Integral inequalities for C-totally real submanifolds in Sasakian space forms

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Abstract

We give two intrinsic integral inequalties for compact minimal C-totally real submanifolds in a Sasakian space form.

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§1. Introduction

Let $\bar{M}^{2m+1}$ be an odd dimensional Riemannian manifold with metric g. Let Φ be a $(1,1)$-tensor field, η a 1-form on $\bar{M}^{2m+1}$ and ζ a vector field, such that

$$\begin{cases} \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi). \end{cases}$$

If, in addition, $d\eta(X,Y) = g(\phi X, Y)$ for all vector fields X, Y on $\bar{M}^{2m+1}$, then $\bar{M}^{2m+1}$ is said to have a *contact metric structure* (ϕ, ξ, η, g), and $\bar{M}^{2m+1}$ is called a *contact metric manifold.* If, moreover, the structure is normal, that is if

$$[\phi X, \phi Y] + \phi^2[X,Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X,Y)\xi,$$

then the contact metric structure is a *Sasakian structure* (normal contact metric structure) and $\bar{M}^{2m+1}$ is called a *Sasakian manifold.* For details and background, see the standard references [4] and [5].

A plane section σ in $\mathrm{T}_P\bar{M}^{2m+1}$ of a Sasakian manifold $\bar{M}^{2m+1}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}(\sigma)$ with respect to a ϕ-section σ is called a *ϕ-section curvature.* If a Sasakian manifold $\bar{M}^{2m+1}$has a constant ϕ-sectional curvature c, then $\bar{M}^{2m+1}$ is called a *Sasakian space form* and is denoted by $\bar{M}^{2m+1}(c)$.

An n-dimensional submanifold M^n of a Sasakian space form $\bar{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\bar{M}^{2m+1}(c)$, if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\phi(TM^n) \subset T^{\perp}M^n$, which means that M^n an anti-invariant submanifold of $\bar{M}^{2m+1}(c)$.

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In [1,2], Cao gave an integral inequality for compact pseudo-umbilical space-like submanifolds in the indefinite space form. In this paper, we prove Cao's result in the case of submanifolds in the Sasakian space. We will prove the following

Theorem 1. *Let M^n be an n-dimensional compact C-totally real submanifold in the Sasakian space form $\bar{M}^{2n+1}(c)$; then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}-\sum R^2_{mj}+\frac{1}{8}\left[2n(n-1)\right](c+3)\,\rho\right\}*1\leq 0$$

Theorem 2. *Let M^n be an n-dimensional compact C-totally real submanifold in the Sasakian space form $\bar{M}^{2n+1}(c)$; then*

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}-\sum R^2_{mj}-\left[\frac{2n(n-1)-1}{8}\right](c+3)|h|^2\right\}*1$$

$$\leq\frac{-2n^2(n-1)+n(n-1)}{32}(c+3)\cdot vol(M^n).$$

In the above theorems, $\sum R^2_{mijk}$ is the square length of Riemannian curvature tensor of M^n and ρ is the scalar curvature of M^n.

§2. Local formulae

We shall give the structure equations of an n-dimensional submanifold M^n of a Sasakian Space form $\bar{M}^{2m+1}(c)$. We choose a local field of orthonormal frames

$$\begin{cases} e_1,e_2,\ldots,e_n,e_{n+1},\ldots,e_m; \quad e_{0*}=\xi,\\ e_{1*}=\phi e_1,\ldots,e_{n*}=\phi e_n; \quad e_{(n+1)^*}=\phi e_{n+1},\ldots,e_{m*}=\phi e_m\end{cases}$$

on $\bar{M}^{2m+1}(c)$ in such a way that, restricted to M^n, the vectors $e_1,e_2,\ldots e_n$ are tangent to M^n, and hence $e_{n+1},\ldots,e_m$, $\xi,e_{1^*},e_{2^*},\ldots e_{m^*}$ are normal to M^n. Let $w_1,w_2,\ldots,w_n,w_{n+1},\ldots,w_m,w_{1^*},w_{2^*},\ldots,w_{m^*},w_{(n+1)^*},\ldots,w_{m^*}$ be the field of dual frames with respect to this frame field of $\bar{M}^{2m+1}(c)$. We shall make use of the following convention on the ranges of indices:

$$\begin{cases} A,B,C,\ldots=1,\ldots,m,0^*,1^*,\ldots,m^*\\ i,j,k,\ldots=1,2,\ldots,n,\\ a,b,c,\ldots=(n+1),\ldots,m,0^*,1^*,\ldots,m^*\end{cases}$$

Then the structure equations of $\bar{M}^{2m+1}$ are given by

$$\begin{cases} dw_A=-\sum w_{AB}\wedge w_B, w_{AB}+w_{BA}=0\\ dw_{AB}=-\sum w_{AC}\wedge w_{CB}+\frac{1}{2}\sum R_{ABCD}w_C\wedge w_D\end{cases}$$

We restrict these forms to M^n. Then $w_a=0$. Since $0=dw_a=-\sum w_{ai}\wedge w_{ii}$, by Cartan's Lemma, we obtain

$$w_{ai}=\sum h^a_{ij}w_j\ ,h^a_{ij}=h^a_{ji}.$$

From these formulas, we obtain the structure equations of M^n

$$\left\{\begin{array}{l} dw_i = -\sum w_{ik} \wedge w_k, w_{ik} + w_{ki} = 0 \\ dw_{ij} = -\sum w_{ik} \wedge w_{kj} + \frac{1}{2} \sum R_{ijkl} w_k \wedge w_l \end{array}\right. \tag{2.1}$$

$$R_{ijkl} = \frac{1}{4}(c+3)\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) + \sum \left(h^a_{ik}h^a_{jl} - h^a_{il}h^a_{jk}\right), \tag{2.2}$$

where R_{ijkl} are the components of the curvature tensor of M^n. We call

$$h = \sum h^a_{ij} w_i \otimes w_j e_a$$

the second fundamental form of M^n. The square length of h is

$$|h|^2 = \sum \left(h^a_{ij}\right)^2$$

and the mean curvature of M^n is $H = \frac{1}{n}\sum h^a_{ii} e_a$. If M^n is minimal, then

$$\sum h^a_{ii} = 0. \tag{2.3}$$

Let h^a_{ijk} and h^a_{ijkl} denote the covariant derivative and second covariant derivative of h^a_{ij} respectively, defined by

$$\left\{\begin{array}{l} \sum h^a_{ijk} w_k = dh^a_{ij} - \sum h^a_{ik} w_{kj} - \sum h^a_{jk} w_{ki} \\ \sum h^a_{ijkl} w_l = dh^a_{ijk} - \sum h^a_{ijl} w_{lk} - \sum h^a_{ilk} w_{lj} - \sum h^a_{ljk} w_{li}. \end{array}\right.$$

Then we have

$$\sum h^a_{ijk} - \sum h^a_{ikj} = 0 \tag{2.4}$$

$$\sum h^a_{ijkl} - \sum h^a_{ijlk} = \sum h^a_{im} R_{mjkl} + \sum h^a_{jm} R_{mikl}. \tag{2.5}$$

The Laplacian Δh^a_{ij} of h^a_{ij} is defined as $\sum h^a_{ijkk}$, and from Lemma 3.3 in [4], (2.3), (2.4) and (2.5), we have (as in [3, 1])

$$\triangle\, h^a_{ij} = \sum h^a_{im} R_{mkjk} + \sum h^a_{km} R_{mijk} \tag{2.6}$$

Proof of Theorem 1. From (2.2), (2.3) and (2.6),

$$\begin{aligned} \sum h^a_{ij} \triangle h^a_{ij} &= \sum h^a_{ij} h^a_{mk} R_{mijk} + \sum h^a_{ij} h^a_{im} R_{mkjk} \\ &= \tfrac{1}{2} \sum \left(h^a_{ij} h^a_{mk} - h^a_{mj} h^a_{ik}\right) R_{mijk} + \sum \left(h^a_{ij} h^a_{im} - h^a_{ii} h^a_{jm}\right) R_{mj} \\ &= \tfrac{1}{2} \sum \left[\tfrac{1}{4}(c+3)\left(\delta_{ij}\delta_{mk} - \delta_{mj}\delta_{ik}\right) - R_{imjk}\right] R_{mijk} + \\ &\quad + \sum \left[\tfrac{1}{4} n(n-1)(c+3)\delta_{ij} - R_{mj}\right] R_{mj} \\ &= \tfrac{1}{2} \sum R^2_{mijk} - \sum R^2_{mj} + \tfrac{1}{8}\left[2n(n-1) - 1\right](c+3)\rho. \end{aligned} \tag{2.7}$$

Using ([3])

$$\int_{M^n} \left\{\sum h^a_{ij} \triangle h^a_{ij}\right\} * 1 \le 0,$$

we get

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}-\sum R^2_{mj}+\frac{1}{8}\left[2n\left(n-1\right)-1\right]\left(c+3\right)\rho\right\}*1\leq 0$$

and Theorem 1 is proved. □

Proof of Theorem 2. From (2.2) and (2.3), we infer

$$\rho=\frac{1}{4}n\left(n-1\right)\left(c+3\right)-\left|h\right|^2. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\int_{M^n}\left\{\frac{1}{2}\sum R^2_{mijk}-\sum R^2_{mj}+\left(\frac{2n\left(n-1\right)-1}{8}\right)\left(c+3\right)\left|h\right|^2\right\}*1$$

$$\leq\frac{-2n^2\left(n-1\right)^2+n\left(n-1\right)}{32}\left(c+3\right)\cdot vol(M^n),$$

which concludes the proof. □

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