A Construction for the Existence of a Flat Induced Connection on the Structure Bundle of an Almost Quaternionic Manifold

Fatma Özdemir

Abstract

Let M be an almost quaternionic manifold, with its almost quaternionic structure defined by a three dimensional subbundle V of $T(M) \otimes T^*(M)$, $\{F, G, H\}$ be a local basis for V and let $[V, V]$ be the (global) (1,2) tensor field defined by $[V, V] = [F, F] + [G, G] + [H, H]$ where [,] denotes the Nijenhuis bracket. it has shown that, if either M is a quaternionic Kahler manifold, or if M is a complex manifold with almost complex structure F, then the vanishing of $[V, V]$ is equivalent to the vanishing of all the Nijenhuis brackets of $\{F, G, H\}$ [8]. For the case of a quaternionic Kahler manifold, we here give a (shorter) construction of a torsion free connection with $\nabla' F = \nabla' G = \nabla' H = 0$ when $[V, V] = 0$. It follows that the bundle V admits a flat connection hence it is trivial.

M.S.C. 2000: 53C55.
Key words: almost quaternionic structure, quternionic Kahler manifold.

§1. Introduction

In this section we will recall certain basic definitions and related theorems. An *almost complex structure* on a real differentiable manifold M is a tensor field J which is, at every point x of M, an endomorphism of the tangent space $T_x(M)$ such that $J^2 = -I$. A manifold with a fixed almost complex structure is called an *almost complex manifold.* A *complex (analytic) manifold* of complex dimension n is a space with a fixed complete atlas compatible with the pseudogroup of transformations of class C^∞ of $\mathbb{C}^n$.

Let $(z^1, \cdots, z^n)$ with $z^j = x^j + iy^j, j = 1, \cdots, n$ be a system of complex analytic coordinates on M and define J by $J\frac{\partial}{\partial x^j} = -\frac{\partial}{\partial y^j}$ and $J\frac{\partial}{\partial y^j} = \frac{\partial}{\partial x^j}$. Then it can be seen that J has the matrix

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \tag{1.1}$$

Differential Geometry - Dynamical Systems, Vol.3, No.2, 2001, pp. 25-30.

with respect to the complex basis $\frac{\partial}{\partial z^j} = \frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}$.

Given two tensor fields A and B of type $(1,1)$ on a manifold M, the ***torsion*** of A and B, which is a tensor field of type $(1,2)$ is defined by

$$\begin{aligned} N(X,Y) &= [A,B](X,Y) = \\ &= [AX,BY] - A[BX,Y] - B[X,AY] + [BX,AY] - \\ &\quad -B[AX,Y] - A[X,BY] + (AB+BA)[X,Y]. \end{aligned} \tag{1.2}$$

An almost complex structure is said to be ***integrable*** if it has no torsion. The famous Newlander-Nirenberg theorem states that an almost complex structure is a complex structure if and only if it has no torsion. In this case, it is also known that there exists a local coordinate system with respect to which J has constant coefficients, hence it can be transformed to the form given in(1.1) [11]. Almost quaternionic structures have been intensively studied ([1]-[10]). An extensive review of the subject can be found in a recent article by Kirichenko and Arseneva [4]. In the early literature, almost quaternionic structures are defined as global (1,1) tensor fields F, G and H satisfying the conditions

$$F^2 = G^2 = H^2 = -I, \quad H = FG, \quad FG + GF = FH + HF = GH + HG = 0, \tag{1.3}$$

where I denotes the identity transformation of $T_x(M)$. Later on it has been realized that such a definition is too restrictive and the following definition is now generally accepted.

Let M be a $4n$-dimensional Riemannian manifold which admits a three dimensional vector bundle V of (1,1) tensors such that on a neighborhood U of each $x \in M$, V has a local base $\{F, G, H\}$. If on each such neigborhood, the tensors F, G and H satisfy the conditions (1.3), then the bundle V is called an ***almost quaternionic structure*** on M.

In [10], it has been shown that the vanishing of any two of the Nijenhuis brackets of F, G and H is a necessary and sufficient condition for the existence of a torsion free connection ∇' such that $\nabla' F = \nabla' G = \nabla' H = 0$. However $[F,F]$, $[G,G]$ and $[H,H]$ are locally defined objects. We shall use the (1-2) tensor field $[V,V]$ defined by

$$[V,V] = [F,F] + [G,G] + [H,H], \tag{1.4}$$

and we use that the vanishing of $[V,V]$ is equivalent to the triviality of V respectively in the cases where M is quaternionic Kahler manifold and where M is a complex manifold with almost complex structure F. In section 2, we shall give an alternative construction of for a torsion free connection ∇' such that $\nabla' F = \nabla' G = \nabla' H = 0$.

If M admits an almost quaternionic structure then at each point x in M there is an orthonormal basis of $T(M)$, of the form

$$\{X_1, \ldots, X_n, FX_1, \ldots, FX_n, GX_1, \ldots, GX_n, HX_1, \ldots, HX_n\}, \tag{1.5}$$

and the set of all such frames at all points $x \in M$ constitute a subbundle of the bundle $\mathcal{O}(M)$ of the orthonormal frames, denoted by $\mathcal{H}(M)$. Such a reduction of the frame bundle is possible if and only if the structure group of the tangent bundle is reducible to $Sp(n)Sp(1)$ [9]. The torsion tensor of a connection ∇ is defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$, and the connection is called "torsion-free" if $T = 0$. A torsion free connection ∇ is compatible with the metric $\langle\ ,\ \rangle$, then $Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$, and this is equaivalent to the reducibility of ∇ to $\mathcal{O}(M)$. If furthermore ∇ is reducible to $\mathcal{H}(M)$ the manifold is called "quaternionic Kahler" [11].

In [10] it has been shown that any two of the equations

$$(1.6)\quad [F,F]=0,\quad [G,G]=0,\quad [H,H]=0,\quad [F,G]=0,\quad [F,H]=0,\quad [G,H]=0$$

hold, the others are satisfied too. It is then shown that there is a torsion-free connection with respect to which F, G and H are covariantly constant, and it follows that V is a trivial bundle.

Our aim is to give a construction of a torsion free connection with $\nabla' F = \nabla' G = \nabla' H = 0$ when $[V,V] = 0$. We study here the case where M is a quaternionic Kahler manifold or where M is a complex manifold. Then we can use connections which are reducible to $\mathcal{H}(M)$ in order to obtain our results. We quote the following theorem [11].

Theorem 1.1. *Let M be an almost quaternionic. A connection ∇ on $\mathcal{O}(M)$ is reducible to a connection on $\mathcal{H}(M)$ if and only if the covariant derivatives of the tensor fields of F, G and H satisfy the following conditions:*

$$(1.7)\qquad \nabla F = aG - bH,\quad \nabla G = -aF + cH,\quad \nabla H = bF - cG.$$

where a, b and c are 1-forms.

By the result of the following theorems it immediately follows that V is flat. In particular if M is a complex manifold, with almost complex structure say F, then $[V,V] = 0$ implies that V is a trivial bundle [8].

Lemma 1.1. *Let M be a $4n$ dimensional manifold, ∇ be a torsion-free connection reducible to $\mathcal{H}(M)$ and a, b and c be as in Theorem 1.1. Then*

$$(1.8)\qquad \begin{array}{lll} [F,F] & = 0 \;\; iff & b(X) = a(FX), \\ [G,G] & = 0 \;\; iff & c(X) = -a(GX), \\ [H,H] & = 0 \;\; iff & c(X) = b(HX). \end{array}$$

It was shown that the vanishing of any mixed Nijenhuis bracket is equivalent to the vanishing of all six.

Theorem 1.2. *Let M be a quaternionic Kahler manifold and ∇ is a torsion-free connection reducible to $\mathcal{H}(M)$ and a, b and c be as in Theorem 1.1. then*

$$(1.9)\qquad [F,F]+[G,G]+[H,H]=0 \quad\Leftrightarrow\quad b(X)=a(FX),\quad c(X)=-a(GX)$$

Theorem 1.3. *Let M be a $4n$ dimensional almost quaternionic manifold with a local basis $\{F,G,H\}$ for its almost quaternionic structure and let*

$$[V,V] = [F,F]+[G,G]+[H,H].$$

If any of the Nijenhuis tensors of F, G and H vanishes and $[V,V]=0$, then the other two also vanish.

§2. A torsion-free connection with $\nabla' F = \nabla' G = \nabla' H = 0$.

From the results of the previous section we know that the vanishing of the tensor field $[V,V]$ implies the existence of a connection on the bundle V with the properties (1.6). Given such a connection, it is known that this connection can be moidified by the addition of a connection difference tensor such that F, G and H are covariantly constant. Namely we have the following theorem [10].

Theorem 2.1. *There is a connection ∇' such that $\nabla' F = \nabla' G = \nabla' H = 0$ and $T_{\nabla'} = \frac{1}{12}\{[F,F]+[G,G]+[H,H]\}$.*

Thus using the previous result we see that the vanishing of $[V,V]$ implies the existence of a torsion-free connection such that $\nabla' F = \nabla' G = \nabla' H = 0$. We will give a proof using the connections reducible to $\mathcal{H}(M)$.

Theorem 2.2. *If $[V,V]=0$, there exists a torsion free connection such that $\nabla' F = \nabla' G = \nabla' H = 0$.*

Proof. We give now an explict construction of a torsion free connection when $[F,F]=[G,G]=0$. Assume that a $4n$ dimensional differentiable manifold V admits an almost quaternionic structure F,G,H and that $[F,F]=0$. We assume that the manifold is complex analytic and it is covered by a system of complex coordinate neighborhoods U on which the local coordinates are $\{z^k, z^{\overline{k}}\}$ $(z^{\overline{k}} = \overline{z}^k)$ $(k,\lambda,\mu,\ldots = 1,2,\ldots,2n;\ \overline{k},\overline{\lambda},\overline{\mu},\ldots = 2n+1,2n+2,\ldots,4n)$ with respect to this coordinates tensor field F has components of the form

$$F = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \tag{2.1}$$

where I is the $2n \times 2n$ unit matrix. F,G,H are real tensor fields. We represent the components of the tensor field G respect to this complex coordinate system by

$$G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}, \tag{2.2}$$

where G_1, G_2, G_3, G_4 are $2n \times 2n$ matrices.

Note that G is a real endomorphism hence the matrix of G with respect to complex basis is Hermitian. Since $G(\overline{e}_i) = (\overline{Ge_i})$ which implies $\overline{G}_2 = G_3$ and G_1, G_4 are Hermitian.

From $FG+GF=0$ we have

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} + \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} = 0, \tag{2.3}$$

that is

$$\begin{pmatrix} iG_1 + iG_1 & iG_2 - iG_2 \\ -iG_3 + iG_3 & -iG_4 - iG_4 \end{pmatrix} = 0 \tag{2.4}$$

which implies $G_1 = 0$ and $G_4 = 0$ thus G has the form $G = \begin{pmatrix} 0 & G_2 \\ \overline{G}_2 & 0 \end{pmatrix}$. Using $H = FG$, we find that H has the form $H = \begin{pmatrix} 0 & iG_2 \\ -i\overline{G}_2 & 0 \end{pmatrix}$ Since $[F, F] = 0$ we know that there exists a symmetric connection ∇ such that $\nabla F = 0$.

We will show that if $[G, G] = 0$, we can define $\nabla' = \nabla + S$ which S is a symmetric tensor so that $\nabla' G = 0$. For this need to show that we can choose S in such a way that the equation $\nabla' F = \nabla' G = 0$ can be solved simultaneously for S. Since $\nabla' F = \nabla F + SF - FS = 0$ and $\nabla' G = \nabla G + SG - GS = 0$ we get

$$[S, F] = -aG + bH \tag{2.5}$$

$$[S, G] = aF - cH \tag{2.6}$$

Recall that if $[F, F] = [G, G] = 0$ then $a(FX) = b(X)$, $a(GX) = -c(X)$ using these we obtain in component notation

$$S^i_{jk} F^k_l - F^i_k S^k_{jl} = -a_j G^i_l + a_k F^k_j H^i_l \tag{2.7}$$

$$S^i_{jk} G^k_l - G^i_k S^k_{jl} = a_j F^i_l + a_k G^k_j H^i_l \tag{2.8}$$

From (2.7) we can solve $S^{\alpha}_{\beta\overline{\gamma}}$, $S^{\alpha}_{\overline{\beta\gamma}}$, $S^{\overline{\alpha}}_{\beta\gamma}$, $S^{\overline{\alpha}}_{\overline{\beta}\gamma}$ and then complex conjugates as

$$\begin{aligned} &S^{\alpha}_{\beta\overline{\gamma}} = -ia_{\beta} G^{\alpha}_{\overline{\gamma}}, \\ &S^{\alpha}_{\overline{\beta\gamma}} = 0, \qquad S^{\overline{\alpha}}_{\beta\gamma} = 0, \\ &S^{\overline{\alpha}}_{\overline{\beta}\gamma} = ia_{\overline{\beta}} G^{\overline{\alpha}}_{\gamma}. \end{aligned} \tag{2.9}$$

Then substituting these in (2.8) we get

$$\begin{aligned} &S^{\alpha}_{\beta\gamma} = ia_{\overline{\eta}} G^{\overline{\eta}}_{\gamma} \delta^{\alpha}_{\beta} + i\delta^{\alpha}_{\gamma} a_{\overline{k}} G^{\overline{k}}_{\beta}, \\ &S^{\alpha}_{\overline{\beta}\gamma} = S^{\alpha}_{\gamma\overline{\beta}} = -ia_{\gamma} G^{\alpha}_{\overline{\beta}}, \\ &S^{\alpha}_{\overline{\beta\gamma}} = 0, \qquad S^{\overline{\alpha}}_{\beta\gamma} = 0, \\ &S^{\overline{\alpha}}_{\overline{\beta}\gamma} = S^{\overline{\alpha}}_{\gamma\overline{\beta}} = -ia_{\overline{\beta}} G^{\overline{\alpha}}_{\gamma}, \\ &S^{\overline{\alpha}}_{\overline{\beta\nu}} = ia_{\gamma} G^{\gamma}_{\overline{\nu}} \delta^{\overline{\alpha}}_{\overline{\beta}} + ia_k G^k_{\overline{\beta}} \delta^{\overline{\alpha}}_{\overline{\nu}}. \end{aligned} \tag{2.10}$$

In particular, we can work with the connection on V induced by the connection ∇' on M constructed above. Then as $\nabla' F = \nabla' G = \nabla' H = 0$, Thus the vanishing of the tensor $[V, V]$ will be a sufficient condition for the triviality of the bundle V.

References

[1] Alekseevsky D.V., Marchiafava, S., *Quaternionic Structures on a Manifold and Subordinated Structures*, Annali di Matematica pura ed applicata (IV), Vol. CLXXI (1996), 205-273.

[2] Doğanaksoy, A., *Doğa-Tr. J. of Mathematics*, 16 (1992), 109-118.

[3] Ishihara, S., *Quaternion Kahlerian Manifolds*, J. Differential Geometry, 9 (1974), 483-500.

[4] Kirichenko, V.F., Arseneva, O.E., *Differential geometry of generalized almost quaternionic structures*, I, dg-ga/9702013.

[5] Obata, M., *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap.J.Math, 26 (1956), 43-47.

[6] Oproiu, V., *Integrability of almost quaternal structures*, An. St. Univ. Al.I Cuza Iaşi, 30 (1984), 75-84.

[7] Özdemir, F., *On almost complex and almost quaternionic substructures and foliations*, Doctor of Philosophy Thesis in Mathematics, Istanbul Technical University, 1996.

[8] Özdemir, F., *A Global Condition for the Triviality of an almost quaternionic structure on complex manifolds*, submitted.

[9] Salamon, S., *Quaternionic Kahler Manifolds*, Invent.Math., 67 (1982), 143-171.

[10] Yano, K., Ako, M., *Integrability conditions for almost quaternion structure*, Hokaido Math. J., 1 (1972), 63-86.

[11] Yano, K., Kon, M., *Stuctures on Manifolds*, World Scientific Publishing Co.Pte.Ltd., 1984.

Author's address:

Fatma Özdemir,
Istanbul Technical University, Faculty of Science and Letters,
Department of Mathematics, 80626 Maslak-Istanbul, Turkey,
E-mail: fozdemir@itu.edu.tr