Note on Skew-Symmetric Vector Fields on Kahlerian Manifolds

G.Caristi and M.Ferrara

Abstract

Let (M, J, Ω, g) be a $2m-$dimensional Kahlerian manifold endowed with the $(1,1)-$structure tensor field J and the structure symplectic form Ω. Let X be a skew symmetric Killing (abrev. S.S.K.) vector field having Y as generative and let $X^{\flat}$ and $Y^{\flat}$ the dual forms of X and Y respectively.

Within this framework, if ${}^{\flat}Y$ is the image of Y via the symplectic isomorphism (i.e., $\Omega^{\flat}(Y) = {}^{\flat}Y$), we prove that the necessary and sufficient condition in order that X define an infinitesimal automorphism of Ω is that the Pfaffian ${}^{\flat}Y$ be closed.

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§1. Results

Let (M, J, Ω, g) be a $2m-$dimensional Kahlerian C^{∞} manifold, where J and Ω are the $(1,1)-$*structure tensor field* and *the symplectic form* respectively, i.e.,

$$J^2 = -1, \qquad d\Omega = 0.$$

Let $O = \{e_A\}_{A=\overline{1,2m}}$ be a local frame field adapted to the structure. Considering ω as an associated dual frame, since Ω is Hermitan, one can write

$$\Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*}, \qquad a^* = a + m,$$

If dp is *the soldering form of* M, then one has

$$dp = \sum_{A=1}^{2m} \omega^A \otimes e_A.$$

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Also, E.Cartan's structure equations for the canonic connection have the form

$$\begin{aligned} \nabla e &= \theta \otimes e \\ d\omega &= -\theta \wedge \omega \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \tag{1}$$

In the above equations θ (resp. Θ) are the local connection forms in the bundle $O(M)$ (resp. the curvature 2−forms). In a Kahlerian manifold, the connection forms satisfy

$$\theta_b^a = \theta_{b^*}^{a^*}, \qquad \theta_b^{a^*} = \theta_a^{b^*}.$$

Let $X = X^A e_A$, $X^A \in \Lambda^0 M$ be a vector field on M, and assume that X is a skew-symmetric Killing (abbrev. S.S.K.) vector field, that is

$$\nabla X = X \wedge Y, \qquad Y \in \mathcal{X}(M), \tag{2}$$

where $\wedge$ is the wedge product of vector fields; also, (2) may be expanded as

$$\nabla X = Y^\flat \otimes X - X^\flat \otimes Y, \tag{3}$$

where $\flat : TM \to T^*M$ is the musical isomorphism defined by g, and $\sharp$ is the inverse of $\flat$. The vector field Y of (2) has bean called in [3] *the generative of* X.

From the structure equations (4) one has, using (3),

$$dX^A + X^B \theta_B^A = X^A Y^\flat - Y^A X^\flat, \tag{4}$$

and since

$$X^\flat = \sum_{A=1}^{2m} X^A \omega^A,$$

one infers from (4) by a standard calculation

$$dX^\flat = 2Y^\flat \wedge X^\flat.$$

This is a reformulation of Rosca's Lemma [3], which shows that $X^\flat$ is an exterior recurrent form ([1]). We deduce that by definition ([3]), the generative $Y^\flat$ of a S.S.K. vector field is a closed 1-form. If ${}^\flat X$ is the symplectic isomorphism 1-form, i.e.

$${}^\flat X = -i_X \Omega,$$

then by (4) one derives straightforward

$$d^\flat X = Y^\flat \wedge {}^\flat X + X^\flat \wedge {}^\flat Y.$$

But the necessary and sufficient condition that X defines an initesimal automorphism of Ω, i.e., $\pounds_X \Omega = 0$, is

$$Y^\flat \wedge {}^\flat X + X^\flat \wedge {}^\flat Y = 0 \tag{5}$$

and since $dY^\flat = 0$, it follows from [3] and (5) that

$$d\,{}^\flat Y = 0 \Leftrightarrow \pounds_X \Omega = 0.$$

Hence we can formulate the following

Theorem. *Let Ω be the symplectic form of a Kahlerian manifold and X be a S.S.K. vector field on M having the vector field Y as generative. Then the necessary and sufficent condition that X defines an infinitesimal automorphism of Ω is that the symplectic isomorphism form ${}^\flat Y$ of Y be closed.*

References

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Authors' addresses:

Giuseppe Caristi and Massimiliano Ferrara
Institute of Mathematics, Faculty of Economics,
University of Messina, Via dei Verdi 75, 98122 Messina - Italy
Email: gcaristi@dipmat.unime.it