Note on Solvable Lie Algebras and Convex Cones of Dimension 8 over $\mathbf{R}$

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Abstract

The aim of this paper is to describe the connection between convex cones and solvable Lie algebras. The first paragraph is a brief introduction. The second paragraph contains basic definitions and results of solvable Lie algebras and convex cones. The last paragraph describes the connection between solvable Lie algebras and convex cones of dimension 8 over $\mathbb{R}$.

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§1. Introduction

Let $D \subseteq \mathbb{R}^n$ be a convex domain and $V(D)$ be a convex cone on the convex domain D. Let ([5])

$$A = \sum_{1 \le i,j \le M} A_{ij}$$

be a T-algebra of rank M provided with an involutive anti-automorphism $*$, and let $\mathbf{g}$ be a solvable Lie algebra included in A ([9])

$$\mathbf{g} = \sum_{1 \le j < i \le M} A_{ij}.$$

§2. Solvable Lie algebras and convex cones

2.1. Solvable Lie algebras. Let $\mathbf{g}$ be a Lie algebra. We call $[\mathbf{g}, \mathbf{g}]$ *the derived algebra* of $\mathbf{g}$ and denote it by $D\mathbf{g}$. If $\mathbf{h}$ is an ideal of $\mathbf{g}$, then so is $D\mathbf{h}$. It can be proved that ([9])

$$[\mathbf{g}, D\mathbf{h}] = [\mathbf{g}, [\mathbf{h}, \mathbf{h}]] \subseteq [\mathbf{h}, [\mathbf{h}, \mathbf{g}]] + [\mathbf{h}, \mathbf{h}[\mathbf{g}, \mathbf{h}]] \subseteq [\mathbf{h}, \mathbf{h}] + [\mathbf{h}, \mathbf{h}] = D\mathbf{h}.$$

We define by induction the following series of subalgebras

$$D^{(0)} = \mathbf{g},\ D^{(1)}y = D\mathbf{g}, \ldots, D^{(n+1)}\mathbf{g} = D(D^{(n)}\mathbf{g}),\ n \in \mathbb{N}.$$

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In this way, we obtain a series of ideals satisfying ([9])

$$D^{(0)}\mathbf{g} \supseteq D^{(1)}\mathbf{g} \supseteq \ldots \supseteq D^{(n)}\mathbf{g} \supseteq \ldots .$$

This series is called *the derived series* of $\mathbf{g}$. If there exists a positive integer n such that $D^{(n)}\mathbf{g} = \{0\}$, then $\mathbf{g}$ is said to be a *solvable Lie algebra.*

We enumerate in brief certain properties of solvable Lie algebras which prove to be useful in the following.

1. Subalgebras of solvable algebras are solvable. Homomorphic images of solvable algebras are solvable (in particular, quotient algebras of solvable algebras are solvable).

2. If $\mathbf{g}$ is a Lie algebra, $\mathbf{h}$ is an ideal of $\mathbf{g}$ and the Lie algebras $\mathbf{h}$ and $\mathbf{g}/\mathbf{h}$ are solvable, then $\mathbf{g}$ is solvable.

3. Direct sums of solvable Lie algebras are solvable.

4. If an ideal $\mathbf{h}$ of a Lie algebra $\mathbf{g}$ is solvable, then it is called a *solvable ideal.* If $\mathbf{g}$ does not contain any solvable ideal except $\{0\}$, then $\mathbf{g}$ is said to be *semisimple.* Conversely, if $\mathbf{h}$ is a non-zero solvable ideal, there then exists a non-negative integer n such that $D^{(n-1)}\mathbf{h} \neq \{0\}$ and $D^{(n)}\mathbf{h} = \{0\}$. Thus $D^{(n-1)}\mathbf{g}$ is a non-zero abelian ideal.

5. The set of all upper triangular matrices of $gl(N, \mathbb{C})$, i.e., matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \ldots & x_{1n} \\ 0 & x_{22} & x_{23} & \ldots & x_{2n} \\ 0 & 0 & x_{33} & \ldots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_{nn} \end{pmatrix}$$

is a solvable Lie algebra ([9]).

2.2. Convex cones

Let $D \subseteq \mathbb{R}^n$ be a convex domain, noninvariant under any affine transformations of $\mathbb{R}^n$. If the group $A(D)$ acts transitively on D, then the domain D is said to be *homogenenous.* From a homogeneous convex domain in D in $\mathbb{R}^n$, we define a *homogeneous convex cone*

$$V = V(D) \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R},$$

as follows ([7])

$$V(D) = \{(t^- x, t)^- \in \mathbb{R}^n \times \mathbb{R};\ x \in D,\ t > 0\}, \tag{2.1}$$

which is called *convex cone on the convex domain D.* Let $G(V)$ be the group of all linear automorphisms of V and g_V the canonical $G(V)$-invariant Riemannian metric on V. Then the natural embedding ([7])

$$\chi : x \in D \to (x, 1) \in V(G) \tag{2.2}$$

is equivariant with respect to the groups $A(D)$ and $G(V)$. Therefore, the Riemannian metric on D

$$g_D = \chi^* g_V$$

induced from (V, g_V) by χ is $A(D)$-invariant. The Riemannian metric g_0 is called the *canonical metric* of D.

We note that the canonical metric g_D is given from the characteristic function φ_V of V as follows. Let us put $\varphi_D = \varphi_V \circ \chi$. Then ([5])

$$g_D = \sum_{1 \le i,j \le n} \frac{\partial^2 \log \varphi_D}{\partial x^i \partial x^j} dx^i dx^j, \tag{2.3}$$

where $(x^1, x^2, \dots, x^n)$ is a system of affine coordinates of $\mathbb{R}^n$.

§3. Main results

Let

$$A = \sum_{1 \le i,j \le 4} A_{ij}$$

be a T-algebra of rank 4 provided with an involutive anti-automorphism $*$. Generally, the elements of A_{ij} will be denoted as $a_{ij}, b_{ij}, \dots$ and an arbitrary element a of A will be written as a matrix $a = (a_{ij})$, where a_{ij} is the A_{ij}-component of a. We define the subsets g, $T(A)$, $V(A)$ and X, as follows:

a)
$$g = \left\{ \lambda = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \in A \;\middle|\; , \; x_{ij} \in \mathbb{R}^* \; 1 \le i,j \le A \right\}$$

which is a solvable Lie algebra dimension 8 over $\mathbb{R}$;

b) $T(A) = \{\lambda = (\lambda_{ij}) \in A \mid \lambda_{ii} > 0,\ 1 \le i \le 4,\ \ \lambda_{ij} = 0,\ 1 \le i,j \le 4\}$
is a subset of g with the diagonal elements positive;

c)
$$V(A) = \left\{ \lambda + \lambda^* = \begin{pmatrix} k_{11} & x & y & z \\ x & k_{22} & 0 & \tilde{x} \\ y & 0 & k_{33} & 0 \\ z & \tilde{x} & 0 & k_{44} \end{pmatrix} \;\middle|\; \begin{cases} k_{11} = x_{11} + x_{11} \\ k_{22} = x_{22} + x_{22} \\ k_{33} = x_{33} + x_{33} \\ k_{44} = x_{44} + x_{44} \\ x = x_{12}, y = x_{13} \\ z = x_{14},\ \tilde{x} = x_{24} \end{cases} , \; \lambda \in T(A) \right\},$$

$V(A) \subset X = \{x \in A,\ x^* = x\}$ is the set of symmetric (Hermitian) matrices.

Then it is known ([8]) that there exists a homogeneous convex cone in the real vector space X and $T(A)$ is a connected Lie group which acts on V simply transitively as a linear transformation by

$$\left(\lambda = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}, \; S + S^* = \begin{bmatrix} S_1 + S_1 & S_{12} & S_{13} & S_{14} \\ S_{12} & S_2 + S_2 & 0 & S_{24} \\ S_{13} & 0 & S_3 + S_3 & 0 \\ S_{14} & S_{24} & 0 & S_4 + S_4 \end{bmatrix} \right) \to$$

$$(\lambda+S)+(\lambda+S)^* = \begin{bmatrix} x_{11}+S_{11} & x_{12}+S_{12} & x_{13}+S_{13} & x_{14}+S_{14} \\ 0 & x_{22}+S_{22} & 0 & x_{24}+S_{24} \\ 0 & 0 & x_{33}+S_{33} & 0 \\ 0 & 0 & 0 & x_{44}+S_{44} \end{bmatrix} +$$

$$+ \begin{bmatrix} x_{11}+S_{11} & 0 & 0 & 0 \\ x_{12}+S_{12} & x_{22}+S_{22} & 0 & 0 \\ x_{13}+S_{13} & 0 & x_{33}+S_{33} & 0 \\ x_{14}+S_{14} & 0 & 0 & x_{44}+S_{44} \end{bmatrix} \in V,$$

that is

$$(\lambda, SS^*) \in T(A) \times V(A) \to (\lambda S)(\lambda S^*) \in V.$$

Conversely, every homogeneous convex cone is realized in this form up to a linear equivalence. We shall use the following notations ([5])

$$\begin{cases} n_{ij} = \dim A_{ji} = \dim A_{ij} \\ n_i = 1 + \dfrac{1}{2}\sum\limits_{k\neq i} n_{ij}, \quad i,j=\overline{1,4}, \\ S_p a = \sum\limits_{1\leq i\leq A} n_i a_{ii}, \quad (a=(a_{ij})\in A). \end{cases} \tag{3.1}$$

Then the numbers n_{ij} satisfy the condition

$$\max\langle n_{ij}, n_{jk}\rangle \leq n_{ik}, \quad \forall i,j,k \in \overline{1,4}, \tag{3.2}$$

for all indices $i<j<k$ with $n_{ij}\cdot n_{jk}\neq 0$.

Moreover, the element $e=(e_{ij})$, $e_{ij}=\delta_{ij}$ (the Kroneker delta), is the unit element of T, and also e is contained in V. Hence, the tangent space $T_e(V)$ of V at the point e may be naturally identified with the ambient space X and also with the Lie algebra λ of T. On the other hand, the solvable Lie algebra λ may be identified with the subspace $\sum\limits_{1\leq i\leq j\leq A} A_{ij}$ of A provided with the bracket product $[a,b]=ab-ba$. A canonical linear isomorphism between λ and X is given by ([5])

$$\xi : a \in \lambda = \sum_{1\leq i\leq j\leq A} A_{ij} \;\to\; a+a^* \in X = T_e(V).$$

Under this identification, by using the canonical Riemannian metric g_V at the point e, we have an inner product, $\langle\ ,\ \rangle$ on λ defined as follows:

$$\langle a,b\rangle = g_{V(e)}\ (\xi(a),\xi(b)) \tag{3.3}$$

for every $a,b\in\lambda$. The inner product $\langle\ ,\ \rangle$ has the following expression ([5])

$$\langle a,b\rangle = S_p(\xi(a),\xi(b)), \tag{3.4}$$

that is

$$n_{11}=n_{22}=n_{33}=n_{44}=n_{12}=n_{13}=n_{14}=n_{24}=1$$

and

$$\begin{cases} n_1 = 1 + \frac{1}{2}\, n_{12} + \frac{1}{2}\, n_{13} + \frac{1}{2}\, n_{14} = \frac{5}{2} \\ n_2 = 1 + \frac{1}{2}\, n_{21} + \frac{1}{2}\, n_{23} + \frac{1}{2}\, n_{24} = \frac{3}{2} \\ n_3 = 1 + \frac{1}{2}\, n_{31} + \frac{1}{2}\, n_{32} + \frac{1}{2}\, n_{34} = 1 \\ n_4 = 1 + \frac{1}{2}\, n_{31} + \frac{1}{2}\, n_{42} + \frac{1}{2}\, n_{43} = 1, \end{cases}$$

whence

$$\begin{aligned} \langle a, b \rangle &= Sp((a + a^*) + (b + b^*)) = Sp((a + b) + (a + b)^*) = \\ &= n_1(a_{11} + b_{11}) + n_2(a_{22} + b_{22}) + n_3(a_{33} + b_{33}) + n_4(a_{44} + b_{44}) = \\ &= \tfrac{S}{2}(a_{11} + b_{11} + \tfrac{3}{2}(a_{22} + b_{22}) + (a_{33} + b_{33}) + (a_{44} + b_{44}). \end{aligned}$$

From (3.2) we finally have

$$\max\langle n_{12}, n_{24} \rangle \leq n_{14}, \ \max\{1, 1\} \leq 1.$$

§4. Conclusions

The connection between solvable Lie algebras and convex cones of dimension 8 was described by a canonical isomorphism ([5])

$$\xi : a \in \lambda = \sum_{1 \leq i \leq j \leq A} A_{ij} \ \rightarrow \ a + a^* \in X = T_e(V).$$

The canonical Riemannian metric g_V at the point e has an inner product $\langle \cdot\,, \cdot \rangle$ on λ defined by (3.3) and (3.4).

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