On a Class of Almost Cosymplectic Manifolds

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Abstract

The authors derive properties of closed torese-forming vector fields, of the Lie derivatives $L_X\Omega$ (where X is the principal vector field associated with the symplectic form Ω) and of the soldering form of the locally-cosymplectic manifold M.

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§1. Preliminaries

Let (M, g) be a Riemannian $C^\infty-$manifold and let ∇ be the Levi-Civita covariant operator defined by the metric tensor g, and assume that M is oriented. Let ΓTM be the set of sections of the tangent bundle and

$$\flat : TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \sharp : TM \xleftarrow{\sharp} T^*M$$

the isomorphisms defined by g (i.e. $\flat$ is the index lowering operator and $\sharp$ is the index raising operator).

Following [3] we denote by

$$A^q\,(M, TM) = \Gamma Hom(\wedge^q TM, TM)$$

the set of vector valued q-forms $(1 < \dim M)$ and we introduce for the covariant derivative operator with respect to ∇ as

$$d^\nabla : A^q\,(M, TM) \to A^{q+1}(M, TM).$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$, then the vector valued $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M and is also called the soldering form of M (on the identity vector valued 1-form). Since ∇ is symmetric one has that $d^\nabla(dp) = 0$.

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Let $O = \{e_A \mid A \in \overline{1,2m} \cup \{0\}\}$ be a local field of orthonormal frames over M and let $O^* = \{\omega^A\}$ the associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

$$\nabla e = \theta \otimes e \tag{1}$$

$$d\omega = -\theta \wedge \omega$$

$$d\theta = -\theta \wedge \theta + \Theta.$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-form on M).

§2. Main results

Let $M\,(\phi, \Omega, \xi, \eta, g)$ be a $(2m+1)-$dimensional locally cosymplectic manifold where ϕ is the (1,1)-tensor field, Ω a 2-form of class $2m$, ξ the real vector field and $\eta = \xi^\flat$ its covector.

In this paper we assume that ξ is a closed torse-forming (abv. TF), ([6], [4], [5]) and we write

$$\nabla\xi = \lambda dp - \lambda\eta \otimes \xi, \qquad \lambda \in \Lambda^0 M. \tag{2}$$

It follows at once from (1) that

$$d\eta = 0,$$

and one has

$$\Omega_o^a = \lambda\omega^a, \qquad a, b \in \overline{1,2m}.$$

We assume

$$d\lambda \wedge \eta = \Omega,$$

and this gives

$$\nabla^2\xi = \left(d\lambda + \lambda^2\eta\right) \wedge dp.$$

Hence following [4], ξ is an exterior concurrent vector, and we may set $d\lambda = f\eta$. In these conditions, if R denotes the Ricci curvature, from (2) it follows

$$R(\xi, Z) = -2m(f + \lambda^2)\eta \wedge dp.$$

Taking now the exterior differential of Ω one derives by a standard calculation

$$d\Omega = 2\lambda\eta \wedge \Omega + \varphi, \tag{3}$$

where φ is a 3-form, which we agree to call the associated 3-form of Ω.

Since by definition $i_\xi\varphi = 0$, taking the Lie differential of Ω with respect to Ω, one derives:

$$L_\xi\Omega = 2\lambda\Omega.$$

This says that ξ defines an infinitesimal conformal transformations of Ω.

Recall now that the cohomology operator λ^ω is defined by [1]

$$d^\omega = de(\omega),$$

when $e(\omega)$ denotes the exterior product by the closed 1-form ω.

Clearly one has

$$d^{\omega} \circ d^{\omega} = 0.$$

Any form $u \in \wedge M$ satisfying $d^{\omega} u = 0$ is said to be d^{ω}-closed and ω is called the cohomology form.

If ω is an exact form, then u is said to be d^{ω}-exact. Hence by (3) one may write

$$d^{-2\lambda\eta}\Omega = \varphi.$$

Definition. A vector field X is said to be $-2\lambda\eta-$closed with respect to the 3-form φ iff

$$d\left(i_X \varphi\right) = 2X\eta \wedge \varphi.$$

In the following we agree to call X *the principal vector field on* M. In these conditions

$$d(L_X\Omega) = 2\lambda\eta \wedge L_X\Omega \Longleftrightarrow (dL_X\Omega) = 0$$

and one may say that $L_X\Omega$ is $2\lambda\eta$-conformal [4].

It follows from above that if X is any horizontal vector field which is $2\lambda\eta$-closed with respect to φ one gets

$$d(L_X\Omega) = 0.$$

This says that $L_X\Omega$ is cohomolocally $-2\lambda\eta$ closed.

Finally making use of the (1.1)-tensor field ϕ one finds

$$d^{\nabla}(\phi dp) = 2\lambda\Omega \otimes \xi + \lambda\eta \wedge \phi dp.$$

Summing up we have the following

Theorem. *Let $M(\phi, \Omega, \xi, \eta, g)$ be a $(2m+1)-$dimensional locally cosymplectic manifold and let the real vector field ξ be a closed torse forming. Let φ be the associated 3-form of Ω and X the principal vector field associated with Ω. One has the following properties:*

(i) ξ is an exterior concurrent vector field and defines an infinitesimal conformal transformation of Ω, i.e.

$$L_X\Omega = 2\lambda\Omega;$$

(ii) the Lie derivative $L_X\Omega$ is $2\lambda\eta$-conformal;

(iii) $L_X\Omega$ is cohomologically $-2\lambda\eta$ closed, i.e.,

$$d(L_X\Omega)^{-2\lambda\eta};$$

(iv) if dp is the soldering form, ϕ is related to Ω by

$$d^{\nabla}(\phi dp) = 2\lambda\Omega \otimes \xi + \lambda\xi \wedge \phi dp.$$

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