KdV equation obtained by Lie groups and Sturm-Liouville problems

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Abstract

In this study, we solve the Sturm-Liouville differential equation, which is obtained by using solutions of the KdV equation.

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§0. Introduction

In [3], Gardner C.S. et al, announced a method for exact solution of the Korteweg-de Vries equation (KdV for short)

$$u_t + uu_x + u_{xxx} = 0.$$

This is a nonlinear partial differential equation and the equation arises in different physical applications. The problems in long waves, in water of relatively shallow depth, for very small amplitudes, would drop the nonlinear term uu_x [6]. Korteweg-de Vries [2] first derived it in the study of long water waves in a channel of finite depth. Other fluid dynamical applications have been studied in [5]. On the other hand, this equation was obtained from a special case of the structure equation of the Lie group $SL(2, \mathbb{R})$ (the special linear group of all (2×2)-real unimodular matrices) by S.S. Chern and C.K. Peng [1].

Also, in [4], Gardner C.S. et al, showed how the initial value problem for the (nonlinear) Korteweg-de Vries equation can be reduced to a sequence of nonlinear problems. It is convenient to raplace u by $-6u$ and thereby transform the KdV equation into

$$u_t - 6uu_x + u_{xxx} = 0.$$

Various distinct sets of boundary conditions can be specified, each posing a KdV equation with initial data $u(x, 0) = u_0(x)$, where $u_0(x)$ is assumed to be bounded and three times continuously differentable. Certain special conditions must also be given to completely specify the problem.

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This paper is organized in the following way. In the first section, it is obtained from $SL(2,\mathbb{R})$ the Korteweg-de Vries equation

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x,$$

which was obtained by Chern and Leng, and in the second section the Sturm-Liouville differential equation is solved and obtained from the solutions of the KdV equation.

§1. Lie groups and the KdV equation

In this section we will recall the notions and terminology used in [1]. Let $SL(2,\mathbb{R})$ be the group of all (2×2)-real unimodular matrices. If we choose $X \in SL(2,\mathbb{R})$, then we have

$$\det X = 1, \tag{1.1}$$

and therefore $\det X \neq 0$. Its right-invariant Maurer-Cartan form is

$$w = dX.X^{-1} = \begin{pmatrix} w_1^1 & w_1^2 \\ w_2^1 & w_2^2 \end{pmatrix},$$

where $w_1^1 + w_2^2 = 0$. The structure (Maurer Cartan) equation of $SL(2,\mathbb{R})$, is

$$dw = w \wedge w. \tag{1.2}$$

Let U be a neighbourhood in the (x,t)-plane and consider a smooth mapping $f : U \to SL(2,\mathbb{R})$. The pull-backs of the Maurer-Cartan forms can be written

$$\begin{cases} w_1^1 = \eta dx + Adt \\ w_1^2 = qdx + Bdt \\ w_2^1 = rdx + Cdt, \end{cases} \tag{1.3}$$

where the coefficients are functions of x and t.

The forms in (1.3) satisfy the equations (1.2). This gives

$$\begin{cases} -\eta_t + A_x - qC + rB = 0 \\ -q_t + B_x - 2\eta B + 2qA = 0 \\ -r_t + C_x - 2rA + 2\eta C = 0. \end{cases} \tag{1.4}$$

We consider the special case when $r = \pm 1$, and η is a parameter indepent of x and t. Writing $q = u(x,t)$, we get from (1.4)

$$\begin{cases} A = +\eta C + \frac{1}{2}C_x \\ B = uC - \eta C_x - \frac{1}{2}C_{xx}. \end{cases}$$

Substitution into the second equation of (1.4) we obtain

$$u_t = K(u) = u_x C + 2uC_x + 2\eta^2 C_x - \frac{1}{2}C_{xxx}. \tag{1.5}$$

As an example we take $C = \eta^2 - \frac{1}{2}u$. Then, (1.5) becomes

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x,$$

which is a well-known KdV equation mentioned in [1].

§2. Spectral theory of Sturm-Liouville problems vs. KdV equation

We consider the problem

$$Ly = -y'' + u_0(x)y = \lambda y, \quad t \neq 1, \quad 0 \leq x \leq 1 \tag{2.1}$$

$$y\,(0) = 0, \tag{2.2}$$

$$y'(1,\lambda) + Hy(1,\lambda) = 0, \quad H \neq 0, \tag{2.3}$$

where

$$u(x,t) = \frac{-2x}{3(1-t)}$$

is a solution of the KdV equation and $\left. u\,(x,t)\right|_{t=0} = u_0\,(x)$. Since u_0 is, the operator L is self-adjoint, has a discret spectrum, which consists of simple eigenvalues. As known, a solution of the problem (2.1)-(2.3) is

$$\varphi\,(x,\lambda) = \frac{\sin sx}{s} + \frac{1}{s}\int_0^s \sin\left\{s\,(x-\tau)\right\} u\,(\tau,\lambda)\,d\tau.$$

On the other hand, if we solve (2.3) equation for λ, we find

$$s_n = n + \frac{1}{2} + \frac{H_1}{n+\frac{1}{2}} + O\left(\frac{1}{n^2}\right), \tag{2.4}$$

where $s = \sqrt{\lambda}$ and

$$H_1 = H + \frac{1}{2}\int_0^1 u\,(\tau,\lambda)\,d\tau.$$

s_n are called the eigenvalues of the problem (2.1)-(2.3). Using the formula (2.4), we will obtain an asymptotic formula for the eigenfunctions x,

$$\varphi_n(x) = \left(n+\frac{1}{2}\right)\sin\left(n+\frac{1}{2}\right)x + O\left(\frac{1}{n^2}\right),$$

and the normalized eigenfunctions are

$$V_n\,(x) = \frac{\varphi_n\,(x)}{\|\varphi_n\,(x)\|}.$$

Now, we obtain a solution of (2.1)-(2.3) using the translation operator. A solution of the problem

$$\begin{cases} -y'' + u_0\,(x)\,y = \lambda y \\ y\,(0) = 0 \end{cases}$$

is $y(x,\lambda)$ and the solution of the problem, where

$$u\,(x,t) = \frac{-2x}{3\,(1-t)}$$

is a solution of the KdV equation and $u(x,t)|_{t=0} = u_0(x)$. A solution for

$$\begin{cases} -y'' = \lambda y \\ y(0) = 0 \end{cases}$$

is $\cos\sqrt{\lambda}x$. These solutions are connected by

$$\chi(\cos\sqrt{\lambda}x) = \cos\sqrt{\lambda}x + \int_0^x K(x,s)\cos\sqrt{\lambda}s ds,$$

where χ is a translation operator. Now, we consider the two operators

$$A = \frac{-\partial^2}{\partial x^2} + u_0(x), \quad B = \frac{-\partial^2}{\partial x^2}.$$

Since $A\chi = \chi B$, we obtain a hyperbolic-type PDE,

$$\frac{\partial^2 K}{\partial x^2} - u_0(x)K = \frac{\partial^2 K}{\partial s^2} \tag{2.5}$$

and the boundary conditions

$$\begin{cases} K(x,0) = 0 \\ K(x,x) = \frac{1}{2}\int_0^x u(r,t)\,dr = \frac{-1}{2}\int_0^x \frac{2r}{3(1-t)}dr = \frac{-1}{6}\frac{x^2}{(1-t)} \end{cases}$$

for $K(x,s)$.

If we consider two characteristics, $\xi = x + s$ and $\eta = x - s$, the equation (2.5) is reduced to the canonical form

$$\frac{\partial^2 K}{\partial \xi \partial \eta} = \frac{1}{4}Ku.$$

The function

$$A(\xi,\eta) = K\left(\frac{\xi + m}{2}, \frac{\eta - m}{2}\right)$$

yields the equation

$$\frac{\partial^2 A}{\partial \xi \partial \eta} = \frac{1}{4}uA. \tag{2.6}$$

and the conditions

$$\begin{cases} A(\eta,\eta) = 0 \\ A(\xi,0) = -\frac{\xi^2}{24(1-t)}. \end{cases}$$

Integration of the equation (2.6) with respect to the variable η from 0 to η gives

$$\frac{\partial A}{\partial \xi} - \left.\frac{\partial A}{\partial \xi}\right|_{\eta=0} = \int_0^\eta \frac{1}{4}A(\xi,\alpha)\,ud\alpha.$$

On the other hand

$$\left.\frac{\partial A}{\partial \xi}\right|_{\eta=0} = -\frac{\xi}{12(1-t)},$$

and therefore,

$$\frac{\partial A}{\partial \xi} = \int_0^{\eta} \frac{1}{4} A(\xi, \alpha)\, u d\alpha - \frac{\xi}{12(1-t)}. \tag{2.7}$$

Integration of the equation (2.7) with respect to ξ from η to ξ gives

$$A(\xi, \eta) - A(\eta, \eta) = \int_{\eta}^{\xi} d\beta \int_0^{\eta} \frac{1}{4} A(\beta, \alpha)\, u d\alpha - \int_{\eta}^{\xi} \frac{\beta}{12(1-t)} d\beta.$$

Since $A(\eta, \eta) = 0$, it follows that

$$A(\xi, \eta) = \int_{\eta}^{\xi} d\beta \int_0^{\eta} \frac{1}{4} A(\beta, \alpha)\, u d\alpha - \int_{\eta}^{\xi} \frac{\beta}{12(1-t)} d\beta.$$

On the other hand,

$$A(\xi, \eta) = K\left(\frac{\xi + m}{2}, \frac{\eta - m}{2}\right) = K(x, s),$$

with

$$K(x, s) = \int_{|x-s|}^{x+s} d\beta \int_0^{|x-s|} \frac{1}{4} K(\beta, \alpha)\, u d\alpha - \frac{xs}{6(1-t)}. \tag{2.8}$$

The equation (2.8) is a Volterra-type integral equation. Therefore, its solution is unique and the solution can be provided by successive approximations.

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