Mond-Weir Duality: Unitary Character and a Converse Duality Theorem for Vector Programs

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Abstract

In this paper the unitary character of the Mond-Weir duality in nonlinear programming is established. Also, a converse duality theorem for the vector programs is given.

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Key words: nonlinear programming, Mond-Weir duality, Wolfe duality, Pareto optimality

§1. The unitary character of Mond-Weir duality

The Mond-Weir duality in nonlinear programming was introduced in 1981 by Mond and Weir [11] as a generalization of the Wolfe duality. It was formulated as follows.

Let the functions $\varphi : \mathbb{R}^n \to \mathbb{R}, f = (f_1, ..., f_p)' : \mathbb{R}^n \to \mathbb{R}^p$, $g = (g_1, ..., g_m)' : \mathbb{R}^n \to \mathbb{R}^m$ and $h = (h_1, ..., h_s)' : \mathbb{R}^n \to \mathbb{R}^s$ be all differentiable on $\mathbb{R}^n$. Consi der the following scalar nonlinear programs:

(P) Minimize $\varphi(x)$ subject to $g(x) \leqq 0$,

(PE) Minimize $\varphi(x)$ subject to $g(x) \leqq 0, \quad h(x) = 0$.

The dual programs in Wolfe's sense associated to these two programs are, respectively

$$(D) \qquad \begin{cases} \text{Maximize} & \varphi(u) + y'g(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) = 0, \quad y \geqq 0, \end{cases}$$

$$(DE) \qquad \begin{cases} \text{Maximize} & \varphi(u) + y'g(u) + z'h(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) + \nabla z'h(u) = 0, \quad y \geqq 0. \end{cases}$$

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Mond and Weir associated to (P) the following dual

$$(D1) \qquad \begin{cases} \text{Maximize} & \varphi(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) = 0 \\ & y'g(u) \geqq 0, \quad y \geqq 0, \end{cases}$$

and they associated to (PE) the following two dual programs,

$$(DE1) \qquad \begin{cases} \text{Maximize} & \varphi(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'g(u) + z'h(u) \geqq 0, \quad y \geqq 0 \end{cases}$$

and

$$(DEG) \qquad \begin{cases} \text{Maximize} & \varphi(u) + y'_{J_0}g_{J_0}(u) + z'_{K_0}h_{K_0}(u) \\ \text{subject to} & \nabla\varphi(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y \geqq 0,\ y'_{J_\alpha}g_{J_\alpha}(u) + z'_{K_\alpha}h_{K_\alpha}(u) \geqq 0, \quad \alpha = \overline{1,r}. \end{cases}$$

In the dual (DEG), the set families $\{J_\alpha\}_{0\leq\alpha\leq r}$ and $\{K_\alpha\}_{0\leq\alpha\leq r}$ are partitions of the sets $M = \{1, 2, ..., m\}$ and $S = \{1, 2, ..., s\}$, and $r = \max\{r_1, r_2\}$, where r_1 and r_2 are the numbers of partitions M and S and $J_\alpha = \varnothing$ or $K_\alpha = \varnothing$ for $\alpha > \min\{r_1, r_2\}$. Also, the notations $y'_{J_\alpha}g_{J_\alpha}(u) = \sum\limits_{i\in J_\alpha} y_i g_i(u)$ and $z'_{K_\alpha}h_{K_\alpha}(u) = \sum\limits_{j\in K_\alpha} z_j h_j(u)$ are used.

For the following partitions of M and S,

$$J_0 = M, \quad J_\alpha = \varnothing\,; \quad \alpha = \overline{1,r}$$
$$K_0 = S, \quad K_\alpha = \varnothing\,; \quad \alpha = \overline{1,r},$$

the dual (DEG) becomes (DE), which is the Wolfe dual of (PE).

Mond and Weir didn't mentioned the existence of some relations between the duals (DEG) and (DE1). Using some hypotheses of generalized convexity they have established a weak duality theorem and another theorem of strong (direct) duality between (P) and (D1), (PE) and (DEG) respectively, establishing a duality framework that later has been named "Mond-Weir duality"' [3], [12] etc.

We mention that Mond and Weir have presented in the same paper [11], some particulary variants for (DEG). But they didn't present the variant that corresponds to the following partitions of M and S, respectively:

$$J_0 = \varnothing\,, \quad J_\alpha = \{\alpha\}\,; \quad \alpha = \overline{1,r}$$
$$K_0 = \varnothing\,, \quad K_\alpha = \{\alpha\}\,; \quad \alpha = \overline{1,r},$$

when the dual (DEG) one reduces to (DE1).

Moreover, if $J_0 = \varnothing\,, \quad J_\alpha = \{\alpha\}\,; \quad \alpha = \overline{1,r}$ for M and $S = \varnothing$, then (DEG) becomes (D1).

These facts justify the unity of the Mond-Weir duality in the framework given by the pair (PEV) - (DEV).

Even from its appearance the Mond-Weir duality has been approached in the vector framework. The general vector minimization problem, that is a general vector program of minimization, is formulated as follows:

$$(PEV)\qquad \begin{cases} \text{Minimize} & f(x) = (f_1(x), ..., f_p(x))' \\ \text{subject to} & g(x) \leqq 0, \quad h(x) = 0. \end{cases}$$

The domain of this problem is the set $D = \left\{ x \in \mathbb{R}^n \middle| g(x) \leqq 0, \quad h(x) = 0 \right\}$. For two vectors, $u = (u_1, ..., u_m)'$ and $v = (v_1, ..., v_m)'$ of $\mathbb{R}^m$, the relations $u = v$, $u < v$, $u \leq v$, $u \leqq v$ are defined respectively by

$$\begin{aligned} &u = v \Longleftrightarrow u_i = v_i, \quad i = \overline{1,m}; \\ &u < v \Longleftrightarrow u_i < v_i, \quad i = \overline{1,m}; \\ &u \leqq v \Longleftrightarrow u_i \leqq v_i, \quad i = \overline{1,m}; \\ &u \leq v \Longleftrightarrow u \leqq v \text{ and } u \neq v. \end{aligned}$$

Definition. A feasible point $x^0 \in D$ of (PEV) $\max_{x\in D} f(x)]$ is said to be a *Pareto minimum [maximum] point* or an *efficient solution* (of minimum [maximum] type) of this program if there exists no other feasible point $x \in D$ such that $f(x) \leq f(x^0)[f(x) \geq f(x^0)]$ (shortly $f(x) \geqq_p f(x^0)[f(x) \leqq_p f(x^0)]$).

The first dual vector programs in Mond-Weir sense were independently introduced in 1987 by Egudo and Hanson [4] and by Weir [14]. Thus Egudo, Hanson and Weir have introduced for (PV) the following dual vector program

$$(DV1)\qquad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) = 0 \\ & y' g(u) \geqq 0, \; y \geqq 0 \\ & t \geq 0, \; t' e = 1, \; e = (1, ..., 1)' \in \mathbb{R}^p, \end{cases}$$

and Weir associated to (PEV) the following dual vector program

$$(DEVG)\qquad \begin{cases} \text{Maximize} & f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0}(u) h_{K_0}(u)] e \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) + \nabla z' h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geqq 0, \; \alpha = \overline{1,r} \\ & y \geqq 0, \; t \geq 0, \; t' e = 1. \end{cases}$$

Later, in 1992 Preda [12] has used for (PEV) the following dual

$$(DEV1)\qquad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t' f(u) + \nabla y' g(u) + \nabla z' h(u) = 0 \\ & y' g(u) \geqq 0, \; z' h(u) = 0 \\ & y \geqq 0, \; t \geq 0, \; t' e = 1. \end{cases}$$

I am supposing now for the vector program (PEV) the following duals

$$(DEV2)\quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t'f(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'g(u) + z'h(u) \geqq 0 \\ & y \geqq 0,\ t \geq 0,\ t'e = 1 \end{cases}$$

(the vector correspondent of (DE1)),

$$(DEV3)\quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t'f(x) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) \geqq 0,\ z'_{K_\alpha} h_{K_\alpha}(u) = 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1, \end{cases}$$

$$(DEV4)\quad \begin{cases} \text{Maximize} & f(u) \\ \text{subject to} & \nabla t'f(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geqq 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1, \end{cases}$$

$$(DEV5)\quad \begin{cases} \text{Maximize} & f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0} h_{K_0}(u)]e \\ \text{subject to} & \nabla t'f(u) + \nabla y'g(u) + \nabla z'h(u) = 0 \\ & y'_{J_\alpha} g_{J_\alpha}(u) \geqq 0,\ z'_{K_\alpha} h_{K_\alpha}(u) = 0,\ \alpha = \overline{1,r} \\ & y \geqq 0,\ t \geq 0,\ t'e = 1. \end{cases}$$

The Mond-Weir duality was developed especially in the papers of Egudo, Hanson, Mond, Weir and Preda. Using various types of differentible generalized convex functions there were established especially, weak and direct (strong) duality theorems, by:

- Egudo [2, 1987], [3, 1989], Egudo, Hanson [4, 1987], Weir [14, 1987] and Weir, Mond [1989]us for the pair of programs (PV)-(DV1);
- Weir [14, 1987], Weir, Mond [16, 1989] and Preda [12] for the pair of programs (PEV)-(DEVG);
- Preda [12, 1992], [13, 1993] for the pair of programs (PEV)-(DEV1).

We remark the presence only of a few converse duality theorems, and these in nonstandard forms, in the papers of Egudo, Hanson [4, 1987], Weir-Mond [16, 1989] and Preda [12, 1992], [13, 1993].

We denote Ω_X the domain of a dual vector program (X). The expressions

$$L(x, y, z) = f(x) + [y'g(x) + z'h(x)]e,$$
$$L_0(x, y, z) = f(x) + [y'_{J_0} g_{J_0}(x) + z'_{K_0} h_{K_0}(x)]e$$

are called the vector Lagrangians associated to the vector program (PEV).

Theorem 1. *We suppose that $D \neq \varnothing$, $\Omega_{DEVG} \neq \varnothing$ and for every $(u,t,y,z) \in \Omega_{DEVG}$ the component u is a Pareto minimum point for the vector Lagrangian $L(\cdot, y, z)$.*
Then, for all $x \in D$ and $(u,t,y,z) \in \Omega_{DEVG}$, the relation $f(x) \leq L_0(u,y,z)$ is false, that is

$$f(x) \overset{\geq}{=}_p L_0(u,y,z), \forall x \in D, \forall (u,t,y,z) \in \Omega_{DEVG}.$$

Proof. Suppose, by absurdum, that there is an $\overline{x} \in D$, $\overline{x} \neq u$ such that for every $(u,t,y,z) \in \Omega_{DEVG}$ one has

$$f(\overline{x}) \leq L_0(u,y,z) = f(u) + [y'_{J_0} g_{J_0}(u) + z'_{K_0} h_{K_0}(u)]e. \tag{1}$$

But $\overline{x} \in D$ and $y \geqq 0$ imply $y'g(\overline{x}) + z'h(\overline{x}) \leqq 0$ and then

$$[y'g(\overline{x}) + z'h(\overline{x})]e \leqq 0. \tag{2}$$

From $y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geqq 0$, $\alpha = \overline{1,r}$ it follows

$$0 \leqq [y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u)]e, \ \alpha = \overline{1,r}. \tag{3}$$

Summing the inequalities (1), (2) and (3), member by member, it follows $L(\overline{x}, y, z) \leq L(u,y,z)$, a relation which contradicts the fact that u is a Pareto minimum point for $L(\cdot, y, z)$. □

Remark 1. For the partitions $J_0 = \varnothing$, $J_\alpha = \{\alpha\}$; $\alpha = \overline{1,r}$, $S = \varnothing$, the dual vector (DEVG) becomes (DV1).

Remark 2. For the following partitions

$$J_0 = \varnothing, \quad J_\alpha = \{\alpha\}; \quad \alpha = \overline{1,r}$$
$$K_0 = \varnothing, \quad K_\alpha = \{\alpha\}; \quad \alpha = \overline{1,r},$$

(DEVG) becomes (DEV2).

Remark 3. $y'g(u) \geqq 0$, $z'h(u) = 0 \Rightarrow y'g(u) + z'h(u) \geqq 0$. Then $\Omega_{DEV1} < \Omega_{DEV2}$ and also

$$\max_{(u,t,y,z) \in \Omega_{DEV1}} f(u) \overset{\leq}{=}_P \max_{(u,t,y,z) \in \Omega_{DEV2}} f(u) \overset{\leq}{=}_P \min_{x \in D} f(x).$$

Consequently, the duality for the pair of programs (PEV)-(DEV1) implies the duality for the pair programs (PEV)-(DEV2). Then see **2**.

Remark 4. (DEV4) is obtained by (DEVG) for $J_0 = \varnothing$ and $K_0 = \varnothing$.

Remark 5. $y'_{J_\alpha} g_{J_\alpha}(u) \geqq 0$, $z'_{K_\alpha} h_{K_\alpha}(u) = 0 \Rightarrow y'_{J_\alpha} g_{J_\alpha}(u) + z'_{K_\alpha} h_{K_\alpha}(u) \geqq 0$ that is $\Omega_{DEV3} \subset \Omega_{DEV4}$ and then

$$\max_{(u,t,y,z) \in \Omega_{DEV3}} f(u) \overset{\leq}{=}_P \max_{(u,t,y,z) \in \Omega_{DEV4}} f(u) \overset{\leq}{=}_P \min_{x \in D} f(x).$$

Then the Mond-Weir duality for the pair of Programs (PEV)-(DEV3) implies the Mond-Weir duality for the pair programs (PEV)-(DEV4). Now see **4**.

Remark 6. According to Remark 5 we have $\Omega_{DEV5} \subset \Omega_{DEVG}$ and

$$\max_{(u,t,y,z)\in\Omega_{DEV5}} L_0(u,y,z) \leqq_P \max_{(u,t,y,z)\in\Omega_{DEVG}} L_0(u,y,z) \leqq_P \min_{x\in D}.$$

Then the Mond-Weir duality for the pair programs (PEV)-(DEV5) implies the Mond-Weir duality for the pair programs (PEV)-(DEVG).

Remarks 1-6 show that the Mond-Weir duality for the pair of programs (PEV)-(DEVG) is a generalized framework for the various variants, above mentioned, of this duality. This fact confirms the unitary character of the Mond-Weir duality in the vector programming too.

§2. A strict converse duality theorem in the Mond-Weir duality

Egudo, and Hanson [2, 1987] and Weir and Mond [16, 1989] gave theorems of converse duality in nonstandard forms for the pair of vector programs (PV)-(DV1) and respectively for the pair of vector programs (PV)-(DV1) and (PEV)-(DEVG). In that follows we shall establish a strict converse duality theorem of Mangasarian type for the pair of vector programs (PEV)-(DEVG).

Let $x^0 \in D$ and $I^0 = \left\{ i \,|g_i\,(x^0) = 0 \right\}$. The following Kuhn-Tucker type theorem and lemma 3.1 by [6] will be needed.

Theorem 2. [10] **(Necessary conditions of efficiency).** *Let x^0 be an efficient solution of (PEV). Suppose that the functions f, g and h are differentiable at x^0 and (PEV) satisfies at x^0 the following constraint qualification*

$$\mathcal{R}(x^0)\begin{cases} \exists v \in \mathbb{R}^n: & v'\nabla g_{I^0}(x^0) \leq 0, \quad v'\nabla h(x^0) = 0 \\ \exists \varepsilon > 0: & h(x^0 + \varepsilon v) = 0. \end{cases}$$

Then there are vectors $t^0 \in \mathbb{R}^p, y^0 \in \mathbb{R}^m$ and $z^0 \in \mathbb{R}^s$ such that

$$(KT) \qquad \begin{cases} \nabla t^{0\prime} f(x^0) + \nabla y^{0\prime} g(x^0) + \nabla z^{0\prime} h(x^0) = 0 \\ y^{0\prime} g(x^0) = 0, y^0 \geqq 0 \\ t^0 \geq 0, t^{0\prime} e = 1. \end{cases}$$

Lemma. (Kanniappan [6]). *Let X be any space and let $F = (F_1, ..., F_n) : X \to \mathbb{R}^m$ be a mapping. Then F has a Pareto optimum (of maximum type) at $x^0 \in X$ if and only if x^0 maximizes each F_i on the constraint set*

$$C_i = \left\{ x \in X \,|F_j(x) \leqq F_j(x^0),\; j \neq i \right\}.$$

Now the main result of this section it follows.

Theorem 3. (Strict converse duality). *Let (x^0, t^0, y^0, z^0) be an efficient solution of the dual vector program (DEVG). We suppose that the following conditions are satisfied:*

(c1) The primal program (PEV) admits the efficient solution $\overline{x}$ where the constraint qualification $\mathcal{R}(\overline{x})$ is verified;

(c2) The vector Lagrangian $L(\cdot, y^0, z^0)$ admits at x^0 a Pareto minimum point.

Then $\overline{x} = x^0$ and x^0 is an efficient solution for the primal program (PEV). Moreover, $f(x^0) = L_0(x^0, t^0, y^0, z^0)$.

Proof. We suppose, by absurdum, that $\overline{x} \neq x^0$ and we will obtain a contradiction. Because (PEV) satisfies at $\overline{x}$ a constraint qualification, there are vectors $\overline{t} \in \mathbb{R}^p, \overline{y} \in \mathbb{R}^m$ and $\overline{z} \in \mathbb{R}^s$ such that the following efficiency conditions are satisfied:

$$\begin{cases} \nabla \overline{t'} f(\overline{x}) + \nabla \overline{y'} g(\overline{x}) + \nabla \overline{z'} h(\overline{x}) = 0 \\ \overline{y'} g(\overline{x}) = 0, \overline{y} \geqq 0 \\ \overline{t} \geq 0, \overline{t'} e = 1. \end{cases} \tag{4}$$

From these relations it results $(\overline{x}, \overline{t}, \overline{y}, \overline{z}) \in \Omega_{DEVG}$ and in addition,

$$0 = \overline{y'}_{J_0} g_{J_0}(\overline{x}) + \overline{z'}_{K_0} h_{K_0}(\overline{x}), \tag{5}$$

$$0 = \overline{y'}_{J_\alpha} g_{J_\alpha(\overline{x}) + \overline{z'}_{K_\alpha} h_{K_\alpha}}(\overline{x}), \alpha = \overline{1, r}. \tag{6}$$

The point x^0 is a Pareto minimum for $L(\cdot, y^0, z^0)$. Then for $x = \overline{x}$ there exists a component $L_i(\cdot, y^0, z^0)$ of $L(\cdot, y^0, z^0)$ such that $L_i(\overline{x}, y^0, z^0) > L_i(x^0, y^0, z^0)$ or, developed,

$$f_i(\overline{x}) + y^{0\prime} g(\overline{x}) + z^{0\prime} h(\overline{x}) > f_i(x^0) + y^{0\prime} g(x^0) + z^{0\prime} h(x^0). \tag{7}$$

Because (x^0, t^0, y^0, z^0) is an efficient solution for (DEVG), according to Lemma it follows that (x^0, t^0, y^0, z^0) is an optimal solution for the following scalar program

$$(P_i) \quad \begin{cases} \text{Maximize } \; f_i(x) + y'_{J_0} g_{J_0}(x) + z'_{K_0} h_{K_0}(x) \\ \text{subject to} \\ f_j(x) + y'_{J_0} g_{J_0}(x) + z'_{K_0} h_{K_0}(x) \leqq f_j(x^0) + y^{0\prime}_{J_0} g_{J_0}(x^0) + z^{0\prime}_{K_0} h_{K_0}(x^0) \\ j \neq i, (x, t, y, z) \in \Omega_{DEVG}. \end{cases}$$

Then, for $(x, t, y, z) =, (\overline{x}, \overline{t}, \overline{y}, \overline{z})$, we obtain

$$f_i(x^0) + y^{0\prime}_{J_0} g_{J_0}(x^0) + z^{0\prime}_{K_0} h_{K_0}(x^0) \geqq f_i(\overline{x}) + \overline{y}'_{J_0} g_{J_0}(\overline{x}) + \overline{z}'_{K_0} h_{K_0}(\overline{x}). \tag{8}$$

Because $(x^0, t^0, y^0, z^0) \in \Omega_{DEVG}$ we have too

$$y^{0\prime}_{J_\alpha} g_{J_\alpha}(x^0) + z^{0\prime}_{K_\alpha} h_{K_\alpha}(x^0) \geqq 0, \; \alpha = \overline{1, r}. \tag{9}$$

Summing now the relations (6), (8) and (9) , side by side, it follows

$$f_i(x^0) + y^{0\prime} g(x^0) + z^{0\prime} h(x^0) \geqq f_i(\overline{x}) + \overline{y}' g(\overline{x}) + \overline{z}' h(\overline{x}). \tag{10}$$

Finally, from relations (7) and (10) it follows the following inequality

$$f_i(\overline{x}) + y^{0\prime}g(\overline{x}) + z^{0\prime}h(\overline{x}) > f_i(\overline{x}) + \overline{y}'g(\overline{x}) + \overline{z}'h(\overline{x}),$$

that is equivalent to $y^{0\prime}g(\overline{x}) > 0$. But this inequality is false because $y^0 \geqq 0$ and $g(\overline{x}) \leqq 0$ imply $y^{0\prime}g(\overline{x}) \leqq 0$. Therefore the supposition, above made by absurdum, is false and then $\overline{x} = x^0$.

From $y^0 \geqq 0$, $g(x^0) \leqq 0$ and $h(x^0) = h(\overline{x}) = 0$, we have

$$y^0_{J_0}g_{J_0}(x^0) + z^0_{K_0}h_{K_0}(x^0) \leqq 0.$$

Using this inequality and (5) we obtain

$$L_0(x^0, y^0, z^0) = f(x^0) + \left[y^{0\prime}_{J_0}g_{J_0}(x^0) + z^{0\prime}_{K_0}h_{K_0}(x^0)\right] e \leq [=]$$

$$\leq [=]\, f(\overline{x}) + 0e = f(\overline{x}) + \left[\overline{y}'_{J_0}g_{J_0}(\overline{x}) + \overline{z}'_{K_0}h_{K_0}(\overline{x})\right] e = L_0(\overline{x}, \overline{y}, \overline{z}).$$

But the relation $L_0(x^0, y^0, z^0) \leq L_0(\overline{x}, \overline{y}, \overline{z})$ is false because (x^0, y^0, z^0) is an efficient solution for (DEVG). Then it remains

$$L_0(x^0, y^0, z^0) = L_0(\overline{x}, \overline{y}, \overline{z}) = f(\overline{x}) = f(x^0).$$

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