Topological Properties of Killing Tensor Fields

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Abstract

Let (M,g) be a compact Riemannian manifold of dimension n. Let $K^q(M,\mathbb{R})$, $q=\overline{2,n-2}$ be the vector space of Killing tensor fields on M. The aim of the present paper is to study if $\dim K^q(M,\mathbb{R})$ has topological meaning.

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§1. Introduction

Let (M,g) be a compact Riemannian manifold of dimension n. The study of the existence of special tensor fields on (M,g) is an open problem. An important problem is also to determine a relation of the topology of M and these tensor fields.

The purpose of the present paper is to find out if dim $K^q(M,\mathbb{R})$, $q=\overline{2,n-2}$ is a topological invariant, where $K^q(M,\mathbb{R})$ is the vector space of Killing tensor fields of order q on M.

The paper contains four paragraphs. Each of them is analyzed as follows.

The first paragraph is the introduction. The basic notions of a Riemannian manifolds are given in the second paragraph. The third paragraph includes the theory of Killing tensor fields of order q. The relation between topology and Killing tensor fields is given in the fourth paragraph.

§2. Basic properties of tensor fields

Let (M,g) be a compact Riemannian manifold of dimension n. Let $D_q(M,\mathbb{R})$ be the vector space of covariant tensor fields of order q on M. It is known that $D_q(M,\mathbb{R})$ is a module over the algebra $D^0(M,\mathbb{R})$ and a vector space over $\mathbb{R}$ of infinite dimension. Let $\Lambda^q(M,\mathbb{R})$ be the subspace of $D_q(M,\mathbb{R})$, which consists of all antisymmetric tensor fields of order q, that is the exterior q-forms on M. The dimension of $\Lambda^q(M,\mathbb{R})$ is $\binom{n}{q}$. We shall study the vector spaces of $\Lambda^q(M,\mathbb{R}), q=\overline{2,n-2}$.

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If $w \in \Lambda^q(M, \mathbb{R})$, then on the chart (U, ϕ) on M with local coordinate system $(x^1, \dots x^n)$, the $q-$form w can take the form

$$w = \frac{1}{q!} w_{i_1 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q} \ , \quad i_1, i_2, \dots i_q = \overline{1, n}.$$

From (1) we can obtain the local norm $||w||^2$ of w, which has the form

$$||w||^2 = \frac{1}{q!} w_{i_1 \dots i_q} w^{i_1 \dots i_q},$$

where

$$w^{i_1 \dots i_q} = g^{i_1 j_1} \dots g^{i_q j_q} w_{j_1 \dots j_q}$$

Hence $||w||^2$ is a function on M. If we apply the Laplace operator Δ on $||w||^2$, we obtain another function $\Delta(||w||^2)$ on M, which is given by

$$\frac{1}{2} \Delta ||w||^2 = < \delta \nabla w, w > - ||\nabla w||^2$$

From the formula (4), by integration, we have

$$\int_M [< \delta \nabla w, w > - ||\nabla w||^2] \, dM \ = 0,$$

where dM is the volume element on the Riemannian manifold (M, g).

Let (U, ϕ) be a chart of (M, g) with local coordinate system $(x^1, \dots x^n)$. Then on (U, ϕ) the $q-$form w can be written

$$w = \frac{1}{q!} w_{i_1 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q},$$

where $\{w_{i_1 \dots i_q}\}$ the components of w on (U, ϕ) and $i_1, i_2, \dots i_q = \overline{1, n}$.

If we apply the Laplace operator Δ on w, we have another $q-$form Δw, that is $\Delta w \in \Lambda^q(M, \mathbb{R})$. The $q-$form Δw on (U, ϕ) has the expression

$$\begin{aligned} \Delta w \ &= -\tfrac{1}{q!} [g^{ij} \nabla_j \nabla_i w_{i_1 \dots i_q} - \textstyle\sum_{s=1}^{q} \rho_{i_s}^{v} w_{i_1 \dots i_{s-1} v i_{s+1} \dots i_q} - \\ &- \textstyle\sum_{t<s}^{1,q} R_{i_t i_s}^{uv} w_{i_1 \dots i_{t-1} u i_{t+1} \dots i_{s-1} v i_{s+1} \dots i_q}] dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_q}. \end{aligned}$$

The components $\{(\Delta w)_{i_1 \dots i_q}\}$ of Δw on the chart (U, ϕ) are the following

$$\begin{aligned} (\Delta w)_{i_1 \dots i_q} \ &= \tfrac{1}{q!} [-g^{ij} \nabla_j \nabla_i w_{i_1 \dots i_q} + \rho_{i_s}^{v} w_{i_1 \dots i_{s-1} v i_{s+1} \dots i_q} + \\ &+ R_{i_t i_s}^{uv} w_{i_1 \dots i_{t-1} u i_{t+1} \dots i_{s-1} v i_{s+1} \dots i_q}]. \end{aligned}$$

The local norm of w and Δw, by means of (8), takes the form:

$$< \Delta w, w > = < \delta \Delta w, w > + \frac{1}{(q-1)!} F_q(w, w),$$

where $F_q(w, w)$ is a quadratic form defined as follows:

$$F_q(w, w) = \rho_{ij} w^{i i_2 \dots i_q} w^{j}_{i_2 \dots i_q} + \frac{q-1}{2} R_{ijkl} w^{ij i_3 \dots i_q} w^{kl}_{i_3 \dots i_q}.$$

The relation (4), by means of (9), takes the form

$$\frac{1}{2}\Delta\|w\|^2 =< \nabla w, w > -\|\nabla w\|^2 - \frac{1}{(q-1)!}F_q(w,w).$$

The relation (11) by integration on the manifold (M, g) becomes

$$\int_M [< \nabla w, w > -\|\nabla w\|^2 - \frac{1}{(q-1)!}F_q(w)]dM = 0.$$

The quadratic form $F_q(w,w)$, given by (10), can be written

$$F_q(w,w) = F_q^1(w,w) + F_q^2(w,w),$$

where

$$F_q^1(w,w) = \rho_{ij} w^{ii_2\dots i_q} w^j_{\;i_2\dots i_q}$$

$$F_q^2(w,w) = R_{ijkl} w^{iji_3\dots i_q} w^{kl}_{\;\;i_3\dots i_q}.$$

The quadratic forms (14) and (15) can be also written

$$F_q^{01}(w,w) = b_{i_1 i_2\dots i_q j_1, j_2\dots j_q} w^{i_1 i_2\dots i_q} w^{j_1 j_2\dots j_q}$$

$$F_q^{02}(w,w) \; = c_{i_1 i_2\dots i_q j_1 j_2\dots j_q} w^{i_1, i_2\dots i_q} w^{j_1 j_2\dots j_q}.$$

The coefficient $b_{i_1 i_2\dots i_q, j_1, j_2,\dots j_q}$ has the form

$$b_{i_1, i_2\dots i_q, j_1, j_2,\dots j_q} = \rho_{i_1 j_1} g_{i_2 j_2} g_{i_3 j_3} \cdots g_{i_q j_q}.$$

When we fix i_1 and j_1 we get

$$b_{i_1 i_2\dots i_q j_1 j_2\dots j_q},$$

while $i_2 \dots i_q$, $j_2 \dots j_q$ run all the integers from 1 to n.

The coefficients $c_{i_1 i_2\dots i_q j_1 j_2\dots j_q}$ are given by

$$c_{i_1 i_2\dots i_q j_1 j_2\dots j_q} \; = \frac{q-1}{2} R_{i_1 i_2 j_1 j_2} g_{i_3 j_3} \cdots g_{i_q j_q}.$$

When we fix (i_1, i_2) and (j_1, j_2), we find

$$c_{i_1 i_2\dots i_q j_1 j_2\dots j_q}$$

while $i_2 \dots i_q$, $j_2 \dots j_q$ run all the integers from 1 to n.

Now, we can prove the following

Proposition 1. *The tensor field $b = (b_{i_1 i_2\dots i_q j_1 j_2\dots j_q})$ is symmetric with respect to any of two indices (i_v, j_v) $v = 1, 2\dots q$ and with respect to $(i_1, i_2, \dots i_q)$ and $(j_1, j_2, \dots j_q)$. Therefore the quadratic form (16) is symmetric.*

Proof. We know that the Ricci tensor field ρ is symmetric. The same is true for the metric tensor field g. From these we have

$$\rho_{i_1 j_1} = \rho_{j_1 i_1}\;, \qquad g_{i_2 j_2} = g_{j_2 i_2}, \;\; \cdots \;\; g_{i_q j_q} = g_{j_q i_q}.$$

From (21) we obtain

$$b_{j_1 j_2 \dots j_q i_1 i_2 \dots i_q} = b_{i_1 i_2 \dots i_q j_1 j_2 \dots j_q},$$

or one of the following relations

$$b_{j_1 i_2 \dots i_q i_1 j_2 \dots j_q} = b_{i_1 i_2 \dots i_q j_1 j_2 \dots j_q}.$$

□

The components $c_{i_1 i_2 \dots i_q j_1 j_2 \dots j_q}$ of the tensor field c, according to the formula (20), have the form

$$c_{i_1 i_2 \dots i_q j_1 j_2 \dots j_q} = \frac{q-1}{2} R_{i_1 i_2 j_1 j_2} g_{i_3 j_3} \cdots g_{i_q j_q}.$$

Some properties of the tensor field c can be discribed by the proposition.

Proposition 2. *The tensor field $c = (c_{i_1 i_2 \dots i_q j_1 j_2 \dots j_q})$ is symmetric with respect to $(i_1 i_2, j_1 j_2)$ and any of two indices $(i_v, j_v,\ \ v = \overline{3,q})$ and as well as with respect to $(i_1, i_2, \dots i_q)$ and $(j_1, j_2, \dots j_q)$. Therefore the quadratic form (17) is symmetric with respect to $(i_1, i_2, \dots i_q)$ and $(j_1, j_2, \dots j_q)$.*

Proof. It is known from the properties of curvature tensor field that we have

$$R_{i_1 i_2 j_1 j_2} = R_{j_1 j_2 i_1 i_2}.$$

We also have

$$g_{i_3 j_3} = g_{j_3 i_3}\ ,\ g_{i_4 j_4} = g_{j_4 i_4},\ \dots\ g_{i_q j_q} = g_{j_q i_q}.$$

From (25) and (26) we conclude that

$$c_{j_1, j_2 \dots j_q i_1 i_2 \dots i_q} = c_{i_1, i_2 \dots i_q j_1 j_2 \dots j_q},$$

or one of the following relations

$$c_{i_2 i_1 i_3 \dots i_q j_2 j_1 j_3 \dots j_q} = c_{i_1 i_2 i_3 \dots i_q j_1 j_2 j_3 \dots j_q}$$

$$c_{i_1 i_2, j_3 i_4 \dots i_q j_1 j_2 i_3 j_4 \dots j_q} = c_{i_1, i_2 i_3 \dots i_q j_1 j_2 j_3 \dots j_q}.$$

□

§3. Killing tensor fields

Let $\Lambda_q(M, \mathbb{R})$ be the vector space of antisymmetric contravariant tensor fields of order q on a compact Riemannian manifold (M, g).

There is an isomorphism between $\Lambda_q(M, \mathbb{R})$ and $\Lambda^q(M, \mathbb{R})$.

Therefore we can exchange the notion of antisymmetric covariant tensor field of order q with of antisymmetric contravariant tensor field of order q and conversely.

Let (U, ϕ) be a chart of M with local coordinate system $(x^1, x^2, \dots x^n)$. An antisymmetric contravariant tensor field X of order q is called *Killing*, if the components $(w_{i_1 i_2 \dots i_q})$ of the corresponding Killing exterior q-form w satisfy the relations

$$\nabla_i w_{j i_2 \dots i_q} + \nabla_j w_{i i_2 \dots i_q} = 0, \quad \nabla_i w^i_{i_2 \dots i_q} = 0.$$

In the formulas (28) we have used the symbol w for the Killing exterior q-form associated to the Killing tensor field X of order q .

Therefore the exterior q-form w is Killing if its covariant derivative ∇w is an exterior $(q+1)-$form.

The conditions (28), for a Killing exterior $q-$form w, can be expressed as follows:

$$(q+1)\nabla w = dw \ , \qquad \delta w = 0.$$

The Killing exterior $q-$form w on the chart (U,ϕ) with local coordinates $(x^1,\dots x^n)$ satisfies the relations

$$g^{ij}\nabla_i\nabla_j w_{i_1\dots i_q} \quad +\tfrac{1}{q}\sum_{s=1}^{q}\rho^t_{i_s} w_{i_1\dots i_{s-1}ti_{s+1}\dots i_q} + \\ +\tfrac{1}{q}\sum_{t<s}^{1\dots q} R^{vu}_{i_s i_t} w_{i_1\dots i_{t-1}vi_{t+1}\dots i_{s-1}ui_{s+1}\dots i_q} = 0$$

and

$$g^{ij}\nabla_j w_{ii_2\dots i_q} = 0.$$

The inner product of w and $\Delta w-(q+1)\delta\nabla w$ satisfies the relation

$$\frac{1}{2}\Delta\|w\|^2 = -\|\nabla w\|^2 + \frac{1}{q!}F_q(w,w) - \frac{1}{q}\langle\Delta w-(q+1)\delta\nabla w, w\rangle,$$

for any exterior $q-$form w. The formula (32) implies by integration

$$\int[<\Delta w-(q+1)\delta\nabla w, w> +q\|\nabla w\|^2 - \frac{1}{(q-1)!}F_q(w,w)]dM = 0.$$

If w is a Killing exterior $q-$form, then from the relations (29) , we have

$$\Delta w = (q+1)\delta\nabla w, \quad \delta w = 0.$$

Therefore the relation (33) by means of (34) becomes

$$\int_M [q\|\nabla w\|^2 - \frac{1}{(q-1)!}F_q(w,w)]dM = 0.$$

Using (13), the equality (35) takes the form

$$\int_M [q\|\nabla w\|^2 - \frac{1}{(q-1)!}(F^1_q(w,w)+F^2_q(w,w)]dM = 0,$$

which by virtue of (16) and (17) rewrites

$$\int_M [q\|\nabla w\|^2 - \frac{1}{(q-1)!}(F^{01}_q(w,w)+F^{02}_q(w,w)]dM = 0.$$

In the following, we use the symbols

$$l_P = \sup\{F^{01}(w_P,w_P)/w_P \ unit\ vector\ in\ \Lambda^q(T_P(M))\}$$

$$L_{\max} = \sup\{l_P/P\in M\}$$

$$d_P = \sup\{F^{02}(w_P, w_P)/w_P \textit{ unit vector in } \Lambda^q(T_P(M)\}$$
$$D_{\max} = \sup\{d_P/P \in M\},$$

and state

Theorem 3. *Let (M, g) be a compact Riemannian manifold of dimension n. Let $K^q(M, \mathbb{R})$ be the vector space of the Killing tensor fields of order q on M. If $L_{\max} + D_{\max} \leq 0$ and if there exist points P_1 and P_2 such that $l_{P_1} < 0$ or $d_{P_2} < 0$, then $dimK^q(M, \mathbb{R}) = 0$. If $L_{\max} + D_{\max} = 0$, then $dimK^q(M, \mathbb{R}) \leq \binom{n}{q}$.*

Proof. From the conditions

$$L_{\max} + D_{\max} \leq 0,$$

we conclude from the formula (37) that

$$\nabla w = 0,$$

that means w is parallel. From the conditions

$$l_{P_1} < 0 \quad or \quad d_{P_2} < 0,$$

we obtain $w = 0$, which infers

$$dimK^q(M, \mathbb{R}) = 0.$$

From the conditions

$$L_{\max} + D_{\max} = 0$$

we conclude that

$$\int \|\nabla w\|^2 \leq 0,$$

which implies $\nabla w = 0$ and therefore w is parallel. Hence we have

$$dimK^q(M, \mathbb{R}) \leq \binom{n}{q}.$$

□

§4. Topological properties of $\mathbf{K}^q(M, \mathbb{R})$

Let (M, g) be a compact orientable Riemannian manifold of dimension n. Let $H^q(M, \mathbb{R})$ be the vector space of harmonic $q-$forms on (M, g). Then $\dim H^q(M, \mathbb{R}) = b_q$, is the $q-$Betti number, which is a topological property of M, that means, it is independent of the Riemannian metric g on M.

We recall the following

Conjecture 4. *Let (M, g) be a compact orientable Riemannian manifold of dimension n. Let $K^q(M, \mathbb{R})$ be the vector space of the Killing tensor fields on (M, g). Is the* $\dim K^q(M, \mathbb{R})$ *a topological invariant?*

This conjecture is not true. This is a consequence of the following theorem.

Theorem 5. *The dimension of $K^q(M, \mathbb{R})$ is not a topological invariant on the compact manifold M.*

Proof. There are constants $\beta(n) > \gamma(n)$ depending only on the dimension n of the compact manifold M ([3]) such that M admits a complete metric g with Ricci curvature $\rho(g)$ satisfying the inequalities

$$-\beta(n) < \rho(g) < -\gamma(n).$$

The inequalities (48) imply that $L_{\max} < 0$. If we take a Riemanniam manifold N having two metrics g_1 and g_2 such that

$$L_{\max}(g_1) < 0, \quad L_{\max}(g_2) > 0,$$

we conclude that the quadratic forms $F_q(w,w)(g_1)$ and $F_q(w,w)(g_2)$ give different results for the formulas (37) and therefore $\dim K^q(N, \mathbb{R}) = p$ depends on the metric. Hence p is not topological invariant. The same is true for $\dim K^q(M, \mathbb{R})$. □

References

[1] P. H. Berard, *Spectral Geometry. Direct and Inverse Problems,* Lecture Notes in Mathematics, 1207, Springer-Verlag, Berlin, 1986.

[2] R.Brooks, *A connection of Metrics of Negative Ricci Curvature,* Journal of Differential Geometry, 29(1989), 85-94.

[3] J. Lohcamp, *Negatively Ricci curved Manifold,* Bulletin of A.M.S., 27(1992), 288-291.

[4] Gr. Tsagas, *A Relation between Killing Tensor Fields and Negative Pinched Riemannian Manifolds,* Proc. A.M.S., 22(1969), 476-478.

[5] Gr. Tsagas, *Some Properties of Negative Pinched Riemannian Manifolds of Dimension 5 and 7,* Journal of Differential Geometry, 3(1969), 523-530.

[6] Gr. Tsagas, *Onthe Killing tensor fields on a compact Riemannian manifold,* Balkan Journal of Geometry and its Applications, 1(1996), No 1, 91-96.

[7] K. Yano and S. Bochner, *Curvature and Betti Numbers,* Princeton University Press, Princeton, 1953.

[8] K. Yano, *On Harmonic and Killing Vector Fields*, Ann. Math., 55(1952), 38-45.

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