On the Generalized Clifford Bundle[♣]

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Abstract

In this note, in the generalized Clifford bundle we obtain certain useful relations between Clifford product and interior and exterior products.

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§1. Introduction.

Let V be a vector space over the field K and $q : V \to K$ a quadratic form on V. By $Cl(V,q)$ is denoted the Clifford algebra of V and q ([1], [2]). As a vector space, the Clifford algebra is isomorphic with the exterior algebra $\Lambda^*(V)$ and, therefore, we have the decomposition

$$Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$$

into even and odd parts. Next we shall assume that $K = \mathbb{R}$, $dimV = n$, and in a fixed base $\{e_i\}_{i=\overline{1,n}}$,

$$q(x) = x_1^2 + .. + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2,$$

$r + s = n$. In this case, $Cl(V,q)$ is simply denoted by $Cl_{r,s}(V)$, and for $s = 0$, $Cl_{n,0}$ is denoted by $Cl_n(V)$. $Cl_{r,s}(V)$ is determined by a q-orthonormal base $\{e_i\}_{i=\overline{1,n}}$ with the properties:

$$e_i \cdot e_j + e_j \cdot e_i = \begin{cases} -2\delta_{ij}, & i \leq r \\ 2\delta_{ij}, & i > r. \end{cases}$$

The Clifford algebra can be extended over $V^c = \mathbb{C} \otimes V$, the complexified of V, and $q^c = \mathbb{C} \otimes q$. Then we can define

$$\mathbb{C}l_{r,s}(V) = Cl(V^c, q^c),$$

which is isomorphic with the complexification of $Cl_{r,s}(V)$. It is known ([4]) that $\mathbb{C}l_{r,s}(V) \simeq \mathbb{C}l_n(V)$, and hence it is possible to consider $Cl_{r,s}(V) \subset \mathbb{C}l_n(V)$.

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Now let us consider an $n-$dimensional real differentiable manifold M, and a pseudoriemannian metric g with constant signature

$$g_x(X) = X_1^2 + .. + X_r^2 - X_{r+1}^2 - \ldots - X_{r+s}^2, \ \forall x \in M.$$

We can discuss about the generalized Clifford bundle $Cl_{r,s}(M)$, which at any point $x \in M$ has as fibre the Clifford algebras $Cl_{r,s}(T_xM)$. This is a more general case than the Clifford bundle considered in [1], where $s = 0$, but, as we shall see further, a many of Clifford bundle properties are preserved.

Based on the isomorphisms

$$Cl_{r,s}(T_xM) \simeq \Lambda^n(T_xM),$$

we have the vector bundle isomorphism

$$Cl_{r,s}(M) \simeq \Lambda^*(TM).$$

The correspondence between vector fields and 1–forms writes: $X \leftrightarrow g_X$, where $g_X(Y) = g(X,Y)$; this gives an isomorphism between $\Lambda^*(TM)$ and $\Lambda^*(M)$, and hence we have

$$Cl_{r,s}(M) \simeq \Lambda^*(M).$$

Identifying $Cl_{r,s}(M)$ and $\Lambda^*(M)$ we can find a correspondence between the Clifford product and the exterior and interior products, as follows. Let $\{e_i\}_{i=\overline{1,n}}$ be a local base of 1–forms; then the next relations hold true:

$$e_i \cdot e_j + e_j \cdot e_i = \begin{cases} -2\delta_{ij} & i \leq r \\ 2\delta_{ij} & i > r. \end{cases} \tag{1}$$

Consider now a vector field $X_i \in \chi(U)$, such that $g_{X_i} = e_i$.

Proposition 1 *The relation*

$$e_i \cdot \varphi = e_i \wedge \varphi - i_{X_i}(\varphi), \ \textit{for all } \varphi \in \Gamma(\Lambda^*(M)),$$

where $\Gamma(E)$ is the module of sections in the vector bundle E.

Proof. Let $\varphi = \sum_{i_1<\ldots<i_k} a^{i_1 \cdots i_k} e_{i_1} \cdot \ldots \cdot e_{i_k}$; then it is sufficient to prove that

$$e_i(e_{i_1} \cdot \ldots \cdot e_{i_k}) = e_i \wedge (e_{i_1} \cdot \ldots e_{i_k}) - i_{X_i}(e_{i_1} \cdot \ldots \cdot e_{i_k}).$$

But

$$e_{i_1} \cdot \ldots \cdot e_{i_k} = e_{i_1} \wedge \ldots \wedge e_{i_k}, \quad \text{for all } i_1 < \ldots < i_k,$$

and consequently the relation above follows directly from (1). □

Corollary. If ω is an 1–form and $\varphi \in \Gamma(\Lambda^*(M))$, then

$$\omega \cdot \varphi = \omega \wedge \varphi - i_{X_\omega}(\varphi),$$

where $X_\omega \in \chi(M), g_{X_\omega} = \omega$.

Proof. Using, e.g., the Gauss method, we can obtain the local base of 1−forms $\{e_i\}_{i=\overline{1,n}}$ at any $x \in M$. If $\omega = \omega^i e_i$ then $X_\omega = \omega^i X_i$. Hence

$$\begin{aligned} \omega \cdot \varphi &= (\omega^i e_i) \cdot \varphi = \omega^i (e_i \cdot \varphi) = \omega^i (e_i \wedge \varphi - i_{X_i}(\varphi)) = \\ &= (\omega^i e_i) \wedge \varphi - i_{\omega^i X_i}(\varphi) = \omega \wedge \varphi - i_{X_\omega}(\varphi). \end{aligned}$$

□

Now, if $X \in \chi(M)$, we get

$$\begin{aligned} i_X(\omega \cdot \varphi) &= i_X(\omega \wedge \varphi - i_{X_\omega}(\varphi)) = i_X(\omega) \wedge \varphi - \omega \wedge i_X(\varphi) - i_X i_{X_\omega}(\varphi) = \\ &= i_X(\omega) \wedge \varphi - (\omega \wedge i_X(\varphi) - i_{X_\omega}(i_X(\varphi)) = i_X(\omega) \cdot \varphi - \omega \cdot i_X(\varphi). \end{aligned}$$

We have proved that

$$i_X(\omega \cdot \varphi) = i_X(\omega) \cdot \varphi - \omega \cdot i_X(\varphi). \tag{2}$$

If $\varphi = \sum_{i_1 < .. < i_k} a^{i_1 \dots i_k} e_{i_1} \cdot \dots \cdot e_{i_k}$, we define

$$\alpha(\varphi) = \sum_{i_1 < \dots < i_k} (-1)^k a^{i_1 \dots i_k} e_{i_1} \cdot \dots \cdot e_{i_k}.$$

Remark. If $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 \in \Gamma(Cl^0_{r,s}(M))$ and $\varphi_1 \in \Gamma(Cl^1_{r,s}(M))$ are the even and respectively the odd parts, then $\alpha(\varphi) = \varphi_0 - \varphi_1$ gives a fundamental spin-tensor ([4], [1]).

Proposition 2 *For any* $\varphi, \psi \in \Gamma(\Lambda^*(M))$ *we have*

$$i_X(\varphi \cdot \psi) = i_X(\varphi) \cdot \psi + \alpha(\varphi) \cdot i_X(\psi).$$

Proof. For $\varphi = \sum_{i_1 < \dots < i_k} a^{i_1 \dots i_k} e_{i_1} \cdot \dots \cdot e_{i_k}$ it is sufficient then to prove that

$$i_X[(e_{i_1} \cdot \dots \cdot e_{i_k}) \cdot \psi] = [i_X(e_{i_1} \cdot \dots \cdot e_{i_k})] \cdot \psi + (-1)^k (e_{i_1} \cdot \dots \cdot e_{i_k}) \cdot i_X(\psi),$$

using induction after $k \geq 1$; for $k = 1$ this is just the relation (2). □

Proposition 2 gives a relation between the interior and the Clifford product, similar to the one between the interior and the exterior product on a manifold.

Theorem 1 *Let us consider* $\varphi, \psi \in \Gamma(\Lambda^*(M))$ *and* ∇ *a metric connection in respect to* g. *Then we have*

$$\nabla(\varphi \cdot \psi) = (\nabla \varphi) \cdot \psi + \varphi \cdot (\nabla \psi).$$

Proof. If ω is an 1−form and $\psi \in \Gamma(\Lambda^*(M))$, then

$$\begin{aligned} \nabla(\omega \cdot \psi) &= \nabla(\omega \wedge \psi - i_{X_\omega}(\psi)) = \\ &= (\nabla \omega) \wedge \psi + \omega \wedge \nabla \psi - i_{X_\omega}(\nabla \psi) - i_{\nabla X_\omega}(\psi) = \\ &= ((\nabla \omega) \wedge \psi - i_{\nabla X_\omega}(\psi)) + (\omega \wedge \nabla \psi - i_{X_\omega}(\nabla \psi)) = \\ &= ((\nabla \omega) \wedge \psi - i_{X_{\nabla \omega}}(\psi)) + \omega \cdot \nabla \psi = (\nabla \omega) \cdot \psi + \omega \cdot \nabla \psi. \end{aligned}$$

Hence, if $\varphi_1 = e_{i_1} \cdot \ldots e_{i_k}$, then

$$\nabla(\varphi_1 \cdot \psi) = (\nabla\varphi_1) \cdot \psi + \varphi_1 \cdot (\nabla\psi),$$

by induction after $k \geq 1$. If a is a differentiable function on M, then

$$\begin{aligned}\nabla((a\varphi_1) \cdot \psi) &= \nabla(a(\varphi_1 \cdot \psi)) = (\nabla a)(\varphi_1 \cdot \psi) + a\nabla(\varphi_1 \cdot \psi) = \\ &= (\nabla a)(\varphi_1 \cdot \psi) + a((\nabla\varphi_1) \cdot \psi + \varphi_1 \cdot (\nabla\psi) = \\ &= ((\nabla a)\varphi_1 + a\nabla\varphi_1) \cdot \psi + a(\varphi_1 \cdot \nabla\psi) = \\ &= [\nabla(a\varphi_1)] \cdot \psi + (a\varphi_1) \cdot \nabla\psi,\end{aligned}$$

and the theorem is proved. □

Corollary. *If ∇ is a metric connection on M, then:*

1) *The following relations hold true:*

$$\nabla(Cl^0_{r,s}(M)) = Cl^0_{r,s}(M)$$
$$\nabla(Cl^1_{r,s}(M)) = Cl^1_{r,s}(M);$$

2) *We have*

$$R(X,Y)(\varphi \cdot \psi) = (R(X,Y)\varphi) \cdot \psi + \varphi \cdot (R(X,Y)\psi),$$

where R is the curvature tensor.

Now let us consider $\mathbb{C}l_n(M)$, the complexified of $Cl_n(M)$. If ∇ is a metric connection on M, then let $\tilde{\nabla}$ be the extension by linearity of ∇ to $\mathbb{C}l_n(M)$.

Proposition 3 *For any $z, \omega \in \Gamma(\mathbb{C}l_n(M))$, we have*

$$\tilde{\nabla}(z \cdot \omega) = (\tilde{\nabla} z) \cdot \omega + z \cdot (\tilde{\nabla}\omega).$$

We know that $(Cl_{r,s}(M))_x \subset (\mathbb{C}l_n(M))_x$; then all the real vector bundles $Cl_{r,s}(M)$ are included in a complex vector bundle.

Remark. From a historical point of view, our generalization of Clifford bundles is natural. Let us recall that the Dirac operator contains the generators of $Cl_{1,3}$ algebra. Therefore, a generalization on Clifford bundles of the Dirac operator is also interesting on the $Cl_{r,s}$ bundles, in particular is obtained the actual case of Cl_n. But this is a subject for a forecoming paper.

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