On the Homogeneous Lift G in the Cotangent Bundle (II)♣

Liviu Popescu

Abstract

In this paper we study the conditions of integrability for the (non)-homogeneous almost product structure on the Hamilton and Cartan spaces.

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Key words: nonlinear connection, adapted basis, homogeneous lift, self-curvature, non-holonomy distorsion, self-torsion, almost product structure.

§1. Introduction

Let (T^*M, π^*, M) be the cotangent bundle of a $C^\infty-$differentiable real, $n-$dimensional real manifold M. If (U, φ) is a local chart on M then the coordinates of a point $u = (x, \tau) \in \pi^{*-1}(U) \in T^*M$ will be denoted (x^i, τ_i). A nonlinear connection N on T^*M is a horizontal distribution of class C^∞, $H : u \in T^*M \to H_u \subset T_uT^*M$, which is supplementary to the vertical distribution V, that is $T_uT^*M = H_u \oplus V_u$.

Let $N_{ij}(x, \tau)$ be the coefficients of N on $\pi^{*-1}(U)$ and the adapted basis in H_u,

$$(\delta_i = \partial/\partial x^i + N_{ij}(x, \tau)\partial^j),\ i, j \in \overline{1, n},$$

where $\partial^j = \partial/\partial\tau_j$. The dual basis of (δ_i, ∂^k) is $(dx^i, \delta\tau_k)$, where $\delta\tau_k = d\tau_k - N_{kj}dx^j$.

The fields of 2-forms

$$\theta = \delta\tau_k \wedge dx^r$$

$$\tau = \frac{1}{2}\tau_{rs}dx^r \wedge dx^s,$$

where $\tau_{rs} = N_{rs} - N_{sr}$, are globally defined on T^*M and θ is a natural almost symplectic structure on T^*M. We have:

$$\Omega(\delta_j, \delta_k) = -v[\delta_j, \delta_k] = -R_{jk(r)}\partial^r,$$

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where $R_{jk(r)} = \delta_j N_{kr} - \delta_k N_{jr}$ is a $d-$tensor field of type $(0,3)$. The horizontal distribution H on T^*M is integrable if and only if $\Omega = 0$. Since

$$d\theta = -d\tau = -\frac{1}{6}\sum_{(ijk)} R_{ijk} dx^i \wedge dx^j \wedge dx^r - \dot{\partial}^a\, \tau_{ij}\delta\tau_a \wedge dx^i \wedge dx^j,$$

the almost symplectic structure θ is integrable if and only if

$$\sum_{(ijk)} R_{ijk} = 0 \qquad \text{and} \qquad \partial^s \tau_{jk} = 0.$$

§2. On the integrability of the homogeneous structure $\overline{P}$

Let $H^n = (M, g^{ij}(x,\tau))$ be a generalized Hamilton space. In the adapted basis we consider the mapping defined by

$$\overline{P}(\delta_k) = g_{kr}\partial^r; \qquad \overline{P}(\partial^r) = g^{rs}\delta_s,$$

where $g_{ir}g^{rk} = \delta_i^r$, will be called the dual almost product structure.

We shall consider the lift to T^*M of the fundamental tensor fields $g^{ij}(x,\tau)$ of the generalized Hamilton space,

$$G = g_{ij}(x,\tau)dx^i \otimes dx^k + g^{rs}(x,\tau)\delta\tau_r \otimes \delta\tau_s.$$

Let also $N_{\overline{P}}$ be the Nijenhuis tensor of the dual almost product structure,

$$N_{\overline{P}}(X,Y) = [\overline{P}X, \overline{P}Y] - \overline{P}[\overline{P}X, Y] - \overline{P}[X, \overline{P}Y] + \overline{P}^2[X,Y]$$

Proposition 1. *The Nijenhuis tensor $N_{\overline{P}}$ has the properties*

(1) $$hN_{\overline{P}}(hX,hY) = hN_{\overline{P}}(\overline{P}hX, \overline{P}hY) = -h\overline{P}N_{\overline{P}}(hX, \overline{P}hY)$$

(2) $$vN_{\overline{P}}(hX,hY) = vN_{\overline{P}}(\overline{P}hX, \overline{P}hY) = -v\overline{P}N_{\overline{P}}(hX, \overline{P}hY).$$

Definition 1. The tensor field $\overset{*}{\Omega}$ defined by

$$\overset{*}{\Omega}_{\overline{P}}(X,Y) = -vN_{\overline{P}}(hX,hY), \quad \forall X,Y \in \mathrm{X}(T^*M)$$

is called the $\overline{P} - self - curvature$ of the nonlinear connection N on T^*M.

Definition 2. The 2- form $\omega = \overset{*}{\Omega}_{\overline{P}} - \Omega$ is called *the non − holonomy distorsion.*

Definition 3. The tensor field $\overset{*}{t}$ defined by

$$\overset{*}{t}(X,Y) = -hN_{\overline{P}}(vX,vY), \quad \forall X,Y \in \mathrm{X}(T^*M)$$

is called the $\overline{P} - self - torsion$ of the nonlinear connection N on T^*M.

Theorem 1. *The $\overline{P}-self-curvature$ $\overset{*}{\Omega}$ depends only on g and N, and is given by*

$$\overset{*}{\Omega}_{\overline{P}}(\delta_k,\delta_l)=-(2g_{ki}\Omega^{li}_{js}\partial^s g_{lr}+R_{jk(r)})\partial^r, \tag{3}$$

where $\Omega^{ls}_{ki}=\frac{1}{2}(\delta^l_k\delta^s_i-g_{ki}g^{ls})$ *is the Obata operator.*

Theorem 2. *The non-holonomy distorsion ω depends only on g.*

For the Nijenhuis tensor $N_{\overline{P}}$ we have the following decomposition:

$$\begin{cases} N_{\overline{P}}(\delta_j,\delta_k)=N^{i}_{\overline{P}jk}\delta_i+N_{\overline{P}jk(s)}\partial^s\\ N_{\overline{P}}(\delta_j,\partial^k)=N^{(k)s}_{\overline{P}j}\delta_s+N^{(k)}_{\overline{P}j(r)}\partial^r\\ N_{\overline{P}}(\partial^r,\partial^k)=N^{(r)(k)i}_{\overline{P}}\delta_i+N^{(r)(k)}_{\overline{P}(s)}\partial^s. \end{cases}$$

Theorem 3. *The following relations hold true:*

$$N^{k}_{\overline{P}jr}=-g_{js}g_{lr}N^{(s)(l)k}_{\overline{P}}=-N^{(s)}_{\overline{P}ji}g^{ik}g_{sr} \tag{4}$$

$$N_{\overline{P}jk(r)}=-g_{js}g_{ki}N^{(s)(i)}_{\overline{P}(r)}=-N^{(s)i}_{\overline{P}j}g_{sk}g_{ir}. \tag{5}$$

Theorem 4. *The dual almost product structure is integrable if and only if the following two conditions are fulfilled:*

$$\delta_j g_{ki}-\delta_k g_{ji}=g_{lr}\left(\partial^r N_{ji}\delta^l_k-\partial^r N_{ki}\delta^l_j\right); \tag{6}$$

$$\Omega=-\omega\Leftrightarrow R_{jk(r)}=-2g_{ki}\Omega^{li}_{js}\partial^s g_{lr}. \tag{7}$$

2.1. The case of the Cartan space

Let (M,C) be a Cartan space with $C=g^{rs}(x)\tau_r\tau_s$ and nonlinear connection $N_{kr}(x,\tau)=\tau_s\Gamma^s_{ks}(x)$ where $\Gamma^s_{ks}(x)$ are the Christoffel symbols of the metric g.

We consider the mapping P_0 given by

$$P_0(\delta_k)=\frac{\sqrt{C}}{r}\overline{P}(\delta_k);\qquad P_0(\partial^r)=\frac{r}{\sqrt{C}}\overline{P}(\partial^r). \tag{8}$$

Theorem 5. *P_0 has the following properties:*

a) P_0 is an almost product structure on T^*M, ($P^2=I$);

b) P_0 depend only by the metric g;

c) P_0 is homogeneous on the fibres of T^*M.

Theorem 6. *The almost product structure P_0 is integrable on T^*M if and only if the Riemannian space $(M,g_j(x))$ is of constant curvature $K=-\frac{1}{r^2}$.*

Remark. For $n=2$, the structure P_0 is integrable if and only if the Riemannian space $(M,g_j(x))$ is locally isometric with a pseudophere of radius r.

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Author's address:

Liviu Popescu
Departament of Mathematics, University of Craiova
13, Al. I. Cuza st., Craiova 1100, Romania
e-mail livpop@hotmail.com