

On lifts of structures on complex manifolds

Mehmet Tekkoyun and Şevket Civelek

Abstract

The present survey paper presents prolongations of basic structures from special almost Hermitian manifolds to the associated tangent spaces.

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§1. Introduction and notations

The lift method has an important role in differentiable geometry, since it permits to extend differentiable structures. The basic classical results can be traced in the well-known papers [12]-[15]. As well, contributions to the study of lifts of complex structures on Hermitian and Kählerian manifolds were provided in [9]. In the following, by using lifts, we recollect the extensions to TM of the basic particular structures of the almost Hermitian manifolds M . Namely, we consider the following classes of almost Hermitian manifolds M (assumed C^∞ and of even dimension $2m$):

$$\text{Kählerian manifolds :} \quad [\nabla_X, J]Y = 0$$

$$\text{Nearly Kählerian manifolds :} \quad [\nabla_X, J]X = 0$$

$$\text{Almost Kählerian manifolds :} \quad d\Phi = 0$$

$$\text{Quasi Kählerian manifolds :} \quad [\nabla_{JX}, J] = -J[\nabla_X, J]$$

$$\text{Hermitian manifolds :} \quad [\nabla_{JX}, J] = J[\nabla_X, J]$$

$$\text{Semi Kählerian manifolds :} \quad \sum_{i=1}^m \{\nabla_{E_i}(J)E_i + \nabla_{JE_i}(J)JE_i\} = 0,$$

for all $X, Y \in \mathcal{X}(M)$, where $\mathcal{X}(M)$ denotes the Lie algebra of C^∞ vector fields on M , ∇ the Riemannian connection, J the almost complex structure, Φ the fundamental 2-form, and $\{E_i, JE_i\}$ the associated local orthonormal frame field ([6,7].)

Throughout the paper, all the mappings and manifolds will be assumed to be C^∞ and the sum will be taken over repeated indices. Also, v and c will denote the vertical and complete lifts to TM of geometric complex structures of M , respectively.

Let M be a complex manifold M and TM the associated tangent space. The basic lifts of geometric objects from M to TM are described below (for basic properties of lifts, we recommend [12-15]).

a) The lift of functions. The *vertical lift* of a function $f \in \mathcal{T}_0^0(M) \equiv \mathcal{F}(M)$ to TM is the function $f^v \in \mathcal{F}(TM)$ given by $f^v = f \circ \tau_M$, where $\tau_M : TM \rightarrow M$ is the canonical projection. We have $rank(f^v) = rank(f)$, since

$$f^v(Z_p) = f(\tau_M(Z_p)) = f(p), \quad \forall Z_p \in T_p M.$$

The *complete lift* of the function $f \in \mathcal{F}(M)$ to TM is the function $f^c \in \mathcal{F}(TM)$ given by

$$f^c = z'^\alpha \left(\frac{\partial f}{\partial z^\alpha} \right)^v + \bar{z}'^\alpha \left(\frac{\partial f}{\partial \bar{z}^\alpha} \right)^v,$$

where we denote by $(z, \bar{z}, z', \bar{z}')$ the local complex coordinates of a chart-domain $TU \subset TM$. Furthermore, for $Z_p \in T_p M$ we have

$$f^c(Z_p) = z'^\alpha(Z_p)(\partial_\alpha f)(p) + \bar{z}'^\alpha(Z_p)(\partial_{\bar{\alpha}} f)(p).$$

b) The lift of vector fields. The *vertical lift* of a vector field $Z \in \mathcal{X}(M)$ to TM is the vector field $Z^v \in \mathcal{X}(TM)$ given by

$$Z^v(f^c) = (Zf)^v, \quad \forall f \in \mathcal{F}(M),$$

and the *complete lift* of vector field $Z \in \mathcal{X}(M)$ to TM is the vector field $Z^c \in \mathcal{X}(TM)$ given by

$$Z^c(f^c) = (Zf)^c, \quad \forall f \in \mathcal{F}(M).$$

c) The lift of 1-forms. The *vertical lift* of a 1-form $\omega \in \mathcal{X}^*(M)$ to TM is the 1-form $\omega^v \in \mathcal{X}^*(TM)$ given by

$$\omega^v(Z^c) = (\omega Z)^v, \quad \forall Z \in \mathcal{X}(M),$$

and the *complete lift* of a 1-form $\omega \in \mathcal{X}^*(M)$ to TM is the 1-form $\omega^c \in \mathcal{X}^*(TM)$ given by

$$\omega^c(Z^c) = (\omega Z)^c, \quad \forall Z \in \mathcal{X}(M).$$

d) The lift of tensor fields of type (1,1). The *vertical lift* of complex tensor field $F \in \mathcal{T}_1^1(M)$ to TM is the complex tensor field $F^v \in \mathcal{T}_1^1(TM)$ given by

$$F^v(Z^c) = (FZ)^v, \quad \forall Z \in \mathcal{X}(M),$$

and the *complete lift* of complex tensor field $F \in \mathcal{T}_1^1(M)$ to TM is the complex tensor field $F^c \in \mathcal{T}_1^1(TM)$ given by

$$F^c(Z^c) = (FZ)^c, \quad \forall Z \in \mathcal{T}_0^1(M).$$

e) The lift of tensor fields of type (0,2). The *vertical lift* of a complex tensor field $g \in \mathcal{T}_2^0(M)$ to TM is the complex tensor field $g^v \in \mathcal{T}_2^0(TM)$ given by

$$g^v(Z^c, W^c) = (g(Z, W))^v, \quad \forall Z, W \in \mathcal{X}(M),$$

and the *complete lift* of a complex tensor field $g \in \mathcal{T}_2^0(M)$ to TM is the complex tensor field $g^c \in \mathcal{T}_2^0(TM)$ given by

$$g^c(Z^c, W^c) = (g(Z, W))^c, \quad \forall Z, W \in \mathcal{X}(M).$$

The general properties of vertical and complete lifts of the covariant derivative are described by

Proposition 1.1. *Let M be a complex manifold. For the complex tensor fields $X, Y, Z \in \mathcal{X}(M)$, $f \in \mathcal{F}(M)$, $\omega \in \mathcal{X}^*(M)$, $K \in \mathcal{T}_s^r(M)$, and the affine connection ∇ , we have*

- i) $\nabla_{X^v}^C f^V = 0, \nabla_{X^v}^C f^C = \nabla_{X^c}^C f^V = (\nabla_X f)^V, \nabla_{X^c}^C f^C = (\nabla_X f)^C$
- ii) $\nabla_{X^v}^C Y^V = 0, \nabla_{X^v}^C Y^C = \nabla_{X^c}^C Y^V = (\nabla_X Y)^V, \nabla_{X^c}^C Y^C = (\nabla_X Y)^C$
- iii) $\nabla_{X^v}^C \omega^V = 0, \nabla_{X^v}^C \omega^C = \nabla_{X^c}^C \omega^V = (\nabla_X \omega)^V, \nabla_{X^c}^C \omega^C = (\nabla_X \omega)^C$
- iv) $\nabla_{X^v}^C K^V = 0, \nabla_{X^v}^C K^C = \nabla_{X^c}^C K^V = (\nabla_X K)^V, \nabla_{X^c}^C K^C = (\nabla_X K)^C$.

Proof. Straightforward, similar to the real case [11]. □

§2. Basic lifts of structures

Let M be a complex manifold endowed with a complex structure J and $\{z^\alpha, \bar{z}^\alpha\}$, $1 \leq \alpha \leq m$ its complex local coordinates. The vertical lift of

$$J = i \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha - i \frac{\partial}{\partial \bar{z}^\alpha} \otimes d\bar{z}^\alpha.$$

to TM is

$$(2.1) \quad J^v = i \frac{\partial}{\partial z'^\alpha} \otimes dz^\alpha - i \frac{\partial}{\partial \bar{z}'^\alpha} \otimes d\bar{z}^\alpha.$$

Since $(J^v)^2 = 0$, J^v is an almost tangent structure for the complex tangent bundle TM . The *complete lift* of the complex structure J to TM is

$$(2.2) \quad J^c = i \frac{\partial}{\partial z^\alpha} \otimes dz^\alpha + i \frac{\partial}{\partial z'^\alpha} \otimes dz'^\alpha - i \frac{\partial}{\partial \bar{z}^\alpha} \otimes d\bar{z}^\alpha - i \frac{\partial}{\partial \bar{z}'^\alpha} \otimes d\bar{z}'^\alpha.$$

Since $(J^c)^2 = -I$, J^c is an almost complex structure for the complex space TM .

Assume that g is an almost Hermitian metric on a complex manifold M . Then the vertical and complete lifts satisfy the relations

$$(2.3) \quad g^v(Z^c, W^c) = (g(Z, W))^v = g^v(J^c Z^c, J^c W^c).$$

An almost Hermitian metric g on M provides the vertical lift g^v given by (2.3) which is Hermitian as well and satisfies

$$(2.4) \quad g^c(Z^c, W^c) = (g(Z, W))^c = g^c(J^c Z^c, J^c W^c).$$

An almost Hermitian metric g on M provides the complete lift g^c given by (2.4) which is Hermitian as well.

If Φ is an almost Kählerian form defined on M , then *the vertical lift* Φ^v of Φ obeys the relations

$$(2.5) \quad \Phi^v(Z^c, W^c) = (\Phi(Z, W))^v = \Phi^v(Z^c, J^c W^c)$$

and is an almost Kählerian form on TM . As well, *the complete lift* Φ^c of Φ satisfies

$$(2.6) \quad \Phi^c(Z^c, W^c) = (\Phi(Z, W))^c = \Phi^c(Z^c, J^c W^c)$$

and is an almost Kählerian form on TM .

In the following are described the extensions to the complex tangent space via lifts of the basic particular structures considered on almost Hermitian manifolds.

Proposition 2.1. *Let M be a Kähler manifold with almost complex structure J . If $[\nabla_{Z^c}^c, J^c] W^c = 0, \forall Z, W \in \mathcal{X}(M)$, then TM is also a Kähler manifold.*

Proof. Consider Z, W complex vector fields and J an almost complex structure on M . Let Z^c, W^c and J^c be respectively the complete lifts of these fields to TM . Then we have

$$[\nabla_{Z^c}^c, J^c] W^c = \nabla_{Z^c}^c(J^c W^c) - J^c(\nabla_{Z^c}^c W^c) = ([\nabla_Z, J] W)^c = 0,$$

which completes the proof. \square

Similarly, one may prove

Proposition 2.2. *Let M be a nearly Kähler manifold with almost complex structure J . If*

$$[\nabla_{Z^c}^c, J^c] Z^c = 0, \quad \forall Z \in \mathcal{X}(M),$$

then TM is also a nearly Kähler manifold.

Proposition 2.3. *Let M be an almost Kähler manifold with almost Kähler form Φ . If $d\Phi^c = 0$, then TM is also an almost Kähler manifold.*

Proof. Let Z, W be two complex vector fields, J the almost complex structure and Φ the almost Kähler form on the almost Kähler manifold M ; let Z^c, W^c, J^c and Φ^c be respectively their complete lifts to TM . Then we have

$$d\Phi^c(Z^c, W^c) = d(\Phi(Z, W))^c = 0. \quad \square$$

Proposition 2.4. *Let M be a quasi-Kähler manifold with the almost complex structure J . If*

$$[\nabla_{J^c Z^c}^c, J^c] = -J^c [\nabla_{Z^c}^c, J^c],$$

then TM is also a quasi-Kähler manifold.

Proof. Let Z, W be complex vector fields and J the almost complex structure on the quasi-Kähler manifold M . Let Z^c, W^c , and J^c be respectively their complete lifts to TM . Then

$$\begin{aligned} [\nabla_{J^c Z^c}^c, J^c] W^c &= \nabla_{J^c Z^c}^c(J^c W^c) - J^c(\nabla_{J^c Z^c}^c W^c) \\ &= (\nabla_{JZ} JW - J\nabla_{JZ} W)^c = ([\nabla_{JZ}, J]W)^c \\ &= (-J[\nabla_Z, J] W)^c = -J^c[\nabla_{Z^c}^c, J^c] W^c. \end{aligned} \quad \square$$

Analogously, one can prove the following

Proposition 2.5. *Let M be a Hermitian manifold M with almost complex structure J . If*

$$[\nabla_{J^c Z^c}^c, J^c] = J^c [\nabla_{Z^c}^c, J^c], \quad Z \in \mathcal{X}(M),$$

then TM is also a Hermitian manifold.

Proposition 2.6. *If J is an almost complex structure and $\{E_i, JE_i\}$ the local orthonormal frame field on a semi-Kähler manifold M , such that*

$$\sum_{i=1}^n \left\{ \nabla_{E_i^c}^c(J^c)E_i^c + \nabla_{J^c E_i^c}^c(J^c)J^c E_i^c \right\} = 0,$$

then TM is also a semi-Kähler manifold.

Proof. Let J^c , E_i^c and $J^c E_i^c$ be respectively the complete lifts of the given tensors to TM . Then, we have

$$\begin{aligned} \sum_{i=1}^n \left\{ \nabla_{E_i^c}^c(J^c)E_i^c + \nabla_{J^c E_i^c}^c(J^c)J^c E_i^c \right\} &= \\ &= \sum_{i=1}^n \left(\nabla_{E_i^c}^c(J^c)E_i^c + \nabla_{J^c E_i^c}^c(J^c)^2 E_i^c \right) = \left(\sum_{i=1}^n [E_i, JE_i] \right)^c = 0, \end{aligned}$$

and hence TM is a semi-Kähler manifold. \square

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Authors' address:

Mehmet Tekkoyun and Şevket Civelek
Pamukkale University, Department of Mathematics, Denizli-Turkey
E-mails: mtekkoyun@ogu.edu.tr, tekkoyun@pamukkale.edu.tr;
scivelek@pamukkale.edu.tr