Generalized Invexity and Vector Optimization on Differentiable Manifolds[♣]

Ştefan Mititelu

Abstract

This paper has two goals: to define the invex, pseudoinvex and quasiinvex vector functions on differentiable manifolds and Riemannian manifolds, and to develop a vector programming on a manifold. A vector program (VP) on a manifold M is a vector problem of Pareto minimum generated by vector functions defined on M. Necessary and sufficient conditions of efficiency (Pareto optimization) for (VP) are established. These conditions are of Karush-Kuhn-Tucker type and some functions of (VP) are generalized invex.

Also, the notion of Pareto saddle point for the vector Lagrangian associated to (VP) is defined and a theorem of Pareto saddle point for (VP) program is given. This theorem generalizes the well-known theorem of saddle point of Kuhn and Tucker for scalar programs in nonlinear programming.

M.S.C. 2000: 26B25, 58A05, 58B20, 90C29, 90C46.
Key words: generalized invexity, manifolds, vector optimization, saddle point.

§1. Introduction

For the convex functions on Riemannian manifolds are known many definitions. In the present framework, a function is *geodesic convex* on a Riemannian manifold (M, g) if its restrictions to all geodesics of M are convex. More precisely, if M is a complete manifold, then the function $f : M \to \mathbf{R}$ is said to be *geodesic convex* if for all geodesic arcs $\gamma : [a, b] \to M$ and all $\lambda \in [0, 1]$ one has

$$f(\gamma(a + \lambda(b - a)) \leqq (1 - \lambda)f(\gamma(a)) + \lambda(\gamma(b))$$

or,

$$(f \circ \gamma)(b) - (f \circ \gamma)(a) \geqq d(f \circ \gamma)_a(b - a), \tag{1}$$

if f is a differentiable function.

[♣] *Presented at the Third Conference of Balkan Society of Geometers, July 31-August 3 2000, Bucharest, Romania.*

Differential Geometry - Dynamical Systems, Vol.3, No.1, 2001, pp. 21-31.

Particularly, for the Euclidian space $\mathbf{R}^n$ the geodesic γ has the form $\gamma(t) = At + c$, $t \in \mathbf{R}$, where A is an $n \times n$ dimensional matrix and $c \in \mathbf{R}^n$. Then the relation $\gamma'(a)(b-a) = \gamma(b) - \gamma(a)$ is true and denoting $u = \gamma(a)$, $x = \gamma(b)$, relation (1) becomes

$$f(x) - f(u) \geqq df_u(x-u).$$

The preceding relation doesn't require any metrical structure and therefore it can be extended to manifolds M like in Pini [9].

Let M be a differentiable manifold endowed with a linear connection and let

$$TM = \bigcup_{p \in M} T_pM$$

be the tangent bundle of M. For an $\varepsilon > 0$ we denote $I_\varepsilon = (-\varepsilon, \varepsilon) \subset \mathbf{R}$.

Definition 1.1. A differentiable curve (α) on M is a differentiable aplication $\alpha : I_\varepsilon \to M$.

We put $\alpha(0) = p \in M$. The tangent vector to the curve (α) at p is $v = \alpha'(0) \in T_{\alpha(0)}M = T_pM$.

Let N be another differentiable manifold and $\varphi : M \to N$ a differentiable application. Consider also the composed application $\beta = \varphi \circ \alpha : I_\varepsilon \to N$. Obviously,

$$\beta'(0) \in T_{\beta(0)}N = T_{\varphi(\alpha(0))}N = T_{\varphi(p)}N.$$

Definition 1.2. The linear application $d\varphi_p : T_pM \to T_{\varphi(p)}N$ defined by $d\varphi_p(v) = \beta'(0)$, where $v = \alpha'(0) \in T_pM$ and $\beta = \varphi \circ \alpha$ is called the differential of φ at the point p.

But $\beta'(0) = \varphi'(\alpha(0)) \cdot \alpha'(0) = \varphi'(p)v$ and then

$$d\varphi_p(v) = \varphi'(p)v, \quad v \in T_pM. \tag{2}$$

Definition 1.3. Let $I \subset \mathbf{R}$ be an open interval and $\gamma : I \to M$ be a differentiable curve. The curve (γ) is called *geodesic* of M if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0.$$

We consider now an application $\eta : M \times M \to TM$ such that $\eta(p,q) \in T_qM$ for every $q \in M$ and any $p \in M$. The function $\eta(p,\cdot)$ is a vector field on M for every $p \in M$ and it is named "point-to-set vector field". Particularly, if $\eta(p,\cdot)$ is smooth, then $\eta(p,\cdot)$ is a smooth vector field on M, for every $p \in M$. Relative to the differentiable function $f : M \to \mathbf{R}$ Pini [9] defined the invexity notion as follows

Definition 1.4. (Pini) The differentiable function f is said to be *η-invex* on M if for any $p, q \in M$,

$$f(p) - f(q) \geqq df_q(\eta(p,q)).$$

This definition admits the following generalizations.

Definition 1.5. The differentiable function f is said to be *η-pseudoinvex* on M if for any $p, q \in M$,

$$df_q(\eta(p,q)) \geqq 0 \Rightarrow f(p) \geqq f(q).$$

Definition 1.6. The differentiable function f is said to be *η-quasiinvex* on M if for any $p, q \in M$,

$$f(p) \leqq f(q) \Rightarrow df_q(\eta(p,q)) \leqq 0.$$

In these definitions according to (2),

$$df_q(\eta(p,q)) = [df(q)]\eta(p,q).$$

If (M, g) is a Riemannian manifolds, then

$$df_q(\eta(p,q)) = g_q(grad\, f(q), \eta(p,q))$$

where $grad\, f(q)$ is the gradient of f at the point q.

In the sequel we present the notions of invexity pseudoinvexity, quasiinvexity and inquasimonotonicity in the *pointwise variants* for the vector function $F : M \to \mathbf{R}^n$. The differential of F at $p \in M$, namely, $dF_p : T_pM \to T_{F(p)}\mathbf{R}^n \equiv \mathbf{R}^n$, is introduced by

$$dF_p(v) = dF(p)v, \quad v \in T_pM$$

and for the Riemannian manifolds (M, g), by

$$dF_p(v) = q_p(dF(p), v), \quad v \in T_pM.$$

Definition 1.7. The differentiable vector function F is said to be *invex at $u \in M$* if there exists an application $\eta : M \times M \to TM$ such that (shortly, F is *η-invex at u* if)

$$F(x) - F(u) \geqq dF_u(x,u), \quad \forall x \in M.$$

Definition 1.8. The differentiable vector function F is said to be *pseudoinvex at $u \in M$* if there exists an application $\eta : M \times M \to TM$ such that (shortly, F is *η-pseudoinvex at u* if)

$$dF_u(\eta(x,u)) \geqq 0 \;\Rightarrow F(x) \geqq F(u), \quad \forall x \in M.$$

Definition 1.9. The differentiable vector function F is said to be *quasiinvex at $u \in M$* if there exists an application $\eta : M \times M \to TM$ such that (shortly, F is *η-quasiinvex at u* if)

$$F(x) \leqq F(u) \Rightarrow dF_u(\eta(x,u)) \leqq 0, \quad \forall x \in M.$$

Definition 1.10. The differentiable vector function F is said to be *inquasimonotonic at $u \in M$* if there exists an application $\eta : M \times M \to TM$ such that (shortly, F is *η-inquasimonotonic at u*, if)

$$F(x) = F(u) \Rightarrow dF_u(\eta(x,u)) = 0, \quad \forall x \in M.$$

If the manifold M is endowed with the Riemannian metric g, then everywhere in Definitions 1.7 - 1.10, we have

$$dF_u(\eta(x,u)) = g_u(dF(u), \eta(x,u)).$$

Also, everywhere in this paper the relations $u = v$, $u < v$, $u \leqq v$, $u \leq v$ etc between the two vectors $u = (u_1, \dots, u_n)^t$ and $v = (v_1, \dots, v_n)^t$ are equivalent to

$$\begin{cases} u = v \Leftrightarrow u_i = v_i, & i = \overline{1,n} \\ u < v \Leftrightarrow u_i < v_i, & i = \overline{1,n} \\ u \leqq v \Leftrightarrow u_i \leqq v_i, & i = \overline{1,n} \\ u \leq v \Leftrightarrow u \leqq v, & u \neq v, \end{cases}$$

respectively, where we denoted by " ^{t} " the transposition.

The (generalized) invex functions have the property that every local minimum point is a global mimimum point [3]. For this reason the (generalized) invex functions are used in the optimization theory.

§2. The purpose of the paper

Let us consider now the vector functions $f = (f_1, \dots, f_p)^t : M \to \mathbf{R}^p$, $g = (g_1, \dots, g_m)^t : M \to \mathbf{R}^m$ and $h = (h_1, \dots, h_s)^t : M \to \mathbf{R}^s$. A minimization vector program on M is the following Pareto extremum problem:

$$(VP) \begin{cases} \text{Minimize } f(x) = (f_1(x), \dots f_p(x))^t \\ \text{subject to } g(x) \leqq 0, \ h(x) = 0, \ x \in M. \end{cases}$$

The domain of this program is the set

$$D = \{x \in M \mid g(x) \leqq 0, \ h(x) = 0\}.$$

Definition 2.1. A feasible point $x^0 \in D$ is said to be a *Pareto minimum point*, or an *efficient solution* (of minimum) of (VP) if there exists no other feasible point $x \in D$ such that $f(x) \leq f(x^0)$ [1].

The paper establishes necessary and sufficient conditions of Karush-Kuhn-Tucker type for the efficiency of a point x in the vector program (VP). Moreover, there is defined the notion of *Pareto Saddle Point* for the vector Lagrangian associated to vector program (VP) and is given a theorem of Pareto Saddle Point which generalizes the well-known theorem of Saddle Point of Kuhn and Tucker for convex scalar programs from the nonlinear programming.

The necessary conditions of efficiency use the follwing:

Lemma 2.1 [4, 5]. *Let X be a locally convex space, Hausdorff separated and let $\theta = (\theta_1, \dots, \theta_p)^t : X \to \mathbf{R}^p$, $\varphi = (\varphi_1, \dots, \varphi_m)^t : X \to \mathbf{R}^m$ and $l = (l_1, \dots, l_s)^t : X \to \mathbf{R}^s$ be vector functions. We suppose that θ and φ are convex functions and $\varphi_j (1 \leqq r < j \leqq m)$, $(r \in \mathbb{N}^*)$ and l are linearly affine functions.*

If the system

$$\begin{cases} \theta_i(x) < 0, & i = \overline{1,p} \\ \varphi_j(x) \leqq 0, & j = \overline{1,m} \\ l_k(x) = 0, & k = \overline{1,s} \end{cases}$$

is incompatible on X *but the following system*

$$\begin{cases} \varphi_j(x) < 0, & j = \overline{1,r} \\ \varphi_j(x) \leqq 0, & j = \overline{r+1,m} \\ l_k(x) = 0, & k = \overline{1,s} \end{cases}$$

is compatible on X, *then there exist vectors* $t = (t^1, \dots, t^n)^t \in \boldsymbol{R}^p$, $y = (y^1, \dots, y^n)^t \in \boldsymbol{R}^m$ *and* $z = (z^1, \dots, z^n)^t \in \boldsymbol{R}^s$ *such that*

$$\begin{cases} \sum_{i=1}^{p} t^i \theta_i(x) + \sum_{j=1}^{m} y^j \varphi_j(x) + \sum_{k=1}^{s} z^k l_k(x) \geqq 0, & \forall x \in X \\ t \geq 0, \quad y \geqq 0. \end{cases}$$

For the sufficient conditions of efficiency the following statements are indispensable.

Definition 2.2 [1]. The point x^0 is said to be *properly efficient* of (VP) if it is efficient for (VP) and if there exist a scalar $S > 0$ such that, for each i,

$$\frac{f_i(x^0) - f_i(x)}{f_j(x) - f_j(x^0)} \leqq S$$

for some j such that $f_j(x) > f_j(x^0)$, whenever x is feasible for (VP) and $f_i(x) < f_i(x^0)$.

An efficient point that is not properly efficient is said to be *improperly efficient* and a such point is excluded in our study.

We consider the following parametric scalar program

$$(P_t) \quad \begin{cases} \text{Minimize } \sum_{i=1}^{p} t^i f_i(x) \\ \text{subject to } g(x) \leqq 0,\ h(x) = 0 \\ x \in M,\ t \geq 0,\ \sum_{i=1}^{p} t^i = 1. \end{cases}$$

The follwing result is true.

Lemma 2.2 (Geoffrion [1]). Let $t \geq 0$ $(\sum_{i=1}^{p} t^i = 1)$ be fixed. If x^0 is optimal in (P_t), then x^0 is properly efficient in (VP).

§3. Efficiency conditions for the vector program (VP)

Let $x^0 \in D$ be an efficient solution of (VP) and we define the set $J^0 = \{j \in \{1,\dots,m\} \mid g_j(x^0) = 0\}$. We suppose that the domain D verifies at x^0 the following constraint qualification of Mangasarian-Fromowitz type:

$$\mathcal{R}(x^0) : \exists v \in TM : d(g_{J^0})_{x^0}(v) \leq 0,\ dh_{x^0}(v) = 0.$$

Here $d(g_{J^0})_{x^0}(v)$ is the vector of components $d(g_j)_{x^0}(v)$, $\forall j \in J^0$, taken in increasing order of j and

$$dh_{x^0}(v) = (d(h_1)_{x^0}(v), \dots, d(h_s)_{x^0}(v))^t.$$

Now it follows necessary conditions of Karush-Kuhn-Tucker type as $x^0 \in D$ be an efficient solution of (VP).

Theorem 3.1 (Efficiency necessary conditions). . *Let $x^0 \in D$ be an efficient solution of (VP). Suppose that the functions f, g and h are differentiable at x^0. Moreover, suppose that g_{J^0} is η-quasiinvex at x^0 and h is η-inquasimonotonic at x^0, where $\eta(\cdot, x^0)$ is surjective. Also (VP) satisfies the constraint qualification $\mathcal{R}(x^0)$.*

Then there exist vectors $t^0 \in \mathbf{R}^p$, $y^0 \in \mathbf{R}^m$ and $z^0 \in \mathbf{R}^s$ such that the following efficiency conditions of Karush-Kuhn-Tucker type at x^0 are satisfied by (VP),

$$(KKT) \quad \begin{cases} t^{0i} df_i(x^0) + y^{0j} g_j(x^0) + z^{0k} dh_k(x^0) = 0 \\ y^{0j} g_j(x^0) = 0,\ y^0 \geqq 0 \\ t^0 \geqq 0,\ \displaystyle\sum_{i=1}^{p} t^{0i} = 1, \end{cases}$$

(throughout the paper we shall use the Einstein summation rule for indices).

Proof. If the vector function g_{J^0} is η-quasiinvex at x^0, from Definition 1.9 it follows

$$\forall x \in D : g_{J^0}(x) \leqq g_{J^0}(x^0) \Rightarrow d(g_{J^0})_{x^0}(\eta(x, x^0)) \leqq 0. \tag{4}$$

The vector function h is η-inquasimonotonic at x^0. Using Definition 1.10 it follows that

$$\forall x \in D : h(x) = h(x^0) \Rightarrow dh_{x^0}(\eta(x, x^0)) = 0 \tag{5}$$

The point x^0 is a Pareto minimum for f. There exists a vector component $\bar{f}$ of f and a function $\lambda(x, x^0) > 0$, $\forall x \in D$ such that for each $\varepsilon_x \in (0, \lambda(x, x^0)]$ one has

$$x^0 + \varepsilon_x \eta(x, x^0) \in M$$

and

$$\bar{f}(x^0 + \varepsilon_x \eta(x, x^0)) \geqq \bar{f}(x^0).$$

The function $\bar{f}$ is differentiable at x^0 and then, for all $x \in D$

$$d\bar{f}_{x^0}(\eta(x, x^0)) = \bar{f}'(x_0; \eta(x, x^0) = \lim_{\varepsilon_x \downarrow 0} \frac{\bar{f}(x^0 + \varepsilon_x \eta(x, x^0)) - \bar{f}(x^0)}{\varepsilon_x} \geqq 0.$$

Therefore, the following system

$$\text{(6)}\qquad \begin{cases} d(f_i)_{x^0}(\eta(x,x^0)) < 0,\ i=\overline{1,p} \\ d(g_j)_{x^0}(\eta(x,x^0)) \leqq 0,\ \forall j \in J_0 \\ d(h_k)_{x^0}(\eta(x,x^0)) = 0,\ k=\overline{1,s} \end{cases}$$

is incompatible on D. But $\eta(\cdot,x^0)$ is surjective and then, the following system

$$\text{(7)}\qquad \begin{cases} d(f_i)_{x^0}(v) < 0,\ i=\overline{1,p} \\ d(g_j)_{x^0}(v) \leqq 0,\ \forall j \in J_0 \\ d(h_k)_{x^0}(v) = 0,\ k=\overline{1,s} \end{cases} \qquad (v \in TM \setminus \{0\})$$

is also incompatible.

From the constraint qualification $\mathcal{R}(x^0)$ it follows that the following system

$$\text{(8)}\qquad \begin{cases} d(g_j)_{x^0}(v) \leq 0,\ \forall j \in J_0 \\ d(h_k)_{x^0}(v) = 0,\ k=\overline{1,s} \end{cases} \qquad (v \in TM \setminus \{0\})$$

is compatible.

But if the system (7) is incompatible and the system (8) is compatible then, according to Lemma 2.1, it follows that there exists vectors $t \in \mathbf{R}^p$, $y_{J^0} \in \mathbf{R}^{|J^0|}$ and $z \in \mathbf{R}^s$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{p} t^i d(f_i)_{x^0}(v) + \sum_{j\in J_0} y^j d(g_j)_{x^0}(v) + \sum_{k=1}^{s} z^k d(h_k)_{x^0}(v) \geqq 0,\ \forall v \in TM\backslash\{0\} \\ t \geq 0,\ y \geqq 0. \end{cases}$$

We have

$$\left(\sum_{i=1}^{p} t^i d(f_i)_{x^0} + \sum_{j\in J_0} y^j d(g_j)_{x^0} + \sum_{k=1}^{s} z^k d(h_k)_{x^0}\right)(v) \geqq 0,\ \forall v \in TM\backslash\{0\}$$

and from this inequality it follows

$$\sum_{i=1}^{p} t^i d(f_i)_{x^0} + \sum_{j\in J_0} y^j d(g_j)_{x^0} + \sum_{k=1}^{s} z^k d(h_k)_{x^0} = 0.$$

Now, we define

$$t^0 = \frac{t}{\sum\limits_{i=1}^{p} t_i},\ y^0 = \left(\frac{y^t_{J^0}}{\sum\limits_{i=1}^{p} t_i}, 0, \ldots, 0\right)^t \in \mathbf{R}^m,\ z^0 = \frac{z}{\sum\limits_{i=1}^{p} t_i}$$

and so, the conditions (KKT) are fulfiled.

In the sequel, the sufficient conditions of Karush-Kuhn-Tucker type for (VP) are established.

Theorem 3.2 (Efficiency sufficient conditions). *Suppose that the functions* f*,* g *and* h *are differentiable at* $x^0 \in D$ *and that the vector program (VP) satisfies at* x^0 *the conditions (KKT). Suppose also that* f *is* η*-pseudoinvex,* g_{J^0} *is* η*-quasiinvex and* h *is* η*-inquasimonotonic, all in* x^0*.*

Then it follows that x^0 *is a properly efficient solution of (VP).*

Proof. The first relation of (KKT) leads to

$$\left[\sum_{i=1}^{p} t^{0i} df_i(x^0) + \sum_{j\in J_0} y^{0j} dg_j(x^0) + \sum_{k=1}^{s} z^{0k} dh_k(x^0)\right] \eta(x,x^0) = 0,\ \forall x \in D. \tag{9}$$

Moreover, there are true the relations (4) and (5). From these two relations and $y^0_{J^0} \geqq 0$ it follows

$$\left[\sum_{j\in J_0} y^{0j} dg_j(x^0) + \sum_{k=1}^{s} z^{0k} dh_k(x^0)\right] \eta(x,x^0) \leqq 0,\ \forall x \in D. \tag{10}$$

Also, from the relations (9) and (10) we obtain

$$\sum_{i=1}^{p} t^{0i} d(f_i)_{x^0}(\eta(x,x^0)) \geqq 0,\ \forall x \in D,$$

or, equivalently,

$$d\left(\sum_{i=1}^{p} t^{0i} f_i\right)_{x^0} (\eta(x,x^0)) \geqq 0,\ \forall x \in D. \tag{11}$$

But the function $\sum_{i=1}^{p} t^{0i} f_i$ is η-pseudoinvex at x^0 and then the relation (11) implies

$$\left(\sum_{i=1}^{p} t^{0i} f_i\right)(x) \geqq \left(\sum_{i=1}^{p} t^{0i} f_i\right)(x^0),\ \forall x \in D.$$

Finally, according to Lemma 2.2, it follows that x^0 is a properly efficient solution of (VP).

§4. Pareto saddle point for the vector program (VP)

The vector function $L : M \times \mathbf{R}^m \times \mathbf{R}^s \to \mathbf{R}^p$, where

$$L(x,y,z) = f(x) + [y^j g_j(x) + z^k h_k(x)]e,\ e = (1,\dots,1)^t \in \mathbf{R}^p$$

is called *the vector Lagrangian associated to the vector program (VP).*

Definition 4.1 (Pareto Saddle Point). A point $(x^0, y^0, z^0) \in M \times \mathbf{R}^m \times \mathbf{R}^s$ is said to be Pareto saddle point for the Lagrangian L (or for the program (VP)) if the following conditions are fulfiled

$$L(x^0, y, z) \leqq L(x^0, y^0, z^0) \leqq L(x, y^0, z^0),\ \forall x \in D,\ \forall y \geqq 0,\ \forall z \in \mathbf{R}^s.$$

A theorem of Pareto saddle points of Kuhn and Tucker type for (VP) it follows.

Theorem 4 (Pareto Saddle Point). *The point $x^0 \in D$ is an efficient solution of the vector program (VP) if there exist vectors $y^0 \geqq 0$ and $z^0 \in \boldsymbol{R}^p$ such that (x^0, y^0, z^0) be a Pareto saddle point for the Lagrangian L. The necessity of theorem uses moreover, the condition $y^{0j} g_j(x^0) = 0$.*

Proof. **Necessity**. We have $g(x^0) \leqq 0$ and $h(x^0) = 0$. Then, for $y \geqq 0$ it follows

$$y^j g_j(x^0) + z^k h_k(x^0) \leqq y^{0j} g_j(x^0) + z^{0k} h_k(x^0),$$

$$f(x^0) + [y^j g_j(x^0) + z^k h_k(x^0)]e \leqq f(x^0) + [y^{0j} g_j(x^0) + z^{0k} h_k(x^0)]e,$$

that is

$$L(x^0, y, z) \leqq L(x^0, y^0, z^0).$$

The inequality

$$L(x^0, y^0, z^0) \leqq L(x, y^0, z^0),\ \forall x \in D$$

is equivalent to

$$f(x^0) - f(x) + y^{0j}[g_j(x^0) - g_j(x)]e + z^{0k}[h_k(x^0) - h_k(x)]e \leqq 0$$

and having $x^0, x \in D$ and $y^{0j} g_j(x^0) = 0$ it becomes

$$f(x^0) - f(x) \leqq [y^{oj} g_j(x)]e \leqq 0,$$

that is true.

Sufficiency. (x^0, y^0) is a Pareto Saddle Point for L. The inequality

$$L(x^0, y, z) \leqq L(x^0, y^0, z^0),\ \forall y \geqq 0,\ \forall z \in \mathbf{R}^s$$

is equivalent to

$$[(y^j - y^{0j}) g_j(x^0)]e + [(z^k - z^{0k})^t] h_k(x^0)]e \leqq 0. \tag{12}$$

Choose y such that for a fixed k,

$$\begin{cases} y^s = y^{0s} + 1 \\ y^j = y^{0s},\ j \neq s \end{cases}$$

and $z = z^0$. Then (12) becomes $g_s(x^0) \leqq 0$. For every $s \in \{1, \ldots, m\}$ it follows $g(x^0) \leqq 0$.

For $y = y^0$ from (12) it follows

$$(z^k - z^{0k})h_k(x^0) \leqq 0, \ \forall z \in \mathbf{R}^s.$$

From this relation we find $h(x^0) = 0$ and so, $x^0 \in D$. Moreover, we have

$$f(x) \geqq L(x, y^0, z^0) \geqq L(x^0, y^0, z^0) \geqq L(x^0, 0, 0) = f(x^0), \ \forall x \in D,$$

that is x^0 is an efficient solution of the program (VP).

Remark. The necessary condition $y^{0j}g_j(x^0) = 0$ is ensured by the constraint qualification $\mathcal{R}(x^0)$ and by the hypothesis of quasiinvexity of g_J and of inquasimonotonicity of h, the both at x^0.

References

[1] A.M. Geoffrion, *Proper efficiency and the theory of vector maximization*, J. Math. Anal. Appl., 22 (1968), 618-630.

[2] E. Miglierina, *Funzioni invesse su varietà differenziabili*, manuscript, 28 April 1998.

[3] Şt. Mititelu, *Generalized invexities and global minimum properties*, Balkan J. of Geometry and Appl., 2(1997), 1, 61-72.

[4] Şt. Mititelu, *O generalizare a teoremei Farkas-Minkowski şi aplicarea ei în optimizarea vectorială netedă*, Lucrările Sesiunii Ştiinţifice a Catedrei de Matematică din U.T.C.B., pag. 9-11, 9.05.1999, Bucureşti.

[5] Şt. Mititelu, *Nonsmooth vector programming in a generalized framework*, Optimization (to appear).

[6] Şt. Mititelu, *Invexity and preinvexity notions are equivalent*, Optimization (to appear).

[7] Şt. Mititelu, I. M. Stancu-Minasian, *Invexity at a point: generalisations and classification*, Bull. Austral Math. Soc., 48 (1993), 117-126.

[8] Şt. Mititelu, I. M. Stancu-Minosian, *Optimality and algorithm for generalized invex Chebyshev problem*, Investigaçaô Operational, 17 (1997), 151-161.

[9] R. Pini, *Convexity along curves and invexity*, Optimization, 29 (1994), 301-309.

[10] V. Preda, *On efficiency and duality for multiobjective programs*, J. Math. Anal. Appl., 166, 2(1992), 365-377.

[11] T. Rapsák, *Geodesic convexity in nonlinear optimization*, J.O.T.A., 69, 1 (1991), 169-183.

[12] C. Udrişte, *Kuhn-Tucker theorem on Riemannian manifolds*, Colloquia Math. Soc. Janos Bolyai, Topics in Diff. Geometry, Debrecen, Hungary (1984), 1247-1259.

[13] C. Udrişte, *Riemannian convexity of functions*, Tensor N. S., 54 (1993), 138-145.

[14] C. Udrişte, *Convex functions and optimization methods on Riemannian manifolds*, Mathematics and Its Applications, 297, Kluwer Academic Publishers, 1994.

Author's address:

Ştefan Mititelu
Technical University of Civil Engineering
Department of Mathematics
Bd. Lacul Tei 124, Sect.2,
72302 Bucharest, Romania