The Unity of Mond-Weir Duality in Nonlinear Programming

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Abstract

In this paper the unitary character of the Mond-Weir duality in nonlinear programming is established.

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The Mond-Weir duality in nonlinear programming was introduced in 1981 by Mond and Weir [8] as a generalization of the Wolfe duality. It was formulated as follows.

Let the functions $\varphi : \mathbb{R}^n \to \mathbb{R}, f = (f_1, ..., f_p)' : \mathbb{R}^n \to \mathbb{R}^p, g = (g_1, ..., g_m)' : \mathbb{R}^n \to \mathbb{R}^m$ and $h = (h_1, ..., h_s)' : \mathbb{R}^n \to \mathbb{R}^s$ be all differentiable on $\mathbb{R}^n$. Consider the following scalar nonlinear programs:

(P) Minimize $\varphi(x)$ subject to $g(x) \leqq 0$,

(PE) Minimize $\varphi(x)$ subject to $g(x) \leqq 0, \quad h(x) = 0$.

The dual programs in Wolfe's sense associated to these two programs are, respectively

$$(D) \begin{cases} \text{Maximize} & \varphi(u) + y'g(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) = 0, \quad y \geqq 0, \end{cases}$$

$$(DE) \begin{cases} \text{Maximize} & \varphi(u) + y'g(u) + z'h(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) + \nabla z'h(u) = 0, \quad y \geqq 0. \end{cases}$$

Mond and Weir associated to (P) the following dual

$$(D1) \begin{cases} \text{Maximize} & \varphi(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) = 0 \\ & y'g(u) \geqq 0, \quad y \geqq 0 \end{cases}$$

and they associated to (PE) the following two dual programs,

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$$(DE1)\begin{cases} \text{Maximize} & \varphi(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'g(u) + z'h(u) \geqq 0, \quad y \geqq 0 \end{cases}$$

and

$$(DEG)\begin{cases} \text{Maximize} & \varphi(u) + y'_{J_0}g_{J_0}(u) + z'_{K_0}h_{K_0}(u) \\ \text{subject to} & \nabla(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y \geqq 0,\ y'_{J_\alpha}g_{J_\alpha}(u) + z'_{K_\alpha}h_{K_\alpha}(u) \geq 0, \quad \alpha = \overline{1,r}. \end{cases}$$

In the dual (DEG) the set families $\{J_\alpha\}_{0\leq\alpha\leq r}$ and $\{K_\alpha\}_{0\leq\alpha\leq r}$ are partitions of the sets $M = \{1, 2, ..., m\}$ and $S = \{1, 2, ..., s\}$, and $r = \max\{\overline{r_1, r_2}\}$ where r_1 and r_2 are the numbers of partitions M and S and $J_\alpha = \varnothing$ or $K_\alpha = \varnothing$ for $\alpha > \min\{r_1, r_2\}$. Also, the notations $y'_{J_\alpha}g_{J_\alpha}(u) = \sum_{i\in J_\alpha} y_i g_i(u)$ and $z'_{K_\alpha}h_{K_\alpha}(u) = \sum_{j\in K_\alpha} z_j h_j(u)$ are used.

For the following partitions of M and S,

$$\begin{aligned} J_0 &= M, \quad J_\alpha = \varnothing, \quad \alpha = \overline{1,r} \\ K_0 &= S, \quad K_\alpha = \varnothing, \quad \alpha = \overline{1,r}, \end{aligned}$$

the dual (DEG) becomes (DE), which is the Wolfe dual of (PE).

Mond and Weir didn't mention the existence of some relations between the duals (DEG) and (DE1). Using some hypotheses of generalized convexity they have established a weak duality theorem and another theorem of strong (direct) duality between (P) and (D1), (PE) and (DEG) respectively, establishing a duality framework that later has been named "Mond-Weir duality" [2], [9] etc.

We mention that Mond and Weir have presented in the same paper [8], some particular variants for (DEG). But they didn't present the variant that corresponds to the following partitions of M and S, respectively:

$$\begin{aligned} J_0 &= \varnothing, \quad J_\alpha = \{\alpha\}; \quad \alpha = \overline{1,r} \\ K_0 &= \varnothing, \quad K_\alpha = \{\alpha\}; \quad \alpha = \overline{1,r} \end{aligned}$$

when the dual (DEG) one reduces to (DE1).

Moreover, if $J_0 = \varnothing$, $J_\alpha = \{\alpha\}$, $\alpha = \overline{1,r}$ for M and $S = \varnothing$ then (DEG) becomes (D1).

These facts justify the unity of the Mond-Weir duality in the framework given by the pair (PEV) - (DEV).

Even from its appearance the Mond-Weir duality has been approached in the vector framework. The general vector minimization problem, that is, a general vector program of minimization, is formulated as follows:

$$(PEV)\quad\begin{cases} \text{Minimize} & f(x) = (f_1(x), ..., f_p(x))' \\ \text{subject to} & g(x) \leqq 0, \quad h(x) = 0. \end{cases}$$

The domain of this problem is the set $D = \left\{ x \in \mathbb{R}^n \middle| g(x) \leqq 0, \quad h(x) = 0 \right\}$. For two vectors, $u = (u_1, ..., u_m)'$ and $v = (v_1, ..., v_m)'$ of $\mathbb{R}^m$, the relations $u = v$, $u < v$, $u \leq v$, $u \leqq v$ etc. are defined respectively by

$$\begin{aligned} u &= v \Longleftrightarrow u_i = v_i, \quad i = \overline{1,m}; \\ u &< v \Longleftrightarrow u_i < v_i, \quad i = \overline{1,m}; \\ u &\leqq v \Longleftrightarrow u_i \leq v_i, \quad i = 1,m; \\ u &\leq v \Longleftrightarrow u \leqq v \text{ and } u \neq v. \end{aligned}$$

Definition. A feasible point $x^0 \in D$ of (PEV) $[\max\limits_{x \in D} f(x)]$ is said to be a *Pareto minimum [maximum] point* or an *efficient solution* (of minimum [maximum] type) of this program if there exists no other feasible point $x \in D$ such that $f(x) \leq f(x^0)[f(x) \geq f(x^0)]$ (shortly $f(x) \geqq_P f(x^0)[f(x) \leqq_P f(x^0)]$).

The first dual vector programs in Mond-Weir sense were independently introduced in 1987 by Egudo and Hanson [3] and by Weir [11]. Thus Egudo, Hanson and Weir have introduced for (PV) the following dual vector program

$$(DV1) \quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) = 0 \\ & y' g(u) \geq 0, \ y \geqq 0 \\ & t \geq 0, \ t'e = 1, \ e = (1, ..., 1)' \in \mathbb{R}^p \end{cases}$$

and Weir associated to (PEV) the following dual vector program

$$(DEVG) \quad \begin{cases} \text{Maximize} & f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0}(u) h_{K_0}(u)]e \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) + \nabla z' h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geq 0, \ \alpha = \overline{1,r} \\ & y \geqq 0, \ t \geq 0, \ t'e = 1. \end{cases}$$

Later, in 1992 Preda [9] has used for (PEV) the following dual

$$(DEV1) \quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) + \nabla z' h(u) = 0 \\ & y' g(u) \geq 0, \ z' h(u) = 0 \\ & y \geqq 0, \ t \geq 0, \ t'e = 1. \end{cases}$$

Consider now for the vector program (PEV) the following duals

$$(DEV2) \quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) + \nabla z' h(u) = 0 \\ & y' g(u) + z' h(u) \geq 0 \\ & y \geqq 0, \ t \geq 0, \ t'e = 1 \end{cases}$$

(the vector correspondent of (DE1)),

$$(DEV3)\quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t'f(x) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) \geq 0,\ z'_{K_\alpha} h_{K_\alpha}(u) = 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1, \end{cases}$$

$$(DEV4)\quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t'f(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geq 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1, \end{cases}$$

$$(DEV5)\quad \begin{cases} \text{Maximize} & f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0} h_{K_0}(u)]e \\ \text{subject to} & \nabla t'f(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) \geq 0,\ z'_{K_\alpha} h_{K_\alpha}(u) = 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1. \end{cases}$$

The Mond-Weir duality was developed especially in the papers of Egudo, Hanson, Mond, Weir and Preda. Using various types of differentible generalized convex functions there were established especially, weak and direct (strong) duality theorems by,

- Egudo [1] (1987), [2] (1989), Egudo, Hanson [3] (1987), Weir [11] (1987) and Weir, Mond [12] (1989) for the pair of programs (PV)-(DV1);
- Weir [11] (1987), Weir, Mond [12] (1989) and Preda [9] for the pair of programs (PEV)-(DEVG);
- Preda [9] (1992), [10] (1993) for the pair of programs (PEV)-(DEV1).

We remark the presence of only a few converse duality theorems and these in nonstandard forms, in the papers of Egudo, Hanson [3] (1987), Weir-Mond [12] (1989) and Preda [9] (1992), [10] (1993).

We denote Ω_X the domain of a dual vector program (X). The expressions

$$\begin{aligned} L(x,y,z) &= f(x) + [y'g(x) + z'h(x)]e, \\ L_0(x,y,z) &= f(x) + [y'_{J_0} g_{J_0}(x) + z'_{K_0} h_{K_0}(x)]e \end{aligned}$$

are called the vector Lagrangians associated to the vector program (PEV).

Theorem. *We suppose that $D \neq \varnothing$, $\Omega_{DEVG} \neq \varnothing$ and for every $(u,t,y,z) \in \Omega_{DEVG}$ the component u is a Pareto minimum point for the vector Lagrangian $L(\cdot,y,z)$.*

Then for all $x \in D$ and $(u,t,y,z) \in \Omega_{DEVG}$ the relation $f(x) \leq L_0(u,y,z)$ is false, that is

$$f(x) \geqq {}_p L_0(u,y,z), \forall x \in D, \forall (u,t,y,z) \in \Omega_{DEVG}.$$

Proof. Suppose, by absurdum, that there is an $\overline{x} \in D$, $\overline{x} \neq u$ such that for every $(u,t,y,z) \in \Omega_{DEVG}$ one has

$$f(\overline{x}) \leq L_0(u,y,z) = f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0} h_{K_0}(u)]e. \tag{1}$$

But $\overline{x} \in D$ and $y \geqq 0$ imply $y'g(\overline{x}) + z'h(\overline{x}) \leq 0$ and then

$$[y'g(\overline{x}) + z'h(\overline{x})]e \leqq 0. \tag{2}$$

From $y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geq 0$, $\alpha = \overline{1,r}$ it results

$$0 \leqq [y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u)]e, \ \alpha = \overline{1,r}. \tag{3}$$

Summing now member by member the inequalities (1), (2) and (3) it results $L(\overline{x}, y, z) \leq L(u, y, z)$, a relation which contradicts the fact that u is a Pareto minimum point for $L(\cdot, y, z)$. □

Remark 1. For the partitions $J_0 = \varnothing$, $J_\alpha = \{\alpha\}$, $\alpha = \overline{1,r}$, $S = \varnothing$, the dual vector (DEVG) becomes (DV1).

Remark 2. For the following partitions

$$\begin{array}{llll} J_0 & = \varnothing, & J_\alpha = \{\alpha\}, & \alpha = \overline{1,r} \\ K_0 & = \varnothing, & K_\alpha = \{\alpha\}, & \alpha = \overline{1,r} \end{array}$$

(DEVG) becomes (DEV2).

Remark 3. $y'g(u) \geq 0$, $z'h(u) = 0 \Rightarrow y'g(u) + z'h(u) \geq 0$. Then $\Omega_{DEV1} < \Omega_{DEV2}$ and also

$$\max_{(u,t,y,z) \in \Omega_{DEV1}} f(u) \leqq_P \max_{(u,t,y,z) \in \Omega_{DEV2}} f(u) \leqq_P \min_{x \in D} f(x).$$

As consequence, the duality for the pair of programs (PEV)-(DEV1) implies the duality for the pair programs (PEV)-(DEV2). Then see *2*.

Remark 4. (DEV4) is obtained by (DEVG) for $J_0 = \varnothing$ and $K_0 = \varnothing$.

Remark 5. $y'_{J_\alpha} g_{J_\alpha}(u) \geq 0$, $z'_{K_\alpha} h_{K_\alpha}(u) = 0 \Rightarrow y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geq 0$ that is $\Omega_{DEV3} \subset \Omega_{DEV4}$ and then

$$\max_{(u,t,y,z) \in \Omega_{DEV3}} f(u) \leqq_P \max_{(u,t,y,z) \in \Omega_{DEV4}} f(u) \leqq_P \min_{x \in D} f(x).$$

Then the Mond-Weir duality for the pair of Programs (PEV)-(DEV3) implies the Mond-Weir duality for the pair programs (PEV)-(DEV4). Now see the previous remark.

Remark 6. According to Remark 5 we have $\Omega_{DEV5} \subset \Omega_{DEVG}$ and

$$\max_{(u,t,y,z) \in \Omega_{DEV5}} L_0(u, y, z) \leqq_P \max_{(u,t,y,z) \in \Omega_{DEVG}} L_0(u, y, z) \leqq_P \min_{x \in D} f(x).$$

Then the Mond-Weir duality for the pair programs (PEV)-(DEV5) implies the Mond-Weir duality for the pair programs (PEV)-(DEVG).

Remarks 1-6 show that the Mond-Weir duality for the pair of programs (PEV)-(DEVG) is a generalized framework for the various variants, above mentioned, of this duality. This fact confirms the unity of the Mond-Weir duality in the vector programming too.

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