Tangent spaces on S-manifolds

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Abstract

In this paper the author presents in detail the concept of $\mathcal{S}-$manifold. An $\mathcal{S}-$manifold is an infinite dimensional manifold modeled on the space of tempered distributions, an infinite dimensional topological vector space. One of the motivations that determined the author to introduce this kind of manifold is the following one: the space of tempered distributions, even though it is an infinite dimensional topological vector space, can be endowed with some algebraic-topological structures that makes it very similar to the finite dimensional Euclidean spaces, in the sense that it is possible to find a sort of bases, the $\mathcal{S}-$bases, that are not Hamel bases but that fulfill the most important properties of the Hamel bases in the finite dimensional vector spaces. We shall prove, that the tangent spaces of the $\mathcal{S}-$manifolds are endowed with a new kind of bases that are the analogous of the $\mathcal{S}-$bases.

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§P1. Introduction.

The theory of infinite dimensional vector spaces is quite unsatisfactory in some its sections. We want to emphasize one lack of this theory.

We know that in $\mathbb{R}^n$ the canonical basis $b = (b_1, ..., b_n)$ allows us to expand a vector $x \in \mathbb{R}^n$ as $x = \sum_{i=1}^{n} x_i b_i$. Thus $(b_1, ..., b_n)$ is a basis in which the i-th coordinate of x is x_i, i.e., just the i-th point-component !

Hence, in this way, a vector x is characterized by the set of its canonical projections $_i(x) = x_i$, where we recall that, for $i \in \mathbb{N}\,(\leq n)$, the i-th projection on $\mathbb{R}^n$ is the function

$$_i(\ ) : \mathbb{R}^n \to \mathbb{R} : x \mapsto x_i,$$

Similarly, we can describe a map $f : \mathbb{R} \to \mathbb{R}$ or yet $f : \mathbb{R}^n \to \mathbb{R}$. The question is:

Can we represent a map $f : \mathbb{R}^n \to \mathbb{R}$ via "components" $(f)_i$ or "projections" $_i(f)$?

We can do that identifing both the "y-th component" of f and the "y-th projection" of f with the value of f in y: $(f)_y = \ _y(f) := f(y)$, for every $y \in \mathbb{R}^n$! The next natural question is:

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What is the basis in which the y-component of f is the value $f(y)$?

The physicists usually prefere bases g such that if $f \in C^0(\mathbb{R}, \mathbb{R})$ one has

$$f = \sum_{y \in \mathbb{R}} f(y)\, g_y,$$

but in the usual theory of infinite dimensional vector spaces this is not possible. Indeed, if $b : I \to X$ is an ordered Hamel basis of a vector space X, then when $x = \sum_{i \in I} a_i b_i$, one has $a_i = 0$ for every $i \in I$ except for a finite number of indices.

On the contrary, it is possible to develop in the space of tempered distribution $\mathcal{S}'_n$, a kind of linear algebra (see [2]), in which there exists a basis δ, such that for each distribution $u \in \mathcal{S}'_n$ one has

$$\text{“}u = \sum_{i \in \mathbb{R}^n} u(i)\, \delta_i\text{”}$$

i.e., u is a combination, in some sense, of the family $(\delta_i)_{i \in \mathbb{R}^n}$ with respect to the “system of coefficients” u; it's clear that, if u is a singular distribution (i.e., if does not exist a function $g \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mathbb{K})$, such that $u(\phi) = \int_{\mathbb{R}^n} g\phi d\mu_n$) this “system of coefficients” has the “strange” property of being “non-locally defined”:

one doesn't know who is the i-th component of u, but it is possible to obtain the “global” combination of the vectors with respect to u,

if $(v_i)_{i \in \mathbb{R}^n}$ is a family of tempered distributions such that

i) the function $v(\phi) : \mathbb{R}^n \to \mathbb{K} : i \mapsto v_i(\phi)$ is of class $\mathcal{S}-$;

ii) the operator $\widehat{v} : \mathcal{S}_n \to \mathcal{S}_n : \phi \mapsto v(\phi)$ is a continuous operator; then one sets

$$\sum_{\mathbb{R}^n} uv = \int_{\mathbb{R}^n} uv : \phi \mapsto u(\widehat{v}(\phi)).$$

One doesn't know each “weight” induced by u, but can build the superposition

$$\int_{\mathbb{R}^n} uv = {}^t(\widehat{v})(u).$$

At this point it is natural to ask if it is possible to build up a theory of differentiable manifolds modeled on the space $\mathcal{S}'_n$ (endowed with these new structures) which, as in the finite dimensional case, gives to the bases a central role. This paper answers to the question. First of all we show as, thanks to the $\mathcal{S}-$linear algebra (see [2]), it is possible to formulate a more efficient differential calculus for maps $f : \Omega \subseteq \mathcal{S}'_n \to \mathcal{S}'_m$, then we define the *space modeled on* $\mathcal{S}'_n$ and the $\mathcal{S}-$differentiable manifolds of Carfì and finally we construct some basic examples.

§P2. Preliminaries and notations on tempered distributions

In this paper we shall use the following notations:

1) n, m are natural numbers;

2) $\mathcal{S}_n := \mathcal{S}-(\mathbb{R}^n,\mathbb{R})$ is the real Schwartz space, the set of all the real smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ in $\mathbb{R}$ *rapidly decreasing at infinity*

$$\mathcal{S}-(\mathbb{R}^n,\mathbb{K}) = \left\{ f \in C^\infty(\mathbb{R}^n,\mathbb{K}) : \lim_{|x|\to\infty} \left| x^\beta D^\alpha f(x) \right| = 0 \ \forall \alpha,\beta \in \mathbb{N}_0^n \right\},$$

and $\mathcal{S}_{(n)}$ is the standard Schwartz topology on $\mathcal{S}_n$ (see [5, p. 91, Example 14]);

3) μ_n is the Lebesgue measure on $\mathbb{R}^n$; $(\cdot)_{(\mathbb{R},\mathbb{C})}$ is the immersion of $\mathbb{R}$ in $\mathbb{C}$ and if X is a non-empty set $\mathbb{I}_X = (\cdot)_X$ is the identity map on X;

4) $\mathcal{S}'_n := \mathcal{S}-'(\mathbb{R}^n,\mathbb{R})$ is the space of tempered distributions from $\mathbb{R}^n$ to $\mathbb{R}$, that is the dual of the topological vector space $(\mathcal{S}_n, \mathcal{S}_{(n)})$ i.e., $\mathcal{S}'_n = (\mathcal{S}_n, \mathcal{S}_{(n)})^*$, where, if X and Y are two topological vector spaces on $\mathbb{K}$, $\mathrm{Hom}(X,Y)$ is the set of all linear operators from X to Y, $\mathcal{L}(X,Y)$ is the set of all linear and continuous operators from X to Y and $X^* = \mathcal{L}(X,\mathbb{K})$ is the dual of the topological vector space X;

5) if $a \in \mathbb{R}^n$, δ_a is the *distribution of Dirac on* $\mathcal{S}_n$ *centered at* a, i.e., the functional:

$$\delta_a : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi(a);$$

6) if $f \in L^1_{loc}(\mathbb{R}^n,\mathbb{K})$, then the map

$$\langle f| = \langle f|_n : \mathcal{D}_n \to \mathbb{K} : g \mapsto \int_{\mathbb{R}^n} fg d\mu_n$$

is the *regular distribution generated by* f, where

$$\mathcal{D}_n = \left\{ f \in C^\infty(\mathbb{R}^n,\mathbb{K}) : \mathrm{supp}\, f \text{ is compact} \right\}.$$

§0. Basic concepts of S-linear algebras

We denote by $s\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ the space of all families in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, i.e., the set of all the maps from $\mathbb{R}^m$ to $\mathcal{S}'_n$. Moreover, if v is one of these families, for each $p \in \mathbb{R}^m$, the distribution $v(p)$ is denoted by v_p. The set $s\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ is a vector space with respect to the following two standard operations:

i) the addition $+ : s\left(\mathbb{R}^m, \mathcal{S}'_n\right)^2 \to s\left(\mathbb{R}^m, \mathcal{S}'_n\right) : (v,w) \mapsto v+w$, where $v+w$ is the family defined by $v+w : \mathbb{R}^m \to \mathcal{S}'_n : p \mapsto v_p + w_p$, i.e., $(v+w)_p = v_p + w_p$;

ii) the multiplication by scalars $\cdot : \mathbb{R} \times s\left(\mathbb{R}^m, \mathcal{S}'_n\right) \to s\left(\mathbb{R}^m, \mathcal{S}'_n\right) : (\lambda, v) \mapsto \lambda v$ where λv is the family: $\lambda v : \mathbb{R}^m \to \mathcal{S}'_n : p \mapsto \lambda v_p$ i.e., $(\lambda v)(p) = (\lambda v)_p = \lambda v_p$.

Definition 0.1 (family of tempered distributions of class $\mathcal{S}$). Let $T \in s(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. The family T is called **family of class $\mathcal{S}-$** or **$\mathcal{S}-$family** if, for each $f \in \mathcal{S}_n$, the map

$$T(f) : \mathbb{R}^m \to \mathbb{C} : T(f)(p) = T_p(f),$$

for each $p \in \mathbb{R}^m$, belongs to the space $\mathcal{S}_m$. The set of all these families is denoted by $\mathcal{S}-(\mathbb{R}^m, \mathcal{S}'_n)$.

Definition 0.2 (operator generated by an $\mathcal{S}$ - family). Let $T \in \mathcal{S}-(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}-$. The **operator generated by the family** T (or **associated with** T) is the operator $\widehat{T} : \mathcal{S}_n \to \mathcal{S}_m$ defined by $\widehat{T}(f)(p) = T_p(f)$, for each f in $\mathcal{S}_n$ and for each p in $\mathbb{R}^m$, i.e., with the notations of the above definition, defined by $\widehat{T}(f) = T(f)$, for each f in $\mathcal{S}_n$.

It's easy to prove the following facts:

i) The set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is a subspace of the vector space $\left(s(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot\right)$ and for each $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ the operator $\widehat{v}$ is linear.

ii) The application $(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}\,(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$ is an injective linear operator, thus one has $(v + \lambda w)^\wedge = \widehat{v} + \lambda \widehat{w}$ and $\widehat{v} = 0 \Rightarrow v = 0$.

Definition 0.4 (linear superpositions of an $\mathcal{S}-$family). Let $a \in \mathcal{S}'_m$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The $\mathcal{S}-$**linear combination of** v **with respect to** (the system of coefficients) a , or **linear superposition of** v **with respect to** (the system of coefficients) a , is the distribution

$$\int_{\mathbb{R}^m} av := a \circ \widehat{v} : \phi \mapsto a\left(\widehat{v}(\phi)\right),$$

i.e., $\int_{\mathbb{R}^m} av = {}^t(\widehat{v})(a)$.

Moreover, if $u \in \mathcal{S}'_n$ and there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$, u is called an $\mathcal{S}-$linear superposition of v. Finally, we define as a linear superposition of v, the distribution $\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} \left\langle 1_{(\mathbb{R}^m, \mathbb{K})} \right| v$, where $\left\langle 1_{(\mathbb{R}^m, \mathbb{K})} \right|$ is the regular distribution generated by the constant functional on $\mathbb{R}^m$ of value 1.

Definition 0.5 ($\mathcal{S}-$**-linear independence**). Let $v \in \mathcal{S} - (\mathbb{R}^m, \mathcal{S}'_n)$. v is called $\mathcal{S}-$ **-linearly independent**, if one has

$$\left(u \in \mathcal{S}'_m \wedge \int_{\mathbb{R}^m} uv = 0_{\mathcal{S}'_n}\right) \Rightarrow u = 0_{\mathcal{S}'_m}.$$

Definition 0.6 (of $\mathcal{S}-$linear span). Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. One defines $\mathcal{S}-'$ **-linear span** of v , and it's denoted by $\mathcal{S}\,\mathrm{uspan}\,(v)$, the set

$$\left\{u \in \mathcal{S}'_n : \exists a \in \mathcal{S}'_m : u = \int_{\mathbb{R}^m} av\right\}.$$

Definition 0.7 (system of $\mathcal{S}-$-generators). Let $T \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. T is defined system of $\mathcal{S}$ **-generators** for $V \subseteq \mathcal{S}'_n$ if and only if $\mathcal{S}\,\mathrm{uspan}\,(T) = V$.

Definition 0.8 (of $\mathcal{S}-$basis). Let $v \in \mathcal{S} - (\mathbb{R}^m, \mathcal{S}'_n)$ and let $V \subseteq \mathcal{S}'_n$. One defines v $\mathcal{S}-$**basis** of V if it is $\mathcal{S}-$linearly independent, and one has

$$\mathcal{S}\,\mathrm{uspan}(v) = V.$$

It's possible to prove that if $u \in \mathcal{S}\,\mathrm{uspan}(v)$ and v is $\mathcal{S}-$linearly independent then there exists a unique $a \in \mathcal{S}'_m$ such that $u = \displaystyle\int_{\mathbb{R}^m} av$.

Definition 0.9 (system of coordinates) . Let $v \in \mathcal{SL}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}-$linearly independent family and $w \in \mathcal{S}\,\mathrm{uspan}(v)$. The only tempered distribution $a \in \mathcal{S}'_m$ such that $w = \int_{\mathbb{R}^m} av$ is denoted by $[w|v]$ and is called the **system of contravariant components of** w **in** v **or the system of coordinates of** w **in** v.

§1. S-differential calculus for operators between S'_n and S'_m

Definition 1.1 (of $\mathcal{S}-$differentiability). Let $\Omega \subseteq \mathcal{S}'_n$ be an open set in the weak* topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, $Y \subseteq \mathcal{S}'_m$, $f : \Omega \to Y$ be a continuous map and $u \in \Omega$. The map f is said $\mathcal{S}-$**differentiable at** u if there exists an $\mathcal{S}-$ linear operator l such that, the map

$$t : \Omega - u \to \mathcal{S}'_m : h \mapsto f(u+h) - f(u) - l(h),$$

is tangent to 0 at $0_{\mathcal{S}'_n}$ (see, for the definition of tangent at 0, [6, p. 6].

For convenience, in the following remark we shall recall the definition of an $\mathcal{S}-$linear operator ([**?**]).

Remark 1.1 (on $\mathcal{S}-$-linearity). A map $L : \mathcal{S}'_n \to \mathcal{S}'_m$ is called $\mathcal{S}-$linear operator if

i) for all $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ one has $L(v) \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ i.e., L is an $\mathcal{S}-$operator;

ii) for any $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$ and for all $a \in \mathcal{S}'_k$ one has

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v).$$

The identity operator on $\mathcal{S}'_n$, $\mathbb{I}_{\mathcal{S}'_n}$ is $\mathcal{S}-$linear, in fact, for each $w \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$, one has

$$\mathbb{I}_{\mathcal{S}'_n}\left(\int_{\mathbb{R}^k} aw\right) = \int_{\mathbb{R}^k} aw = \int_{\mathbb{R}^k} a\mathbb{I}_{\mathcal{S}'_n}(w).$$

The set of all the $\mathcal{S}-$linear operators from $\mathcal{S}'_n$ to $\mathcal{S}'_m$ is denoted by $\mathcal{S}\,\mathrm{Hom}(\mathcal{S}'_n, \mathcal{S}'_m)$.

Theorem 1.1 (**$\mathcal{S}-$linearity and linearity).** *Let $L : \mathcal{S}'_m \to \mathcal{S}'_n$ be an $\mathcal{S}-$linear operator. Then L is linear and weakly* continuous. In particular a map $\mathcal{S}-$ differentiable at u is differentiable at u (see, for the definition of differentiability,* [6, p. 6].

Proof. For each $u, v \in \mathcal{S}'_m$, one has

$$L(u) = L\left(\int_{\mathbb{R}^m} u\delta\right) = \int_{\mathbb{R}^m} uL(\delta) = {}^t\left((L(\delta))^\wedge\right)(u),$$

where $\delta : \mathbb{R}^m \to \mathcal{S}'_m$ is the Dirac basis of $\mathcal{S}'_m$ i.e., the family such that $\forall \phi \in \mathcal{S}_m$ and $\forall p \in \mathbb{R}^m$ one has $\delta(p)(\phi) = \delta_p(\phi) = \phi(p)$. It's easy to see that $\int_{\mathbb{R}^m} u\delta = u$, in fact, for each $\phi \in \mathcal{S}_m$, one has

$$\left(\int_{\mathbb{R}^m} u\delta\right)(\phi) = u\left(\widehat{\delta}(\phi)\right) = u(\phi),$$

this because $\widehat{\delta}(\phi) = \phi$. So L is the transpose of a linear and continuous operator and then is linear and weakly* continuous (see [5, p. 256 Prop. 3]): one has $\mathcal{S}\,\mathrm{Hom}\,(\mathcal{S}'_m, \mathcal{S}'_n) \subseteq \mathcal{L}\,(\mathcal{S}'_m, \mathcal{S}'_n)$. □

Remark 1.2 (Gateaux differentiability). A map $f : \Omega \subseteq \mathcal{S}'_n \to Y \subseteq \mathcal{S}'_m$ is called *Gateaux differentiable at* $u \in \Omega$, if for all $v \in \mathcal{S}'_n$ there exists the following limit

$$(\mathbb{R}, \sigma\,(\mathcal{S}'_m, \mathcal{S}_m))\text{-}\lim_{h\to 0} \frac{f\,(u + hv) - f(u)}{h},$$

where the map *difference quotient with respect to* v *at* u

$$R : A \subseteq \mathbb{R} \to \mathcal{S}'_m : h \mapsto \frac{f\,(u + hv) - f(u)}{h}$$

is defined in a convenient open set A of $\mathbb{R}$, more precisely let

$$r : \mathbb{R} \to \mathcal{S}'_n : h \mapsto u + hv,$$

then, being r continuous, the "natural" domain of R is $A = r^{-}(\Omega) \setminus \{0\}$. In this case the value of the above limit is called *the Gateaux derivative of* f *at* u *with respect to* v and it is denoted by $\partial_v(f)(u)$ or by $\partial(f, v)(u)$.

Now, the following result follows straightforward:

Theorem 1.2 (characterization of Sdifferentiability). *Let* $\Omega \subseteq \mathcal{S}'_n$ *be an open set in the weak* topology* $\sigma\,(\mathcal{S}'_n, \mathcal{S}_n)$, $Y \subseteq \mathcal{S}'_m$, $u \in \Omega$ *and* $f : \Omega \to Y$ *be a map. Then* f *is* $\mathcal{S}-$*differentiable at* u *if and only if* f *is differentiable at* u *and the Frèchet differential*

$$d_u f = df(u) : \mathcal{S}'_n \to \mathcal{S}'_m : v \mapsto \partial_v(f)(u) = \partial\,(f, v)\,(u)$$

is $\mathcal{S}-$*linear (we recall that a differentiable map at* u *is Gateaux differentiable at* u *and moreover* $\partial_v(f)(u) = df(u)$*).*

Example 1.1 (of $\mathcal{S}-$differentiable maps). Let $A \in \mathcal{S} - \mathrm{uHom}\,(\mathcal{S}'_n, \mathcal{S}'_m)$, then A is $\mathcal{S}-$differentiable at every $u \in \mathcal{S}'_n$ and $d_u A = A$. Indeed, for all $h \in \mathcal{S}'_n$ one has

$$A\,(u + h) - A(u) - A(h) = 0_{\mathcal{S}'_m},$$

so A is $\mathcal{S}-$ **-**differentiable, by definition, with $d_u A = A$, for every $u \in \mathcal{S}'_n$.

Now, we shall prove a theorem that shows the "naturalness" of the preceding concepts.

Notation 1.1. Let a be a tempered distribution and $A : \mathcal{S}'_n \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_m\right)$, we put

$$\int_{\mathbb{R}^n} aA : \mathcal{S}'_n \to \mathcal{S}'_m : u \mapsto \int_{\mathbb{R}^n} aA(u).$$

Moreover, if f is an $\mathcal{S}-$differentiable map we put $\partial_p(f) = \partial_{\delta_p}(f)$, for each $p \in \mathbb{R}^n$, $\partial(f) = (\partial_p(f))_{p\in\mathbb{R}^n}$, and $\nabla(f) :\ \mathcal{S}'_n \to \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_m\right) : u \mapsto (\partial_p(f)(u))_{p\in\mathbb{R}^n}$. The last definition is good: in fact, the family $(\nabla f)\,(u)$ lies in $\mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_m\right)$; indeed $df(u)$ is an $\mathcal{S}-$operator and

$$(\nabla f)\,(u) = (\partial_p(f)(u))_{p\in\mathbb{R}^n} = \left(\partial_{\delta_p}(f)(u)\right)_{p\in\mathbb{R}^n} = (df(u)\,(\delta_p))_{p\in\mathbb{R}^n} = df(u)\,(\delta)\,.$$

Theorem 1.3. *Let $\Omega \in \sigma(\mathcal{S}'_n, \mathcal{S}_n)$, $Y \subseteq \mathcal{S}'_m$ and $f : \Omega \to Y$ be an $\mathcal{S}-$differentiable map. Then, for each $v \in \mathcal{S}'_n$, one has*

$$\partial_v(f) = \int_{\mathbb{R}^n} v\nabla(f).$$

Proof. For every $u \in \Omega$, one has

$$\begin{aligned}\partial_v(f)(u) &= df(u)(v) = df(u)\left(\int_{\mathbb{R}^n} v\delta\right) = \int_{\mathbb{R}^n} v df(u)(\delta) = \\ &= \int_{\mathbb{R}^n} v\nabla(f)(u) = \left(\int_{\mathbb{R}^n} v\nabla(f)\right)(u). \ \square\end{aligned}$$

§2. $\mathcal{S}-$differentiable manifolds

Definition 2.1 (space modeled on $\mathcal{S}'_n$). Let $S = (X, \tau)$ be a Hausdorff topological space. S is called a **space modeled on $\mathcal{S}'_n$** if each point of S has a neighborhood homeomorphic to an open subset of the topological space $(\mathcal{S}'_n, \sigma(\mathcal{S}'_n, \mathcal{S}_n))$. If φ is a homeomorphism of an open set $U \in \tau$ onto an open subset $A \in \sigma(\mathcal{S}'_n, \mathcal{S}_n)$, φ is called **$\mathcal{S}-$coordinate map on** S or **$\mathcal{S}-$coordinate system on** U or an **$\mathcal{S}-$chart on** S, the pair (U, φ) is called an **$\mathcal{S}-$coordinate open set of** S or an **$\mathcal{S}-$coordinate neighborhood of** S . Moreover, a coordinate system φ on U is said **centered at** $P \in U$ if $\varphi(P) = 0_{\mathcal{S}'_n}$. The set of all the $\mathcal{S}-$ charts on S modeled on $\mathcal{S}'_n$,i.e., the union $\bigcup \{\mathrm{Homeo}(\Omega, A) : \Omega \in \tau, A \in \sigma(\mathcal{S}'_n, \mathcal{S}_n)\}$ is denoted by $\mathcal{S} - \mathrm{chart}(S, \mathcal{S}'_n)$ ..

Definition 2.2 (of composition). Let $f : A \to B$ and $g : C \to D$ two maps. The composition of f and g is the map

$$g \circ f : f^-(B \cap C) \to D : x \mapsto g(f(x)),$$

where $f^-(B \cap C) = \{x \in A : f(x) \in B \cap C\}$ (we shall not exclude the case in which $f^-(B \cap C) = \varnothing$ (see [7, p. 3]).

Let now φ and ψ be two bijective maps, one has $\varphi : \mathrm{dom}\,\varphi \to \mathrm{im}\,\varphi$ and $\psi^- :$ $\mathrm{im}\,\psi \to \mathrm{dom}\,\psi$, hence

$$\varphi \circ \psi : \left\{x \in \mathrm{dom}\,\psi^- : \psi^-(x) \in \mathrm{dom}\,\varphi\right\} \to \mathrm{im}\,\varphi,$$

and obviously one has

$$\begin{aligned}\mathrm{dom}(\varphi \circ \psi) &= \{x \in \mathrm{im}\,\psi : x \in \psi(\mathrm{dom}\,\varphi)\} = \\ &= \mathrm{im}\,\psi \cap \psi(\mathrm{dom}\,\varphi) = \psi(\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi);\end{aligned}$$

moreover

$$\begin{aligned}\mathrm{im}(\varphi \circ \psi) &= \{x \in \mathrm{im}\,\psi : x \in \varphi(\mathrm{dom}\,\psi)\} = \mathrm{im}\,\varphi \cap \varphi(\mathrm{dom}\,\psi) = \\ &= \varphi(\mathrm{dom}\,\varphi) \cap \varphi(\mathrm{dom}\,\psi) = \varphi(\mathrm{dom}\,\psi \cap \mathrm{dom}\,\psi).\end{aligned}$$

Definition 2.3 (of $\mathcal{S}-$atlas). Let $S = (X, \tau)$ be a Hausdorff topological space and let $\Phi \subseteq \mathcal{S} - \mathrm{chart}\,(S, \mathcal{S}_n)$. The set Φ is called **$\mathcal{S}-$atlas on** S modeled on $\mathcal{S}_n'$ if

i) the set $\{\mathrm{dom}\,\varphi\}_{\varphi\in\Phi}$ covers X: $\bigcup_{\varphi\in\Phi} \mathrm{dom}\,\varphi = X$;

ii) Φ is $\mathcal{S}-$ **compatible** i.e., for each $\varphi, \psi \in \Phi$ such that $\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi \neq \varnothing$, one has that the composition

$$\varphi \circ \psi^- : \psi\,(\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi) \to \mathrm{im}\,\varphi$$

is an $\mathcal{S}-$differentiable map.

Remark 2.1. The composition in the above definition is defined in a $\sigma\,(\mathcal{S}_n', \mathcal{S}_n)$-open set, in fact, because $\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi \in \mathrm{rel}\,(\tau, \mathrm{dom}\,\psi)$, and because $\psi \in \mathrm{Homeo}\,(\mathrm{dom}\,\psi, \mathrm{im}\,\psi)$, one has $\psi\,(\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi) \in \sigma\,(\mathcal{S}_n', \mathcal{S}_n)$.

Example 2.1 (of $\mathcal{S}-$atlas). Let $\sigma\,(\mathcal{S}_n', \mathcal{S}_n)$ be the weak* topology on $\mathcal{S}_n'$ and $I : \mathcal{S}_n' \to \mathcal{S}_n'$ be the identity map on $\mathcal{S}_n'$ (i.e., such that $\hat{e}$ is an $\mathcal{S}_{(n)}$-topological isomorphism), then the set $\{I\}$ is an $\mathcal{S}-$atlas on $(\mathcal{S}_n', \sigma\,(\mathcal{S}_n', \mathcal{S}_n))$, in fact $I \circ I^- = I$ and I is an $\mathcal{S}-$differentiable map that is also an homeomorphism.

Example 2.2 (of $\mathcal{S}-$atlas). Let $e \in \mathcal{S}_{\mathrm{inv}}\left(\mathbb{R}^n, \mathcal{S}_n'\right)$ be an invertible family of $\mathcal{S}_n'$, then the operator $[\cdot \mid e] : \mathcal{S}_n' \to \mathcal{S}_n' : u \mapsto [u \mid e]$ is a $\sigma\,(\mathcal{S}_n', \mathcal{S}_n)$-automorphism, in fact, for each $u \in \mathcal{S}_n'$ one has

$$\begin{aligned} u &= \int_{\mathbb{R}^n} [u \mid e]\, e = [u \mid e] \circ \hat{e} = \\ &= {}^t(\hat{e})\,([u \mid e]) = ({}^t(\hat{e}) \circ [\cdot \mid e])\,(u), \end{aligned}$$

so ${}^t(\hat{e}) \circ [\cdot \mid e] = \mathbb{I}_{\mathcal{S}_n'}$, and thus one has $[\cdot \mid e] = ({}^t(\hat{e}))^-$; concluding, $[\cdot \mid e]$ is a topological isomorphism (we recall that $f : \mathcal{S}_n \to \mathcal{S}_n$ is a topological isomorphism if and only if ${}^t f : \mathcal{S}_n' \to \mathcal{S}_n'$ is a topological isomorphism). Moreover, the set $\{[\cdot \mid e]\}$ is an $\mathcal{S}-$atlas on $(\mathcal{S}_n', \sigma\,(\mathcal{S}_n', \mathcal{S}_n))$, indeed, $[\cdot \mid e] \in \mathrm{Homeo}\,(\sigma\,(\mathcal{S}_n', \mathcal{S}_n)\,, \sigma\,(\mathcal{S}_n', \mathcal{S}_n))$, and, making reference to Definition 2.2, one has i) $\mathrm{dom}\,[\cdot \mid e] = \mathcal{S}_n'$ and ii) $[\cdot \mid e] \circ [\cdot \mid e]^- = \mathbb{I}_{\mathcal{S}_n'}$ that is an $\mathcal{S}-$differentiable map.

Definition 2.4 (of transition maps). Let $S = (X, \tau)$ be a Hausdorff topological space, Φ be an $\mathcal{S}-$atlas on S and $\varphi, \psi \in \Phi$ be two coordinate systems. The map

$$t_{(\psi,\varphi)} : \psi\,(\mathrm{dom}\,\varphi \cap \mathrm{dom}\,\psi) \to \varphi\,(\mathrm{dom}\,\psi \cap \mathrm{dom}\,\psi) : y \mapsto \varphi\left(\psi^-\,(y)\right)$$

is called **transition map from** ψ **to** φ. Moreover, for every $P \in X$, the two distributions $\varphi(P)$, $\psi(P)$ are called the **representations of** P **in the systems** φ and ψ respectively. So the transition map $t_{(\psi,\varphi)}$ associates to the ψ-representation of P the φ-representation of P: $t_{(\psi,\varphi)}\,(\psi(P)) = \varphi\,(\psi^-\,(\psi(P))) = \varphi(P)$.

Theorem 2.1 (characterization of an $\mathcal{S}-$atlas). *Let $S = (X, \tau)$ be a Hausdorff topological space and let $\Phi \subseteq \mathrm{uchart}\,(S, \mathcal{S}_n)$. Then Φ is an $\mathcal{S}-$atlas on S modeled on $\mathcal{S}_n'$ if and only if Φ is a Frèchet-differentiable atlas on S modeled on the topological vector space $(\mathcal{S}_n', \sigma\,(\mathcal{S}_n', \mathcal{S}_n))$ and, for every $\varphi, \psi \in \Phi$, the transition map from ψ to φ is an $\mathcal{S}-$differentiable map.*

Definition 2.5 (of $\mathcal{S}-$differentiable structure). *Let $S = (X, \tau)$ be a Hausdorff topological space and $\Phi \subseteq \mathrm{uchart}\,(S, \mathcal{S}'_n)$ be an $\mathcal{S}-$atlas on S. Φ is said an* $\mathcal{S}-$**differentiable structure on** S *if for each coordinate system $\varphi \in \mathcal{S}-\mathrm{chart}\,(S, \mathcal{S}'_n)$ such that $\varphi \circ \psi^-$ and $\psi \circ \varphi^-$ are $\mathcal{S}-$differentiable for each $\psi \in \Phi$ one has $\varphi \in \Phi$. Two coordinate systems φ, ψ are said* $\mathcal{S}-$**compatible** *if $\varphi \circ \psi^-$ and $\psi \circ \varphi^-$ are $\mathcal{S}-$differentiable. Moreover, the* $\mathcal{S}-$**structure generated by** Φ *is the collection*

$$\langle \Phi \rangle = \{\varphi \in \mathrm{uchart}\,(S, \mathcal{S}'_n) : \varphi \textit{ is } \mathcal{S} - \textit{compatible with each } \psi \in \Phi\}\,.$$

The pair (S, Φ) is called $\mathcal{S}-$**differentiable manifold modeled on** $\mathcal{S}'_n$ **of support** S **and Satlas** Φ*, if Φ is an $\mathcal{S}-$differentiable structure.*

Remark 2.2. Obviously if Φ is an $\mathcal{S}-$atlas on S then $(S, \langle \Phi \rangle)$ is an $\mathcal{S}-$differentiable manifold.

Definition 3.1 (of $\mathcal{S}-$diffeomorphism). Let $\Omega, A \in \sigma\,(\mathcal{S}'_n, \mathcal{S}_n)$, and $f : \Omega \to A$. f is called $\mathcal{S}-$**diffeomorphism** if

i) $f \in \mathrm{Homeo}\,(\Omega, A)$;

ii) f and f^- are $\mathcal{S}-$differentiable.

The set of all the $\mathcal{S}-$ diffeomorphisms from Ω to A is denoted by $\mathcal{S} - \mathrm{Diff}\,(\Omega, A)$.

Example 2.3. Let $I : \mathcal{S}'_n \to \mathcal{S}'_n$ be the identic map on $\mathcal{S}'_n$, then

$$\langle \{I\} \rangle = \bigcup \{\mathcal{S}\,\mathrm{Diff}\,(\Omega, A) : \Omega, A \in \sigma\,(\mathcal{S}'_n, \mathcal{S}'_n)\}\,.$$

Moreover, since $\forall e \in \mathcal{S}_{\mathrm{inv}}\left(\mathbb{R}^n, \mathcal{S}'_n\right)$ (a family is called invertible when the associated operator is invertible), $[\cdot \mid e]$ is an $\mathcal{S}-$linear operator (it's the transpose of the linear and continuous operator $(\hat{e})^-$), it's easy to see that $[\cdot \mid e] \in \langle \{I\} \rangle$.

Example 2.4. An open set in $\mathcal{S}'_n$, say A, is a $\mathcal{S}-$**differentiable manifold modeled on** $\mathcal{S}'_n$ with atlas $\langle \{I_A\} \rangle$ where I_A is the identic map on A.

Example 2.5. Let $f : U \subseteq \mathcal{S}'_n \to \mathbb{R}$ be an $\mathcal{S}-$**differentiable function,** its graph is an $\mathcal{S}-$**manifold** with atlas $\langle \{F\} \rangle$, where $F : gr(f) \to \mathcal{S}'_n : (u, y) \mapsto u$ (the differentiability implies in this case the existence of a tangent vector).

Remark. (what is new in the $\mathcal{S}-$ manifolds ?). The peculiarity of $\mathcal{S}-$manifolds, that makes the difference with the usual manifolds modeled on the space of tempered distributions, is obviously the fact that the differentials of the transition maps are $\mathcal{S}-$linear operators, this peculiarity allow us to define in the tangent spaces of this manifolds the new operation of $\mathcal{S}-$linear superposition, and so to transfer locally on the manifolds the new structures of the $\mathcal{S}-$linear algebra, in addition to the usual linear structures already existent on the tangent spaces.

§3. The tangent space at a point of an $\mathcal{S}-$manifold

Definition 3.1 (of tangent vector). Let (X, Φ) be an $\mathcal{S}-$ -manifold, $x_0 \in X$ and

$$\Phi_{x_0} = \{\varphi \in \Phi : x_0 \in \mathrm{dom}\,\varphi\}\,,$$

be the set of all the charts at x_0 . A tangent vector at x_0 , say v , is a map

$$v : \Phi_{x_0} \to \mathcal{S}'_n$$

such that, for every φ , $\psi \in \Phi_{x_0}$ one has

$$v(\psi) = d_{\varphi(x_0)}\left(\psi \circ \varphi^-\right)(v(\varphi)).$$

Remark. We underline that, although the definition of tangent vector is the classical one, in the $\mathcal{S}-$manifolds the tangent vectors inherit all the characteristic structures introduced in the space of tempered distributions ([2]), as we shall see.

First of all we note that in the above definition we can write

$$\begin{aligned} v(\psi) &= d_{\varphi(x_0)}(\psi \circ \varphi^-)(v(\varphi)) = \partial_{v(\varphi)}(\psi \circ \varphi^-)(\varphi(x_0)) = \\ &= \int_{\mathbb{R}^m} v(\varphi)\left(\partial_i(\psi \circ \varphi^-)(\varphi(x_0))\right)_{i \in \mathbb{R}^n}. \end{aligned}$$

Definition 3.2. (of tangent space). Let (X, Φ) be an $\mathcal{S}-$ -manifold and $x \in X$. The tangent space at $x \in X$ is the set of all tangent vectors at x . It is denoted by

$$T_x(X, \Phi).$$

We define addition and scalar multiplication on $T_x(X, \Phi)$ as follows: let v , $w \in T_x(X, \Phi)$ and let $\lambda \in \mathbb{R}$, then define

$$(v + w)(\varphi) = v(\varphi) + w(\varphi)$$

and

$$(\lambda v)(\varphi) = \lambda v(\varphi),$$

for every $\varphi \in \Phi_x$.

Now we can introduce the first tool that is proper of the $\mathcal{S}-$manifolds and not of a generic differential manifold: in the tangent space at a point it is possible to define the superposition of certain families of tangent vectors indexed by a non-denumerable set:

Definition. Let $a \in \mathcal{S}'_m$ and let $v = (v_i)_{i \in \mathbb{R}^m}$ be a family of tangent vectors such that $(v_i(\varphi))_{i \in \mathbb{R}^m}$ is an $\mathcal{S}-$ -family for every $\varphi \in \Phi_x$, we call such a family $\mathcal{S}-$family of tangent vectors. We define the superposition of v under a, denoted by $\int_{\mathbb{R}^m} av$ the map

$$\int_{\mathbb{R}^m} av : \Phi_x \to \mathcal{S}'_n,$$

such that, for every $\varphi \in \Phi_x$, we have

$$\int_{\mathbb{R}^m} av(\varphi) = \int_{\mathbb{R}^m} a(v_i(\varphi))_{i \in \mathbb{R}^m}.$$

We must show that $v + w$, λv and $\int_{\mathbb{R}^m} av$ are actually tangent vectors.

Indeed, we have

$$v_\psi = d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(v(\varphi))$$

and

$$w_\psi = d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(w(\varphi))$$

so

$$\begin{aligned}(v+\lambda w)_\psi &= v_\psi + \lambda w_\psi = \\ &= d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(v(\varphi)) + d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(\lambda w(\varphi)) = \\ &= d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(v_\varphi + \lambda w_\varphi) = d_{\varphi(x)}\left(\psi \circ \varphi^-\right)((v+\lambda w)(\varphi)).\end{aligned}$$

Concerning the superposition we have, thanks to the $\mathcal{S}-$linearity of $d_{\varphi(x)}(\psi \circ \varphi^-)$,

$$\begin{aligned}\left(\int_{\mathbb{R}^m} av\right)_\psi &= \int_{\mathbb{R}^m} av_\psi = \int_{\mathbb{R}^m} a d_{\varphi(x)}\left(\psi \circ \varphi^-\right)(v_\varphi) = \\ &= d_{\varphi(x)}\left(\psi \circ \varphi^-\right)\left(\int_{\mathbb{R}^m} av_\varphi\right) = d_{\varphi(x)}\left(\psi \circ \varphi^-\right)\left(\left(\int_{\mathbb{R}^m} av\right)_\varphi\right).\end{aligned}$$

Remark. So in the tangent space of an $\mathcal{S}-$manifold, in addition to the standard algebraic operations of addition and scalar multiplication, it's possible to define a superposition operation as in $\mathcal{S}'_n$. Moreover, we have the following

Theorem. 3.1. *The set $T_x(X,\Phi)$ is a vector space with respect to the above operations of addition and scalar multiplication. Moreover, let $\varphi \in \Phi_x$ and $L_{(x,\varphi)}$ the map*

$$L_{(x,\varphi)} : T_x(X,\Phi) \to \mathcal{S}'_n$$

defined by

$$L_{(x,\varphi)}(v) = v(\varphi),$$

then $L_{(x,\varphi)}$ is a linear isomorphism and it is $\mathcal{S}-$linear in the sense that

$$L_{(x,\varphi)}\left(\int_{\mathbb{R}^m} av\right) = \int_{\mathbb{R}^m} aL_{(x,\varphi)}(v).$$

Proof. We prove only the $\mathcal{S}-$linearity:

$$\begin{aligned}L_{(x,\varphi)}\left(\int_{\mathbb{R}^m} av\right) &= \left(\int_{\mathbb{R}^m} av\right)(\varphi) = \\ &= \int_{\mathbb{R}^m} av(\varphi) = \int_{\mathbb{R}^m} aL_{(x,\varphi)}(v). \qquad \square\end{aligned}$$

So $T_x(X,\Phi)$ is endowed not only with a linear structure but also with an "$\mathcal{S}-$structure" that is "$\mathcal{S}-$isomorphic" to that of $\mathcal{S}'_n$, according to the following definition.

Definition 3.3. (of $\mathcal{S}-$structure). Let X be a non empty set and let, for every positive integer m ,S_m be a set of families of X indexed by $\mathbb{R}^m$ and let

$$s : \cup_{m\in\mathbb{N}}(\mathcal{S}'_m \times S_m) \to X$$

be a map. We say that the pair $((S_m)_{m\in\mathbb{N}}, s)$ is an $\mathcal{S}-$structure of dimension n on X if there exists a bijection L from X to $\mathcal{S}'_n$ such that, setting $L(v) = (L(v_k))_{k\in\mathbb{R}^m}$ for every family v in X indexed by $\mathbb{R}^m$ and for every m , we have

$$L(S_m) = \mathcal{S} - (\mathbb{R}^m, \mathcal{S}'_n),$$

and moreover

$$L(s(a,v)) = \int_{\mathbb{R}^m} aL(v).$$

In this condition, setting $0_X := L(0)$, $v \in S_n$ is said to be an $\mathcal{S}-$basis if for every $x \in X$ one has

$$x = s(a,v)$$

for some $a \in \mathcal{S}'_n$ and if $s(a,v) = 0_X$ implies $a = 0_{\mathcal{S}'_n}$.

Now, we see that $T_x(X,\Phi)$ is endowed with an $\mathcal{S}$-basis as $\mathcal{S}'_n$.

Theorem 3.2. (of existence of an $\mathcal{S}-$-basis). *Let $k \in \mathbb{R}^n$ put*

$$\left.\frac{\partial}{\partial(\varphi,k)}\right|_x = \left(L_{(x,\varphi)}\right)^{-1}(\delta_k).$$

Then $\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n}$ is an $\mathcal{S}$-basis of $T_x(X,\Phi)$.

Proof. Let $v \in T_x(X,\Phi)$ and let $z = L_{(x,\varphi)}(v)$, one has

$$\begin{aligned} v &= \left(L_{(x,\varphi)}\right)^{-1}(z) = \left(L_{(x,\varphi)}\right)^{-1}\left(\int_{\mathbb{R}^n} z\delta\right) = \\ &= \int_{\mathbb{R}^n} z\left(L_{(x,\varphi)}\right)^{-1}(\delta) = \int_{\mathbb{R}^n} z\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n} = \\ &= \int_{\mathbb{R}^n} L_{(x,\varphi)}(v)\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n} \end{aligned}$$

so, the family $\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^m}$ is such that every $v \in T_x(X,\Phi)$ is an $\mathcal{S}-$combination of v, under the system of coefficients $L_{(x,\varphi)}(v)$. Moreover, if

$$\int_{\mathbb{R}^n} a\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}_n} = 0_{T_x(X,\Phi)},$$

one has

$$L_{(x,\varphi)}\left(\int_{\mathbb{R}^n} a\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n}\right) = L_{(x,\varphi)}\left(0_{T_x(X,\Phi)}\right),$$

that is, by the $\mathcal{S}-$linearity of $L_{(x,\varphi)}$,

$$\int_{\mathbb{R}^n} aL_{(x,\varphi)}\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n} = 0_{\mathcal{S}'_n},$$

finally, because $L_{(x,\varphi)}\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right) = \delta_k$, we get $\int_{\mathbb{R}^n} a\delta = 0_{\mathcal{S}'_n}$. Hence, because δ is $\mathcal{S}-$linearly independent, one has $a = 0_{\mathcal{S}'_n}$ and so also $\left(\left.\frac{\partial}{\partial(\varphi,k)}\right|_x\right)_{k\in\mathbb{R}^n}$ is $\mathcal{S}-$linearly independent and thus an $\mathcal{S}-$basis of $T_x(X,\Phi)$. □

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