Over the optimal control theory in investment strategies

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Abstract

The present paper develops an (optimal) numerical method for the problem of dividing the resources of a small company between the funds allocated for investment and the funds allocated for usual expanses (salaries, etc.). This is accomplished by using the Pontryagin principle within the frame of the optimal control theory to model a new investment strategy for small companies.

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§1. Introduction

The problem studied in this article improves the investment strategy of small companies. The strategy of modelling the financial phenomenon is inspired and adapted from the problem of biological nature of finding optimal life cycles strategies for a given organism [4]. The biological model from [4] uses Pontryagin's maximum principle to find the optimal allocation energy between growth and reproduction for a given organism. Here we will use the same principle within the frame of the optimal control theory to model a new investment strategy for small companies.

§2. Statement of the problem

We will treat our actuarial problem within the frame of the optimal control theory. Let's denote by $u(t)$ the control variable, and by $w(t)$ the number of employees at time t. Let's denote with T the last moment of investment (we assume T finite). This is a quantity similar the maximum age of reproduction in the analogical biological problem [4]. The fitness of the investment strategy is defined through its increasing rate r:

$$1 = \int_0^T e^{-rt} l(t) b(t) dt, \tag{2.1}$$

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where $l(t)$ is a measure of the financial evolution of the company from its establishing moment to the present moment t, given by

$$l(t) = e^{-\int\limits_0^t \mu(x)dx},$$

$b(u, w)$ is the rate of the establishment of companies, and $\mu(w)$ is the bankruptcy rate. Let's denote by $P(w)$ the total resources of a company available for allocation, with w employees. The function $P(w)$ will depend on the specific company studied. Usually, $P(w)$ is depending upon funds necessary for investment and the funds allocated for salaries. In order to simplify the discussion, let's denote by

$$L(t) = e^{-rt}l(t), \quad L(0) = 1,$$

another amount which measures the financial evolution of the company from its establishing moment to the present moment t. Then $L(t)$ can be seen as a factor which characterizes the investment funds in (2.1). In the following, u will denote the proportion of resources allocated to investment and the rest of the resources will be allocated to salaries.

One can notice that the investments are decreasing in time because of the increasing of the fitness rate r, and the decrease in development probability with the rate $\mu(w)$.

In the following we will state the optimal control theory, based on a model similar to the biological one developed in [4]. So, we choose u to maximize r subject to the following constraints

$$\dot{w} = (1 - u(t))P(w), \quad w(0) = given, \tag{2.2}$$

$$\dot{L} = -(r + \mu(w))L, \quad L(0) = 1 \tag{2.3}$$

$$\dot{\theta} = \frac{kuP(w)L}{w(0)}, \quad \theta(0) = 0, \quad \theta(T) = 1 \tag{2.4}$$

$$\dot{r} = 0, \tag{2.5}$$

with k given constant and $0 \leq u \leq 1$. We observe from (2.4) that the relation (2.1) becomes

$$\int\limits_0^T \frac{kuP(w)}{w(0)} L dt = 1.$$

In order to apply Pontryagin's principle we define the Hamiltonian

$$H = \lambda_0 \frac{ku(t)L(t)P(w)}{w(0)} + \lambda_1(1 - u(t))P(w) - \lambda_2(r + \mu(w))L(t).$$

Thus, the adjoint equations are

$$\dot{\lambda}_0 = 0 \tag{2.6}$$

$$\dot{\lambda}_1 = \lambda_2 g' L - \frac{ku\lambda_0 LP'}{w(0)} - (1 - u(t))\lambda_1 P', \quad \lambda_1(T) = 0 \tag{2.7}$$

$$\dot{\lambda}_2 = \lambda_2 g(w) - \frac{ku\lambda_0 P(w)}{w(0)}, \quad \lambda_2(T) = 0 \tag{2.8}$$

$$\dot{\lambda}_3 = \lambda_2 L(t), \quad \lambda_3(0) = 0, \quad \lambda_3(T) = 1, \tag{2.9}$$

where $g(w) = r + \mu(w)$, and we denote with prime the differentiation with respect to w.

Let's consider a two phase solution of the problem in which the control u switches from 0 to 1. Thus, to find the switching point we write

$$\frac{\partial H}{\partial u} = \frac{k\lambda_0 LP}{w(0)} - \lambda_1 P = 0. \tag{2.10}$$

Hence,

$$\hat{u} = 0 \text{ if } \frac{k\lambda_0 L}{w(0)} < \lambda_1 \text{ and } \hat{u} = 1 \text{ if } \frac{k\lambda_0 L}{w(0)} > \lambda_1.$$

An exact solution r can be found immediately following the two phase strategy, like in [1].

§3. The numerical method

In the following we will solve the stated problem for an arbitrary control vector u. In order to avoid using interpolation, we will solve the state equations and the equations for the adjoint variables for the same step size, since they are not independent. To optimize the solution we will use the projected gradient method.

We will solve the state equations for a given control u. Following the idea from [5], we apply an iterative trapezoidal rule to the relation (2.2)

$$w_{i+1}^{n+1} = w_i + \frac{h}{2}((1-u_i)P(w_i) + (1-u_{i+1})P(w_{i+1}^n)), \quad i = \overline{0,n},$$

where $h = \frac{T}{n}$, $n$is the number of steps, w_0 is known and $w_{i+1}^1 = w_i$ at each step. The iterations will stop when $w_{i+1}^{n+1} - w_{i+1}^n \leq tol$. It can be shown easily that if we assume that P satisfies the Lipschitz condition with the Lipschitz constant K, then the method converges for $h \leq \frac{2}{K}$.

To solve the equations (2.3) and (2.4) we consider the trapezoidal rule

$$L_{i+1} = \frac{1 - \frac{h}{2}(r^n + \mu(w_i))}{1 + \frac{h}{2}(r^n + \mu(w_{i+1})}L_i, \quad L_0 = 1$$

$$\theta_{i+1} = \theta_i + \frac{kh}{2w(0)}(u_i P(w_i)L_i + u_{i+1}P(w_{i+1})L_{i+1}), \quad \theta_0 = 0.$$

For the solution of the equation (2.5), we consider a sequence of iterations for r having two initial guesses. We will evaluate L and θ for each guess. We will use a secant method to impose the condition $\theta(T) = 1$, recalculating r by

$$r^{n+1} = r^n - \frac{s(r^n)(r^n - r^{n-1})}{s(r^n) - s(r^{n-1})},$$

where $s(r^n) = \theta(T) - 1$. The iteration process will stop when $r^{n+1} - r^n \leq tol$. Up to this moment we've solved the state equations. To solve the adjoint equations, we will use the same technique as for solving (2.3), (2.4) and (2.5). So, we consider again a trapezoidal scheme for (2.8) (2.9) working from $t = T$ to $t = 0$,

$$\lambda_{2_i} = \frac{(1 - \frac{h}{2}(r + \mu(w_{i+1}))\lambda_{2_{i+1}} + \frac{kh\lambda_0}{2w(0)}(u_i P(w_i) + u_{i+1}P(w_{i+1}))}{1 + \frac{h}{2}(r + \mu(w_i))}$$

$$\lambda_{3_i} = -\frac{h}{2}(\lambda_{2_i} L_i + \lambda_{2_{i+1}} L_{i+1}) + \lambda_{3_{i+1}}.$$

As before, we will define an iterative process to find λ_0 through two initial guesses and we use the end condition for λ_3 in the secant method for λ_0,

$$\lambda_0^{n+1} = \lambda_0^n - \frac{\lambda_3^n(0)(\lambda_0^n - \lambda_0^{n-1})}{\lambda_3^n(0) - \lambda_3^{n-1}(0)}.$$

Finally we solve the equation (2.7), using the same trapezoidal idea:

$$\lambda_{1_i} = \frac{(1 + \frac{h}{2}(1 - u_{i+1})P'(w_{i+1}))\lambda_{1_{i+1}}}{(1 - \frac{h}{2}(1 - u_i)P'(w_i))} - \frac{h(\lambda_{2_i}\mu'(w_i)L_i + \lambda_{2_{i+1}}\mu'(w_{i+1})L_{i+1})}{2(1 - \frac{h}{2}(1 - u_i)P'(w_i))} + \\ + \frac{kh\lambda_0(u_i L_i P'(w_i) + u_{i+1}L_{i+1}P'(w_{i+1}))}{w(0)(1 - \frac{h}{2}(1 - u_i)P'(w_i))}.$$

In order to find the optimal control u, assuming that this exists, we need to optimise the solution of the equations (2.2)-(2.5) and (2.6)-(2.9). Hence, we will adapt an projected gradient algorithm from [3], and maximize the first variation of inner product functional $\langle \frac{\partial H}{\partial u}, u - \tilde{u} \rangle$ for all choices of u. A sketch of the algorithm is presented:

Begin
repeat

$u_{old} := u_{new}$;
$r_{old} := r_{new}$;
$\frac{\partial H}{\partial u_{old}} = \frac{\partial H}{\partial u_{new}}$
repeat
$u_{new} := u_{old} + h * \frac{\partial H}{\partial u_{old}}$

if $u_{new} < 0.0$ then $u_{new} = 0.0$
else if $u_{new} > 1.0$ then $u_{new} = 1.0$
$h := \frac{h}{2}$
until $r_{new} > r_{old}$
compute $\frac{\partial H}{\partial u_{new}}$ by (2.10)
if $\frac{\partial H}{\partial u_{new}} > 0$ then $\tilde{u} = 1$
else $\tilde{u} = 0$
compute $c = \langle \frac{\partial H}{\partial u_{new}}, \tilde{u} - u_{new} \rangle$
until $c \leq tol$
$\hat{u} = u_{new}$
End.

§4. Conclusions

The present problem improves the investment strategy of small companies. The strategy of modelling the financial phenomenon is inspired and adapted from the problem of biological nature of finding optimal life cycles strategies for a given organism [4]. By means of Pontryagin's maximum principle we find the optimal allocation between the funds dedicated to investment and the funds dedicated to usual expenses (salaries, etc.).

As future work, we shall study the errors of the method, comparing it with real company data.

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