Kenmotsu manifold carrying a skew symmetric Killing vector field

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Abstract

In the present note we use a Kenmotsu manifold M$(\Phi, \Omega, \eta, \xi, f)$, (in the sense of Z.Olszak and R.Roşca [3]) carrying a skew symmetric Killing vector field X (in the sense of R.Rosca [7], to show that X, under certain conditions, defines a relative conformal transformation of a structure 2-form Ω.

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§1. Main results

Let (M, g) be a Riemannian C^∞manifold and let ∇ be the covariant differential operator produced by the metric tensor g. We assume that M is oriented. Define $\Gamma(TM) = \mathcal{X}M$ the set of vector fields on M and $TM \overset{\flat}{\underset{\sharp}{\rightleftharpoons}} T^*M$ be the musical isomorphisms defined by g, and

$$\Omega^\flat : TM \to T^*M;$$

the symplectic isomorphism defined by Ω.

Following Poor [4] we set $A^q(M, TM) = HM(\wedge^q TM, TM)$ and notice that elements of $A^q(M, TM)$ are vector valued q-forms.

The field of orthonormal frames of an n-dimensional Riemannian manifold is denoted by

$$O = \{e_A; A = 1, ..., n\}$$

and the associated coframe by

$$O^* = \{\omega^A; A = 1, ..., n\}.$$

The canonical vector valued 1-form dp of M is called soldering form and is expressed by

$$dp = \omega^A \otimes e_A.$$

Then E. Cartan's structure equations in index-full notation are written as

$$\nabla e = \theta \otimes e$$

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$$\begin{cases} d\omega = -\theta \wedge \omega \\ d\theta = -\theta \wedge \theta + \Theta, \end{cases}$$

where the 1-forms θ and the 2-forms Θ are the connection forms in the tangent bundle TM and the curvature forms respectively.

A vector field X whose covariant differential ∇X satisfies

$$\nabla X = X \wedge Y,$$

where $\wedge$ is the wedge product of vector fields (e.g., $\nabla X = Y^\flat \otimes X - X^\flat \otimes Y$) is defined by [7] as a skew symmetric vector field.

Let $M(\Phi, \Omega, \eta, \xi, f)$ be a generalized Kenmotsu manifold (in the sense of Olszak and Rosca [3]. Then, as is known, the structure tensors $\Phi, \Omega, \eta, \xi, f)$ satisfy (see also [2]):

$$\begin{cases} \phi^2 = -I + \eta \otimes \xi, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1 \\ \nabla\xi = f(dp - \eta \otimes \xi) \\ g(Z, Z') = g(\phi Z, \phi Z') + \eta(Z)\eta(Z') \\ (\nabla_Z, \phi)Z = -f(\eta(Z)\phi Z' + g(\phi Z, Z')\xi) \\ \Omega(Z, Z') = g(\phi Z, Z'). \end{cases} \tag{1.1}$$

If $\{e_A\}$are the vectors of an orthonormal basis and $\{\omega^A\}$the associated cobasis, then as is known $dp = \omega^A \otimes e_A$ means the soldering form [1] of M, i.e., the canonical vector valued 1-form of M, (see also [2]).

We assume in the present note, that M carries a skew symmetric Killing vector field X (see SSK) in the sense of [7] having ξ as generative, i.e., satisfying

$$\nabla X = X \wedge \xi,$$

where $\wedge$ is the wedge product. Hence one may also write

$$\nabla X = \eta \otimes X - \alpha \otimes \xi, \tag{1.2}$$

where α is the dual form of X, i.e.,

$$\alpha = X^\flat.$$

Since the operator ∇ is recurrent one derives from (1.2), taking account of (1.1)

$$\nabla^2 X = f\alpha \wedge dp - (d\alpha + f\alpha \wedge \eta + \alpha \wedge \eta) \otimes \xi.$$

Assuming that X is an exterior concurrent vector field 4in the sense of [6], (see also [2]) that is $\nabla^2 X = (*)\alpha \wedge dp$ we derives by a standard calculation

$$f = -1, \quad dX + \alpha \wedge \eta = 0.$$

Hence one has

$$\nabla^2 X = -\alpha \wedge dp, \tag{1.3}$$

and using [2], (1.3) implies

$$R(X, Z) = 2mf(X, Z), \quad Z \in \mathcal{X}(M),$$

where R is the Ricci tensor field of the Levi-Civita operator ∇.

In an other order of ideas, relative to the basis

$$\{e_A : A = 0, 1, ..., 2m\},$$

one may write

$$X = X^\alpha e_a + X^0\xi, \qquad a \in \{1, ..., 2m\}$$

and taking the covariant differential ∇X of X one finds by the structure equations (1.1), since $f = -1$,

$$\nabla X = (dX^a + X^\flat \omega_b^a) \otimes e_a - X^0 \omega^a \otimes e_a + \alpha \otimes \xi = \eta \otimes X - \alpha \otimes \xi, \tag{1.4}$$

where in (1.4), $\alpha = \sum X^a \omega^a + X^0 \eta = X^\flat$.

The second fundamental form Ω of the manifold M under consideration is expressed by ([2])

$$\Omega = \sum \omega^i \wedge \omega^{i^*}; \quad i = 1, ..., m, \quad i^* = i + m.$$

Taking the inner product of Ω of X (i.e., $i_X\Omega$, where $Z \to -i_Z\Omega$, $Z \in XM$ which is also called *the symplectic isomorphism*), we agree to set

$$\beta = i_X\Omega = \sum (X^i \omega^{i^*} - X^{i^*} \omega^i)$$

and one derives

$$d\beta = \eta \wedge \beta + 2\Omega.$$

Next taking the Lie derivative of Ω with respect to the S.S.K. vector field X, one deduces

$$\mathcal{L}_X\Omega = 3\eta \wedge \beta + (2 - X^0)\Omega.$$

Since as known [2]

$$d\Omega = 2\eta \wedge \Omega,$$

then clearly η is closed (i.e., $d\eta = 0$), and from (11) one also derives

$$d(\mathcal{L}_X\Omega) = (2(-X^0)\eta - dX^0 \wedge \Omega.$$

Hence following [5], the above equation says that the vectors field X defines a relative conformal transformation of the structure 2-form Ω, [5, 7].

Consequently, we obtain the following

Theorem. *Let $M(\phi, \Omega, \eta, \xi, f)$ be a $(2m + 1)$-dimensional manifold. Then the necessary and sufficient conditions that M carries a skew symmetric Killing vector field X is that $F = -I$ and that $X^\flat(X^\flat = \alpha)$ be an exterior recurrent form with F as recurrence scalar and η as recurrence scalar. In this case X defines a relative conformal transformation of the structure 2-form Ω.*

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