On the infinitesimal affine rigidity of ellipsoid

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Abstract

In this work infinitesimal affine rigidity of ellipsoids in the $(n+1)$-dimensional real affine space are considered. It is shown that every ellipsoid E in A^{n+1} is infinitesimally H_r rigid. As a consequence of this theorem each ellipsoid E in A^{n+1} is infinitesimally affinely rigid.

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§1. Introduction.

Let $f : M \to A^{n+1}$ be a connected compact nondegenerate hypersurface immersion of $n-$ dimensional differentiable manifold M $(n \geq 2)$ into $(n+1)$-dimensional real affine space A^{n+1} (i.e., $f(M)$ is an ovaloid). An Ellipsoid in A^{n+1} is known as a closed, compact locally strongly convex elliptic proper affine hypersphere without boundary. It is defined by

$$\langle f(p), f(p)\rangle = 1, \tag{1.1}$$

where $\langle\ ,\ \rangle$ is auxiliary inner product in A^{n+1} and $f(p)$ is the position vector field of $f(M)$ with respect to $o \in f(M)$.

In [4] ellipsoids are the only affinely rigid hypersurfaces among the ovaloids in A^{n+1}. Since any hypersurface $f(M)$ in A^{n+1} is infinitesimally affinely rigid if the only deformation vector field of $f(M)$ satisfying $\delta G_{ij} = 0$ is trivial, where G_{ij} are the components of the Blaschke metric tensor G on $f(M)$. Using the algebraic integrability conditions for ellipsoids in A^n $(n > 2)$, K. Leichtweiss proved in [2] that every ellipsoid E in $n-$ dimensional affine space A^n is infinitesimally S-rigid and as a consequence of this theorem every ellipsoid E in A^n is fortiori infinitesimally rigid, where S is the scalar curvature function on the ellipsoid.

An-Min Li [3] generalized the Minkowski integral formulas to affine hypersurfaces and many characterization of ellipsoids are obtained by using $r-$ th affine mean curvature H_r $(1 \leq r \leq n)$. Recall that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the affine principal curvatures of a locally strongly convex hypersurface $f : M \to A^{n+1}$, the $r-$ th affine mean curvature of hypersurface is defined by

$$H_r = \frac{1}{\begin{pmatrix} n \\ r \end{pmatrix}} \sum_{i_1 < i_2 < \ldots < i_r} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_r}.$$

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In particular, $K = H_n$ is the Gauss curvature. The affine mean curvature $H = H_1$ is defined by

$$H = \frac{1}{n} B_{ij} G^{ij}.$$

It is obvious that for an ellipsoid $f(M)$, we have $H^r = H_r$.

§2. The main theorem

Theorem 1 *Every ellipsoid E in A^{n+1} $(n \geq 2)$ is infinitesimally H_r- rigid, in the sense that any normal allowable deformation of E satisfying*

$$\delta H_r = 0 \tag{2.2}$$

is trivial.

Proof. Let $F : I \times M \to A^{n+1}$, $I = (-\varepsilon, \varepsilon)$ be a normal allowable deformation of the ellipsoid E. Such a normal allowable deformation of E is given by a smooth one-parameter family $F_t : M \to A^{n+1}$ with $F_0 = f$. Let

$$x = F_t\left(u^1, u^2, ...u^n\right).$$

We consider the tangent vector of the curve at $\alpha(0)$, $\alpha : I \to A^{n+1}$, $\alpha(t) = F(t, p)$. Denote $\alpha'(0) = (\delta x)_p$. Then $(\delta x)_p$ is the initial velocity of the orbit $f(p)$ under F. The section δx is called the deformation vector field of E. That is,

$$\delta x = \frac{\partial F_t}{\partial t}\,|_{t=0}\,.$$

From [1, p.113, (4.6)] for some local parameter system $u = \left(u^1, u^2, ...u^n\right)$ defined in the neighbourhood of p in M, we have

$$\delta x = \nu y - G^{ij} \nu_i f_j \tag{2.3}$$

where ν is the normal component of δx, y is the affine normal (Blachke normal) vector field, $\nu_i = \frac{\partial \nu}{\partial u^i}, f_j = \frac{\partial f}{\partial u^j}$. In this case, the equalities (1.1) and (2.3) imply

$$\nu = \left\langle (\delta x)_p\,, f(p) \right\rangle$$

where $y = -f(p)$. From [5, Corollary 5.16] we have

$$\delta H_r = -\frac{r}{n} H^{r-1} \delta H. \tag{2.4}$$

Since $H \neq 0$, $\delta H_r = 0$ implies $\delta H = 0$. Now we consider the Calabi formula [1, Eq. (4.21)]

$$\begin{aligned} \delta H &= \frac{1}{n}(\Delta\left(\Delta\nu + nH\nu\right) + nH\left(\Delta\nu + nH\nu\right) \\ &\quad + (n+2)\, B_{ij} G^{jk} (G^{il} \nu_{l,k} + A_{mkl} G^{im} G^{lh} \nu_h + B_{kl} G^{il} \nu)), \end{aligned}$$

where A_{mkl} and B_{ij} is the cubic differential form and symmetric bilinear form respectively. For simplicity, in the last equation we assume that $H = 1$, $B_{ij} = G_{ij}$. Thus for an ellipsoid E we have,

$$\delta H = \frac{1}{n}\left(\Delta^2\nu + (3n+2)\,\Delta\nu + \left(2n^2+2n\right)\nu\right). \tag{2.5}$$

From the equalities (2.2), (2.4) and (2.5) we have,

$$\Delta^2\nu + (3n+2)\,\Delta\nu + \left(2n^2+2n\right)\nu = 0.$$

The last equality may be written in the form,

$$\Delta\left(\Delta\nu + 2\,(n+1)\,\nu\right) + n\left(\Delta\nu + 2\,(n+1)\,\nu\right) = 0 \tag{2.6}$$

or equivalently,

$$\Delta\left(\Delta\nu + n\nu\right) + 2\,(n+1)\left(\Delta\nu + n\nu\right) = 0. \tag{2.7}$$

The equation (2.6) means that $\Delta\nu + 2\,(n+1)\,\nu$ is eigenfunction to the eigenvalue n of the Laplace operator Δ and therefore harmonic function of degree 1. In the same manner in (2.7) $\Delta\nu + n\nu$ is eigen function to the eigenvalue $2\,(n+1)$ of Δ and therefore a spherical harmonic function of degree 2.

Denote the above harmonic functions as follows

$$K_1 = \Delta\nu + 2\,(n+1)\,\nu$$

and

$$K_2 = \Delta\nu + n\nu.$$

We obtain that

$$\nu = \frac{1}{n+2}\left(K_1 - K_2\right). \tag{2.8}$$

By the harmonicity of K_1 and K_2, we have

$$K_1 = \langle a, f(p)\rangle\,, a \in A^{n+1} \tag{2.9}$$

and

$$K_2 = \langle Af(p)\,, f(p)\rangle\,, \quad tr\ A = 0. \tag{2.10}$$

By setting (2.9), (2.10) in (2.8) we have,

$$\begin{aligned}
\nu &= \frac{1}{n+2}\left(\langle a, f(p)\rangle - \langle Af(p)\,, f(p)\rangle\right)\\
&= \left\langle \frac{a}{n+2} - \frac{A}{n+2} f(p)\,, f(p)\right\rangle\\
&= -\left\langle \frac{A}{n+2} f(p) - \frac{a}{n+2}, f(p)\right\rangle\\
&= -\left\langle (\delta x)_p\,, f(p)\right\rangle = \langle Bf(p) + b, f(p)\rangle\,, trB = 0.
\end{aligned}$$

Hence the only deformation vector field of an ellipsoid E such that $\delta H = 0$ is trivial. By the equalities (2.4) the only deformation vector field of the ellipsoid satisfying $\delta H_r = 0$ is trivial. This completes the proof. □

Corollary 1 *Every ellipsoid E in A^{n+1} is infinitesimally affinely rigid.*

Proof. We assume that

$$\delta G_{ij} = 0. \tag{2.11}$$

From [1, p. 118], we have

$$\begin{aligned} G^{ij}\delta G_{ij} &= -n\left(\Delta\nu + nH\nu\right) - (n+2)\left(\Delta\nu + nH\nu\right) \\ &= -\left(2n+2\right)\Delta\nu - n\left(2n+2\right)H\nu \\ &= -\left(2n+2\right)\left(\Delta\nu + nH\nu\right) = 0, \end{aligned}$$

and considering the equalities [1, Eq.4.20], we infer

$$\delta B_i^h = \Delta\left(\Delta\nu + nH\nu\right) + nH\left(\Delta\nu + nH\nu\right) + (n+2)\left(\Delta\nu + nH\nu\right). \tag{2.12}$$

The equation (2.11) gives us

$$\Delta\nu + nH\nu = 0$$

and from (2.12) we obtain that $\delta B_i^h = 0$. Finally, the equation (2.4) implies

$$\delta H_r = 0.$$

By the main theorem, the only variation vector field of E satisfying $\delta G_{ij} = 0$ is trivial. □

References

[1] E. Calabi, *Hypersurfaces with maximal affinely invariant area,* Amer. J. Math. 104 (1982), 1, 91-126.

[2] K. Leichtweiss, *On the infinitesimal affine rigidity of ellipsoids in the n-space*, Arch. Math. 72 (1999), 315-320.

[3] An-Min Li, *A characterization of ellipsoids*, Results in Mathematics, 20 (1991), 657-659.

[4] K. Schneider, *Zur affinen differential geometrie im grossen*, I. Math. Z. 101 (1967), 375-406.

[5] U. Simon, *Hypersurfaces in equiaffine differential geometry*, Geom. Dedicata, 17 (1984), 157-168.

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