Null Helices in $\mathbf{R}_1^3$

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Abstract

The main purpose of this paper is to give some characterizations of null helices in $\mathbb{R}_1^3$ and to provide examples which illustrate the results.

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§1. Introduction

The helices are well known in Riemannian geometry [4]. It is known that a curve α is a helix if and only if ratio k_1 and k_2 of α is constant [4].

On the other hand, the theory of degenerate submanifolds is relatively new geometry. The general theory has been developed by D.N.Kupeli[3] and K.L.Duggal and A.Bejancu [2]. In [2], the authors constructed the principal vector bundles related to a degenerate submanifold in a semi-Riemannian manifold and obtained several fundamental properties of this submanifolds. A.Bejancu studied null curves in Lorentz manifolds and obtained Frenet frame for null curves. We present all of these results in section two.

In this paper, we study null helices which are 1-dimensional lightlike submanifolds in $\mathbb{R}_1^3$. We have given some characterizations for the null helices of $\mathbb{R}_1^3$ similar to the Euclidean setting. In fact, we have proved that a null curve is a null helix if and only if the ratio of k_1 and k_2 of the curve is constant.

§2. Preliminaries

Let (M, g) be a real $(m+2)$ dimensional proper semi-Riemannian manifold of index q and α be a smooth curve of M with immersion $i : \alpha \to M$. Suppose U is a coordinate neighborhood on α and t is the corresponding local parameter. Then, the tangent vector field to α in U is

$$\frac{d}{dt} = \left(\frac{dx^0}{dt}, ..., \frac{dx^{m+1}}{dt}\right)$$

where $x^i = x^i(t), i \in \{0, ..., m+1\}, t \in \mathbb{R}$.

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The smooth curve α is said be a null curve if tangent vector to α any point is a null vector. Hence, α is a null curve if and only if locally a each point of U we have

$$g\left(\frac{d}{dt},\frac{d}{dt}\right)=0.$$

Denote by $T\alpha$ the tangent bundle of α, and let

$$T\alpha^{\perp}=\underset{x\in\alpha}{\cup}T_x\alpha^{\perp},T_x\alpha^{\perp}=\{V_x\in T_xM|g(V_x,\xi_x)=0\},$$

where ξ_x is null vector tangent to α at x. We note that since $g(\xi_x,\xi_x)=0$, the tangent bundle $T\alpha$ of α is a vector subbundle of $T\alpha^{\perp}$. Now we consider a complementary vector bundle $S\left(T\alpha^{\perp}\right)$ to $T\alpha$ in $T\alpha^{\perp}$. We have the direct orthogonal sum

$$T\alpha^{\perp}=T\alpha\perp S\left(T\alpha^{\perp}\right)$$

here $S\left(T\alpha^{\perp}\right)$ is a nondegenerate vector bundle; therefore we have

$$TM|_{\alpha}=S\left(T\alpha^{\perp}\right)\perp S\left(T\alpha^{\perp}\right)^{\perp}$$

where $S\left(T\alpha^{\perp}\right)^{\perp}$ is a complementary orthogonal vector bundle to $S\left(T\alpha^{\perp}\right)$ in $TM|_{\alpha}$.

Theorem 2.1. [Duggal-Bejancu] *Let α be a null curve, a proper semi-Riemannian manifold (M,g) and $S\left(T\alpha^{\perp}\right)$ be a screen vector bundle of α. Then there exists a unique vector bundle $ntr(\alpha)$ over α of rank 1 such that on each coordinate neighborhood $U\subset\alpha$ there is a unique $N\in\Gamma(ntr(\alpha))$ satisfying*

$$g\left(\frac{d}{dt},N\right)=1 \tag{2.1}$$

and

$$g(N,N)=g(N,X)=0,\forall X\in\Gamma\left(S\left(T\alpha^{\perp}\right)\right). \tag{2.2}$$

Considering now

$$tr(\alpha)=ntr(\alpha)\perp S\left(T\alpha^{\perp}\right),$$

hence we have the following sum

$$TM|_{\alpha}=T\alpha\oplus tr(\alpha)=(T\alpha\oplus tr(\alpha))\perp S\left(T\alpha^{\perp}\right).$$

Now, let α be a null curve of a 3−dimensional Lorentz manifold and $S\left(T\alpha^{\perp}\right)$ be a screen vector bundle of α. Then on $U\subset\alpha$ we have the Frenet frame $F=\left\{\frac{d}{dt},N,W\right\}$. In case s is a distinguished parameter on α set [1]

$$\frac{d}{ds}=\xi,-N=n,W=u,$$

hence we have [1]

$$\xi^{'}=k_1u \tag{2.3}$$

(2.4) $$n' = -k_2 u$$

(2.5) $$u' = -k_2 \xi + k_1 n$$

where k_1 and k_2 are smooth functions on U which are called curvature functions of α with respect to F.

§3. Characterization of null helices in $\mathbf{R}_1^3$

Let $(\mathbb{R}_1^3, g)$ be the 3−dimensional Lorenz space, where g is given by

(3.1) $$g(X,Y) = -x_1 y_1 + x_2 y_2 + x_3 y_3$$

for any $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$.

Definition. We say that α is a null helix in $\mathbb{R}_1^3$, if there exists fixed direction V such that

$$g(\xi, V) = \lambda\,(\text{constant}).$$

Theorem 3.1. *Let α be a null curve in $\mathbb{R}_1^3$. Then α is a null helix if and only if*

$$\frac{k_1}{k_2} = \textit{constant}.$$

Proof. ($\Rightarrow$) Let α be a null helix; then from definition we have

$$g(\xi, V) = \lambda(\text{constant}),$$

whence, by differentiation we get

$$g(\xi', V) = 0.$$

From (2.3) we derive

(3.2) $$k_1 g(u,V) = 0 \quad \Rightarrow \quad g(u,V) = 0.$$

Thus we conclude $V \in Sp\,\{\xi, n\}$, i.e.,

$$V = \alpha_1 \xi + \beta n,$$

where $\alpha_1 = g(V,n)$ and $\beta = \lambda$. Thus from (2.3) and (2.4) we obtain

$$k_1 g(V,n) = \lambda k_2$$

or

$$\frac{k_1}{k_2} = \frac{\lambda}{g(V,n)}.$$

On the other hand, differentiating $g(V,n)$, we get

$$\begin{aligned} g(V,n)' &= g(V',n) + g(V,n') = g(V,n') \\ &= -k_2 g(V,u) = 0, \end{aligned}$$

that is, $g(V,n)$ is a constant. Then

$$\frac{k_1}{k_2} = \text{constant}.$$

($\Leftarrow$). We suppose that $\frac{k_1}{k_2} = \lambda = \frac{\lambda_1}{\lambda_2}$(constant) such that $\lambda_1, \lambda_2 \in \mathbb{R}$. Then we have

$$V = \lambda_2 \xi + \lambda_1 n.$$

Thus from (2.3) and (2.4) we obtain

$$V' = \lambda_2 \xi' + \lambda_1 n' = \lambda_2 k_1 u - \lambda_1 k_2 u = (\lambda_2 k_1 - \lambda_1 k_2) u = 0.$$

Thus V is a constant vector.

Example 3.1. We consider an example from [1], namely the curve α in $\mathbb{R}_1^3$ given by

$$x_0 = \sinh s, x_1 = s, x_2 = \cosh s, \quad s \in \mathbb{R}.$$

Then we choose the Frenet frame $F = \left\{\frac{d}{ds}, N, W\right\}$ as follows,

$$\begin{aligned} \tfrac{d}{ds} &= (\cosh s, 1, \sinh s) \\ N &= \tfrac{1}{2}(-\cosh s, 1, -\sinh s) \\ W &= (\sinh s, 0, \cosh s). \end{aligned}$$

Thus from (2.3),(2.4) and (2.5) we get $k_1 = 1$ and $k_2 = -\frac{1}{2}$. Hence $\frac{k_1}{k_2} = -2$. On the other hand

$$V = \xi - 2n = (0, 2, 0).$$

Hence $g(V, \xi) = 2$, i.e., α is a null helix.

Example 3.2. Another example from [1, Proposition 3.3], is a curve α in $\mathbb{R}_1^3$ given by the equations

$$x_0 = s, x_1 = \frac{1}{b}\sin(bs + a) + c, x_2 = -\frac{1}{b}\cos(bs + a) + d,$$

where $a, b \neq 0, c, d$ are real constants. Then we have

$$\begin{aligned} \tfrac{d}{ds} &= (1, \cos(bs + a), \sin(bs + a)) \\ u &= (0, -\sin(bs + a), \cos(bs + a)) \\ N &= \tfrac{1}{2}(-1, \cos(bs + a), \sin(bs + a)). \end{aligned}$$

Using (2.3),(2.4) and (2.5), we get $k_2 = \frac{b}{2}$ and $k_1 = b$. Hence we have

$$\frac{k_1}{k_2} = 2.$$

We note that as curves given by Example 3.2 are known circular helices [4]. Duggal and Bejancu have given a complete characterization of null circular helices.

Example 3.3. Consider a null curve α of $\mathbb{R}_1^3$ given by the equations

$$x_1 = \frac{1}{2}s\sqrt{s^2+1} + \frac{1}{2}\ln\left|\sqrt{s^2+1}+s\right|, \quad x_2 = \frac{1}{2}s^2, \quad x_3 = s, \qquad s \in \mathbb{R}.$$

Then by using (2.1) , (2.2) and (3.1), we choose the Frenet frame $F = (\xi, N, W)$ as follows

$$\begin{aligned} \xi &= \left(\sqrt{s^2+1}, s, 1\right) \\ W &= \left(\tfrac{s}{\sqrt{s^2+1}}, 1, 0\right) \\ N &= \left(-\tfrac{1}{2}\sqrt{s^2+1}, -\tfrac{1}{2}s, \tfrac{1}{2}\right). \end{aligned}$$

Thus from (2.3) , (2.4) and (2.5) we have $k_1 = 1$ and $k_2 = \frac{-1}{2}$. Hence α is a null helix.

We note that the null curve in the Example.3.3 is not a circular helix, contrary to the Example.3.1 and Example.3.2

From Theorem 3.1 we have the following corollaries.

Corollary 3.1. *Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{R}_1^3$. Then α is a null helix if and only if*

$$\det\left(\frac{dn}{ds}, \frac{d^2n}{ds^2}, \frac{d^3n}{ds^3}\right) = 0.$$

Proof. From(2.4) we have

$$\frac{dn}{ds} = -k_2 u$$

and

$$\begin{aligned} \tfrac{d^2n}{ds^2} &= k_2^2\xi - k_2k_1n - k_2'u, \\ \tfrac{d^3n}{ds^3} &= (3k_2'k_2)\,\xi + \left(-\left(k_2k_1\right)' - k_2'k_1\right)n + \left(2k_1k_2^2 - k_2''\right)u. \end{aligned}$$

Thus, summing up these equations we have

$$\det\left(\frac{dn}{ds}, \frac{d^2n}{ds^2}, \frac{d^3n}{ds^3}\right) = -k_2^5\left(\left(\frac{k_1}{k_2}\right)'\right),$$

hence, we have assertion of the corollary.

Corollary 3.2. *Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{R}_1^3$. Then α is a null helix if and only if*

$$\det\left(\frac{d^2\alpha}{ds^2}, \frac{d^3\alpha}{ds^3}, \frac{d^4\alpha}{ds^4}\right) = 0.$$

Proof. The proof the corollary similar to the previous corollary.

We consider the null curve from Example 3.3. Then its easy check that

$$\det\left(\frac{dn}{ds}, \frac{d^2n}{ds^2}, \frac{d^3n}{ds^3}\right) = 0$$

and

$$\det\left(\frac{d^2\alpha}{ds^2}, \frac{d^3\alpha}{ds^3}, \frac{d^4\alpha}{ds^4}\right) = 0.$$

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