

# Geometrical Interpretation of the Principal Directions and Principal Curvatures of Submanifolds $\clubsuit$

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## Abstract

In recent papers [4, 6] are introduced principal directions of locally imbedded  $n$ -dimensional manifold in Euclidean spaces and Riemannian spaces of arbitrary codimension. Using some recent results of [7] concerning the  $n$ -scalar products, in this paper are given two geometric interpretations for the principal directions and principal curvatures: one concerning the submanifolds of Euclidean spaces and the other one for the general case of submanifolds of Riemannian spaces. The presented results yield to a natural introduction of principal directions and principal curvatures for submanifolds imbedded in Hilbert spaces.

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## §1. Preliminaries

In this paper is given a geometric interpretation for the principal directions and the principal curvatures for submanifolds imbedded in Euclidean spaces [4] and Riemannian spaces [6] with arbitrary codimension. It is done by using the methods in [7]. We recall the necessary definitions and results from [4], [6] and [7].

In the paper [4] are considered locally embedded  $n$ -dimensional manifold  $M$  in the Euclidean space  $R^{n+k}$ . Let  $\{Y_i\}(1 \leq i \leq n)$  be an arbitrary orthonormal system of tangent vector fields on the manifold, and let  $\{N_\alpha\}(1 \leq \alpha \leq k)$  be an arbitrary orthonormal system of vector fields orthogonal to the tangent space. Considering an arbitrary tensor field  $T$  on  $M$ , it can be prolonged locally in an  $(n+k)$ -dimensional neighborhood such that

$$(1.1) \quad \nabla_N T \equiv 0,$$

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for each orthogonal unit vector  $N$ , where  $\nabla$  denotes the covariant differentiation with respect to the Euclidean metric. Usually, by  $N$  we will denote a unit vector or vector field which is orthogonal to the basic manifold. For arbitrary vector field  $X$ , it implies

$$(1.2) \quad [N, X] = \nabla_N X - \nabla_X N = -\nabla_X N$$

and hence  $N \perp [N, X]$ . For arbitrary normal vector field  $N$  and arbitrary tangent vector fields  $X$  and  $Y$ , we also have  $X \cdot [N, Y] = Y \cdot [N, X]$ .

In the paper [4] are introduced the principal directions and the so called principal normal vectors of the manifold  $M$  imbedded in  $R^{n+k}$ . The following tensor field  $T_{ij}$ , defined by

$$(1.3) \quad T_{ij} = \sum_{\alpha=1}^k \sum_{r=1}^n (Y_i \cdot [N_\alpha, Y_r]) (Y_r \cdot [N_\alpha, Y_j]).$$

is symmetric and invariant of the orthonormal basis  $Y_1, \dots, Y_n$ . If  $(\lambda_1, \dots, \lambda_n)$  is an arbitrary eigenvector of the matrix  $T_{ij}$ , then the vector  $\lambda_1 Y_1 + \dots + \lambda_n Y_n$  is eigenvector for the corresponding (1.1) tensor field  $T_j^i$ , and it does not depend on the bases  $Y_1, \dots, Y_n$  or  $N_1, \dots, N_k$ . The eigenvectors are orthogonal and they determine the *principal directions*. Their eigenvalues are non-negative real numbers. These eigenvalues are squares of the *principal curvatures*. To avoid the sign problem, in [4] are introduced the *principal normal vectors* as follows.

Assume that  $Z_1, \dots, Z_n$  are orthonormal vectors which determine the principal directions. Then we define the corresponding *principal normal vectors*  $U_i (i \in \overline{1, n})$ , by

$$(1.4) \quad U_i = \sum_{\alpha=1}^k (Z_i \cdot [N_\alpha, Z_i]) N_\alpha,$$

and also define the vectors

$$(1.5) \quad U_{ij} = \sum_{\alpha=1}^k (Z_i \cdot [N_\alpha, Z_j]) N_\alpha,$$

which are canonical, i.e., they do not depend on the choice of the orthonormal system  $\{N_\alpha\}$  and do not depend on the signs of the vectors  $Z_i$ . The following Proposition holds.

**Proposition 1.1.** *The sums  $\sum_{j=1}^n \|U_{ij}\|^2$  for  $i \in \overline{1, n}$  are the eigenvalues of  $T$ .*

These results are generalized in [6] for Riemannian spaces. Let  $(M', g)$  be an  $n$ -dimensional submanifold of an  $n+k$ -dimensional Riemannian manifold  $(M, g)$ . We denote by  $\nabla_Y$  the covariant derivative with respect to the metric  $g$  in direction of the tangent vector  $Y$ . It verifies that the linear mapping

$$(1.6) \quad \varphi(X) = -\nabla_X N$$

is a symmetric operator, i.e.,

$$(1.7) \quad g(\varphi(X), Y) = g(X, \varphi(Y)).$$

Let  $N_1, \dots, N_k$  be arbitrary  $k$  orthonormal vector fields on  $U$ , and let  $Y_1, \dots, Y_n$  be arbitrary  $n$  orthonormal vector fields on  $U$ . The tensor field

$$(1.8) \quad T_{ij} = \sum_{\alpha=1}^k \sum_{r=1}^n g(Y_r, \nabla_{Y_i} N_\alpha) g(Y_r, \nabla_{Y_j} N_\alpha)$$

does not depend on the choice of the orthonormal system  $N_\alpha$ . In special case if  $(M, g)$  is an Euclidean space, then this field  $T$  coincides with the field  $T$  defined by (1.3). If we consider new orthonormal base  $Y'_i$  which is obtained from the basis  $Y_i$  by an orthogonal transformation  $C$ , then we have

$$(1.9) \quad T' = C \cdot T \cdot C^{-1}.$$

Thus the eigenvalues of  $T$  are invariant on the base tangent vectors and since  $T$  is symmetric, its eigenvalues are real numbers and the corresponding eigenvectors are orthogonal. If  $(t_1, \dots, t_n)$  is an arbitrary eigenvector of  $T_{ij}$ , then  $t_1 Y_1 + \dots + t_n Y_n$  is eigenvector of the corresponding (1.1) tensor field  $T_j^i$  and it does not depend on the base tangent vectors  $Y_1, \dots, Y_n$ . The eigenvectors of  $T_j^i$  determine the *principal directions*. The eigenvalues which are non-negative scalars are squares of the *principal curvatures*. It is more convenient to define the *principal normal vectors*  $U_i$  by

$$(1.10) \quad U_i = \sum_{\alpha=1}^k -g(Z_i, \nabla_{Z_i} N_\alpha) N_\alpha,$$

and also the vectors

$$(1.11) \quad U_{ij} = \sum_{\alpha=1}^k -g(Z_i, \nabla_{Z_j} N_\alpha) N_\alpha,$$

where  $Z_1, \dots, Z_n$  are orthonormal tangent vectors which determine the principal directions. The Proposition 1.1 also holds in this case, where the norm is taken from the Riemannian metric  $g$ .

Finally we define [7] an angle between two  $n$ -dimensional subspaces  $\Sigma_1$  and  $\Sigma_2$  of  $R^m$ , or of any vector space endowed with inner product, where  $m = n + k$ . Assume that  $\Sigma_1$  is generated by the  $n$ -form  $\omega_1 = a_1 \wedge \dots \wedge a_n$ , i.e.,  $a_1, \dots, a_n$  is a linearly independent set which generates the subspace  $\Sigma_1$  and assume that  $\Sigma_2$  is generated by the  $n$ -form  $\omega_2 = b_1 \wedge \dots \wedge b_n$ . Then the angle  $\varphi$  between  $\Sigma_1$  and  $\Sigma_2$  is defined by

$$(1.12) \quad \cos \varphi = \frac{\omega_1 \cdot \omega_2}{\|\omega_1\| \cdot \|\omega_2\|},$$

where

$$(1.13) \quad \omega_1 \cdot \omega_2 = \begin{vmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_n \\ a_2 \cdot b_1 & a_2 \cdot b_2 & \dots & a_2 \cdot b_n \\ \dots & \dots & \dots & \dots \\ a_n \cdot b_1 & a_n \cdot b_2 & \dots & a_n \cdot b_n \end{vmatrix},$$

$\|\omega_1\| = \sqrt{\omega_1 \cdot \omega_1}$ ,  $\|\omega_2\| = \sqrt{\omega_2 \cdot \omega_2}$ . It is proven that  $\varphi(\Sigma_1, \Sigma_2) = \varphi(\Sigma_1^*, \Sigma_2^*)$ , where  $\Sigma_1^*$  and  $\Sigma_2^*$  are the subspaces of  $R^m$  which are orthogonal to  $\Sigma_1$  and  $\Sigma_2$  respectively.

Moreover, the function

$$(1.14) \quad d(\Sigma_1, \Sigma_2) = \inf_{|\lambda|=1} \left\| \frac{\omega_1}{\|\omega_1\|} - \lambda \frac{\omega_2}{\|\omega_2\|} \right\| = \inf_{|\lambda|=1} \sqrt{2 - 2\lambda \frac{\omega_1 \cdot \omega_2}{\|\omega_1\| \cdot \|\omega_2\|}}$$

where  $\Sigma_1$  and  $\Sigma_2$  are generated by  $\omega_1$  and  $\omega_2$ , defines a metric in the Grassmann manifold of all  $n$ -dimensional subspaces of  $R^m$ .

## §2. Geometric interpretation of the principal directions and principal curvatures of submanifolds of Euclidean spaces

First we prove the following theorem.

**Theorem 2.1.** *If  $P_1, \dots, P_n$  and  $P_1 + \Delta P_1, \dots, P_n + \Delta P_n$  are two orthonormal systems of vectors in  $R^m$ , then for the angle  $\varphi$  between the corresponding two  $n$ -dimensional subspaces holds*

$$(2.1) \quad (\Delta\varphi)^2 = (\Delta P_1)^2 + (\Delta P_2)^2 + \dots + (\Delta P_n)^2 - 2 \sum_{1 \leq i < j \leq n} (P_i \cdot \Delta P_j)^2 + o((\Delta P_1)^2 + \dots + (\Delta P_n)^2).$$

**Proof.** By neglecting the small increments of higher order than 2, using the orthonormality of the both systems of vectors, from (1.12) and (1.13) it follows that

$$1 - \frac{(\Delta\varphi)^2}{2} = \begin{vmatrix} 1 + P_1 \cdot \Delta P_1 & P_1 \cdot \Delta P_2 & \dots & P_1 \cdot \Delta P_n \\ P_2 \cdot \Delta P_1 & 1 + P_2 \cdot \Delta P_2 & \dots & P_2 \cdot \Delta P_n \\ \dots & \dots & \dots & \dots \\ P_n \cdot \Delta P_1 & P_n \cdot \Delta P_2 & \dots & 1 + P_n \cdot \Delta P_n \end{vmatrix}.$$

Since

$$1 = (P_i + \Delta P_i) \cdot (P_i + \Delta P_i) = 1 + 2P_i \cdot \Delta P_i + \Delta P_i \cdot \Delta P_i$$

we obtain

$$(2.2) \quad P_i \cdot \Delta P_i = -\frac{1}{2}(\Delta P_i)^2.$$

Further, using that

$$0 = (P_i + \Delta P_i) \cdot (P_j + \Delta P_j) = P_i \cdot \Delta P_j + P_j \cdot \Delta P_i + \Delta P_i \cdot \Delta P_j,$$

for  $i \neq j$ , we obtain that

$$(2.3) \quad P_j \cdot \Delta P_i = -P_i \cdot \Delta P_j,$$

by neglecting the quantities of higher order.

By substituting of (2.2) and (2.3) in the above determinant, we obtain the equality (2.1).  $\square$

*Remark.* Note that (2.1) can also be written in the following form

$$(2.4) \quad (\Delta\varphi)^2 = (\Delta P_1)^2 + (\Delta P_2)^2 + \dots + (\Delta P_n)^2 - \sum_{i=1}^n \sum_{j=1}^n (P_i \cdot \Delta P_j)^2 +$$

$$+o((\Delta P_1)^2 + \dots + (\Delta P_n)^2),$$

because  $(P_i \cdot \Delta P_i)^2 = \frac{1}{4}(\Delta P_i)^4 = o((\Delta P_1)^2 + \dots + (\Delta P_n)^2)$  and (2.3).

Now let us assume that  $Y_1, \dots, Y_n$  be linearly independent vector fields on an  $n$ -dimensional manifold  $M'$  locally imbedded into  $R^m$  and let  $Y_{n+1}, \dots, Y_m$  be a basis of the normal space. Let their coordinates be  $Y_i = (Y_i^1, Y_i^2, \dots, Y_i^m)$  for  $1 \leq i \leq m$ . Without loss of generality we assume that  $Y_j^i = \delta_j^i$  for  $1 \leq i, j \leq m$  at a chosen point of the submanifold  $M'$ .

Now using that

$$\Delta Y_i^p = \sum_{j=1}^n \sum_{r=1}^m \Gamma_{rj}^p Y_i^r dx^j = \sum_{j=1}^n \Gamma_{ij}^p dx^j,$$

according to (2.4) for the angle  $\Delta\varphi$  between the tangent spaces of the chosen point and a close point of it, we get

$$\begin{aligned} (\Delta\varphi)^2 &= \sum_{i=1}^n \sum_{p=1}^m \left( \sum_{j=1}^n \Gamma_{ij}^p dx^j \right) \left( \sum_{t=1}^n \Gamma_{it}^p dx^t \right) - \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{p=1}^m \sum_{t=1}^n Y_i^p \Gamma_{jt}^p dx^t \right)^2 \\ (\Delta\varphi)^2 &= \sum_{i=1}^n \sum_{p=1}^m \left( \sum_{j=1}^n \Gamma_{ij}^p dx^j \right) \left( \sum_{t=1}^n \Gamma_{it}^p dx^t \right) - \sum_{i=1}^n \sum_{j=1}^n \left( \sum_{t=1}^n \Gamma_{jt}^i dx^t \right)^2, \\ (\Delta\varphi)^2 &= \sum_{i=1}^n \sum_{p=1}^m \sum_{j=1}^n \sum_{t=1}^n \Gamma_{ij}^p \Gamma_{it}^p dx^j dx^t - \sum_{i=1}^n \sum_{j=1}^n \sum_{p=1}^m \sum_{q=1}^n \Gamma_{ip}^j \Gamma_{iq}^j dx^p dx^q, \\ (\Delta\varphi)^2 &= \sum_{i=1}^n \sum_{p=n+1}^m \sum_{j=1}^n \sum_{t=1}^n \Gamma_{ij}^p \Gamma_{it}^p dx^j dx^t, \end{aligned}$$

and hence

$$(2.5) \quad \left( \frac{\Delta\varphi}{\Delta s} \right)^2 = \sum_{i=1}^n \sum_{p=n+1}^m \sum_{j=1}^n \sum_{t=1}^n \Gamma_{ij}^p \Gamma_{it}^p \frac{dx^j}{ds} \frac{dx^t}{ds} = \sum_{u=1}^n \sum_{v=1}^n A_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds},$$

where

$$(2.6) \quad A_{uv} = \sum_{i=1}^n \sum_{p=n+1}^m \Gamma_{iu}^p \Gamma_{iv}^p.$$

Now we introduce a *geometrical definition* of the principal directions and principal curvatures. Indeed, *principal directions* are those directions on the tangent space at the considered point such that the expression  $\left( \frac{\Delta\varphi}{\Delta s} \right)^2$  takes minimal or maximal value, where  $\Delta\varphi$  is the angle between the tangent spaces of the considered point and a point close to it. The square of the *principal curvatures* in chosen principal direction is just the value  $\left( \frac{\Delta\varphi}{\Delta s} \right)^2$  of this direction. Now, according to (2.5) and (2.6) we obtain that the principal directions (in geometric sense) are the eigenvectors of the matrix  $A = [A_{uv}]$  and the squares of the principal curvatures are the corresponding values of  $\left( \frac{\Delta\varphi}{\Delta s} \right)^2$ .

In order to prove that the principal directions and the principal curvatures defined now coincide with those defined using the tensor  $T$  from (1.3), it is sufficient to verify that the matrix  $T$  coincides with the matrix  $A$  in the considered special case  $Y_j^i = \delta_j^i$  at the considered point. According to (1.3) we obtain

$$T_{ij} = \sum_{p=n+1}^m \sum_{r=1}^n (-Y_i \cdot \nabla_{Y_r} Y_p) (-Y_j \cdot \nabla_{Y_r} Y_p).$$

Using that for  $p > n$  it holds

$$\begin{aligned} Y_i \cdot \nabla_{Y_r} Y_p &= -Y_p \cdot \nabla_{Y_r} Y_i = -Y_p \cdot (\nabla_{Y_i} Y_r - [Y_i, Y_r]) = \\ &= -Y_p \cdot \nabla_{Y_i} Y_r = -\sum_{j=1}^m Y_p \cdot \Gamma_{ri}^j Y_j = -\Gamma_{ri}^p, \end{aligned}$$

and hence we obtain

$$T_{ij} = \sum_{p=n+1}^m \sum_{r=1}^n \Gamma_{ri}^p \Gamma_{rj}^p = A_{ij}.$$

Since  $\varphi(\Sigma_1, \Sigma_2) = \varphi(\Sigma_1^*, \Sigma_2^*)$  ([7]), the previous geometric interpretation also holds if the tangent space is replaced by the orthogonal space of the tangent space. Indeed, the angle  $\Delta\varphi$  between the normal spaces of the chosen point and a close point of it we get

$$(\Delta\varphi)^2 = \sum_{i=n+1}^m \sum_{p=1}^m \left( \sum_{j=1}^n \Gamma_{ij}^p dx^j \right) \left( \sum_{t=1}^n \Gamma_{it}^p dx^t \right) - \sum_{i=n+1}^m \sum_{j=n+1}^m \left( \sum_{p=1}^m \sum_{t=1}^n Y_i^p \Gamma_{jt}^p dx^t \right)^2.$$

Analogously to (2.5) and (2.6) we obtain

$$(2.7) \quad \left( \frac{\Delta\varphi}{\Delta s} \right)^2 = \sum_{u=1}^n \sum_{v=1}^n B_{uv} \frac{dx^u}{ds} \frac{dx^v}{ds}$$

where

$$(2.8) \quad B_{uv} = \sum_{i=n+1}^m \sum_{p=1}^n \Gamma_{iu}^p \Gamma_{iv}^p.$$

We will prove that both symmetric  $n \times n$  matrices  $A$  and  $B$  are equal.

The Christoffel's symbols  $\Gamma_{jk}^i$  can be calculated by

$$\Gamma_{jk}^i = -\sum_{u=1}^m X_u^i \frac{\partial Z_j^u}{\partial x^k}, \quad i, j = \overline{1, m}, \quad k = \overline{1, n},$$

where  $X_1, \dots, X_m$  are arbitrary  $m$  vector fields such that the first  $n$  of them are tangent to  $M^1$  and  $Z_j^i$  is  $m \times m$  matrix which is inverse of  $X_j^i$  such that  $(X_j^1, \dots, X_j^m)$  are the coordinates of  $X_j$  for  $j = \overline{1, m}$ . Now we are able to prove that the matrices  $A$  and  $B$  are equal. It is sufficient to prove it for a special choice of the connection.

Indeed, assuming that  $X_1, \dots, X_m$  is an orthonormal system and  $X_j^i = \delta_j^i$  at the considered point, we get

$$Z_j^i = \delta_j^i, \quad \Gamma_{rj}^p = -\frac{\partial Z_r^p}{\partial x^j}.$$

Hence

$$\Gamma_{rj}^p = -\frac{\partial Z_r^p}{\partial x^j} = \frac{\partial Z_p^r}{\partial x^j} = -\Gamma_{pj}^r$$

and now it is obvious that  $A_{uv} = B_{uv}$ .

### §3. Geometric interpretation of the principal directions and principal curvatures of submanifolds of Riemannian spaces

In the previous section we calculated the angle between two different tangent spaces. This can be determined only if the manifold is imbedded into a space with vanishing curvature tensor. In the general case, if the manifold is imbedded into a Riemannian manifold, we should require another geometrical representation.

Let  $Y_1, \dots, Y_m$  be an orthonormal system of vectors in the Riemannian space, and assume that  $Y_1, \dots, Y_n$  are tangent vectors to the imbedded manifold. Let denote

$$\omega = Y_1 \wedge \dots \wedge Y_n.$$

We denote by  $\nabla_s$  the derivative  $\nabla_{Y_s}$ . Now we will prove that the tensor (1.8) can be written in the following form

$$(3.1) \quad T_{st} = g(\nabla_s \omega, \nabla_t \omega).$$

Without loss of generality we can assume that at the chosen point we have

$$Y_i = \frac{\partial}{\partial x^i}, \quad i = \overline{1, n}.$$

Since

$$\begin{aligned} \nabla_s \omega &= \nabla_s(Y_1 \wedge \dots \wedge Y_n) \\ &= \sum_{i=1}^n Y_1 \wedge \dots \wedge \nabla_s Y_i \wedge \dots \wedge Y_n = \\ &= \sum_{i=1}^n Y_1 \wedge \dots \wedge \sum_{j=1}^m \Gamma_{is}^j Y_j \wedge \dots \wedge Y_n = \\ &= \sum_{i=1}^n \sum_{j=n+1}^m \Gamma_{is}^j Y_1 \wedge \dots \wedge Y_{i-1} \wedge Y_j \wedge Y_{i+1} \wedge \dots \wedge Y_n + \sum_{i=1}^n \Gamma_{is}^i \omega = \\ &= \sum_{i=1}^n \sum_{j=n+1}^m \Gamma_{is}^j Y_1 \wedge \dots \wedge Y_{i-1} \wedge Y_j \wedge Y_{i+1} \wedge \dots \wedge Y_n. \end{aligned}$$

Further, it is easy to verify that

$$\begin{aligned} &g(Y_1 \wedge \dots \wedge Y_{i-1} \wedge Y_j \wedge Y_{i+1} \wedge \dots \wedge Y_n, \\ &Y_1 \wedge \dots \wedge Y_{r-1} \wedge Y_l \wedge Y_{r+1} \wedge \dots \wedge Y_n) = \delta_{jl} \delta_{ir}, \end{aligned}$$

and thus we obtain

$$\begin{aligned} g(\nabla_s \omega, \nabla_t \omega) &= g\left(\sum_{i=1}^n \sum_{j=n+1}^m \Gamma_{is}^j Y_1 \wedge \dots \wedge Y_{i-1} \wedge Y_j \wedge Y_{i+1} \wedge \dots \wedge Y_n, \right. \\ &\quad \left. \sum_{r=1}^n \sum_{l=n+1}^m \Gamma_{rt}^l Y_1 \wedge \dots \wedge Y_{r-1} \wedge Y_l \wedge Y_{r+1} \wedge \dots \wedge Y_n\right) = \\ &= \sum_{i=1}^n \sum_{j=n+1}^m \sum_{r=1}^n \sum_{l=n+1}^m \Gamma_{is}^j \Gamma_{rt}^l \delta_{jl} \delta_{ir} = \sum_{i=1}^n \sum_{j=n+1}^m \Gamma_{is}^j \Gamma_{it}^j. \end{aligned}$$

Further, using that

$$\nabla_{Y_i} Y_r = \Gamma_{ri}^j Y_j,$$

we obtain

$$\begin{aligned} T_{ij} &= \sum_{\alpha=1}^k \sum_{r=1}^n g(Y_r, \nabla_{Y_i} N_\alpha) g(Y_r, \nabla_{Y_j} N_\alpha) = \\ &= \sum_{t=n+1}^m \sum_{r=1}^n g(Y_r, \sum_{p=1}^m \Gamma_{ti}^p Y_p) g(Y_r, \sum_{q=1}^m \Gamma_{tj}^q Y_q) = \\ &= \sum_{t=n+1}^m \sum_{r=1}^n \Gamma_{ti}^r \Gamma_{tj}^r = \sum_{t=n+1}^m \sum_{r=1}^n (-\Gamma_{ri}^t) (-\Gamma_{rj}^t) = g(\nabla_i \omega, \nabla_j \omega). \end{aligned}$$

Thus the basic tensor  $T$  used to find the principal directions and principal curvatures is reduced in geometric form.

Moreover, instead of the principal normal vectors  $U_i$  and  $U_{ij}$ , now we have the following  $n$ -form. Let  $Z_1, \dots, Z_n$  be the unit tangent orthonormal vectors which determine the principal directions. Then

$$\nabla_1 \omega = \nabla_{Z_1} \omega, \dots, \nabla_n \omega = \nabla_{Z_n} \omega$$

are  $n$ -forms, called *principal  $n$ -forms*. Note that  $\nabla_s \omega$  is orthogonal to  $\omega$ , i.e.  $g(\nabla_s \omega, \omega) = 0$ , since  $g(\omega, \omega) = 1$ . It means that at least one vector of  $\nabla_s \omega$  is orthogonal to  $\omega$ . Obviously,  $g(\nabla_s \omega, \nabla_s \omega)$  are the eigenvalues of  $T$ . This is result analogous to Proposition 1.1. Note also that  $g(\nabla_s \omega, \nabla_t \omega) = 0$  if  $s \neq t$ .

Finally, we shall prove that

$$(3.2) \quad T_{st} = g(\nabla_s \omega^*, \nabla_t \omega^*)$$

where  $\omega^*$  is the unit  $(m-n)$ -form which generates the orthogonal space of the tangent space. This gives the dual interpretation of the principal directions and curvatures.

According to the previous assumptions of this section we can set

$$\omega^* = Y_{n+1} \wedge \dots \wedge Y_m.$$

Then

$$\begin{aligned} \nabla_s \omega^* &= \nabla_s (Y_{n+1} \wedge \dots \wedge Y_m) = \sum_{u=n+1}^m Y_1 \wedge \dots \wedge \nabla_s Y_u \wedge \dots \wedge Y_n = \\ &= \sum_{u=n+1}^m Y_{n+1} \wedge \dots \wedge \sum_{j=1}^m \Gamma_{us}^j Y_j \wedge \dots \wedge Y_m = \\ &= \sum_{u=n+1}^m \sum_{j=1}^n \Gamma_{us}^j Y_{n+1} \wedge \dots \wedge Y_{u-1} \wedge Y_j \wedge Y_{u+1} \wedge \dots \wedge Y_m + \sum_{u=n+1}^m \Gamma_{us}^u \omega = \\ &= \sum_{u=n+1}^m \sum_{j=1}^n \Gamma_{us}^j Y_{n+1} \wedge \dots \wedge Y_{u-1} \wedge Y_j \wedge Y_{u+1} \wedge \dots \wedge Y_m. \end{aligned}$$

Further, it is easy to verify that

$$\begin{aligned} g(Y_{n+1} \wedge \dots \wedge Y_{u-1} \wedge Y_j \wedge Y_{u+1} \wedge \dots \wedge Y_m, \\ Y_{n+1} \wedge \dots \wedge Y_{v-1} \wedge Y_l \wedge Y_{v+1} \wedge \dots \wedge Y_m) = \delta_{jl} \delta_{uv} \end{aligned}$$

and thus we obtain

$$\begin{aligned} g(\nabla_s \omega^*, \nabla_t \omega^*) &= g(\sum_{u=n+1}^m \sum_{j=1}^n \Gamma_{us}^j Y_{n+1} \wedge \dots \wedge Y_{i-1} \wedge Y_j \wedge Y_{u+1} \wedge \dots \wedge Y_m, \\ &= \sum_{v=n+1}^m \sum_{l=1}^n \Gamma_{vt}^l Y_1 \wedge \dots \wedge Y_{v-1} \wedge Y_l \wedge Y_{v+1} \wedge \dots \wedge Y_m) = \\ &= \sum_{u=n+1}^m \sum_{j=1}^n \sum_{v=n+1}^m \sum_{l=1}^n \Gamma_{us}^j \Gamma_{vt}^l \delta_{jl} \delta_{uv} = \sum_{j=1}^n \sum_{u=n+1}^m \Gamma_{us}^j \Gamma_{ut}^j = \\ &= \sum_{j=1}^n \sum_{u=n+1}^m (-\Gamma_{js}^u) (-\Gamma_{jt}^u) = \sum_{j=1}^n \sum_{u=n+1}^m \Gamma_{js}^u \Gamma_{jt}^u = g(\nabla_s \omega, \nabla_t \omega). \end{aligned}$$

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