On Bertrand Curves and Their Characterization

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Abstract

Bertrand curves have been investigated in E^n, n-dimensional Euclidean space and many characterizations are given [3], [7].

In this paper, the definition of n-dimensional Bertrand curves in Lorentzian space is given by comparing a well-known Bertrand couple of curves in n-dimensional Euclidean space. It is shown that the distance between corresponding points of Bertrand couple of curves and the angle between the tangent vector fields of these points are constant. Moreover, Schell and Manheim theorems are given in the Lorentzian space.

Also, a linear relationship is obtained between the first and second curvatures of a Bertrand couple in Lorentzian space. Also if there are more than one Bertrand mates of a curve in L^n, it is shown that the first harmonic curvature of the curve is constant. Then a necessary condition is given for a Bertrand curve to be a general helix.

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§1. Preliminaries

1.1 Symmetric Bilinear Forms

Let V be a real n-dimensional vector space with a symmetric bilinear function $g : V \times V \to R$. A symmetric bilinear form g on V is:

(a) positive [negative] definite provided $v \neq 0$ implies $g(v, v) > 0$ [< 0],

(b) positive [negative] semi definite provided $g(v, v) \geq 0$ [≤ 0],

(c) nondegenerate provided $g(v, w) = 0$ for all $w \in V$ implies $v = 0$ [5].

If g is symmetric bilinear form on V, then for any subspace W of V the restriction $g|_{W\times W}$ denoted merely by $g|_W$, is again symmetric and bilinear.

If g is [semi-] definite, so is $g|_W$.

The index ν of a symmetric bilinear form g on V is the largest integer, that is, the dimension of subspace $W \subset V$ on which $g|_W$ is negative definite.

Thus $0 \leq \nu \leq \dim V$ and $\nu = 0$ if and only if g is positive semidefinite [5].

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A scalar product space $V \neq \{0\}$ has an orthonormal basis.
Let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis for V, with $\varepsilon_i = g(e_i, e_j)$. Then each $v \in V$ has a unique expression [5],

$$v = \sum_{i=1}^{n} \varepsilon_i g(v, e_i) e_i.$$

We use $<,>$ as an alternative notation for g, writing $g(v,w) = \langle v, w \rangle$. For an integer ν with $0 \leq \nu \leq n$, changing the first ν plus signs above to minus we obtain a metric tensor

$$\langle v_p, w_p \rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j = w^j$$

of index ν.

The resulting Semi-Euclidean Space R^n_ν reduces to R^n if $\nu = 0$. For $n \geq 2$, R^n_1 is called Minkowski n-space.

Fix the notation:

$$\varepsilon_{i-1} = \begin{cases} -1 & \text{for } 0 \leq i-1 \leq \nu - 1 \\ 1 & \text{for } \nu \leq i-1 \leq n-1 \end{cases}$$

A Lorentz vector space is a scalar product of index 1 and dimension ≥ 2 [5].

1.2 Curves

A curve in a Lorentzian space L^n is a smooth mapping, $\alpha : I \to L^n$, where I is an open interval in the real R. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $t \in I$ is

$$\alpha'(t) = \frac{d\alpha(u)}{du}|_t.$$

A curve in Lorentzian space L^n is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$ is said to be space-like if its velocity vectors α' are space-like for all $t \in I$, similarly for time-like and null curves.

If α is a space-like (or time-like) curve, we can reparametrize it such that $\langle \alpha'(t), \alpha'(t) \rangle = \varepsilon_0 = +1$ ($\varepsilon_0 = -1$). In this case α is said to be unit speed or it has arc length parametrization [5], [8].

Definition 1.2.1. Let M be a curve L^n, parametrized by it own arc length. Denoting the Frenet vector fields of this curve by $V_1(s), V_2(s), \ldots, V_r(s)$ and assuming:

$$\frac{dV_i(s)}{ds} = \sum_{j=1}^{r} k_{ij}(s) V_j(s), \qquad i,j = 1, \ldots, r$$

the functions defined by

$$k_{ij}(s) = \varepsilon_{j-1} \left\langle \frac{dV_j(s)}{ds}, V_j(s) \right\rangle$$

are called the higher ordered curvatures of the curve M, and [1]

$$k_{ij}(s) = -k_{ji}(s) \quad (i < j).$$

Theorem 1.2.1. Let $M \subset L^n$ be a regular curve coordinate neighborhood (I, α) and $\{V_1(s), V_2(s), \ldots, V_r(s)\}$ be the Frenet r-frame at $\alpha(s)$ with $s \in I$. Then we have [1]

(a) $V_1'(s) = k_{12}(s)V_2(s)$,

(b) $V_i'(s) = \varepsilon_{i-2}\varepsilon_{i-1}k_{i(i-1)}(s)V_{i-1}(s) + k_{i(i+1)}(s)V_{i+1}(s)$,

(c) $V_r'(s) = \varepsilon_{r-2}\varepsilon_{r-1}k_{r(r-1)}(s)V_{r-1}(s)$.

§2. Bertrand Curves in Lorentzian Space

Definition 2.1. Let $\alpha(s)$ and $\beta(s^*)$ be two regular time-like curves in L^n. $\{V_1(s), V_2(s), \ldots, V_r(s)\}$ and $\{V_1^*(s^*), V_2^*(s^*), \ldots, V_r^*(s^*)\}$ are Frenet r-frames, respectively, on these curves. $\alpha(s)$ and $\beta(s^*)$ are called *Bertrand curves* if $V_2(s)$ and $V_2^*(s^*)$ are linearly dependent.

We say that $\beta(s^*)$ is a *Bertrand mate* for $\alpha(s)$ and $\beta(s^*)$ are Bertrand curves. And (α, β) is called a *Bertrand couple.*

Theorem 2.1. *Let (α, β) be a Bertrand couple in L^n and α, β are given (I, α), (I, β) coordinate neighborhoods, respectively. Then*

$$d(\alpha(s), \beta(s^*)) = constant, \quad \forall s \in I.$$

Proof. From Definition 2.1 we can write,

$$\beta(s^*) = \alpha(s) + \lambda(s)V_2(s); \tag{2.1}$$

from (2.1) we obtain

$$\begin{aligned} V_1^* &= \frac{d\beta(s^*)}{ds^*} = \alpha'(s)\frac{ds}{ds^*} + \lambda'(s)\frac{ds}{ds^*}V_2(s) + \lambda(s)V_2'(s)\frac{ds}{ds^*} \\ \tfrac{ds^*}{ds}V_1^* &= V_1(s) + \lambda'(s)V_2(s) + \lambda(s)V_2'(s) \end{aligned} \tag{2.2}$$

and from Theorem 1.2.1. we can write

$$V_2'(s) = \varepsilon_0\varepsilon_1 k_{21}(s)V_1(s) + k_{23}(s)V_3(s)$$

and

$$\frac{ds^*}{ds}V_1^* = (1 + \lambda(s)\varepsilon_0\varepsilon_1 k_{21}(s))V_1(s) + \lambda'(s)V_2 = (s) + \lambda(s)k_{23}(s)V_3(s).$$

If we multiply last equation with $V_2(s)$, we get $\lambda'(s)\varepsilon_1 = 0$, whence $\lambda(s) =$ constant. Thus,

$$d(\alpha(s), \beta(s)) = \|\beta(s) - \alpha(s)\| = \|\lambda(s)\| = \text{costant}.$$

Theorem 2.2. *The measure of the angle between the vector fields of Bertrand curves is constant.*

Proof. We show that

$$\langle V_1(s), V_1^*(s^*)\rangle = \mid \cosh\theta \mid = \text{constant}.$$

If we take the derivative of $\langle V_1(s), V_1^*(s^*)\rangle$ we get

$$\begin{aligned} \frac{d}{ds}\langle V_1(s), V_1^*(s^*)\rangle &= \langle V_1'(s), V_1^*(s^*)\rangle + \langle V_1(s), {V_1^*}'(s^*)\frac{ds^*}{ds}\rangle \\ &= \langle k_{12}(s)V_2(s), V_1^*(s^*)\rangle + \langle V_1(s), k_{12}^*(s^*)V_2^*(s^*)\rangle = 0. \end{aligned}$$

Thus,

$$\langle V_1(s), V_1^*(s^*)\rangle = \text{constant}.$$

Theorem 2.3. *Let α, β be two regular time-like curves in L^n. Then α and β are Bertrand curves if and only if*

$$\lambda_1(s)k_{21}(s) + \lambda_2(s)k_{23}(s) = 1,$$

where λ_1 and λ_2 are constant.

Proof. $(\Rightarrow)$. If α and β are Bertrand curves, we can write

$$\beta(s^*) = \alpha(s) + \lambda(s)V_2(s).$$

Differentiating, we get

$$V_1^*(s^*) = \frac{d\beta(s^*)}{ds^*} = \alpha'(s)\frac{ds}{ds^*} + \lambda'(s)\frac{ds}{ds^*}V_2(s) + \lambda(s)V_2'(s)\frac{ds}{ds^*}$$

$$\begin{aligned} V_1^*(s^*) &= \{V_1(s) + \lambda(s)\left(\varepsilon_0\varepsilon_1 k_{21}(s)V_1(s) + k_{23}(s)V_3(s)\right)\}\frac{ds}{ds^*} \\ &= \{\left(1 + \varepsilon_0\varepsilon_1 k_{21}(s)\lambda(s)\right)V_1(s) + k_{23}(s)\lambda(s)V_3(s)\}\frac{ds}{ds^*} \end{aligned} \tag{2.3}$$

as $\{V_2(s), V_2^*(s^*)\}$ is a linearly dependent set, we have

$$V_1^* = \varepsilon_0 \cosh\theta V_1 + \varepsilon_2 \sinh\theta V_3; \tag{2.4}$$

from (2.3) and (2.4) we get,

$$\varepsilon_0 \cosh\theta = (1 + \varepsilon_0\varepsilon_1\lambda k_{21})\frac{ds}{ds^*} \tag{2.5}$$

$$\varepsilon_2 \sinh\theta = \lambda k_{23}\frac{ds}{ds^*}. \tag{2.6}$$

If we divide the terms of the equation (2.5) by the ones of the equation (2.6), we obtain

$$\frac{\varepsilon_0 \cosh\theta}{\varepsilon_2 \sinh\theta} = \frac{(1 + \varepsilon_0\varepsilon_1\lambda k_{21})\frac{ds}{ds^*}}{\lambda k_{23}\frac{ds}{ds^*}} \tag{2.7}$$

is written. If we take $c = \dfrac{\varepsilon_0 \cosh\theta}{\varepsilon_2 \sinh\theta}$, we get

$$c\lambda k_{23} = 1 + \varepsilon_0\varepsilon_1\lambda k_{21}, \qquad c\lambda k_{23} - \varepsilon_0\varepsilon_1\lambda k_{21} = 1.$$

If we take $\lambda_1 = -\varepsilon_0\varepsilon_1\lambda$ and $\lambda_2 = c\lambda$. Thus,

$$\lambda_1 k_{21} + \lambda_2 k_{23} = 1.$$

($\Leftarrow$). It is clearly shown that $V_2(s)$ and $V_2^*(s^*)$ are linearly dependent.

Now, if we interchange the position of curves $\alpha(s)$ and $\beta(s^*)$, we can write

$$\alpha(s) = \beta(s^*) - \lambda V_2^*(s^*). \tag{2.8}$$

From this, we infer

$$\begin{aligned} \frac{d\alpha(s)}{ds} &= \frac{d\beta(s^*)}{ds^*}\frac{ds^*}{ds} - \lambda(s)\frac{dV_2^*}{ds^*}\frac{ds^*}{ds} \\ V_1(s) &= (1 - \lambda\varepsilon_0\varepsilon_1 k_{21}^*)V_1^*\frac{ds^*}{ds} - \lambda k_{23}^* V_3^*\frac{ds^*}{ds} \end{aligned} \tag{2.9}$$

and

$$V_1(s) = \varepsilon_0 \cosh\theta V_1^* + \varepsilon_2 \sinh\theta V_3^*\ ; \tag{2.10}$$

from (2.9) and (2.10) we get

$$\varepsilon_0 \cosh\theta = 1 - \lambda\varepsilon_0\varepsilon_1 k_{21}^* = \frac{ds^*}{ds} \tag{2.11}$$

$$\varepsilon_2 \sinh\theta = -\lambda k_{23}^* \frac{ds^*}{ds} \tag{2.12}$$

and from (2.11) and (2.12),

$$\frac{\varepsilon_0 \cosh\theta}{\varepsilon_2 \sinh\theta} = \frac{(1 - \varepsilon_0\varepsilon_1\lambda k_{21}^*)\frac{ds^*}{ds}}{-\lambda k_{23}^*\frac{ds^*}{ds}}. \tag{2.13}$$

Theorem 2.4. (Schell's Theorem) *Let (α, β) be a Bertrand couple in L^n. The product of higher ordered curvature functions k_{23} and k_{23}^* at the corresponding points of the Bertrand curves is constant, where the higher ordered curvature k_{23} belong to α and the higher ordered curvature k_{23}^* belong to β.*

Proof. Multiplying both sides of the equation (2.6) by the corresponding sides of equation (2.12),

$$\sinh^2\theta = -\lambda^2(s) k_{23} k_{23}^*$$

and

$$k_{23}k_{23}^* = -\frac{\sinh^2\theta}{\lambda^2(s)} = \text{constant}.$$

Theorem 2.5. (Manhiem's Theorem) *Let (α,β) be a Bertrand couple in L^n. If the points P and P^* are two corresponding points of (α,β) and A and A^* are the curvature centers at these points, then the ratio*

$$\frac{\|P^*A\|}{\|PA\|}:\frac{\|P^*A^*\|}{\|PA^*\|}$$

is constant. (This ratio is the cross ratio of the points P, P^*, A, A^*).

Proof. From (2.5) and (2.11) we have that

$$(1+\varepsilon_0\varepsilon_1\lambda k_{21})(1-\lambda\varepsilon_0\varepsilon_1 k_{21}^*)=\varepsilon_0^2\cosh^2\theta=\text{constant},$$

where ρ is the curvature radius. The following relation can be written

$$\|P^*A\|=\lambda-\rho=\lambda-\frac{\varepsilon_0\varepsilon_1}{k_{12}}=\frac{\lambda k_{12}-\varepsilon_0\varepsilon_1}{k_{12}},$$
$$\|PA\|=\rho=\frac{\varepsilon_0\varepsilon_1}{k_{12}},\qquad \|P^*A^*\|=\rho^*=\frac{\varepsilon_0\varepsilon_1}{k_{12}^*},$$
$$\|PA^*\|=\lambda+\rho^*=\lambda+\frac{\varepsilon_0\varepsilon_1}{k_{12}^*}=\frac{\lambda k_{12}^*+\varepsilon_0\varepsilon_1}{k_{12}^*}.$$

Let us write the cross ratio

$$\begin{aligned}(P^*P,AA^*)&=\tfrac{\|P^*A\|}{\|PA\|}:\tfrac{\|P^*A^*\|}{\|PA^=*\|}=\frac{\frac{\lambda k_{12}-\varepsilon_0\varepsilon_1}{k_{12}}}{\frac{\varepsilon_0\varepsilon_1}{k_12}}:\frac{\frac{\varepsilon_0\varepsilon_1}{k_{12}^*}}{\frac{\lambda k_{12}^*+\varepsilon_0\varepsilon_1}{k_{12}^*}}=\\&=(1+\lambda\varepsilon_0\varepsilon_1 k_{12}^*)(\lambda\varepsilon_0\varepsilon_1 k_{12}-1)=\\&=-(1+\lambda\varepsilon_0\varepsilon_1 k_{12})(1-\lambda\varepsilon_0\varepsilon_1 k_{12}^*)=\text{constant}.\end{aligned}$$

Theorem 2.6. *Assume that γ and γ^* are time-like Bertrand curves in L^n. There are two constants λ,μ such that,*

$$\mu(k_{23}-k_{23}^*)+\varepsilon_0\varepsilon_1\lambda(k_{12}+k_{12}^*)=0,$$

where the curvatures k_{12},k_{23} belong to γ and the curvatures k_{12}^*,k_{23}^* belong to γ^*.

Proof. From (2.7) and (2.13), we know that

$$\frac{\varepsilon_0\cosh\theta}{\varepsilon_2\sinh\theta}=\frac{(1+\varepsilon_0\varepsilon_1\lambda k_{21})\frac{ds}{ds^*}}{\lambda k_{23}\frac{ds}{ds^*}}=\text{constant}$$

and

$$\frac{\varepsilon_0\cosh\theta}{\varepsilon_2\sinh\theta}=\frac{(1-\lambda\varepsilon_0\varepsilon_1 k_{21}^*)\frac{ds^*}{ds}}{-\lambda k_{23}\frac{ds^*}{ds}}=\text{constant}.$$

Assuming $-\frac{\varepsilon_0\cosh\theta}{\varepsilon_2\sinh\theta}\lambda=\mu$, the following results

$$\mu k_{23}+\varepsilon_0\varepsilon_1\lambda k_{12}=1 \tag{2.14}$$

and

$$\mu k_{23}^{*} - \varepsilon_0\varepsilon_1\lambda k_{12}^{*} = 1 \tag{2.15}$$

are obtained. From (2.14) and (2.15), the linear relationship

$$\mu(k_{23} - k_{23}^{*}) + \varepsilon_0\varepsilon_1\lambda(k_{12} + k_{12}^{*}) = 0$$

is obtained.

Theorem 2.7. *Let γ be a time-like curve in L^n. If γ has more than one Bertrand mates, then the first harmonic curvature H_1 of the curve is constant.*

Proof. Assume that γ^{*} and γ^{**} are Bertrand mates of γ in L^n. Let us denote the first and second curvatures of these curves by $k_{12}, k_{23}, k_{12}^{*}, k_{23}^{*}, k_{12}^{**}, k_{23}^{**}$, respectively.

Since γ^{*} is a bertrand mate of γ, according to the Theorem 2.3 and Theorem 2.6 there exist two constants λ, μ such that

$$\begin{array}{rcl} \mu k_{23} + \varepsilon_0\varepsilon_1\lambda k_{12} & = & 1 \\ \mu k_{23}^{*} + \varepsilon_0\varepsilon_1\lambda k_{12}^{*} & = & 1 \end{array} \tag{2.16}$$

since γ^{**} is another Bertrand mate of γ there exist again two constants u, v such that

$$\begin{array}{rcl} v k_{23} + \varepsilon_0\varepsilon_1 u k_{12} & = & 1 \\ v k_{23}^{*} + \varepsilon_0\varepsilon_1 u k_{12}^{*} & = & 1; \end{array} \tag{2.17}$$

From (2.16) and (2.17)

$$k_{12} = \frac{v - \mu}{\varepsilon_0\varepsilon_1(\lambda v - \mu u)} = \text{constant}, \qquad k_{23} = \frac{\lambda - u}{\lambda v - \mu u} = \text{constant}$$

are obtained. Hence the harmonic curvature is $H_1 = \varepsilon_0\varepsilon_1 \dfrac{k_{12}}{k_{23}} = \text{constant}.$

Corollary 2.1. *If there are more than one Bertrand mates of a curve, then the curve is a general helix.*

Proof. This is clear from Theorem 2.7.

Theorem 2.8. *Assume that γ and γ^{*} are Bertrand curves in L^n. The angle between V_1, V_1^{*} is θ and the first harmonic curvature of γ is H_1. If $k_{34} = 0$ and $H_1 = \tanh\theta$, then γ is a general helix.*

Proof. Since γ and γ^{*} are Bertrand curves in L^n, θ is constant and we have $V_1^{*}\varepsilon_0 \cosh\theta V_1 + \varepsilon_2 \sinh\theta V_3$. Also the relation $H_1 = \tanh\theta$ leads to

$$\varepsilon_0\varepsilon_1 \frac{k_{12}}{k_{23}} = \frac{\sinh\theta}{\cosh\theta} \quad \Rightarrow \quad \varepsilon_0 k_{12}\cosh\theta = -\varepsilon_1 k_{32}\sinh\theta. \tag{2.18}$$

For $V_1^{*} = r$, we have

$$\frac{dr}{ds} = \varepsilon_0 \cosh\theta \frac{dV_1}{ds} + \varepsilon_2 \sinh\theta \frac{dV_3}{ds}.$$

Using Theorem 1.2.1, we get

$$\frac{dr}{ds} = \varepsilon_0 \cosh\theta(k_{12}V_2) + \varepsilon_2 \sinh\theta(\varepsilon_1\varepsilon_2 k_{32}V_2 + k_{34}V_4)$$

and using $k_{34} = 0$,

$$\frac{dr}{ds} = \varepsilon_0 \cosh\theta(k_{12}V_2) + \varepsilon_2 \sinh\theta(\varepsilon_1\varepsilon_2 k_{32}V_2)$$

which rewrites

$$\frac{dr}{ds} = (\varepsilon_0 \cosh\theta k_{12} + (\varepsilon_1 \sinh\theta k_{32})V_2$$

Using (2.18), we get $r =$ constant. Thus

$$\langle r, V_1\rangle = \langle V_1^*, V_1\rangle = \cosh\theta = \text{constant}.$$

Also $Sp\{V_1^*\}$ is the slope axis. By the definition of general helix, we conclude that γ is a general helix.

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