$d-$Connections Compatible with a Class of Metrical Almost $2-\pi$ Structures on TM [♣]

Adrian Sandovici

Abstract

In the paper [7] we defined and studied two classes of metrical almost $2-\pi$ structures: $(G_{a,b,c,d},\ \Phi_{a,b,c,d})$ and $(G_{a,a,c,c},\ \Phi_{a,c})$. Here we determine the set of all $d-$connections compatible with the metrical structure $G_{a,b,c,d}$ in Section 2. In Section 3 we obtain the set of all $d-$connections compatible with the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$. Moreover we raise the (a,c)-geometrical model of the Riemannian space (M,γ) with respect to the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$.

M.S.C. 2000: 53C60.
Key words: metrical almost 2-π structure, $d-$connections compatible with a metrical almost $2-\pi$ structure.

§1. Introduction

Let $R^{(n)}=(M,\gamma)$ be a Riemannian space with a smooth, real manifold M and a Riemannian structure γ. Here $x=(x^i)$ are coordinates on M and $(x,y)=(x^i,y^i)$ are coordinates on the tangent manifold TM projected on M by τ. The indices $i,j,k,\ldots.$ will run from 1 to $n=dimM$ and the Einstein convention on summation is implied. The geometrical objects on TM whose local components change like on M will be called *$d-$objects* as in [6]. The kernel of the differential $\tau^T: TTM\rightarrow TM$ is a vector subbundle of TTM called *the vertical distribution on TM*. The local vector fields $\left\{\frac{\partial}{\partial y^i}\right\}$ determine a local frame in VTM. A *nonlinear connection* in the tangent bundle $\tau: TM\rightarrow M$ is a distribution HTM on TM supplementary to the vertical distribution, that is

$$TTM=HTM\oplus VTM \tag{1.1}$$

The position of the subspace H_uTM, $u\in TM$ can be given by n local vector fields $\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N_i^k(x,y)\frac{\partial}{\partial y^k}$. The real differentiable functions $\left(N_i^k(x,y)\right)$ completely determines a nonlinear connection which will be denoted simply N. For exemple we

[♣] *Presented at the Third Conference of Balkan Society of Geometers, July 31-August 3 2000, Bucharest, Romania.*

Differential Geometry - Dynamical Systems, Vol.2, No.1, 2000, pp. 36-42.

can take $N^i_j(x,y)=\gamma^i_{jk}(x)y^k$, where $\gamma^i_{jk}(x)$ are the Christoffel symbols of the Levi–Civita connection. Therefore a nonlinear connection N determines a basis $\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$ adapted to the decomposition (1.1).

We recall that the Sasaki lift of the Riemannian structure γ on TM is

$$G_S=\gamma_{ij}(x)dx^i\otimes dx^j+\gamma_{ij}(x)\delta y^i\otimes\delta y^j. \tag{1.2}$$

Next we consider a $(h,v)-$ metrical structure on TM given by

$$G(x,y)=g_{ij}(x,y)dx^i\otimes dx^j+h_{ij}(x,y)\delta y^i\otimes\delta y^j, \tag{1.3}$$

where

$$\begin{aligned} g_{ij}(x,y)&=\tfrac{a^2}{F^2}\gamma_{ij}(x)+\tfrac{b^2-a^2}{F^4}y_iy_j\\ h_{ij}(x,y)&=\tfrac{c^2}{F^2}\gamma_{ij}(x)+\tfrac{d^2-c^2}{F^4}y_iy_j, \end{aligned} \tag{1.4}$$

and $F^2=\gamma_{ij}(x)y^iy^j, y_i=\gamma_{ij}(x)y^j$ and $a,b,c,d:\mathrm{Im}(F^2)\subseteq\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are differentiable functions with $b\geq a>0$, $d\geq c>0$.

The study of this $(h,v)-$ metrical structure was done in [7]. We notice that

a) For $a=F$, $b=F$, $c^2=\frac{F^2}{1+F^2}$, $d^2=F^2$, the metrical structure $G_{a,b,c,d}(x,y)$ is the *Cheeger – Gromoll metric*

$$G_{CG}(x,y)=\gamma_{ij}(x)dx^i\otimes dx^j+\tfrac{1}{1+F^2}(\gamma_{ij}(x)+y_iy_j)\delta y^i\otimes\delta y^j. \tag{1.5}$$

b) If $a=F$, $b=F$, $c^2=\varphi' F^2$, $d^2=F^2(\varphi'+2\varphi'' F^2)$, where $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ with $\varphi'(t)\neq 0, t\in\mathrm{Im}(F^2)$, one obtains *the Antonelli – Hrimiuc metrical structure*

$$G_{AH}(x,y)=\gamma_{ij}(x)dx^i\otimes dx^j+(\varphi'\gamma_{ij}(x)+2\varphi'' y_iy_j)\delta y^i\otimes\delta y^j. \tag{1.6}$$

c) For $a=F$, $b=F$ one obtains *the Anastasiei metrical structure*

$$G_A(x,y)=\gamma_{ij}(x)dx^i\otimes dx^j+(\tfrac{c^2}{F^2}\gamma_{ij}(x)+\tfrac{d^2-c^2}{F^4}y_iy_j)\delta y^i\otimes\delta y^j. \tag{1.7}$$

d) If $a=F$, $b=F$, $c=d=k^2$, $k\in\mathbb{R}$, one obtains *the Miron metrical structure*

$$G_M(x,y)=\gamma_{ij}(x)dx^i\otimes dx^j+\tfrac{k^2}{F^2}\gamma_{ij}(x)\delta y^i\otimes\delta y^j. \tag{1.8}$$

§2. $d-$connections compatible with $G_{a,b,c,d}$

Let us assume TM is endowed with a nonlinear connection given by the local coefficients $N^i_j(x,y)=\gamma^i_{jk}(x)y^k$.

Definition 2.1. A $d-$connection D on TM is called *compatible with the metrical structure G* if it satisfies the condition

$$D_XG=0\ ,\ \forall\ X\in\chi(TM). \tag{2.1}$$

In the next considerations we shall determine the set of all connections compatible with the metrical structure $G_{a,b,c,d}$. In the adapted basis, any $d-$connection on TM can be represented in the following way:

$$D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{(H)^i}{F}_{jk} \frac{\delta}{\delta x^i} \quad , D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = \overset{(V)^i}{F}_{jk} \frac{\partial}{\partial y^i}$$
(2.2)
$$D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = \overset{(H)^i}{C}_{jk} \frac{\delta}{\delta x^i} \quad , D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = \overset{(V)^i}{C}_{jk} \frac{\partial}{\partial y^i},$$

in which the system of functions $(\overset{(H)^i}{F}_{jk}, \overset{(V)^i}{F}_{jk}, \overset{(H)^i}{C}_{jk}, \overset{(V)^i}{C}_{jk})$ represents the local coefficients of the above $d-$connection D. It is not difficult to prove:

Theorem 2.1. *There are $d-$metrical connections on TM with respect to the metrical structure $G_{a,b,c,d}$. One of them has the following coefficients*:

(2.3) $$\overset{(H,c)^i}{F}_{jk} = \overset{(V,c)^i}{F}_{jk} = \gamma^i_{jk}$$

(2.4) $$\overset{(H,c)^i}{C}_{jk} = A \cdot \delta^i_j \cdot y_k + A_1 \cdot \delta^i_k \cdot y_j + A_2 \cdot y^i \cdot \gamma_{jk} + A_3 \cdot y^i \cdot y_j \cdot y_k$$

(2.5) $$\overset{(V,c)^i}{C}_{jk} = B \cdot \Lambda^i_{jk} + B_1 \cdot y^i \cdot \gamma_{jk} + B_2 \cdot y^i \cdot y_j \cdot y_k$$

where:

(2.6) $$\Lambda^i_{jk} = \delta^i_j \cdot y_k + \delta^i_k \cdot y_j - y^i \cdot \gamma_{jk}$$

(2.7) $$A = \frac{2a'F^2 - a}{aF^2} \quad , A_1 = \frac{b^2 - a^2}{2a^2F^2}$$

(2.8) $$A_2 = \frac{b^2 - a^2}{b^2F^2} \quad , \quad A_3 = \frac{a^4 - b^4 + 4ab(ab' - ba')F^2}{2a^2b^2F^4}$$

(2.9) $$B = \frac{2c'F^2 - c}{cF^2} \quad , B_1 = \frac{2c'\left(d^2 - c^2\right)}{cd^2}$$

(2.10) $$B_2 = \frac{2\left(c^2c' + cdd' - 2d^2c'\right)}{cd^2}$$

Theorem 2.2. *The set of all $d-$connections compatible with the metrical structure $G_{a,b,c,d}$ is given by the following coefficients:*

(2.11) $$\overset{(H)^i}{F}_{jk} = \overset{(H,c)^i}{F}_{jk} + O^{ei}_{jm} \cdot X^m_{ek}$$

(2.12) $$\overset{(V)^i}{F}_{jk} = \overset{(V,c)^i}{F}_{jk} + \tilde{O}{}^{ei}_{jm} \cdot Y^m_{ek}$$

(2.13) $$\overset{(H)^i}{C}_{jk} = \overset{(H,c)^i}{C}_{jk} + O^{ei}_{jm} \cdot U^m_{ek}$$

(2.14) $$\overset{(V)^i}{C}_{jk} = \overset{(V,c)^i}{C}_{jk} + \tilde{O}{}^{ei}_{jm} \cdot V^m_{ek}$$

with X^m_{ek} , Y^m_{ek} , U^m_{ek} , V^m_{ek} arbitrary $d-$tensor fields and

$$O^{ei}_{jm} = \frac{1}{2}\left(\delta^e_j \cdot \delta^i_m - g_{jm} \cdot g^{ei}\right)$$
(2.15)
$$\tilde{O}{}^{ei}_{jm} = \frac{1}{2}\left(\delta^e_j \cdot \delta^i_m - h_{jm} \cdot h^{ei}\right)$$

the Obata operators.

Theorem 2.3. *The set of all d−connections compatible with the metrical structure $G_{a,a,c,c}$ is given by the following local coefficients*

$$\overset{(H)}{F}{}^{i}_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek} \tag{2.16}$$

$$\overset{(V)}{F}{}^{i}_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot Y^m_{ek} \tag{2.17}$$

$$\overset{(H)}{C}{}^{i}_{jk} = A \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek} \tag{2.18}$$

$$\overset{(V)}{C}{}^{i}_{jk} = B \cdot \Lambda^i_{jk} + \Omega^{ei}_{jm} \cdot V^m_{ek} \tag{2.19}$$

with X^m_{ek} , Y^m_{ek} , U^m_{ek} , V^m_{ek} arbitrary d−tensor fields and

$$\Omega^{ei}_{jm} = \tfrac{1}{2}\left(\delta^e_j \cdot \delta^i_m - \gamma_{jm} \cdot \gamma^{ei}\right) \tag{2.20}$$

the Obata operator of the Riemannian structure γ.

§3. d−connections compatible with a class of metrical almost $2-\pi$ structures on TM

Recall that an *almost* $2-\pi$ *structure* is a tensor field of type (1,1) on TM, which, in adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ is given by

$$\Phi_S(\tfrac{\delta}{\delta x^i}) = -\lambda \cdot \tfrac{\partial}{\partial y^i} \ , \ \Phi_S(\tfrac{\partial}{\partial y^i}) = \lambda \cdot \tfrac{\delta}{\delta x^i} \ , \ \lambda \in C \tag{3.1}$$

Also, a *metrical almost* $2-\pi$ *structure* is a pair (G_S, Φ_S) on TM for which

$$\tfrac{1}{\lambda^2} G_S(\Phi_S X, \Phi_S Y) = G_S(X,Y) \ , \ \forall X, Y \in \chi(TM), \tag{3.2}$$

and the 2-form $\Omega_S(X,Y) = G_S(\Phi_S(X), Y)$ is closed.

The pair (G, Φ_S), where the metrical structure G is defined by (1.3), (1.4), is *not* a metrical almost 2-π structure. In [7] we proved that there exists a class of almost 2-π structures Φ such that the pair (G, Φ) is a metrical almost $2-\pi$ structure. In fact we proved that the pairs $(G_{a,b,c,d}, \Phi_{a,b,c,d})$ and $(G_{a,a,c,c}, \Phi_{a,c})$ are metrical almost $2-\pi$ structures on the tangent bundle. The almost $2-\pi$ structures $\Phi_{a,b,c,d}$ and $\Phi_{a,c}$ are given by

$$\Phi_{a,b,c,d} = \lambda A^k_i \tfrac{\partial}{\partial y^k} \otimes dx^i + \lambda B^k_i \tfrac{\delta}{\delta x^k} \otimes \delta y^i \tag{3.3}$$

and:

$$\Phi_{a,c} = \lambda \tilde{A}{}^k_i \tfrac{\partial}{\partial y^k} \otimes dx^i + \lambda \tilde{B}{}^k_i \tfrac{\delta}{\delta x^k} \otimes \delta y^i \tag{3.4}$$

where:

$$A^k_i = -\tfrac{a}{c}\delta^k_i + \tfrac{ad+bc}{dcF^2} y_i y^k \quad , \quad \tilde{A}{}^k_i = -\tfrac{a}{c}\delta^k_i \tag{3.5$_1$}$$

$$(3.5)_2 \qquad B_i^k = \tfrac{c}{a}\delta_i^k - \tfrac{ad+bc}{abF^2} y_i y^k \quad , \quad \tilde{B}{}_i^k = \tfrac{c}{a}\delta_i^k$$

Definition 3.1. a) A linear connection on TM is said to be *compatible with the almost* $2-\pi$ *structure* Φ iff it satisfies the condition

$$(3.6) \quad D_X\Phi = 0 \ , \forall \ X \in \chi(TM).$$

b) A linear connection on TM is said to be compatible with the metrical almost $2-\pi$ structure (G, Φ), *iff it satisfies the conditions*

$$(3.7) \quad D_X G = 0 \quad , D_X\Phi = 0 \quad , \forall \ X \in \chi(TM).$$

The first step consists in the investigation of the linear connections compatible with the almost $2-\pi$ structure $\Phi_{a,b,c,d}$. In the adapted basis, any linear connection D on TM can be represented in the following way:

$$(3.8)_1 \qquad \begin{aligned} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= \overset{(H)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{F}{}^i_{jk} \frac{\partial}{\partial y^i} \\ D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} &= \overset{(2)}{F}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{F}{}^i_{jk} \frac{\partial}{\partial y^i} \end{aligned}$$

$$(3.8)_2 \qquad \begin{aligned} D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} &= \overset{(H)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(1)}{C}{}^i_{jk} \frac{\partial}{\partial y^i} \\ D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} &= \overset{(2)}{C}{}^i_{jk} \frac{\delta}{\delta x^i} + \overset{(V)}{C}{}^i_{jk} \frac{\partial}{\partial y^i} \end{aligned}$$

In local coordinates, the condition (3.6) is equivalent with the following set of relations:

$$(3.9) \qquad A^p_j \cdot \overset{(V)}{F}{}^i_{pk} - A^i_q \cdot \overset{(H)}{F}{}^q_{jk} = 0$$

$$(3.10) \qquad \frac{\partial A^i_j}{\partial y^k} + A^p_j \cdot \overset{(V)}{C}{}^i_{pk} - A^i_q \cdot \overset{(H)}{C}{}^q_{jk} = 0$$

$$(3.11) \qquad A^p_j \cdot \overset{(2)}{F}{}^i_{pk} - B^i_q \cdot \overset{(1)}{F}{}^q_{jk} = 0$$

$$(3.12) \qquad A^p_j \cdot \overset{(2)}{C}{}^i_{pk} - B^i_q \cdot \overset{(1)}{C}{}^q_{jk} = 0$$

If we investigate only $d-$connections compatible with the almost $2-\pi$ structure $\Phi_{a,b,c,d}$, then it is necessary to take into consideration (3.9) and (3.10) only. Our purpose is to find $d-$connections compatible with the metrical almost $2-\pi$ structure $(G_{a,b,c,d}, \Phi_{a,b,c,d})$. For this end in (3.9) we replace the coefficients $(\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ with those from (2.11)–(2.14). The relation (3.9) can be written, in this case, in the following way:

$$(3.13) \qquad A^i_q \cdot O^{eq}_{jm} \cdot X^m_{ek} - A^p_j \cdot \tilde{O}{}^{ei}_{pm} \cdot Y^m_{ek} = \beta \cdot y^p \cdot y_q \cdot E^{qi}_{jpk},$$

where

$$E^{qi}_{jpk} = \delta^q_j \cdot \gamma^i_{pk} - \delta^i_p \cdot \gamma^q_{jk} \quad , \beta = \tfrac{ad+bc}{dcF^2}. \tag{3.14}$$

We notice that the left hand of the relation (3.13) has a $d-$tensorial character. The fact that E^{qi}_{jpk} is not a d-tensor field involves that the relation (3.13) is an impossible one. Therefore we can assert that there are no $d-$connections compatible with the metrical almost $2-\pi$ structure $(G_{a,b,c,d},\ \Phi_{a,b,c,d})$.

Next we study the $d-$connections which are compatible with the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$. Repeating for this one the above considerations, one obtains the following conditions which should necessarily be satisfied:

$$\overset{(H)^i}{F}_{jk} = \overset{(V)^i}{F}_{jk} \quad , \quad \overset{(H)^i}{C}_{jk} = \overset{(V)^i}{C}_{jk} + 2 \cdot \delta^i_j \cdot \tfrac{a'c-ac'}{ac} \cdot y_k. \tag{3.15}$$

After some calculations with respect to (3.15), one obtains the following result:

Theorem 3.1. *The set of all $d-$connections compatible with the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$ is given by the following local coefficients:*

$$\overset{(H)^i}{F}_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek} \tag{3.16}$$

$$\overset{(V)^i}{F}_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek} \tag{3.17}$$

$$\overset{(H)^i}{C}_{jk} = A \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek} \tag{3.18}$$

$$\overset{(V)^i}{C}_{jk} = B \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek} \tag{3.19}$$

with X^m_{ek} , U^m_{ek} arbitrary $d-$tensor fields and Ω^{ei}_{jm} the Obata operator of the Riemannian structure γ.

Particular cases.

1^0. In the case $X^m_{ek} = U^m_{ek} = 0$ one obtains a $d-$connection compatible with the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$, which depends only on the Riemannian structure γ and the functions a, c. The local coefficients of this $d-$connection are as follows:

$$\overset{(H)^i}{F}_{jk} = \overset{(V)^i}{F}_{jk} = \gamma^i_{jk} \quad , \quad \overset{(H)^i}{C}_{jk} = A \cdot \delta^i_j \cdot y_k \quad , \quad \overset{(V)^i}{C}_{jk} = B \cdot \delta^i_j \cdot y_k. \tag{3.20}$$

The simplicity of this $d-$connection and the fact that it is determined only by the Riemannian structure γ and by the functions a, c allows to call it the canonical $d-$connection of the space $(\widetilde{TM},\ G_{a,a,c,c},\ \Phi_{a,c})$.

2^0. If $a = F$, $c = k$, $k \in R^*$, one obtains the so called homogeneous metrical almost $2-\pi$ structure $(\overset{(0)}{G}\ ,\ \overset{(0)}{\Phi})$ where the metrical structure $\overset{(0)}{G}$ is the Miron metrical structure from (1.8) and the almost $2-\pi$ structure $\overset{(0)}{\Phi}$ is defined by

$$\overset{(0)}{\Phi} = -\lambda \cdot \frac{F}{k} \cdot \frac{\partial}{\partial y^i} \otimes dx^i + \lambda \cdot \frac{k}{F} \cdot \frac{\delta}{\delta x^i} \otimes \delta y^i. \tag{3.21}$$

The canonical d-connection of the space $(\widetilde{TM}, \overset{(0)}{G}, \overset{(0)}{\Phi})$ is given as follows:

$$\overset{(H)}{F}{}^i_{jk} = \overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk} \quad , \quad \overset{(H)}{C}{}^i_{jk} = 0 \quad , \quad \overset{(V)}{C}{}^i_{jk} = -\frac{1}{F^2} \cdot \delta^i_j \cdot y_k. \tag{3.22}$$

Remarks. The space $\overset{(2-\pi)}{M}{}^{2n} = (\widetilde{TM}, G_{a,a,c,c}, \Phi_{a,c})$ is called *the $(a, c)-$geometrical model of the Riemannian space (M, γ) with respect to the metrical almost $2-\pi$ structure $(G_{a,a,c,c}, \Phi_{a,c})$*. It can be used in a gauge theory of the Riemannian metric defined by the metrical structure $G_{a,a,c,c}$. The investigation of the solution of the Einstein equations or of the Einstein–Yang Mills equations for certain relevant cases will be the subject of a forecoming paper.

Acknowledgements: We are very indebted to *Professor Dr. R. Miron* and *Professor Dr. M. Anastasiei* for their suggestions and comments on the subject of this paper.

References

[1] M.Anastasiei M., H.Shimada, *Deformations of Finsler Metrics*, in vol. Finslerian Geometries. A Meetings of Minds. Ed. by P. L. Antonelli, Kluwer Academic Publishers, FTPH 109, 2000, 53–66.

[2] M.Anastasiei, *Locally Conformal Kähler Structures on Tangent Manifold of a Space Form,* to appear.

[3] V.Blănuţă, M.Yawata, *Infinitesimal Transformation of Metrical 2–π Structure*, Tehn. Inst. of Chiba, Japan, 41(1994), 17–24.

[4] R.Miron, *The Homogeneous Lift of a Riemannian Metric*, An. Şt. Univ."Al. I. Cuza" Iaşi, to appear.

[5] R.Miron, *The Geometry of Higher–Order Finsler Spaces*, Hadronic Press, USA, 1998.

[6] R.Miron, M.Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, FTPH, no 59, 1994.

[7] A.Sandovici, V.Blănuţă, A Class of Metrical Almost $2-\pi$ Structures on Tangent Bundle, to appear.

Author's address:

Adrian Sandovici
High School "Gh. Cartianu"
5600, Piatra Neamţ
E–mail: adi.sandovici@mail.ctcepn.logicnet.ro