Scalar Curvature of C-totally Real Submanifolds in Sasakian Space Forms

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Abstract

We establish a sharp inequality between the squared mean curvature and the scalar curvature for a C-totally real submanifold of maximum dimension in a Sasakian space form. The equality case is investigated.

M.S.C. 2000: 53C25.
Key words: Sasakian space form, Riemannian manifold, sectional curvature.

§1. Introduction

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the *scalar curvature* τ at p is defined by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) \ .$$

If M is a submanifold of a Riemannian manifold $\widetilde{M}$ and $p \in M$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM, we denote by H the *mean curvature vector*, that is

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r)$$

$$||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$$

Here, we rewiew briefly some known facts on *Jacobi's elliptic functions* for later use ([2]).

Put

$$u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{1}$$

Differential Geometry - Dynamical Systems, Vol.4, No.1, 2002, pp. 5-11.

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where we first suppose that x and k satisfy $0 < k < 1$, $-1 \leq x \leq 1$. Equation (1) defines u as an odd function of x which is positive, increasing from 0 to K as x increases from 0 to 1. Inversely, the same equation defines x as an odd function of u which increases from 0 to 1 as u increases from 0 to K; this function is known as a Jacobi's elliptic function, denoted by $sn(u, k)$ (or simply by $sn(u)$), so that we can put

$$u = sn^{-1}(x), \qquad x = sn(u).$$

The other two main Jacobi's functions $cn(u, k)$ and $dn(u, k)$ (simply denoted respectively by $cn(u)$ and $dn(u)$) are defined by

$$cn(u) = \sqrt{1 - sn^2(u)}, \qquad dn(u) = \sqrt{1 - k^2 sn^2(u)},$$

the square roots are positive so long as u is confined to the interval $-K < u < K$, so that $cn(u)$ and $dn(u)$ are even functions of u. Let $k' = \sqrt{1-k^2}$ be the complementary modulus. Then $dn(u) \geq k' > 0$.

§3. Submanifolds of a Sasakian space form

Let $(\tilde{M}, g)$ be a $(2m+1)$-dimensional Riemannian manifold endowed with an endomorphism φ ($(1,1)$-tensor field) of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η such that

$$\begin{cases} \varphi^2 X = -X + \eta(X)\xi\,,\ \varphi\xi = 0\,,\ \eta \circ \varphi = 0\,,\ \eta(\xi) = 1\,, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,,\ \eta(X) = g(X, \xi)\,, \end{cases}$$

for all vector fields $X, Y \in \Gamma(T\tilde{M})$.

If, in addition, $d\eta(X, Y) = g(\varphi X, Y)$, then $\tilde{M}$ is said to have *a contact Riemannian structure* (φ, ξ, η, g). If, moreover, the structure is normal, i.e. if $[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \phi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$, then the contact Riemannian structure is called a *Sasakian structure* and $\tilde{M}$ is called a *Sasakian manifold.* For more details and background, we refer to the standard references [2], [8].
A plane section σ in $T_p\tilde{M}$ of a Sasakian manifold $\tilde{M}$ is called a *φ-section* if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}(\sigma)$ w.r.t. a φ-section σ is called a φ-sectional curvature. If a Sasakian manifold $\tilde{M}$ has constant φ-sectional curvature c, then it is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$.
The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]:

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \tfrac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \\ &+ \tfrac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z), \end{aligned}$$

for any tangent vector fields X, Y, Z to $\tilde{M}(c)$.

An n-dimensional submanifold M of a Sasakian space form $\tilde{M}(c)$ is called a $C-$***totally real submanifold*** if ξ is a normal vector field on M. A direct consequence of this definition is that $\varphi(TM) \subset T^{\perp}M$, i.e. that M is an anti-invariant submanifold of $\tilde{M}(c)$, (hence their name of "contact"-totally real submanifolds); see e.g. [6].

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures.

Let a be a number > 1. Put

$$\mu_a = \frac{\sqrt{a^2-1}}{\sqrt{2}} cn \left(ax, \frac{\sqrt{a^2-1}}{\sqrt{2}} \right).$$

Denote by P_a^n the ***warped product*** $I \times_{\mu_a} S^{n-1}(\frac{a^4-1}{4})$ with ***warped function*** μ_a for $n \geq 3$ and P_a^2 the warped product $I \times_{\mu_a} \mathbb{R}$.

The metric tensor g of P_a^n is then given by

$$g = dx^2 + \mu_a^2 g_0,$$

where x is the canonical coordinate of I and g_0 the canonical metric tensor of $S^{n-1}(\frac{a^4-1}{4})$ or of $\mathbb{R}$, respectively. Since $S^{n-1}(\frac{a^4-1}{4})$ is conformally flat, there exist local coordinate systems $\{u_2, \ldots, u_n\}$ such that the metric tensor g_0 is given by

$$g_0 = E^2(du_2^2 + du_3^2 + .. + du_n^2)$$

(when $n = 2$, we choose $E = 1$).

Therefore the metric tensor of P_a^n is given by

$$g = dx^2 + \mu_a^2 E^2(du_2^2 + du_3^2 + .. + du_n^2).$$

Let

$$e_1 = \frac{\partial}{\partial x}, e_2 = \frac{1}{E\mu_a}\frac{\partial}{\partial u_2}, \ldots, e_n = \frac{1}{E\mu_a}\frac{\partial}{\partial u_n}.$$

Then $\{e_1, \ldots, e_n\}$ forms an orthonormal frame field on P_a^n.

We define a symmetric bilinear function $\sigma : TP_a^n \times TP_a^n \longrightarrow TP_a^n$, by putting

$$\sigma(e_1, e_1) = 3\mu_a e_1, \quad \sigma(e_1, e_i) = \sigma(e_i, e_1) = \mu_a e_i, \quad \sigma(e_i, e_j) = \delta_{ij}\mu_a e_1, \quad i, j = \overline{2, n}. \tag{2}$$

Then $g(\sigma(X, Y), Z)$ is both linear and totally symmetric in X, Y, Z.

Then there exists a C-totally real isometric immersion $p_a : P_a^n \longrightarrow S^{2n+1}$ whose second fundamental form is given by $h = \varphi\sigma$, where σ is defined by (2).

Applying Gauss' equation and (2), we obtain the equality case for C-totally real isometric immersions.

An orthonormal frame field $e_1, \ldots, e_n, e_{1*}, \ldots, e_{n*}, e_{2n+1}$ is called an *adapted frame field* if $e_1, \ldots, e_n$ are orthonormal tangent vector fields and $e_{1*}, \ldots, e_{n*}, e_{2n+1}$ are normal vector fields given by

$$e_{1*} = \varphi e_1, \ldots, e_{n*} = \varphi e_n, e_{2n+1} = \xi$$

§3. Main results

Theorem 1. *If M^n is a C-totally real submanifold of a Sasakian space form $\tilde{M}^{2n+1}(c)$, then the mean curvature H and the scalar curvature τ of M satisfy*

$$\|H\|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \left(\frac{n+2}{n}\right)\left(\frac{c+3}{4}\right). \tag{3}$$

Moreover the equality sign holds if and only if, with to respect an adapted frame field $e_1,\dots,e_n,e_{1*},\dots,e_{n*},e_{2n+1}=\xi$ with e_{1*} parallel to H, the second fundamental form of M^n in $\tilde{M}^{2n+1}(c)$ takes the following form :

$$h(e_1,e_1)=3\lambda e_{1*}\ ,\quad h(e_2,e_2)=\ldots=h(e_n,e_n)=\lambda e_{1*},$$

$$h(e_1,e_j)=\lambda e_{j*}\ ,\quad h(e_j,e_k)=0\ ,\ 2\leq j\neq k\leq n,$$

with $\lambda \in C^\infty(M)$.

Proof. Let M^n be a C-totally real submanifold of a Sasakian space forms $\tilde{M}^{2n+1}(c)$, and $e_1,\dots,e_n,e_{1*},\dots,e_{n*},e_{2n+1}=\xi$ a local adapted frame field on M^n.

Put $h^i_{jk}=g\left(h(e_j,e_k),e_{i*}\right)$.

Then, by

$$A_{\varphi X}Y=-\varphi h(X,Y)=A_{\varphi Y}X\quad \forall X,Y\in\Gamma(TM), \tag{4}$$

we have

$$h^i_{jk}=h^j_{ik}=h^k_{ij}\ ,\ i,j,k=\overline{1,n}.$$

From the definition of the mean curvature function we have

$$n^2\|H\|^2=\sum_i\left(\sum_j\left(h^i_{jj}\right)^2+2\sum_{j<k}h^i_{jj}h^i_{kk}\right).$$

From the equation of Gauss we have

$$2\tau=n(n-1)\left(\frac{c+3}{4}\right)+n^2\|H\|^2-\|h\|^2=n(n-1)\left(\frac{c+3}{4}\right)+n^2\|H\|^2-\sum_{i,j,k=1}^{n}\left(h^i_{jk}\right)^2.$$

Thus, by applying precedent relations, we obtain

$$\tau=\frac{n(n-1)}{2}\left(\frac{c+3}{4}\right)+\sum_i\sum_{j<k}h^i_{jj}h^i_{kk}-\sum_{i\neq j}\left(h^i_{jj}\right)^2-3\sum_{i<j<k}\left(h^i_{jk}\right)^2.$$

Let $m=\dfrac{n+2}{n-1}$. Then, we get

$$n^2\|H\|^2-m\left(2\tau-n(n-1)\left(\frac{c+3}{4}\right)\right)=\sum_i\left(h^i_{ii}\right)^2+(1+2m)\sum_{i\neq j}\left(h^i_{jj}\right)^2$$

$$+6m\sum_{i<j<k}\left(h^i_{jk}\right)^2-2(m-1)\sum_i\sum_{j<k}h^i_{jj}h^i_{kk}$$

$$= \sum_i \left(h^i_{ii}\right)^2 + 6m \sum_{i<j<k} \left(h^i_{jk}\right)^2 + (m-1) \sum_i \sum_{j<k} \left(h^i_{jj} - h^i_{kk}\right)^2$$

$$+ (1 + 2m - (n-2)(m-1)) \sum_{j \neq i} \left(h^i_{jj}\right)^2 - 2(m-1) \sum_{j \neq i} h^i_{jj} h^i_{jj}$$

$$= 6m \sum_{i<j<k} \left(h^i_{jk}\right)^2 + (m-1) \sum_{i \neq j,k} \sum_{j<k} \left(h^i_{jj} - h^i_{kk}\right)^2$$

$$+ \frac{1}{n-1} \sum_{j \neq i} \left(h^i_{ii} - (n-1)(m-1) h^i_{jj}\right)^2 \geq 0$$

which implies inequality (3). We see that the equality sign of (3) holds if and only if $h^i_{ii} = 3h^i_{jj}, h^i_{jk} = 0$, for distinct i, j, k. In particular, if choose $e_1, \ldots, e_n$ in such way that φe_1 is parallel to the mean curvature vector H, we also have $h^j_{kk} = 0$ for $j > 1, k = \overline{1, n}$. □

Theorem 2. *Let $i : M^n \longrightarrow S^{2n+1}$ be a C-totally real isometric immersion satisfying the equality case*

$$\|H\|^2 = \frac{2(n+2)}{n^2(n-1)} \tau - \left(\frac{n+2}{n}\right).$$

Then either M is a totally geodesic submanifold and hence M is a locally isometric to the real projective space $\mathbb{R}P^n(1)$ or the set U of non-totally geodesic points in M is a dense subset of M, U is an open portion of a P^n_a with $a > 1$ and, up to rigid motions of S^{2n+1}, the immersion i is given by p_a.

Proof. It follows from Theorem 1 that the function $\phi = \left(\frac{n}{n-2} \right)^2 \|H\|^2 = \lambda^2$ is a well-defined function on M. If the function ϕ vanishes identically, then M is a totally geodesic submanifold of S^{2n+1}. So, for simplicity, we may assume from now on that M is non-totally geodesic, i.e. $\phi \neq 0$. Thus, $U = \{p \in M \mid \phi(p) \neq 0\}$ is a non-empty open subset of M.

Let $\omega^1, \ldots, \omega^n$ denote the dual 1-forms of $e_1, \ldots, e_n$ and denoted by (ω^A_B), $A, B = 1, \ldots, n, 1*, \ldots n*, 2n+1$, the connection forms on M defined by

$$\widetilde{\nabla} e_i = \sum_{j=1}^n \omega^j_i e_j + \sum_{j=1}^n \omega^{j*}_i e_{j*}, \quad \widetilde{\nabla} e_{i*} = \sum_{j=1}^n \omega^j_{i*} e_j + \sum_{j=1}^n \omega^{j*}_{i*} e_{j*},\ i = 1, \ldots, n,$$

where $\omega^j_i = -\omega^i_j$, $\omega^{j*}_{i*} = -\omega^{i*}_{j*}$

For a C-totally real submanifold M^n of a $S^{2n+1}(c)$, (4) yields

$$\omega^{i*}_j = \omega^{j*}_i, \omega^j_i = \omega^{j*}_{i*},\ \omega^{i*}_j = \sum_{k=1}^n h^i_{jk} \omega^k.$$

We find

$$\omega^{1*}_1 = 3\lambda\omega^1,\ \omega^{1*}_i = \lambda\omega^i,\ \omega^{i*}_i = \lambda\omega^1,\ \omega^{i*}_j = 0,\ 2 \leq i \neq j \leq n. \tag{5}$$

By applying the equation of Codazzi, we obtain

$$e_1\lambda = \lambda\omega_1^2(e_2) = \ldots = \lambda\omega_1^n(e_n),\ e_2\lambda = \ldots = e_n\lambda = 0, \tag{6}$$

$$\omega_1^j(e_k) = 0,\ 1 < j \neq k \leq n. \tag{7}$$

By precedent formulas, one gets

$$\omega_1^j = e_1(\ln\lambda)\omega^j,\ j = \overline{2,n}. \tag{8}$$

Let $\mathcal{D}$ denote the distribution spanned by φH and $\mathcal{D}^\perp$ denote the orthogonal complementary distribution of $\mathcal{D}$ on U. Then $\mathcal{D}$ and $\mathcal{D}^\perp$ are spanned by $\{\varphi H\}$ and $\{e_2, \ldots, e_n\}$, respectively.

Lemma 3. *On U we have*
(1) the integral curves of φH (or, equivalently, of e_1) are geodesics of M,
(2) the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable,
(3) there exist local coordinate systems $\{x_1, \ldots, x_n\}$ such that :
(a) $\mathcal{D}$ is spanned by $\{\frac{\partial}{\partial x}\}$ and $\mathcal{D}^\perp$ is spanned by $\{\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}$,
(b) $e_1 = \frac{\partial}{\partial x}$, $\omega^1 = dx$,
(c) the metric tensor g takes the form : $g = dx^2 + \sum_{j,k=2}^n g_{jk}dx_j dx_k$, where $x = x_1$,
(4) λ is a function of x and it satisfies

$$\frac{d^2\lambda}{dx^2} + 2\lambda^3 + \lambda = 0$$

Proof. (8) and Cartan's structure equations imply $d\omega^1 = 0$ and $\nabla_{e_1}e_1 = 0$.

Therefore, the integral curves of e_1 are geodesics. This proves statement (1).

For any $j, k > 1$, (7) implies $\langle [e_j, e_k], e_1\rangle = \omega_j^1(e_j) - \omega_j^1(e_k) = 0$, which shows that the distribution $\mathcal{D}^\perp$ is integrable. The integrability of $\mathcal{D}$ is obvious, since $\mathcal{D}$ is a 1-dimensional distribution. This proves (2).

Since $\mathcal{D}$ is 1-dimensional, there exists a local coordinate system $\{y_1, \ldots, y_n\}$ such that $e_1 = \frac{\partial}{\partial y_1}$. Because $\mathcal{D}^\perp$ is integrable, there also exists a local coordinate system $\{z_1, \ldots, z_n\}$ such that $\{\frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\}$ span $\mathcal{D}^\perp$. Put

$$x_1 = y_1, x_2 = z_2, \ldots, x_n = z_n.$$

Then $\{x_1, \ldots, x_n\}$ is a local coordinate system satisfying the conditions given in (3).

From (6) and statement (3), we see that the function λ depends only on $x(= x_1)$; thus $\lambda = \lambda(x)$. Let λ' and λ'' denote the first and second derivates of λ with respect to x. By taking the exterior differentiation of (8) and using (5), (8) and Cartan's structure equations, we find

$$(\ln\lambda)'' + (\ln\lambda)'^2 = -1 - 2\lambda^2$$

which is equivalent to (4).

Come back to the proof of Theorem 2.

By reference to [4], the solution of the equation (4) is

$$\lambda = \frac{\sqrt{a^2-1}}{\sqrt{2}} cn \left(ax + b, \frac{\sqrt{a^2-1}}{\sqrt{2}} \right),$$

where a, b are constants with $a > 1$.

The rigidy theorem for C-totally real immersions in S^{2n+1} achieves the proof. □

In a forthcoming paper, we investigate submanifolds in $\mathbb{R}^{2n+1}$ satisfying the equality case of Theorem 1.

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