Dynamical Systems on Lagrange Spaces and Associated Fields♣

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Abstract

A differentiable manifold M endowed with a structure of Lagrange space is considered. For a non-autonomus spray (dynamical system) on $\mathbb{R} \times TM$ it is associated a potential, field and current, field equations, propagation equations and waves as solutions of the propagation equations.

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1. Let M be a m-dimensional differentiable manifold of class C^∞ and let $E = \mathbb{R} \times TM$ be the evolution space of a dynamical system. A local chart (U, φ) of M with the local coordinates (x^i) induces a local chart $(p^{-1}(U), \Phi)$ of E with the local coordinates (t, x^i, y^i). In this local chart we consider the distinguished contact forms having the following local expressions:

$$\theta^i = dx^i - y^i dl. \tag{1}$$

2. A non-autonomus spray is a vector field $S \in \mathcal{X}(E)$ which verifies the properties:

$$dt(S) = 1, \tag{2}$$

$$\theta^i(S) = 0.$$

The local expression of S is

$$S = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + S^i \frac{\partial}{\partial y^i}, \quad S^i \in C^\infty(E). \tag{3}$$

The differential equations of S are

$$\frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} = S^i(t, x, y). \tag{4}$$

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Their projections on M,

$$\frac{d^2x^i}{dt^2} = S^i\left(t, x, \frac{dx}{dt}\right), \tag{5}$$

are called second order differential equations and define the dynamics of the given system.

3. By aid the system (4) we associated to the spray (3), respectively to the dynamical system (5) the 2-form.

$$\Omega_0 = dy^i \wedge dx^i + (S^i dx^i - y^i dy^i) \wedge dl. \tag{6}$$

Ω_0 is called the Lagrange form and its characteristic are the trajectories of (4).

Theorem. [4] *To each dynamical system* (4) *one associates a closed* 2*-form*

$$\Omega = g_{ij} dy^i \wedge dx^i + (E_i dx^i + P_i dy^i) \wedge dt + \frac{1}{2} B_{ij} dx^i \wedge dx^j + \frac{1}{2} Q_{ij} dy^i \wedge dy^j, \tag{7}$$

such that the characteristic of Ω *coincide with the trajectories of the dynamical system* (5).

The characteristic of Ω are given by the system:

$$\begin{aligned} &B_{ij} dx^j - g_{ij} dy^j + E_i dt = 0, \\ &g_{ij} dx^j + Q_{ij} dy^j + P_i dt = 0. \end{aligned} \tag{8}$$

The system (4) and (8) admit the same solutions (i.e. they are equivalent) if and only if

$$\det \begin{Vmatrix} B_{ij} & g_{ij} \\ g_{ij} & Q_{ij} \end{Vmatrix} \neq 0, \tag{9}$$

and the Lorentz conditions are satisfied:

$$\begin{aligned} E_i &= B_{ji} y^j + g_{ji} S^j, \\ P_i &= -(g_{ij} y^j + Q_{ij} S^j). \end{aligned} \tag{10}$$

In virtue of the relations (10), the family of these 2-forms is constitued by the antisymmetric coefficients B_{ij}, Q_{ij} and the property of Ω to be closed is transmitted to its coefficients which verify the Maxwell type equations:

$$\begin{aligned} &a)\ \frac{\partial g_{jh}}{\partial y^i} - \frac{\partial g_{ih}}{\partial y^j} + \frac{\partial Q_{ij}}{\partial x^h} = 0; \qquad b)\ \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{hi}}{\partial x^j} - \frac{\partial B_{ij}}{\partial y^h} = 0; \\ &c)\ \frac{\partial g_{ij}}{\partial t} + \frac{\partial E_j}{\partial y^i} - \frac{\partial P_i}{\partial x^j} = 0; \qquad d)\ \frac{\partial B_{ij}}{\partial t} + \frac{\partial E_j}{\partial x^i} - \frac{\partial E_i}{\partial x^j} = 0; \\ &e)\ \frac{\partial Q_{ij}}{\partial t} + \frac{\partial P_j}{\partial y^i} - \frac{\partial P_i}{\partial y_j} = 0; \qquad f)\ \sum_{(i,j,h)} \frac{\partial B_{ij}}{\partial x^h} = 0; \\ &g)\ \sum_{(i,j,h)} \frac{\partial Q_{ij}}{\partial y^h} = 0, \end{aligned} \tag{11}$$

where (i, j, h) signifies a permutation of $\{i, j, h\}$.

The closed 2-form Ω is said to be a *field form* and the Maxwell type equations (11) are called *field equations.*

4. The 2-form Ω is locally exact because Ω is closed. Therefore there is a 1-form λ on E such that $d\lambda = \Omega$. Such a 1-form

$$\lambda = -Vdt + A_i dx^i + C_i dy^i, \tag{12}$$

is called a potential form and λ defines the field by the following equations:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}, \quad P_i = \frac{\partial V}{\partial y^i} - \frac{\partial C_i}{\partial t}, \tag{13}$$

$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}, \quad Q_{ij} = \frac{\partial C_j}{\partial y^i} - \frac{\partial C_i}{\partial y^j},$$

$$g_{ij} = \frac{\partial A_j}{\partial y^i} - \frac{\partial C_i}{\partial x^j}.$$

If the field Ω is given, then the potential λ is determined up to the addition of an exact total differential. A (gauge) potential change of the form $\overline{\lambda} = \lambda + d\Phi$ allows the choice of the function Φ such that $V = \dfrac{\partial \Phi}{\partial t}$ and therefore a better determination of the potential.

5. The previous formalism is called the Gallissot-Souriau (G.-S.) formalism [1], [5]. The *direct problem* of the G.-S. formalism is a closed presymplectic 2-form Ω of rank $2m$ on $\mathbb{R} \times TM$ and to determine a spray (a field of forces) for which Ω is the Lagrange form. The solution of the problem consists in the writting of the equations of characteristics (8) and in the setting of these ones in an equivalent Newtonian form (5). Practically in the most general case it is given a closed 2-form (7) of rank $2m$. In the assumption (9) it is expressed S^i from the equation (10) then the Newton equations (5) can be write. The *inverse problem* of the G.-S. formalism is to give the equations (5) and to determine a closed presymplectic 2-form of rank $2m$ such that its characteristics are exactly the solutions of the Newton equations.

The result is that every nondegenerate Newton system admits a G.-S. representation. One associates a 2-form Ω_0, (6), to the Newton system. In the equivalence class of Ω_0 there is at least a form which is closed. If the system (11) admits two solutions, then any linear combination of them with real coefficients is also a solution of (11).

6. In order to construct field theories it is necessary to dispose by a metric on $E = \mathbb{R} \times TM$. We shall construct such a metric by aid of a Lagrange structure ([3]) defined on M.

Consider a structure of Lagrange space on M defined by a function $L = L(x, y)$ to which it is associated the fundamental tensor $g_{ij}(x, y) = \dfrac{\partial^2 L}{\partial y^i \partial y^j}$ with the inverse $g^{ij}(x, y)$. This structure induces a natural nonlinear connection having the coefficients

$$N^i_j = -\frac{1}{2}\frac{\partial S^i}{\partial y^j}, \text{ where } S^i = -g^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^h} y^h - \frac{\partial L}{\partial x^j}\right), \tag{14}$$

and at the same time the family of adapted co-bases

$$(dx^i, \delta y^i = dy^i + N^i_j dx^j). \tag{15}$$

The fundamental tensor $g_{ij}(x, y)$ of the Lagrange structure on M lifts on TM to a metric defined by

$$G_{ab}dz^a dz^b = g_{ij}(dx^i dx^j + \delta y^i \delta y^j), \tag{16}$$

where dz^a denotes the co-basis $(dx^i, \delta y^i)$.

The metric G has the coefficients

$$(G_{ab}) = \begin{pmatrix} g_{ij} + g_{hk}N^h_i N^k_j & g_{ih}N^h_j \\ g_{ih}N^h_i & g_{ij} \end{pmatrix},$$

with respect to the natural basis and

$$(G_{ab}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix},$$

with respect to the adapted basis. It follows that G is symmetric, nondegenerate and positive definite (if g_{ij} has this property). With respect to this metric the spaces VTM and HTM (corresponding to the non-linear connection N) are orthogonal.

The space $E = \mathbb{R} \times TM$ may be endowed with the pseudo-Riemannian metric

$$\gamma = dt^2 - G. \tag{17}$$

7. Let now a dynamical system (a spray) S on $\mathbb{R} \times TM$ and the corresponding 2-form α of the form (7), which is closed and characteristic for the system. The *field form* α generates a field theory as follows:

1^0 Locally there is a 1-form λ such that $d\lambda = \alpha$ *called potential form.*

2^0 With the Hodge-de Rham operator it is constructed the $(m-2)$-form $\beta = *\alpha$. The forms α and β define together what is called a *field.* Each of these forms determines the other one by $*$.

3^0 To the form λ one associates a $(m-1)$-form ξ by $\lambda = *\xi$. The pair (λ, ξ) defines what is called the *field potential.*

4^0 In general the form β is not closed. It holds the relation $d\beta + \gamma = 0$, where γ is a $(m-1)$-form which is said a *form current* together with $\theta = *\gamma$.

5^0 From the previous definitions it results the following Maxwell equations: $d\xi = 0$, $d\lambda = \alpha$, for potential, $d\alpha = 0$, $d\beta + \gamma = 0$, for field, $d\gamma = 0$, $d\theta - \nu = 0$, for current, etc.

6^0 By using the Laplace-d'Alembert operator Δ one arrives to the following *propagation equations*: $\Delta\xi = d\delta\xi$ for potential, $\Delta\alpha = d\delta\alpha$ for field and $\Delta\gamma = d\delta\gamma$ for current etc.

7^0 The equation $d\gamma = 0$ is known as the *continuity equation.*

8^0 The Maxwell type equations and the propagation equations defined above are respectively equivalent with other equations which use the codifferential [4].

In conclusion to each dynamical system (spray) defined on a Lagrange space corresponds a closed, 2-form Ω. Ω plays the role of the field form on $R \times TM$. Therefore it is associated a field theory, a field and waves as solutions of the propagation equations.

8. If the dynamical system under study is given by equations of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = Q_i(x, y), \quad \frac{dx^i}{dt} = y^i, \tag{18}$$

where $L = L(x, y)$, then (18) furnishes both the Lagrange structure g_{ij} (via the function L) and the Lagrange form. All the construction of the associated field is canonically obtained from the equations (18).

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