# On the product Riemannian manifolds

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**Abstract**

In this paper, we discuss the Riemannian curvature tensor and the Riemannian-Christoffel curvature tensor of a product Riemannian manifold. We show that the Riemannian curvature tensor and the Riemannian-Christoffel curvature tensor of the product Riemannian manifold can be written respectively as the sum of the Riemannian curvature tensor and the Riemannian-Christoffel curvature tensor of each Riemannian manifold. Furthermore, making use of these results some theorems regarding the local symmetry, flatness and sectional curvature are given.

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## §1. Introduction

By $S$ and $R$ we denote respectively the Ricci tensor and the Riemannian curvature tensor of an $m$-dimensional Riemannian manifold $(M, g)$. Then $S$ and $R$ are defined by

$$S(X,Y) = \sum_{i=1}^{m} g(R(e_i, X)Y, e_i),$$

and

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where $\{e_1, e_2, ..., e_m\}$ are orthonormal basis vector fields in $TM$, $X, Y, Z \in TM$ and $\nabla$ is the connection on $M$. We note that $S$ is the independent from the choice of basis in $TM$.

Let $X$ and $Y$ be two linearly independent vectors at a point $p \in M$ and $\gamma(X,Y)$ be the plane section spanned by $X$ and $Y$. The sectional curvature $k(\gamma)$ for $\gamma$ is defined by

$$k(\gamma) = \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

It is easy see that this $k(\gamma)$ is uniquely determined by the plane section $\gamma$ and is independent the choice of $X$ and $Y$ on $M$.

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If $k(\gamma)$ is a constant for all plane sections $\gamma$ in the tangent space $T_M(p)$ at $p$ and for all points $p \in M$, then $M$ is called a *space of constant curvature* or we say that it has constant sectional curvature ([1]).

**Lemma 1.** *Let $M$ be an $m$-dimensional Riemannian manifold ($m > 2$). If we have*

$$K(X, Y, Z, X) = 0$$

*for any orthonormal vector fields $X, Y, Z$ in $M$, then $M$ has constant sectional curvature, where $K$ is the Riemannian-Christoffel curvature tensor of $M$ [1].*

Conversely, if $M$ has constant sectional curvature $c$, then for all $X, Y, Z, W \in TM$

$$K(X, Y, Z, W) = c\{g(Z, Y)g(X, W) - g(Z, X)g(Y, W)\}.$$

If $X, Y, Z$ are arbitrary orthonormal vector fields in $M$,

$$K(X, Y, Z, X) = c\{g(Z, Y)g(X, X) - g(Z, X)g(Y, X)\} = 0,$$

where $K$ and $g$ are the Riemannian-Christoffel curvature tensor and Riemannian metric of $M$, respectively. Furthermore, A Riemannian manifold of constant sectional curvature is said to be elliptic, hyperbolic or flat, according as the constant sectional curvature is positive, negative or zero, respectively. For a flat manifold the Riemannian curvature tensor is zero [5].

**Definition.** A Riemannian manifold is called a locally symmetric manifold if its Riemannian curvature tensor $R$ is parallel with respect to $\nabla$, i.e., $\nabla R = 0$ [2].

Now, we can give the following

**Proposition.** *Let $M$ be a totally umbilical submanifold of a Riemannian manifold $\overline{M}$ which has constant sectional curvature $c$. Then $M$ has also constant sectional curvature $c + \|H\|^2$, where $H$ is the mean curvature vector field of $M$ in $\overline{M}$ [1].*

Let $(U, g_1)$ and $(V, g_2)$ be Riemannian manifolds, with dimensions $n$ and $m$, respectively. $U \times V$ is the Riemannian product of $U$ and $V$. We denote by $P$ and $Q$ the projection mappings of $T(U \times V)$ to $TU$ and $TV$ respectively, that is,

$$P : T(U \times V) \to TU, \quad Q : T(U \times V) \to TV.$$

Then we have

$$P + Q = I, P^2 = P, Q^2 = Q, PQ = QP = 0.$$

If we put $J = P - Q$, we can easily see that $J^2 = I$, where $I$ is the identity transformation of $T(U \times V)$.

We define a Riemannian metric of $U \times V$ by

$$g(X, Y) = g_1(PX, PY) + g_2(QX, QY)$$

for all $X, Y \in T(U \times V)$. Hence $(U \times V, g)$ is a Riemannian manifold, which is called *product Riemannian manifold*.

From the definition of $g$, we can get that $U$ and $V$ are all totally geodesic submanifolds of $U \times V$. The Riemannian metric $g$ satisfies the condition $g(JX, Y) = g(X, JY)$, which is equivalent to $g(X, Y) = g(JX, JY)$.

By $\nabla$, we denote the Levi-Civita connection of the metric $g$. Then we can easily see that ([3])

$$\nabla P = \nabla Q = \nabla J = 0.$$

## §2. On the Curvatures of Product Riemannian Manifolds

In this section, we will prove the main theorems of the paper.

By $R$ and $S$ we denote the Riemannian curvature tensor and Ricci tensor of product Riemannian manifold $(U \times V, g)$. Then we have the following

**Lemma 2.** *Let $(U \times V, g)$ be a product Riemannian manifold of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$. Then $R$ and $S$ satisfy the following conditions.*

*a)* $R(X,Y)JZ = JR(X,Y)Z$

*b)* $R(JX, JY) = R(X,Y)$

*c)* $S(JX, JY) = S(X,Y)$

*for all $X, Y, Z \in T(U \times V)$.*

*Proof.* a) Using the parallelism of $J$, that is, $\nabla J = 0$, we obtain

$$\begin{aligned} R(X,Y)JZ &= \nabla_X \nabla_Y JZ - \nabla_Y \nabla_X JZ - \nabla_{[X,Y]} JZ \\ &= \nabla_X J \nabla_Y Z - \nabla_Y J \nabla_X Z - J \nabla_{[X,Y]} Z \\ &= J \nabla_X \nabla_Y Z - J \nabla_Y \nabla_X Z - J \nabla_{[X,Y]} Z \\ &= J(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\ &= JR(X,Y). \end{aligned}$$

b) Similarly, we get

$$\begin{aligned} g(R(JX,JY)Z,W) &= K(JX,JY,Z,W) = K(Z,W,JX,JY) \\ &= g(R(Z,W)JX,JY) = g(JR(Z,W)X,JY) \\ &= g(R(Z,W)X,Y) = K(Z,W,X,Y) \\ &= K(X,Y,Z,W) = g(R(X,Y)Z,W) \end{aligned}$$

So, $R(JX, JY) = R(X,Y)$.

c)

$$\begin{aligned} S(JX,JY) &= \sum_{i=1}^{n+m} g(R(e_i,JX)JY,e_i) = \sum_{i=1}^{n+m} g(R(Je_i,X)Y,Je_i) \\ &= \sum_{i=1}^{n+m} g(R(e_i,X)Y,e_i) = S(X,Y). \end{aligned}$$

This completes the proof of the Lemma. □

From Lemma 2, we have $R(X_1, X_2) = 0$ and $S(X_1, X_2) = 0$ for all $X_1 \in TU$ and $X_2 \in TV$. Since $\nabla P = 0$ we have

$$\begin{aligned} R(X,Y)PZ &= \nabla_X \nabla_Y PZ - \nabla_Y \nabla_X PZ - \nabla_{[X,Y]} PZ \\ &= \nabla_X P \nabla_Y Z - \nabla_Y P \nabla_X Z - P\nabla_{[X,Y]} Z \\ &= P\nabla_X \nabla_Y Z - P\nabla_Y \nabla_X Z - P\nabla_{[X,Y]} Z \\ &= P(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\ &= PR(X,Y)Z \end{aligned}$$

for all $X, Y, Z \in T(U \times V)$. So $R(X,Y)PZ \in TU$. In the same way, $R(X,Y)QZ \in TV$. Thus we have

$$R(X,Y) = R(X_1, Y_1) + R(X_2, Y_2),$$

where $X = X_1 + X_2, Y = Y_1 + Y_2 \in T(U \times V)$. Making use of the first Bianchi identity, we obtain by direct calculations

$$\begin{aligned} R(X,Y)Z &= R(X_1, Y_1)Z + R(X_2, Y_2)Z \\ &= PR(X_1, Y_1)Z + QR(X_1, Y_1)Z + PR(X_2, Y_2)Z + QR(X_2, Y_2)Z \\ &= R(X_1, Y_1)PZ + R(X_2, Y_2)QZ + Q(-R(Y_1, Z)X_1 - R(Z, X_1)Y_1) \\ &\quad + P(-R(Y_2, Z)X_2 - R(Z, X_2)Y_2) \\ &= R(X_1, Y_1)Z_1 + R(X_2, Y_2)Z_2 - R(Y_1, Z)QX_1 - R(Z, X_1)QY_1 \\ &\quad - R(Y_2, Z)PX_2 - R(Z, X_2)PY_2 \\ &= R(X_1, Y_1)Z_1 + R(X_2, Y_2)Z_2. \end{aligned}$$

If we put $R_1(X_1, Y_1)Z_1 = R(X_1, Y_1)Z_1$ and $R_2(X_2, Y_2)Z_2 = R(X_2, Y_2)Z_2$, then we can easily see that $R_1$ and $R_2$ are also Riemannian curvature tensors of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$, respectively. Therefore we have

$$R(X,Y)Z = R_1(X_1, Y_1)Z_1 + R_2(X_2, Y_2)Z_2 \tag{2.1}$$

**Theorem 1.** *Let $(U \times V, g)$ be a product Riemannian manifold of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$. Then the product Riemannian manifold $(U \times V, g)$ is a locally symmetric manifold if and only if $(U, g_1)$ and $(V, g_2)$ are locally symmetric manifolds.*

*Proof.* If $(U \times V, g)$ is locally symmetric, then $\nabla R = \nabla R_1 + \nabla R_2 = 0$. From $J = P - Q$, $JP = P$ and $JQ = -Q$ we have

$$JR = JR_1 + JR_2 = JPR + JQR = PR - QR = R_1 - R_2.$$

$$\nabla R = \nabla R_1 + \nabla R_2 = 0, \tag{2.2}$$

Since $J$ is parallel,

$$\begin{aligned} J\nabla R &= J\nabla R_1 + J\nabla R_2 \\ \nabla JR &= \nabla JR_1 + \nabla JR_2 \\ &= \nabla R_2 - \nabla R_2 = 0. \end{aligned} \tag{2.3}$$

Equations (2) and (3) imply that $\nabla R_1 = \nabla R_2 = 0$. So $(U, g_1)$ and $(V, g_2)$ are locally symmetric manifolds.

Conversely, we suppose that $(U, g_1)$ and $(V, g_2)$ are locally symmetric manifolds, that is, $\nabla R_1 = \nabla R_2 = 0$ Then we can easily see that $\nabla R = 0$. So $(U \times V, g)$ is a locally symmetric manifold. □

Now, we can generalize Theorem 1 with the following

**Theorem 2.** *Let $(U \times V, g)$ be a product Riemannian manifold of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$. Then the product Riemannian manifold $(U \times V, g)$ is flat if and only if the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$ are flat.*

*Proof.* Follows from the equation (1). □

Using the Lemma 2, by a direct calculation we obtain

$$S(X, Y) = S(X_1, Y_1) + S(X_2, Y_2).$$

If we put $S(X_1, Y_1) = S_1(X_1, Y_1)$ and $S(X_2, Y_2) = S_2(X_2, Y_2)$, then we can easily see that $S_1$ and $S_2$ are the Ricci tensors of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$, respectively, where $X = X_1 + X_2, Y = Y_1 + Y_2 \in T(U \times V)$.

**Theorem 3.** *Let $(U \times V, g)$ be a product Riemannian manifold of the Riemannian manifold of $(U, g_1)$ and $(V, g_2)$. The product Riemannian manifold $(U \times V, g)$ is a Ricci flat manifold if and only if the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$ are Ricci flat manifolds.*

*Proof.* The product Riemannian manifold $(U \times V, g)$ is Ricci flat; then

$$S(X, Y) = S_1(X_1, Y_1) + S_2(X_2, Y_2) = 0.$$

On the other hand,

$$S(X, JY) = S_1(X_1, Y_1) - S_2(X_2, Y_2) = 0.$$

Thus we have $S_1(X_1, Y_1) = S_2(X_2, Y_2) = 0$, that is, $(U, g_1)$ and $(V, g_2)$ are Ricci flat manifolds. The converse is obvious. □

By $K$ we denote the Riemannian-Christoffel curvature tensor of the product Riemannian manifold $(U \times V, g)$. Then we can give the following

**Lemma 3.** *Let $(U \times V, g)$ be a product Riemannian manifold of the Riemannian manifolds $(U, g_1)$ and $(V, g_2)$. Then $K$ satisfies the following conditions.*
*a)* $K(JX, JY, JZ, JW) = K(JX, JY, Z, W) = K(X, Y, JZ, JW) = K(X, Y, Z, W)$
*b)* $K(X, JY, JZ, W) = K(X, JY, Z, JW) = K(JX, Y, JZ, W)$
*for all* $X, Y, Z, W \in T(U \times V)$.

*Proof.* a) Since $J$ is parallel, from Lemma 2, we get

$$\begin{aligned}
K(JX, JY, JZ, JW) &= g(R(JX, JY)JZ, JW) \\
&= g(JR(JX, JY)Z, JW) = g(R(JX, JY)Z, W) \\
&= g(R(X, Y)Z, W) = K(X, Y, Z, W) \\
K(JX, JY, JZ, JW) &= g(R(JX, JY)JZ, JW) \\
&= g(JR(JX, JY)Z, JW) = g(R(JX, JY)Z, W) \\
&= K(JX, JY, Z, W) = K(JX, JY, Z, Y) \\
&= g(R(JX, JY)Z, W) = g(JR(X, Y)Z, JW) \\
&= g(R(X, Y)JZ, JW) = K(X, Y, JZ, JW).
\end{aligned}$$

b) Similarly, using the properties of $R$ and $K$ we have

$$\begin{aligned} K(X,JY,JZ,W) &= g(R(X,JY)JZ,W) = g(JR(X,JY)Z,W) \\ &= K(R(X,JY)Z,JW) = K(X,JY,Z,JW) \\ K(X,JY,JZ,W) &= K(R(JZ,W)X,JY) = g(R(JZ,W)JX,Y) \\ &= K(JZ,W,JX,Y) = K(JX,Y,JZ,W) \\ &= K(JX,Y,Z,JW). \end{aligned}$$

□

Let $X$ be an arbitrary tangent vector field of $U \times V$. Then we can write $X$ in the following way:

$$X = PX + QX,$$

where $PX$ and $QX$ are tangent vector fields of $U$ and $V$, respectively. We consider Lemma 2, Lemma 3 and the equation (1), and using the properties of $K$ and $R$, by direct calculations we obtain

$$K(X,Y,Z,W) = K(PX,PY,PZ,PW) + K(QX,QY,QW,QZ).$$

It is easily seen that $K(PX,PY,PZ,PW)$ and $K(QX,QY,QZ,QW)$ are the Riemannian-Christoffel curvature tensors of the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$, respectively.

If we put $X = X_1 + X_2$, $Y = Y_1 + Y_2$, $Z = Z_1 + Z_2$, $W = W_1 + W_2 \in T(U \times V)$, then we have

$$K(X,Y,Z,W) = K_1(X_1,Y_1,Z_1,W_1) + K_2(X_2,Y_2,W_2,Z_2), \tag{2.4}$$

where $K_1$ and $K_2$ are also the Riemannian-Christoffel curvature tensor of the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$, respectively.

Now we can give the following

**Theorem 4.** *Let $(U \times V, g)$ be the product Riemannian manifold of Riemannian manifolds $(U,g_1)$ and $(V,g_2)$. Then the product Riemannian manifold $(U \times V, g)$ has constant sectional curvature if and only if the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$ have constant sectional curvatures.*

*Proof.* If the product Riemannian manifold $U \times V$ has constant sectional curvature, then

$$K(X,Y,Z,X) = 0$$

for any orthonormal vector fields $X,Y,Z$ in $U \times V$.

If $\{X_1,Y_1,Z_1\}$ and $\{X_2,Y_2,Z_2\}$ are any orthonormal vector fields in $U$ and $V$ respectively, then

$$\{X = \frac{1}{\sqrt{2}}(X_1 + X_2), Y = \frac{1}{\sqrt{2}}(Y_1 + Y_2), Z = \frac{1}{\sqrt{2}}(Z_1 + Z_2)\}$$

are orthonormal vector fields in $U \times V$. Moreover, from the definition of the Riemannian metric $g$, $\{X,Y,JZ\}$ are also orthonormal vector fields in $U \times V$. So, we have from the equation (4)

$$K(X,Y,Z,X) = \frac{1}{4}K_1(X_1,Y_1,Z_1,X_1) + \frac{1}{4}K_2(X_2,Y_2,Z_2,X_2) = 0$$

and

$$K(X,Y,JZ,X)=\frac{1}{4}K_1(X_1,Y_1,Z_1,X_1)-\frac{1}{4}K_2(X_2,Y_2,Z_2,X_2)=0,$$

which imply that

$$K_1(X_1,Y_1,Z_1,X_1)=K_2(X_2,Y_2,Z_2,X_2)=0$$

for any orthonormal vector fields $X_1,Y_1,Z_1$ and $X_2,Y_2,Z_2$ in $U$ and $V$ respectively. So, $(U,g_1)$ and $(V,g_2)$ have constant sectional curvatures.

Conversely, we suppose that $(U,g_1)$ and $(V,g_2)$ have constant sectional curvatures; then we have

$$K_1(X_1,Y_1,Z_1,X_1)=K_2(X_2,Y_2,Z_2,X_2)=0$$

for any orthonormal vector fields $\{X_1,Y_1,Z_1\}$ and $\{X_2,Y_2,Z_2\}$ in $U$ and $V$, respectively. We get from the equation (4)

$$4K(X,Y,Z,X)=K_1(X_1,Y_1,Z_1,X_1)+K_2(X_2,Y_2,Z_2,X_2)=0$$

for any orthonormal vector fields

$$\{X=\frac{1}{\sqrt{2}}(X_1+X_2),\quad Y=\frac{1}{\sqrt{2}}(Y_1+Y_2),\quad Z=\frac{1}{\sqrt{2}}(Z_1+Z_2)\}$$

in $U\times V$. Thus $(U\times V,g)$ has also constant sectional curvature. □

Because the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$ are totally geodesic submanifolds of the product Riemannian manifold $(U\times V,g)$, we infer from the Proposition and Theorem 4 that the Riemannian manifolds $(U,g_1)$, $(V,g_2)$ and $(U\times V,g)$ have the same constant sectional curvature. So we can give the following

**Corollary.** *Let $(U\times V,g)$ be the product Riemannian manifold of the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$. In this case, product Riemannian manifold $(U\times V,g)$ is elliptic (resp. hyperbolic or flat) if and only if the Riemannian manifolds $(U,g_1)$ and $(V,g_2)$ are elliptic (resp. hyperbolic or flat).*

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