

# On Pseudo Ricci-Symmetric Manifolds \*

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## Abstract

In the present study we consider pseudo Ricci-symmetric manifolds in the sense of M.C.Chaki. We show that pseudo Ricci-symmetric manifolds satisfying  $divR = 0$  (resp.  $divC = 0$ ) property are Einstein (resp. Ricci flat) manifolds.

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**Key words:** R-harmonic manifold, pseudo Ricci-symmetric manifold.

## §1. Introduction

Let  $M$  be an  $n$ -dimensional ( $n \geq 3$ ) Riemannian manifold. For the vector fields  $X, Y, Z \in \chi(M)$  and the the Levi-Civita connection  $\nabla$  of  $M$  the *curvature tensor*  $\mathcal{R}$  and the *Ricci operator*  $S$  of  $M$  are defined by  $\mathcal{R}(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$ , and  $S(X, Y) = g(SX, Y)$  respectively. Furthermore for the vector field  $W$  the Riemannian Christoffel curvature tensor  $R$  of  $M$  is defined by  $R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W)$  [4].

Let  $\Pi$  be a non-degenerate tangent plane to  $M$  at  $p \in M$  given by  $X, Y \in \chi(M)$ . Then the *sectional curvature*  $K(\Pi)$  of  $\Pi$  defined by  $K(X, Y)\{g(X, X)g(Y, Y) - g(X, Y)^2\} = g(\mathcal{R}(X, Y)Y, X)$  which is independent of the choice of the basis  $X, Y$  for  $\Pi$ .

A tensor field  $R$  of type (1, 2) on  $M$  is called *algebraic curvature tensor field* if it has symmetric properties of the curvature tensor field of Riemannian manifolds.

The curvature tensor  $\mathcal{R}$  satisfies the second *Bianchi identity* if

$$(1.1) \quad (\nabla_X \mathcal{R})(Y, Z, W) + (\nabla_Y \mathcal{R})(X, Z, W) + (\nabla_Z \mathcal{R})(X, Y, W) = 0.$$

Let  $R$  be an algebraic curvature tensor field which satisfies the second Bianchi identity. If  $S$  is the associated Ricci tensor field then

$$(1.2) \quad (div \mathcal{R})(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z).$$

For a Riemannian manifold  $M$  if the Ricci tensor  $S$  is of the form  $S = \lambda g$  then it is called *Einstein space* [4]. If  $S = 0$  then  $M$  is called *Ricci-flat*.

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The Weyl conformal curvature tensor  $C$  is defined by

$$\begin{aligned} C(X, Y, Z, W) = & R(X, Y, Z, W) - \frac{1}{n-2} \{g(X, W)S(Y, Z) + g(Y, Z)S(X, W) - \\ & -g(X, Z)S(Y, W) - g(Y, W)S(X, Z)\} + \\ & \frac{\tau}{(n-1)(n-2)} \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \end{aligned}$$

An algebraic curvature tensor field  $R$  is harmonic (or Codazzi type in the sense of [7]) if  $(divR)(X, Y, Z) = 0$ . A Riemannian manifold  $M$  is called  $R$ -harmonic if its curvature tensor field  $R$  is harmonic [1]

The divergence  $divC$  of the Weyl conformal curvature tensor  $C$  is defined by

$$(1.3) \quad \begin{aligned} (divC)(X, Y, Z) = & \frac{n-3}{n-2} \{(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\} - \\ & - \frac{1}{2(n-1)} \{g(Y, Z)\nabla_X \tau - g(X, Z)\nabla_Y \tau\}. \end{aligned}$$

In the present study we consider pseudo Ricci-symmetric submanifolds and also hypersurfaces.

The notion of pseudo Ricci-symmetric (PRS) manifolds were introduced by M. C. Chaki in 1987. A non-flat Riemannian manifold  $(M^n, g)$  ( $n > 2$ ) is called *pseudo Ricci-symmetric* if its Ricci tensor  $S$  is not identically zero and satisfies

$$(1.4) \quad (\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(Y, X),$$

where  $\alpha$  is a 1-form which is non-zero for every  $X, Y, Z \in \chi(M)$  and  $\nabla$  being operator of covariant differentiation with respect to the metric  $g$  [2]. In [3] the authors considered conformally flat pseudo Ricci-symmetric manifolds, see also [5] for the case  $M$  is a contact manifold.

In the present study we consider pseudo Ricci-symmetric manifolds. We show that pseudo Ricci-symmetric manifolds satisfying  $divR = 0$  (resp.  $divC = 0$ ) property are Einstein (resp. Ricci flat) manifolds.

## §2. Pseudo Ricci-Symmetric manifolds

Let  $(M, g)$ , ( $n \geq 3$ ), be an  $n$ -dimensional Riemannian manifold and  $e_i, e_j$  ( $1 \leq i, j \leq n$ ) orthonormal vector fields tangent to  $M$  and  $K_{ij}$  is the sectional curvature of a plane spanned by the vectors  $e_i$  and  $e_j$ . Then by definition of  $S$  we have

$$(2.1) \quad S(e_i, e_i) = \sum_{k=1}^n g(\mathcal{R}(e_k, e_i)e_i, e_k) = \sum_{k=1}^n K_{ik},$$

$$(2.2) \quad S(e_j, e_j) = \sum_{k=1}^n K_{jk}, \quad S(e_i, e_j) = 0.$$

First we prove the following result.

**Proposition 2.1.** *Let  $M^n$  be a  $n$ -dimensional Riemannian manifold. If  $M$  is pseudo Ricci-symmetric then*

$$(2.3) \quad \sum_{k=1}^n (K_{ik} - K_{jk})g(e_j, \nabla_{e_i} e_i) = \alpha(e_j) \sum_{k=1}^n K_{ik},$$

$$(2.4) \quad e_i \left[ \sum_{k=1}^n K_{ik} \right] = 4\alpha(e_i) \sum_{k=1}^n K_{ik},$$

$$(2.5) \quad e_i \left[ \sum_{k=1}^n K_{jk} \right] = 2\alpha(e_i) \sum_{k=1}^n K_{jk}.$$

*Proof.* Let  $e_i, e_j$  be orthonormal vector fields tangent to  $M$ . Combining (2.1)-(2.2) and (1.4) we find

$$(2.6) \quad (\nabla_{e_i} S)(e_i, e_j) = \alpha(e_j)S(e_i, e_i),$$

$$(2.7) \quad (\nabla_{e_i} S)(e_i, e_i) = 4\alpha(e_i)S(e_i, e_i),$$

$$(2.8) \quad (\nabla_{e_i} S)(e_j, e_j) = 2\alpha(e_i)S(e_j, e_j).$$

Moreover, from the covariant differentiation of  $S$  we have

$$(2.9) \quad (\nabla_{e_i} S)(e_i, e_j) = -S(\nabla_{e_i} e_i, e_j) - S(e_i, \nabla_{e_i} e_j),$$

$$(2.10) \quad (\nabla_{e_i} S)(e_i, e_i) = \nabla_{e_i} S(e_i, e_i),$$

$$(2.11) \quad (\nabla_{e_i} S)(e_j, e_j) = \nabla_{e_i} S(e_j, e_j) - 2S(\nabla_{e_i} e_j, e_j).$$

By the use of (2.1)-(2.2) we get

$$(2.12) \quad S(\nabla_{e_i} e_i, e_j) = \sum_{k=1}^n g(\mathcal{R}(e_k, \nabla_{e_i} e_i)e_j, e_k) = \sum_{k=1}^n K_{jk}g(e_j, \nabla_{e_i} e_i),$$

$$(2.13) \quad S(e_i, \nabla_{e_i} e_j) = \sum_{k=1}^n K_{jk}g(e_j, \nabla_{e_i} e_i), \quad S(\nabla_{e_i} e_i, e_i) = 0.$$

Combining (2.12), (2.13) and (2.9) we obtain

$$(2.14) \quad (\nabla_{e_i} S)(e_i, e_j) = \left( \sum_{k=1}^n (K_{ik} - K_{jk}) \right) g(e_j, \nabla_{e_i} e_i).$$

Forthermore differentiating (2.1) and (2.2) covariantly we have

$$(2.15) \quad (\nabla_{e_i} S)(e_i, e_i) = e_i \left[ \sum_{k=1}^n K_{ik} \right],$$

$$(2.16) \quad (\nabla_{e_i} S)(e_j, e_j) = e_i \left[ \sum_{k=1}^n K_{jk} \right] \left( \text{or } (\nabla_{e_j} S)(e_i, e_i) = e_j \left[ \sum_{k=1}^n K_{ik} \right] \right).$$

Since the left hand sides of the equations (2.6)-(2.8) are equal to the left hand sides of (2.14)-(2.16) we get the result.  $\square$

**Theorem 2.2.** *Let  $M$  be a  $n$ -dimensional pseudo Ricci-symmetric manifold. If  $M$  is  $R$ -harmonic (i.e.  $\operatorname{div}R = 0$ ) then it is Ricci-flat.*

*Proof.* Let  $M$  is  $R$ -harmonic so  $\operatorname{div}R = 0$ . Using (1.2) we have

$$(2.17) \quad (\nabla_{e_i}S)(e_i, e_j) - (\nabla_{e_j}S)(e_i, e_i) = 0.$$

Making use of (2.7), (2.8) and (2.1) the equation (2.17) reduces to  $\alpha(e_j)S(e_i, e_i) = 0$ . Since  $\alpha$  is a non-zero one form then  $S(e_i, e_i) = 0$ . Thus  $M$  is Ricci flat this completes the proof.  $\square$

**Theorem 2.3.** *Let  $M$  be a  $n$ -dimensional pseudo Ricci-symmetric manifold. If  $\operatorname{div}C = 0$  then  $M^n$  is an Einstein manifold.*

*Proof.* Suppose  $\operatorname{div}C = 0$ . Then by (1.3) we have

$$(2.18) \quad (\nabla_{e_i}S)(e_i, e_j) - (\nabla_{e_j}S)(e_i, e_i) + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0.$$

where  $\nabla_{e_j}\tau = e_j[\tau]$ . Substituting (2.14) and (2.16) into (2.18) we obtain

$$(2.19) \quad \sum_{k=1}^n (K_{ik} - K_{jk})g(e_j, \nabla_{e_i}e_i) - e_j\left[\sum_{k=1}^n K_{ik}\right] + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0.$$

Moreover, substituting (2.3) and (2.5) into (2.19) we get

$$-\alpha(e_j) \sum_{k=1}^n K_{ik} + \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] = 0.$$

By the use of (2.1) the above equation becomes

$$(2.20) \quad \alpha(e_j)S(e_i, e_i) = \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau].$$

On the other hand

$$(2.21) \quad \sum_{i=1}^n \alpha(e_j)S(e_i, e_i) = \frac{1}{2} \frac{n-2}{(n-1)(n-3)} e_j[\tau] \sum_{i=1}^n g(e_i, e_i),$$

which implies

$$(2.22) \quad e_j[\tau] = \frac{2(n-1)(n-3)}{n(n-2)} \alpha(e_j)\tau.$$

However combining (2.22) and (2.21) one can get  $S(e_i, e_i) = \frac{1}{n}\tau$ . Thus  $M$  is an Einstein manifold.  $\square$

**Theorem 2.4.** *Let  $M$  be a  $n$ -dimensional pseudo Ricci-symmetric manifold whose one of family of curvature lines consists of geodesic (i.e.  $\nabla_{e_i}e_i = 0$ ). Then  $M^n$  is Ricci flat.*

*Proof.* Putting  $\nabla_{e_i} e_i = 0$  into (2.3) we get  $\alpha(e_i) \sum_{k=1}^n K_{ik} = 0$ . Since the one form  $\alpha$  is non-zero then by (2.1) one can get  $S(e_i, e_i) = 0$ , which means that  $M$  is Ricci flat.  $\square$

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