On some second order properties of torse forming vector fields

Dedicated to Professor Radu Rosca

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Abstract

It is shown that if the a vector field $W = \omega^\sharp$ of a torse forming vector field (TF) is a skew symmetric Killing vector field, then the TF is an exterior concurrent vector field. Further the existence of an horizontal TF on a Kenmotsu manifold (ωK -manifold) $M\,(\phi, \Omega, \eta, \xi, g)$ ([3]) is proved by a closed differential system. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two such TF having both the structure 1 -form η as associated Pfaffian, it is proved that M is foliated by surfaces tangent to $\mathcal{T}_1$ and $\mathcal{T}_2$. Other properties of torse forming vector fields are also discussed.

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§1. Introduction

Torse forming vector fields, or shortly torse forming (abbr. TF) have been defined by K. Yano ([9]). Later some other authors as for instance [8] and [6] have outlined some additional properties of TF. If dp denotes the *soldering form* [2] of a differentiable C^∞ manifold (i.e. the canonical vector valued 1-form) and ∇ the covariant differential operators, then a TF may be defined as

$$\nabla \mathcal{T} = s dp + \omega \otimes \mathcal{T}, \qquad s \in \Lambda^0 M \tag{1.1}$$

where $\omega \in \Lambda^1 M$ is the associated Pfaffian with $\mathcal{T}$, and $\frac{1}{2}\|\mathcal{T}\|^2$ is the energy of $\mathcal{T}$.

On the other hand any vector field X such that

$$\nabla^2 X = \lambda \cdot X^\flat \wedge dp, \tag{1.2}$$

where λ is a certain scalar,

$$\flat : TM \to T^* M$$

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is the musical isomorphism defined by g, and

$$\sharp : T^*M \to TM$$

its inverse is defined as an exterior concurrent vector field [6] (see also [3]). One proves the following:

Theorem. *If the vector field $W = \omega^\sharp$ of a TF is a skew symmetric Killing vector field, then the TF $\mathcal{T}$ is an exterior concurrent vector field.*

Further the existence of an horizontal TF on a Kenmotsu manifold (or K-manifold) $M(\phi, \Omega, \eta, \xi, g)$ [3] is proved by a closed differential system.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two such TF having both the structure 1 -form η as associated Pfaffian, it is proved that M is foliated by surfaces tangent to $\mathcal{T}_1$ and $\mathcal{T}_2$. Some other properties are also discussed.

§2. Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian manifold and let ∇ be the covariant differential operator with respect to the metric tensor g. We assume that M is oriented and ∇ is the Levi-Civita connection. Let ΓMT and

$$\flat : TM \to T^*M$$

be the set of sections of the tangent bundle TM and the *associated isomorphism* [5] defined by g, respectively. As is known one denote by $\sharp : T^*M \to TM$ the inverse of $\flat$. Following [5], we set

$$A^q(M, TM) = \text{ Hom }(\Lambda^0 TM, TM)$$

and notice that the elements of $A^q(M, TM)$ are vector-valued q-forms ($q \leq \dim M$).

Denote by

$$A^q(M, TM) \to A^{q+1}(M, TM)$$

the exterior covariant derivative operator with respect to ∇. It should be noticed that generally

$$d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$$

unlike

$$d^2 = d \circ d = 0.$$

If $p \in M$ then the vector valued 1 -form

$$dp \in A^q(M, TM)$$

is the canonical vector 1-form of M and is called the *soldering* form [2] of M. Since V is symmetric, one has

$$d^\nabla(dp) = 0.$$

Any vector field $Z \in \Gamma TM$ such that

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M, TM)$$

for some 1-form $\mathcal{T}$ is said to be an *exterior concurrent* vector field [6], [4]. The 1-form M, which is called the concurrence form, is given by $TL = \lambda Z^\flat$; $\lambda \in C^\infty M$.

Let $\mathcal{O}(M)$ be the bundle of orthonormal frames over M and let

$$e = vect\,\{e_A, A = 1, ..., n\}$$

a moving orthonormal frame and

$$\omega = covect\,\{\omega^A\}$$

its associated coframe. Then the soldering form dp and E. Cartan's structured equations are expressed by

$$dp = \omega^A \otimes e_A \tag{2.1}$$

and

$$\nabla e = \theta \otimes e \tag{2.2}$$

$$d\omega = -\theta \wedge \omega, \tag{2.3}$$

$$d\theta = -\theta \wedge \theta + \Theta \tag{2.4}$$

respectively. In the above θ (resp. Θ) are the scalar connection forms in the bundle $\mathcal{O}(M)$ (resp. the curvature forms on M.)

§3. First result

Torse forming vector fields has been defined by Yano [9], and also studied by some other authors as [8] and [6]. If T is such a vector field then its covariant differential may be expressed as

$$\nabla\mathcal{T} = s dp + \omega \otimes \mathcal{T}. \tag{3.1}$$

In (3.1) s is a scalar, ω the associated Pfaffians (abbr. AP) of ∇ and $\frac{1}{2}\|\mathcal{T}\|^2$ the *energy* of $\mathcal{T}$ ([8]). If $\mathcal{T}^\flat$ is the dual form of $\mathcal{T}$, then Rosca established the following relation (called also Rosca's lemma [3])

$$d\mathcal{T}^\flat = \omega \wedge \mathcal{T}^\flat \tag{3.2}$$

This proves the significant fact that $\mathcal{T}^\flat$ is an exterior recurrent form [1], having as *recurrence form.* Next we agree to denominate the vector field $W = \omega^\sharp$ the associated vector field with $\mathcal{T}$ and let

$$f = g(WT). \tag{3.3}$$

In addition by a standard calculation one derives from (3.1)

$$d\,\|\mathcal{T}\|^2 = s\mathcal{T}^\flat + \|\mathcal{T}\|^2\,\omega. \tag{3.4}$$

In this conditions making use of the Lie derivative of $\mathcal{T}^\flat$ with respect to $\mathcal{T}$ (self Lie derivation) one deduces after calculation

$$\mathcal{L}_\mathcal{T}\mathcal{T}^\flat = (s + f)\,\mathcal{T}^\flat. \tag{3.5}$$

This shows the meaning meaningful fact that any T.F. $\mathcal{T}^\flat$ in a Lie self conformal form (abbr. LSC). We notice from (3.5) that the necessary and sufficient condition in orders that $\mathcal{T}^\flat$ be Lie self-invariant is that the scalar product of $\mathcal{T}$ and w be up to the sign, equated by the associated scalar s of $\mathcal{T}$.

On the other hand by differentiation of (3.4) and the help of (3.2) a standard calculation gives

$$ds \wedge \mathcal{T}^\flat + \|\mathcal{T}\|^2\, d\omega = 0. \tag{3.6}$$

If the associated Pfaffian ω is closed we agree to say that $\mathcal{T}$ is a closed T.F. and one has

$$ds \wedge \mathcal{T}^\flat = 0 \tag{3.7}$$

Assume now that W is a skew symmetric Killing vector (abbr. S.S.K.) in the sense of [6], having $\mathcal{T}$ as generative, that is

$$\nabla W = W \wedge \mathcal{T}, \qquad \wedge : \textit{wedge product} \tag{3.8}$$

As is known, since $\wedge$ is the wedge product one may write (3.8) as:

$$\nabla W = \mathcal{T}^\flat \otimes W - W^\flat \otimes \mathcal{T}. \tag{3.9}$$

Recall now that since TY is a S.S.K. vector field one has Rosca's lemma

$$d\omega = 2T^\flat \wedge \omega \tag{3.10}$$

and by (3.2), a short calculation gives

$$\omega \wedge \mathcal{T}^\flat = 0. \tag{3.11}$$

Operating now on (3.1) by the operator d^∇ one derives

$$d^\nabla\,(\nabla\mathcal{T}) = \nabla^2\mathcal{T} = (ds - s\omega) \wedge dp, \tag{3.12}$$

which proves the relevant fact that T is an exterior concurrent vector field (1.2) and one may set

$$\omega = t\mathcal{T}^\flat, \qquad ds = s'\mathcal{T}^\flat, \qquad t, s' \in \Lambda^0 M$$

which moves (3.12) to the standard form

$$\nabla^2\mathcal{T} = (s' - st)\,\mathcal{T}^\flat \wedge dp. \tag{3.13}$$

Consequently by reference to [4], one may write

$$\mathcal{R}\,(\mathcal{T}, Z) = -\,(n-1)\,(s' - st)\,g\,(\mathcal{T}, Z)\,, \tag{3.14}$$

where $\mathcal{R}$ means the Ricci tensor field of ∇, Z is any vector field on M and $n = \dim M$.

Hence we may formulate the following:

Theorem. *Any torse forming $\mathcal{T}$, whose associated vector field W is a skew symmetric Killing field, is an exterior concurrent vector field.*

§4. Main result

Let now $M(\phi, \Omega, \eta, \xi, g)$ be a $(2m+1)$-dimensional Kenmotsu manifold (or shortly a K-manifold). As is known ([3]), the structure tensors of $(\phi, \Omega, \eta, \xi)$ satisfy the following equations

$$(4.1)\qquad \begin{cases} \phi^2 = -Id + \eta \otimes \xi \qquad \phi\xi = 0 \qquad \eta(\xi) = 1 \\ g(Z, Z') = g(\phi Z, \phi Z') + \eta(Z)\eta(Z') \qquad \eta(Z) = g(\xi, Z) \\ \nabla\xi = f(dp - \eta \otimes \xi) \Leftrightarrow \nabla_Z \xi = Z - \eta(Z)\xi \\ (\nabla\phi) Z = \eta(Z)\phi dp - (\phi Z)^\flat \otimes \xi \qquad \Omega(Z, Z') = g(\phi Z, Z') \end{cases}$$

and

$$(4.2)\qquad d\eta = 0, \qquad d\Omega = 2\eta \wedge \Omega \qquad \Omega^m \wedge \eta \neq 0.$$

From the above it is seen that a K-manifold is endowed with a locally conformal cosympletic structure defined by the covector of Reeb η and the structure 2-form Ω.

Assume that M carries an horizontal torse forming, that is, such that $\eta(\mathcal{T}) = 0$. By reference to (4.1) and to the trivial equation (4.1), one derives with the help of (2.2)

$$(4.3)\qquad dT^a + \mathcal{T}^b \theta^a = s\omega^a + \mathcal{T}^a \omega, \qquad a, b \in \{1, ..., 2m\}$$

and setting $\lambda = -s/f$ one derives by a short calculation

$$(4.4)\qquad dT^b = \frac{d\lambda}{\lambda} \wedge \mathcal{T}^b.$$

Hence one finds

$$(4.5)\qquad ds \wedge \frac{d\lambda}{\lambda} = \lambda^2 d\left(\frac{f}{\lambda}\right) \wedge \eta$$

and since $d\eta = 0$ it follows that (4.4) and (4.5) define a closed differential system.

This being, assume that a K-manifold $M(\phi, \Omega, \eta, \xi, g)$ carries two TF vector fields $\mathcal{T}_1$, $\mathcal{T}_2$ having both as associated scalar the structure 1-form η of M. Accordingly, one may write

$$(4.6)\qquad \begin{cases} \nabla\mathcal{T}_1 = s_1 dp + \eta \otimes \mathcal{T}_1 \\ \nabla\mathcal{T}_2 = s_2 dp + \eta \otimes \mathcal{T}_2 \end{cases}$$

and taking the Lie bracket of the pairing $(\mathcal{T}_1, \mathcal{T}_2)$, one derives

$$(4.7)\qquad [\mathcal{T}_1, \mathcal{T}_2] = s_2\mathcal{T}_1 - s_1\mathcal{T}_2.$$

If we denote now by $\mathcal{D}_\mathcal{T}$ the 2-distribution defined by $\mathcal{T}_1$ and $\mathcal{T}_2$ we agree to agree to denominate $\mathcal{D}_\mathcal{T}$ the torse forming 2-distribution. Next let

$$(4.8)\qquad \varphi = \mathcal{T}_1^\flat \wedge \mathcal{T}_2^\flat$$

be the simple unit 2-form corresponding to $\mathcal{D}_\mathcal{T}$. By (4.4) exterior differentiation of φ gives

$$(4.9)\qquad d\varphi = 2\eta \wedge \varphi,$$

which shows that φ is an exterior recurrent 2-form, having η as concurrence form. Hence on behalf of Frobenius theorem, we deduce that the K-manifold under consideration is foliated by surfaces tangent to $\mathcal{D}_\mathcal{T}$. In addition operating $\nabla\mathcal{T}_1$ and $\nabla\mathcal{T}_2$ by d^∇, one finds by (3.3)

$$(4.10)\qquad \begin{cases} \nabla^2\mathcal{T}_1 = (s_1' - s_1)\,\eta \wedge dp \\ \nabla^2\mathcal{T}_2 = (s_2' - s_2)\,\eta \wedge dp, \end{cases}$$

which shows that both $\mathcal{T}_1$ and $\mathcal{T}_2$ are exterior concurrent vector fields. Since as is known [3], this property is invariant by linearly one may say that $\mathcal{D}_\mathcal{T}$ is an exterior differential distribution. Hence by reference to (3.14) one may write

$$\mathcal{R}\left(\mathcal{T}, Z\right) = -2m\left(s' - s\right) g\left(\mathcal{T}, Z\right),$$

where $\mathcal{T}$ is any TF of M. One may formulate

Theorem. *Let M be a K-manifold which carries 2 horizontal TF, $\mathcal{T}_1$, $\mathcal{T}_2$, and let $\mathcal{D}_\mathcal{T}$ be the 2-distribution defined by $\mathcal{T}_1$ and $\mathcal{T}_2$. Then M is foliated by surfaces $M_\mathcal{T}$ tangent to $\mathcal{D}_\mathcal{T}$ and such that for any vector field $\mathcal{T}$ of $M_\mathcal{T}$ one has*

$$\mathcal{R}\left(\mathcal{T}, Z\right) = -2m\left(s' - s\right) g\left(\mathcal{T}, Z\right) \qquad s, s' \in \Lambda^0 M,$$

where Z is any vector field on M.

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