \xi^a \right] - \xi^i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) + \rho \frac{\partial L}{\partial \dot{x}^i} \right]$$

is a conservation law.

In order to give other solutions of equations (11) and (12) we will make some supplementary suppositions about the restrictions (9) and about the utility function U .

2⁰ Now we consider the case when the functions $F_a(x, \dot{x})$ are 1-st-degree homogeneous (with respect to x^j and \dot{x}^j) and the function $U(x, \dot{x})$ is r -th-degree homogeneous (with respect to x^j and \dot{x}^j).

Observation. The equation (12) is satisfied for $\xi^i = \dot{x}^i$.

Proposition 2. If the functions F_a are 1-st-degree homogeneous and U is r -th-degree homogeneous, then $\xi^\alpha = ((r-1)\lambda^a, \dot{x}^i)$ is a solution for (11) and (12).

Theorem 6. If the functions F_a are 1-st-degree homogeneous, U is r -th-degree homogeneous with respect to x^i and \dot{x}^i , and on the solution the integral (8) is extremum, then

$$(15) \quad \Omega = r \left(-\lambda^a \dot{x}^i \frac{\partial F_a}{\partial \dot{x}^i} + e^{-\rho t} \left(\dot{x}^i \frac{\partial U}{\partial \dot{x}^i} - U \right) \right) - \rho \dot{x}^i \left(\lambda^a \frac{\partial F_a}{\partial \dot{x}^i} + e^{-\rho t} \frac{\partial U}{\partial \dot{x}^i} \right)$$

is a conservation law.

§4. Application

We consider the harmonic oscillator with damped oscillation

$$(16) \quad \ddot{x} + c\dot{x} + \omega^2 x = 0,$$

for which the Lagrange function [6] is known:

$$(17) \quad L = \frac{1}{2} e^{-ct} [(\dot{x})^2 - \omega^2 x^2],$$

This is obviously extreme on the solutions of the system.

Applying the preceding considerations we will obtain conservation laws for the oscillator.

In this sense we will make the change of notation $x = x^1$, $\dot{x} = \dot{x}^2$, so that the equation (16) is equivalent with the system

$$(18) \quad \begin{cases} \dot{x}^1 = \dot{x}^2 \\ \dot{x}^2 = -\omega^2 x^1 - c\dot{x}^2, \end{cases}$$

while on the solutions of the system, the integral

$$(19) \quad \int_0^T e^{-ct} \left(\frac{(\dot{x}^2)^2}{2} - \frac{\omega^2 (x^1)^2}{2} \right) dt$$

is extremum (where c is a positive constant).

Using the notations from the section 3 we have:

$$(20) \quad \begin{aligned} F_1 &= \dot{x}^1 - x^2 = 0 \\ F_2 &= \dot{x}^2 + cx^2 + \omega^2 x^1 = 0 \\ U &= \frac{(x^2)^2}{2} - \frac{\omega^2 (x^1)^2}{2}, \end{aligned}$$

where F_1 and F_2 are 1-st-degree homogeneous, while U is 2-nd-degree homogeneous.

We construct the Lagrangian

$$(21) \quad L = e^{-ct} \left(\frac{(x^2)^2}{2} - \frac{\omega^2 (x^1)^2}{2} \right) + \lambda^1 (\dot{x}^1 - x^2) + \lambda^2 (\dot{x}^2 + cx^2 + \omega^2 x^1),$$

$\lambda^i = \lambda^i(t)$, $i = 1, 2$, for the moment undetermined.

We write the Euler-Lagrange equation for (21) and we obtain the system

$$(22) \quad \begin{cases} \dot{\lambda}^1 = \lambda^2 \omega^2 - e^{-ct} \omega^2 x^1 \\ \dot{\lambda}^2 = -\lambda^1 + c\lambda^2 + e^{-ct} x^2 \\ \dot{x}^1 = x^2 \\ \dot{x}^2 = -\omega^2 x^1 - cx^2. \end{cases}$$

The conservation law (15) is written in the form

$$(15') \quad 2\lambda^1 x^2 + e^{-ct} ((x^2)^2 - \omega^2 (x^1)^2) - c\lambda^1 x^1 - 2\lambda^2 x^1 \omega^2 - 2\lambda^2 x^2 c - cx^2 \lambda^2 = K,$$

where $\lambda^1, \lambda^2, x^1, x^2$ are the solutions of the system (22) and K constant.

a) In the case in which $c < 2\omega$, the solution of the system (22) is given by

$$(23) \quad \begin{cases} x^1 = c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t} \\ x^2 = c_1 (\alpha+i\beta) e^{(\alpha+i\beta)t} + c_2 (\alpha-i\beta) e^{(\alpha-i\beta)t} \\ \lambda^1 = -k_1 (\alpha+i\beta) e^{(-\alpha+i\beta)t} + k_2 (-\alpha+i\beta) e^{(-\alpha-i\beta)t} + \\ \quad + \frac{\omega^2 c_1}{-4\alpha-2i\beta} e^{(3\alpha+i\beta)t} + \frac{\omega^2 c_2}{-4\alpha+2i\beta} e^{(3\alpha-i\beta)t} \\ \lambda^2 = k_1 e^{(-\alpha+i\beta)t} + k_2 e^{(-\alpha-i\beta)t} + \\ \quad + \frac{c_1}{2} \frac{\alpha+i\beta}{2\alpha+\beta i} e^{(3\alpha+i\beta)t} + \frac{c_2}{2} \frac{-\alpha+i\beta}{-2\alpha+i\beta} e^{(3\alpha-i\beta)t}, \end{cases}$$

where $\alpha = -\frac{c}{2}$, $\beta = \frac{\sqrt{4\omega^2 - c^2}}{2}$, $c_1, c_2, k_1, k_2 \in \mathbb{R}$.

These solutions are unique in the given initial conditions $x^1(0) = x_0^1$, $x^2(0) = x_0^2$, $\lambda^1(0) = \lambda_0^1$, $\lambda^2(0) = \lambda_0^2$.

By substitution of the functions $x^1, x^2, \lambda^1, \lambda^2$, from by (23) in (15'), we obtain for the system (18), with $c < 2\omega$,

$$(24) \quad K = k_1 c_2 \left(-4\omega^2 + c^2 + 2c\sqrt{4\omega^2 - c^2}i \right) + c_1 k_2 \left(c^2 - 4\omega^2 - 2c\sqrt{4\omega^2 - c^2}i \right),$$

with $k_1, k_2, c_1, c_2 \in \mathbb{R}$.

b) In the case in which $c > 2\omega$ the solution of the system (22) is given by

$$(25) \quad \begin{cases} x^1 = c_1 e^{(\alpha+\beta)t} + c_2 e^{(\alpha-\beta)t} \\ x^2 = c_1(\alpha + \beta)e^{(\alpha+\beta)t} + c_2(\alpha - \beta)e^{(\alpha-\beta)t} \\ \lambda^1 = k_1(-\alpha - \beta)e^{(-\alpha+\beta)t} + k_2(-\alpha + \beta)e^{(-\alpha-\beta)t} + \\ \quad + \frac{\omega^2 c_1}{2(-2\alpha - \beta)}e^{(3\alpha+\beta)t} + \frac{\omega^2 c_2}{2(-2\alpha + \beta)}e^{(3\alpha-\beta)t} \\ \lambda^2 = k_1 e^{(-\alpha+\beta)t} + k_2 e^{(-\alpha-\beta)t} + \\ \quad + \frac{c_1}{2} \frac{\alpha + \beta}{2\alpha + \beta} e^{(3\alpha+\beta)t} + \frac{c_2}{2} \frac{-\alpha + \beta}{-2\alpha + \beta} e^{(3\alpha-\beta)t}, \end{cases}$$

and the conservation law for the system (18), when on the solution the integral (19) is extremum, is given by (15'), where

$$(26) \quad K = c_1 k_2 \left(-4\omega^2 + c^2 - 2c\sqrt{c^2 - 4\omega^2} \right) + k_1 c_2 \left(-4\omega^2 + c^2 + 2c\sqrt{c^2 - 4\omega^2} \right),$$

with $k_1, k_2, c_1, c_2 \in \mathbb{R}$.

The constant (26) was obtained by substitution of the functions $x^1, x^2, \lambda^1, \lambda^2$ from (25) in (15'), and calculations.

c) In the case in which $c = 2\omega$ the solution of the system (22) is given by:

$$(27) \quad \begin{cases} x^1 = (c_1 + c_2 t)e^{-\omega t} \\ x^2 = (-\omega c_1 + c_2 - \omega c_2 t)e^{-\omega t} \\ \lambda^1 = (-k_2 + \omega k_1 + \omega k_2 t)e^{\omega t} + \left(\frac{2\omega c_1 + c_2}{8} + \frac{\omega c_2}{4} t \right) e^{-3\omega t} \\ \lambda^2 = (k_1 + k_2 t)e^{\omega t} + \left(\frac{2\omega c_1 - c_2}{8\omega} + \frac{c_2}{4} t \right) e^{-3\omega t}, \end{cases}$$

and the conservation law is represented by (15'), where

$$K = 4\omega c_1 k_2 - 4\omega c_2 k_1 - 2c_2 k_2, \quad k_1, k_2, c_1, c_2 \in \mathbb{R}.$$

K was obtained by substitution of the functions given by (27) in (15).

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Author's address:

Mihaela Neamțu
West University of Timișoara
Faculty of Economic Sciences
16 Pestalozzi Str.
Email: nmihaela@lycos.com