On torse-forming vector valued 1-forms

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Abstract

The notion of torse-forming vector valued 1-forms on a Riemannian manifold (M, g) have been defined by Rosca [6]. In this paper the authors derive results regarding the wedge product of torse forming vector fields, which are applied to the case of Kenmotsu manifolds.

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§1. Preliminaries

Let M be an m-dimensional C^∞ Riemannian manifold with metric tensor g, and let TM be the tangent bundle over which is assumed to be not trivial. Let $\Gamma TM = \chi M$ be the set of sections of the tangent bundle and let $\flat : TM \to T^*M$ be the musical isomorphism defined by g and $\sharp$ the inverse of $\flat$, i.e., $\sharp : T^*M \to TM$. Following W. A. Poor [4], we denote

$$A^q\,(M, TM) = \Gamma Hom\,(\wedge^q TM, TM)$$

the set of vector valued q-forms and by

$$d^\nabla : A^q\,(M, TM) \to A^{q+1}(M, TM)$$

the exterior covariant derivative with respect to ∇. Notice that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d \circ d = 0$.

Next $dp \in A^q\,(M, TM)$ stands for the soldering form of M [1] (dp is the canonical vector valued 1-form and d^∇ $(dp = 0)$.

A (non parallel) vector field on a Riemannian (or pseudo-Riemannian) manifold is said to the exterior concurrent (abr. EC) [5], [3], if

$$\nabla^2 X = \pi^\xi \wedge dp \tag{1.1}$$

for some 1-form π^ξ on M; this 1-form π^ξ is called the concurrence form associated with X.

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This definition is a natural extension of the concept of concurrent vector fields (in this case $\xi\nabla$ is used instead of ∇^2). For any E.C. vector field X, the Ricci tensor R of ∇ satisfies

$$R(X,Z) = -(n-1)\, f g(X,Z) \Longrightarrow f = -\frac{1}{n-1} Ric(X), \tag{1.2}$$

where $Ric(X)$ is the Ricci curvature of M with respect to X.

A vector field T such that

$$\nabla T = s dp + \alpha \otimes T \qquad s \in \wedge^0 M, \quad \alpha \in \wedge^1 M \tag{1.3}$$

is defined as torse forming (abr. TF) vector field [7].

The 1-form α is called the generative of σ, and one has the relation (Rosca's lemma)

$$dT^\flat = \alpha \wedge T^\flat, \tag{1.4}$$

which proves that $T^\flat$ is exterior recurrent [1] and has α as recurrence form.

Let $\theta = \{e_A \mid A = 1, ..., n\}$ be a local field of adapted vectorial frames over M and let $\vartheta^* = \left\{\omega^A\right\}$ be its associated coframe field.

Then Cartan's structure equations written in indexless form are

$$\nabla e = \theta \otimes e \tag{1.5}$$

$$d\omega = -\theta \wedge \omega \tag{1.6}$$

$$d\Theta = -\theta \wedge \theta + \Theta. \tag{1.7}$$

In the above equations θ (resp Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms on M).

§2. Torse forming vector valued 1-forms

Let F be a vector valued 1-form in a C^∞-manifold M and let d^∇ be the exterior covariant derivative ξ on M, and dp the soldering form on M.

If d^∇ denote the covariant derivative operator i.e. if

$$A^q(M, TM) = \Gamma Hom\left(\wedge^q \Gamma M, TM\right),$$

then $d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$ means the exterior covariant derivative with respect to the Levi-Civita connection with respect to g, then if F is a vector valued 1-form, then a torse-forming vector valued (abr. TFVF) 1-form is defined by

$$d^\nabla F = \omega \wedge dp + \alpha \wedge F \qquad \alpha, \omega \in \wedge^1 M, \tag{2.8}$$

where α is called the generating form and ω the associated form of F.

We assume in this paper that F is defined by the wedge product $\wedge$ of two vector fields U and V, that is

$$F = U \wedge V = V^\flat \otimes U - U^\flat \otimes V. \tag{2.9}$$

Operating on F by the operator d^∇, one has

$$d^\nabla F = dV^\flat \otimes U - V^\flat \wedge \nabla U - dU^\flat \otimes V + U^\flat \wedge \nabla V. \tag{2.10}$$

Consider now the case when the vector fields U and V are both torse forming vector fields. Consequently by (1.3), the covariant differentials of U and V satisfy

$$\begin{cases} \nabla U = adp + \alpha \otimes U \\ \nabla V = bdp + \beta \otimes V, \end{cases} \tag{2.11}$$

where a, b are two scalar and the Pfaffians α and β are the generating forms of U and V respectively.

Taking account of lemma (1.4) one has

$$\begin{cases} dU^\flat = \alpha \wedge U^\flat \\ dV^\flat = \beta \wedge V^\flat. \end{cases} \tag{2.12}$$

Then operating on (2.9) by the operator d^∇, one infers after some calculations

$$d^\nabla F = \left(bU^\flat - aV^\flat\right) \wedge dp + (\beta + \alpha) \wedge F. \tag{2.13}$$

Consequently we derive

Theorem 1. *The wedge product of two torse forming vector fields defines a torse forming 1-form F. If the sum of the torse forming vanishes, then F moves to a concurrent vector valued 1-form.*

§2. The f-Kenmotsu manifold case

Consider now a f-Kenmotsu manifold $M(\Phi, \Omega, \eta, \xi, h)$ in the sense of Z. Olszak and R. Rosca [2]. For a f-Kenmotsu manifold (abr. f.K), one has the following structure equations

$$\begin{cases} \Phi^2 = I + \eta \otimes \xi, \quad \Phi\xi = 0, \quad \eta(\xi) = 1 \\ \nabla \xi = f(d\rho - \eta \otimes \xi) \\ g(Z, Z') = g(\Phi Z, \Phi Z') + \eta(Z)\eta(Z') \\ (\nabla_Z, \Phi)Z = -f[\eta(Z)\Phi Z' + g(\Phi Z, Z')\xi] \\ \Omega(Z, Z') = g(\Phi Z, Z'). \end{cases} \tag{3.14}$$

By the second equation (3.14) it is seen that the structure vector field ξ is a TF vector field.

Assume now that M^ξ carries a second TF, say U such that

$$\nabla U = adp + \alpha \otimes U. \tag{3.15}$$

Since in the case under discussion one may write

$$dp = \omega^a e_a + \eta\xi; \qquad a, b \in \{1, \ldots, 2m\} \tag{3.16}$$

and making use of (1.5) one derives the relations

$$dU^a + U^b \theta^a_b b^a = a\omega^a + fU^a \eta, \tag{3.17}$$

(3.18)
$$dU^o - dU^\flat = (a + fU^o)\,\eta,$$
where the index "^{o}" corresponds to ξ, i.e., $\theta^a_p = f\omega^a$. From (3.18) we infer

(3.19)
$$dU^o = fU^\flat + (a + fU^o)\,\eta$$
and since by (2.12) one has $dU^\flat = \alpha \wedge U^\flat$.

By (3.14) we get $d\eta = 0$, and derive by exterior differentiation

(3.20)
$$d\alpha \wedge U^\flat + \eta \wedge \alpha \wedge U^\flat = 0.$$
By the above equation we can see that the existence of TF vector field U on a f-Kenmotsu manifold is determined by a closed differential system.

Hence we yield

Theorem 2. *On any f-Kenmotsu manifold with structure vector field ξ, there exists an infinity of TF vector fields U such that the wedge product $U \wedge \xi$ defines a torse forming 1-form on M.*

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