Contact Metric Manifolds with Cyclic-Parallel Ricci Tensor

Cihan Özgür

Abstract

In this study we consider contact metric manifolds which have cyclic-parallel Ricci tensor under the condition that the characteristic vector field ξ belongs to the (k,μ) nullity distribution.

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§1. Introduction

A differentiable manifold $M^{2n+1}$is said to be a *contact manifold* if it admits a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}.

Given a contact form η, one has a unique vector field ξ, which is called the *characteristic vector field*, satisfying

$$\eta(\xi) = 1, \quad d\eta(\xi, X) = 0, \tag{1}$$

for any vector field X.

It is well-known that, there exists a Riemannian metric g and a (1,1)-tensor field φ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \; \varphi^2 X = -X + \eta(X)\xi, \tag{2}$$

where X and Y are vector fields on M.

From (2) it follows that

$$\varphi\xi = 0 \;\;, \;\; \eta \circ \varphi = 0 \;\;, \;\; g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{3}$$

A differentiable manifold M^{2n+1} equipped with the structure tensors (φ, ξ, η, g) satisfying (2) is said to be a *contact metric manifold* and is denoted by $M = (M^{2n+1}, \varphi, \xi, \eta, g)$.

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On a contact metric manifold M, we can define a (1,1)-tensor field h by $h = \frac{1}{2}L_\xi\varphi$, where L denotes Lie differentiation . Then we may observe that h is symmetric and satisfies

(4) $$h\xi = 0 \text{ and } h\varphi = -\varphi h, \nabla_X\xi = -\varphi X - \varphi hX,$$

where $\widetilde{\nabla}$ is Levi-Civita connection [3].

For a contact metric manifold M one may define naturally an almost complex structure on $M \times \mathbb{R}$. If this almost complex structure is integrable then M is said to be a *Sasakian manifold* . A Sasakian manifold is characterized by the condition

(5) $$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X,$$

for all vector fields X and Y on M [2].

A contact metric manifold M is Sasakian if and only if

(6) $$R(X,Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X and Y [2].

A contact metric manifold M is said to be *η-Einstein* if

(7) $$\mathcal{S} = aI_d + b\eta \otimes \xi,$$

where $\mathcal{S}$ is the Ricci operator and a, b are smooth functions on M [2]. The Ricci tensor S of M is defined by $S(X,Y) = g(\mathcal{S}X,Y)$ for the vector fields X and Y on M.

§2. Contact Metric Manifolds ξ Belonging to the $(k,\mu)-$Nullity Distribution

Let M be a contact metric manifold. The *(k,μ)-nullity distribution* of M for the pair (k,μ) is a distribution

$$N(k,\mu) \quad : \quad p \to N_p(k,\mu) = \{Z \in T_pM \mid R(X,Y)Z = k[g(Y,Z)X\text{-}g(X,Z)Y]$$

(8) $$+\mu[g(Y,Z)hX\text{-}g(X,Z)hY]\},$$

where $k, \mu \in \mathbb{R}$ and $k \leq 1$ (see [3]).

If $k = 1$ then $h = 0$ and M is a Sasakian manifold. So if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution then we have

(9) $$R(X,Y)\xi = k\left[\eta(Y)X - \eta(X)Y\right] + \mu\left[\eta(Y)hX - \eta(X)hY\right]$$

(see [3]).

In this paper, sometimes we call this type manifolds as (k,μ)-contact. Firstly we give some lemmas which we use in the future.

Lemma 2.1 *[3]. Let M be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then:*

i) $(\nabla_X h)Y = [(1\text{-}k)g(X,\varphi Y) + g(X,h\varphi Y)]\,\xi + \eta(Y)h(\varphi X + \varphi hX)\text{-}\mu\eta(X)\varphi hY,$

ii) $h^2 = (k\text{-}1)\varphi^2$, $k \leq 1$ *and* $k = 1$ *if and only if* M *is Sasakian,*

iii) $R(\xi,X)Y = k\left[g(X,Y)\xi - \eta(Y)X\right] + \mu\left[g(hX,Y)\xi - \eta(Y)hX\right],$

iv) $\mathcal{S}\xi = 2nk\xi$.

where X and Y are any vector fields on M and $k, \mu \in \mathbb{R}$.

Lemma 2.2 *[3]. Let M be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution $k<1$. For any vector field X, the* Ricci operator $\mathcal{S}$ *is given by*

(10) $\mathcal{S}X = [2(n\text{-}1)\text{-}n\mu]X + [2(n\text{-}1)+\mu]hX + [2(1\text{-}n)+n(2k+\mu)]\eta(X)\xi;\ \ n\geq 1.$

As a consequence of Lemma 2.2 we have the following Lemma.

Lemma 2.3 *[1]. Let M be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution, $k<1$, then*

$$\begin{aligned}(\nabla_X S)(Y,Z) &= [2(n-1)+\mu]g((\nabla_X h)Y,Z)\\ &+[2(1-n)+n(2k+\mu)]\left\{g(Y,\nabla_X\xi)\eta(Z)+g(Z,\nabla_X\xi)\eta(Y)\right\}.\end{aligned} \tag{11}$$

§3. Contact Metric Manifolds with Cyclic-Parallel Ricci Tensor

The Ricci tensor S of a Riemannian manifold M is said to be cyclic-parallel if $C\nabla S=0$, namely

$$(\nabla_Z S)(X,Y)+(\nabla_X S)(Y,Z)+(\nabla_Y S)(Z,X)=0 \tag{12}$$

for all vector fields X,Y,Z.

Let M be an η-Einstein manifold whose Ricci tensor S of the form

$$S(X,Y)=Ag(X,Y)+B\eta(X)\eta(Y), \tag{13}$$

where A,B are non-zero real numbers and X,Y are vector fields on M. So we have;

Theorem 3.4 *Let (M^{2n+1},g) be a (k,μ)-contact, η-Einstein manifold of the form (13). If the Ricci tensor S of M is cyclic parallel then M is a K-contact manifold.*

Proof. Let us consider M is a (k,μ)-contact, η-Einstein manifold of the form (13). If the Ricci tensor S of M is cyclic parallel then replacing Z with ξ in (12) we can write

$$(\nabla_\xi S)(X,Y)+(\nabla_X S)(Y,\xi)+(\nabla_Y S)(\xi,X)=0. \tag{14}$$

Using (13) after some computations we get

$$(\nabla_X S)(Y,Z)=B\left[\eta(Z)g(Y,\nabla_X\xi)+\eta(Y)g(Z,\nabla_X\xi)\right]$$

which implies

$$(\nabla_\xi S)(X,Y)=0, \tag{15}$$

$$(\nabla_X S)(Y,\xi)=B\left[g(Y,\nabla_X\xi)+\eta(Y)g(\xi,\nabla_X\xi)\right], \tag{16}$$

and

$$(\nabla_Y S)(\xi,X)=B\left[g(X,\nabla_Y\xi)+\eta(X)g(\xi,\nabla_Y\xi)\right]. \tag{17}$$

So substituting (15)-(17) in (14) and using (3), (4) the equation (14) becomes

$$g(\varphi X+\varphi hX,Y)+g(\varphi Y+\varphi hY,X)=0 \tag{18}$$

which implies

$$g(\varphi Y,hX)=0. \tag{19}$$

Replacing Y with φY, the equation (19) can be written as $g(\varphi^2 Y,hX)=0$. So by the use of Lemma 2.1 (ii), (2) and (4), we have

$$g(Y,hX)=0. \tag{20}$$

for all vector fields X and Y and hence we have $h=0$ which implies M is K-contact. □

Theorem 3.5 *Let (M^{2n+1},g), $(n\neq 1)$, be a non-Sasakian (k,μ)-contact metric manifold. If the Ricci tensor S of M is cyclic parallel then M is either K-contact or $k=-\frac{1}{4}\frac{\mu^2+4n\mu}{n}$.*

Proof. Let M be a (k,μ)-contact metric manifold. Then by [3], $k\leq 1$. But if $k=1$ then M is Sasakian. Since we suppose M is non-Sasakian we have $k<1$. So by the use of (11) we have

$$\begin{aligned}(\nabla_\xi S)(X,Y)&=[2(n-1)+\mu]g((\nabla_\xi h)X,Y)\\ &+[2(1-n)+n(2k+\mu)]\left\{g(X,\nabla_\xi\xi)\eta(Y)+g(Y,\nabla_\xi\xi)\eta(X)\right\}.\end{aligned} \tag{21}$$

Then using (4) the equation (21) can be written as

$$(\nabla_\xi S)(X,Y)=[2(n-1)+\mu]g((\nabla_\xi h)X,Y). \tag{22}$$

But by making use of Lemma 2.1 (i) we get

$$(\nabla_\xi h)X=-\mu\varphi hX \tag{23}$$

and so substituting (23) into (22) we obtain

$$(\nabla_\xi S)(X,Y)=[2(n-1)+\mu]\mu g(hX,\varphi Y). \tag{24}$$

Similarly, using (11), (4) and Lemma 2.1 (i), by a straightforward computations we get

$$\begin{aligned}(\nabla_X S)(Y,\xi)&=[\mu+2k-k\mu+n\mu]\,g(X,\varphi Y)\\ &[\mu+n(2k+\mu)]\,g(hX,\varphi Y)\end{aligned} \tag{25}$$

and using $g(\varphi hY,hX)=-g(h^2X,\varphi Y)=(k-1)g(\varphi Y,X)$ we have

$$\begin{aligned}(\nabla_Y S)(\xi,X)&=[k\mu-\mu-n\mu-2k]\,g(\varphi Y,X)\\ &[\mu+n(2k+\mu)]\,g(hX,\varphi Y).\end{aligned} \tag{26}$$

So substituting (24), (25) and (26) into (14) we obtain

$$\left[\mu^2+4n\mu+4nk\right]g(hX,\varphi Y)=0. \tag{27}$$

Suppose $g(hX, \varphi Y) = 0$. Then replacing Y with φY, the last equation becomes $g(hX, \varphi^2 Y) = 0$. So using (2) we get $g(hX, Y) = 0$ for all vector fields X and Y and hence; we have $h = 0$ which gives us M is K-contact (note that M is non-Sasakian since $n \neq 1$). If $\mu^2 + 4n\mu + 4nk = 0$ then we get $k = -\frac{1}{4}\frac{\mu^2+4n\mu}{n}$. □

Corollary 3.6 *Let (M^{2n+1}, g) be a non-Sasakian manifold with ξ belonging to k-nullity distribution. If M is not K-contact and the Ricci tensor S of M is cyclic parallel then M is locally isometric to the product $\mathbf{E}^{n+1} \times S^n(4)$.*

Proof. Since ξ belongs to k-nullity distribution then $\mu = 0$. Hence from Theorem 3. 2, we get $k = 0$. So by [2], M is locally isometric to the product $\mathbf{E}^{n+1} \times S^n(4)$. □

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Author's address:

Cihan Özgür
Balikesir University, Department of Mathematics,
10100 Balikesir, Turkey
e-mail: cozgur@balikesir.edu.tr