On the classification of certain curves up to projective tranformations

Mehdi Nadjafikhah

Abstract

The purpose of this paper is to classify the curves in the form $y^3 = c_3x^3+c_2x^2+c_1x+c_0$, with $c_3 \neq 0$, up to projective transformations, and then show that, in the regular case, the necessary and sufficient condition for the two curves $y^3 = c_3x^3+c_2x^2+c_1x+c_0$ and $y^3 = \bar{c}_3x^3+\bar{c}_2x^2+\bar{c}_1x+\bar{c}_0$ with $c_3\bar{c}_3 \neq 0$, are equivalece to a projective transformation is that

$$\frac{27c_0c_3^2 - 9c_1c_3|c_2| + 2|c_2|^3}{(3c_1c_3 - c_2^2)^3} = \frac{27\bar{c}_0\bar{c}_3^2 - 9\bar{c}_1\bar{c}_3|\bar{c}_2| + 2|\bar{c}_2|^3}{(3\bar{c}_1\bar{c}_3 - \bar{c}_2^2)^3}.$$

Furthermore, several special cases are considered.

M.S.C. 2000: 53C10.
Key words: Lie transformation theory, equivalence of submanifolds, symmetry, differential equations.

§1. Introduction

Let $\mathbb{R}^2$ have the standard structure with the identity chart (x, y) and $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$ are the standard vector fields on it. Let

$$\mathcal{C}\colon\ y^3 = c_3x^3 + c_2x^2 + c_1x + c_0 \tag{1.1}$$

have the induced differentiable structure from $\mathbb{R}^2$. Let

$$P(2,\mathbb{R}) \cong GL(3,\mathbb{R})/\{\lambda I_3 \mid \lambda \neq 0\} \tag{1.2}$$

be the Lie group of real projective transformations in the plan; that is, the transformation in the form

$$(x, y) \mapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}\right) \tag{1.3}$$

such that $a_{ij} \in \mathbb{R}$ and $\det[a_{ij}] \neq 0$. The $\mathcal{L}(P(2,\mathbb{R}))$ Lie algebra of this Lie group is spanned by

$$\text{span}_{\mathbb{R}} \{\partial_x\, , \partial_y, x\partial_x\, , y\partial_x\, , y\partial_y\, , x^2\partial_x + xy\partial_y\, , xy\partial_x + y^2\partial_y\} \tag{1.4}$$

Differential Geometry - Dynamical Systems, Vol.6, 2004, pp. 14-22.

over $\mathbb{R}$. $P(2,\mathbb{R})$ as a Lie group acts on $\mathbb{R}^2$ and so, on $\mathcal{M}$ the set of all sub-manifolds in the form (1.1) of $\mathbb{R}^2$.

The two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in the form (1.1) are said to be equivalence if there exists an element T of $P(2,\mathbb{R})$ as (1.3) such that $T(\mathcal{C}_1)=\mathcal{C}_2$. This relation partition $\mathcal{M}$ into disjoint cossets.

The main idea of this paper is to show that there exists a bijective relation between $\mathcal{M}/\mathrm{P}(2,\mathbb{R})$ and $\mathbb{R}^* := \mathbb{R}\cup\{\infty\}$. Then, it is shown that

Main Theorem.

a) *If $c_3\neq 0$ and $3c_1c_3-c_2^2\neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3\neq 0$, $3\bar{c}_1\bar{c}_3-\bar{c}_2^2\neq 0$ and*

$$\frac{27c_0c_3^2-9c_1c_3|c_2|+2|c_2|^3}{(3c_1c_3-c_2^2)^3}=\frac{27\bar{c}_0\bar{c}_3^2-9\bar{c}_1\bar{c}_3|\bar{c}_2|+2|\bar{c}_2|^3}{(3\bar{c}_1\bar{c}_3-\bar{c}_2^2)^3}.$$

b) *If $c_3\neq 0$ and $3c_1c_3-c_2^2=0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3\neq 0$ and $3\bar{c}_1\bar{c}_3-\bar{c}_2^2=0$.*

c) *If $c_3=0$ and $c_2\neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3=0$ and $\bar{c}_2\neq 0$.*

d) *If $c_3=c_2=0$ and $c_1\neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3=\bar{c}_2=0$ and $\bar{c}_1\neq 0$.*

e) *If $c_3=c_2=c_1=0$ and $c_0\neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3=\bar{c}_2=\bar{c}_1=0$ and $\bar{c}_0\neq 0$.*

f) *If $c_3=c_2=c_1=c_0=0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3=\bar{c}_2=\bar{c}_1=\bar{c}_0=0$.*

We done this by studying the structure of symmetries of the equations of an special differential equation which $\mathcal{M}$ is it's solution.

§2. Forming the problem

In this section, we convert the given problem, which in fact, is a classification problem of special submanifolds of the Cartesian plane R^2 up to projective transformations, to the problem of classifying the solutions of a special differential equation. By calculating the fourth order derivative of the equation (1.1), we find that

$$\mathcal{E}\;:\;6y'^2y''+3yy''^2+4yy'y^{(3)}+y^2y^{(4)}=0 \tag{2.1}$$

which is an ordinary differential equation. Solving this equation, it turn out that $\frac{d^3}{dx^3}(y^3)=0$, and then, y^3 is a third order polynomial in x, and belong to $\mathcal{M}$. Therefore, $\mathcal{M}$ is just the set of solution of the equation $\mathcal{E}$. Hence, the problem of classifying the integral curves of the equation (1.1) up to projective transformation, is equivalent to the problem of classifying the curves in the form (1.1) up to projective transformation. That is, the problem of classifying the $P(2,\mathbb{R})-$invariant solutions of the equation $\mathcal{E}$.

§3. Symmetry group of the equation $\mathcal{E}$

In order to find the symmetries of the differential equation $\mathcal{E}$, we use the method, which is described in the page 104 of [4]. Let G be the symmetry group of the equation $\mathcal{E}$, and $\mathcal{G}:=\mathcal{L}(G)$ be it's Lie algebra. Assume $X:=\xi(x,y)\partial_x+\eta(x,y)\partial_y$ be an arbitrary element of $\mathcal{G}$, and prolong it to the $\mathcal{G}^{(4)}$, which is acting on the $\mathcal{E}\subset J^4\mathbb{R}^2$. We find that:

$$\begin{aligned}
&-360\Big(y^2\xi_{yyyy}+8y\xi_{yyy}+12\xi_{yy}\Big)(y')^5\\
&+24\Big(12\eta_{yy}-24\xi_{xy}+8y\eta_{yyy}-24\xi_{xyy}+y^2\eta_{yyyy}-4y^2\xi_{xyyy}\Big)(y')^4\\
&-12\Big(5(y^2\xi_{yyy}+6y\xi_{yy}+6\xi_y)y''+6\xi_{xx}-12\eta_{xy}\\
&\qquad+12\xi_{xxy}\eta_{xyyy}-12y\eta_{xyy}-y^2+3y^2\xi_{xxyy}\Big)(y')^3\\
&-4\Big(5(4y\xi_y+y^2\xi_{yy})y^{(3)}+3(8\xi_x-6\eta_y+16y\xi_{xy}\\
&\qquad-6y\eta_{yy}-y^2\eta_{yyy}+4y^2\xi_{xyy})y''-(6\eta_{xx}+12y\eta_{xxy}\\
&\qquad-4y\xi_{xxx}-12y^2\xi_{xxy}+3y^2\eta_{xxyy})\Big)(y')^2\\
&+\Big((-5y^2\xi_y)y^{(4)}+4(2\eta-8y\xi_x+4\eta_y+y^2\eta_{yy}-4y^2\xi_{xy})y^{(3)}\\
&\qquad+30y(y\xi_{yy}+4\xi_y)(y'')^2+6(4\eta_x+8y\eta_{xy}-6y\xi_{xx}\\
&\qquad-3y^2\xi_{xxy}+2y^2\eta_{xyy})y''+y(8\eta_{xxx}+4y\eta_{xxy}-y\xi_{xxxx})\Big)y'\\
&+\Big(y(2\eta+y\eta_y-4y\xi_x)y^{(4)}+2y((4\eta_x-3y\xi_{xx}+2y\eta_{xx})\\
&\qquad-5y\xi_yy'')y^{(3)}+6(4y\eta_y-8y\xi_x+2\eta+y^2\eta_{yy}-4y^2\xi_{xy})(y'')^2\\
&\qquad+2y(6\eta_{xx}+3y\eta_{xxy}-2\xi_{xxx})y''+y^2\eta_{xxxx}\Big)=0
\end{aligned}\tag{3.1}$$

Taking in to account that y', y'', $y^{(3)}$ and $y^{(4)}$ are independent in the fourth order jet space $J^4\mathbb{R}^2$ of $\mathbb{R}^2$, we find a system of partial differential equations from the equation (3.2), such as

$$y^2\xi_{yyyy}+8y\xi_{yyy}+12\xi_{yy}=0\tag{3.2}$$

$$12\eta_{yy}-24\xi_{xy}+8y\eta_{yyy}-24\xi_{xyy}y^2+\eta_{yyyy}-4y^2\xi_{xyyy}=0\tag{3.3}$$

$$6\xi_{xx}-12\eta_{xy}+12\xi_{xxy}-12y\eta_{xyy}-2y^2\eta_{xyyy}+3y^2\xi_{xxyy}=0\tag{3.4}$$

$$y^2\xi_{yyy}+6y\xi_{yy}+6\xi_y=0\tag{3.5}$$

$$6\eta_{xx} + 12y\eta_{xxy} - 4y\xi_{xxx} - 12y^2\xi_{xxy} + 3y^2\eta_{xxyy} = 0 \tag{3.6}$$
$$8\xi_x - 6\eta_y + 16y\xi_{xy} - 6y\eta_{yy} - y^2\eta_{yyy} + 4y^2\xi_{xyy} = 0 \tag{3.7}$$
$$4\xi_y + y\xi_{yy} = 0 \tag{3.8}$$
$$8\eta_{xxx} + 4y\eta_{xxy} - y\xi_{xxxx} = 0 \tag{3.9}$$
$$4\eta_x + 8y\eta_{xy} - 6y\xi_{xx} - 3y^2\xi_{xxy} + 2y^2\eta_{xyy} = 0 \tag{3.10}$$
$$y\xi_{yy} + 4\xi_y = 0 \tag{3.11}$$
$$2\eta - 8y\xi_x + 4\eta_y + y^2\eta_{yy} - 4y^2\xi_{xy} = 0 \tag{3.12}$$
$$\xi_y = 0 \tag{3.13}$$
$$2\eta + y\eta_y - 4y\xi_x = 0 \tag{3.14}$$
$$4\eta_x - 3y\xi_{xx} + 2y\eta_{xx} = 0 \tag{3.15}$$
$$6\eta_{xx} + 3y\eta_{xxy} - 2\xi_{xxx} = 0 \tag{3.16}$$
$$4y\eta_y - 8y\xi_x + 2\eta + y^2\eta_{yy} - 4y^2\xi_{xy} = 0 \tag{3.17}$$
$$\eta_{xxxx} = 0 \tag{3.18}$$

By solving the equation (3.13) we conclude that $\xi = f(x)$ is a function of x. Then, the equations (3.5), (3.8) and (3.11) are automatically satisfied, and provide no advantages. Now, by solving the equation (3.14) we find that there exists a function g such that $\eta = \frac{4}{3}yf'(x) + \frac{1}{y^2}g(x)$. Using the equation (3.4), it turn out that $f''(x) = 0$, or $f(x) = Ax + B$ for arbitrary numbers A and B. Further, by the equation (3.18) it is conclude that $g^{(4)}(x) = 0$, or $g(x) = Cx^3 + Dx^2 + Ex + F$ for the arbitrary constants C, D, E and F. Furthermore, the other equations (3.2) to (3.18) are satisfied, and the this system are consistent. Therefore, The most general vector field $X \in G$ is in the form

$$X = \left(a_1x^2 + a_2x + a_3\right)\partial_x + \left(a_4 + a_1xy + \frac{a_5x^3 + a_6x^2 + a_7x + a_8}{y^2}\right)\partial_y$$

for the arbitrary constants a_i, $i = 1, \cdots, 8$. In this manner, we have

Theorem 1. *The Lie algebra of symmetry group of the equation $\mathcal{E}$ is an 8 dimensional $\mathbb{R}-$Lie algebra which is spanned by the vector fields*

$$\begin{array}{lll} X_1 := \partial_x & X_2 := x\partial_x & X_3 := y\partial_y \\ X_4 := x^2\partial_x + xy\partial_y & X_5 := \dfrac{1}{y^2}\partial_y & X_6 := \dfrac{x}{y^2}\partial_y \\ X_7 := \dfrac{x^2}{y^2}\partial_y & X_8 := \dfrac{x^3}{y^2}\partial_y & \end{array}$$

and have the commutator table

(3.19)

	X_1	X_2	X_3	X_4
X_1	0	X_1	0	$2X_2+X_3$
X_2	$-X_1$	0	0	X_4
X_3	0	0	0	0
X_4	$-2X_2-X_3$	$-X_4$	0	0
X_5	0	0	$3X_5$	$3X_6$
X_6	$-X_5$	$-X_6$	$3X_6$	$2X_7$
X_7	$-2X_6$	$-2X_7$	$3X_7$	X_8
X_8	$-3X_7$	$-3X_8$	$3X_8$	0

	X_5	X_6	X_7	X_8
X_1	0	X_5	$2X_6$	$3X_7$
X_2	0	X_6	$2X_7$	$3X_8$
X_3	$-3X_5$	$-3X_6$	$-3X_7$	$-3X_8$
X_4	$-3X_6$	$-2X_7$	$-X_8$	0
X_5	0	0	0	0
X_6	0	0	0	0
X_7	0	0	0	0
X_8	0	0	0	0

Theorem 2. *Let* $\mathcal{A} := \text{span}_{\mathbb{R}}\Big\{X_1, X_2, X_3\Big\}$ *and* $\mathcal{B} := \text{span}_{\mathbb{R}}\Big\{X_4, X_5, X_6, X_7, X_8\Big\}$, *then*

(a) *$\mathcal{A}$ is a Lie subalgebra of $\mathcal{G}$, and at the same time the Lie algebra of $\mathcal{L}(P(2,\mathbb{R}))$.*

(b) *The connected Lie subgroup A of $P(2,\mathbb{R})$ corresponding to $\mathcal{A}$, is consist of the transformation in the form*

$$TA_{\alpha_1,\alpha_2,\alpha_3} : (x,y) \mapsto \Big(\alpha_1 + \alpha_2 x, \alpha_3 y\Big) \tag{3.20}$$

where α_1, α_2 and α_3 are arbitrary constants with $\alpha_2\alpha_3 \neq 0$.

(c) *$\mathcal{B}$ is an ideal of $\mathcal{G}$ which is the semi-product of $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{G} = \mathcal{A} \propto_s \mathcal{B}$.*

(d) *$\mathcal{G} \cap \mathcal{L}(P(2,\mathbb{R})) = \mathcal{A}$.*

(e) *The connected Lie subgroup B of $P(2,\mathbb{R})$ corresponding to $\mathcal{B}$, is consist of the transformation in the form*

$$TB_{\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8} : (x,y) \mapsto \Bigg(x + \frac{x}{1-x\alpha_4}, \frac{y}{1-x\alpha_4} + \sqrt[3]{y^3 + 3t(\alpha_5 x^3 + \alpha_6 x^2 + \alpha_7 x + \alpha_8)}\Bigg) \tag{3.21}$$

where $\alpha_4, \cdots \alpha_8$ arbitrary constants, and α_4 is small enough.

Proof: By the table (3.19), we have $[\mathcal{A},\mathcal{A}]\subset\mathcal{A}$, $[\mathcal{G},\mathcal{B}]\subset\mathcal{B}$ and $\mathcal{G}=\mathcal{A}\oplus\mathcal{B}$ as vector spaces. Therefore, (a) and (c) are satisfied. In order to proving (b), it is enough to find the integral curves of the each vector fields $\tilde{X}_1:=\partial_{\tilde{x}}$, $\tilde{X}_2:=\tilde{x}\partial_{\tilde{x}}$ and $\tilde{X}_3:=\tilde{y}\partial_{\tilde{y}}$ on the manifold $\{(\tilde{x},\tilde{y})\,|,\tilde{x},\tilde{y}\in\mathbb{R}\}$, which are initialized in the point (x,y). We find out that these are $(x,y)\mapsto(x+t,y)$, $(x,y)\mapsto(e^tx,y)$ and $(x,y)\mapsto(x,e^ty)$, respectively. Because the Lie bracket of the each X_i's with the generators (1.4) is not belongs to $\mathcal{L}(P(2,\mathbb{R}))$, (d) is also satisfied. For (e), we proceed as (b) and find the integral curves corresponding to X_4, X_5, X_6, X_7 and X_8, respectively. Therefore, we have

$$\begin{array}{ll}(x,y)\mapsto\left(\dfrac{x}{1-xt},\dfrac{y}{1-xt}\right) & (x,y)\mapsto\left(x,\sqrt[3]{y^3+3t}\right)\\ (x,y)\mapsto\left(x,\sqrt[3]{y^3+3tx}\right) & (x,y)\mapsto\left(x,\sqrt[3]{y^3+3tx^2}\right)\\ (x,y)\mapsto\left(x,\sqrt[3]{y^3+3tx^3}\right) & \end{array}$$

and proof is complete. □

§4. The action on the $\mathcal{M}$

In this section we describe the action of G, and then the Lie group $P(2,\mathbb{R})$ on the manifold $\mathbb{R}^2$, and then prolong it to an action on $\mathcal{M}$. First, we need a chart $\varphi:\mathcal{M}\to\mathbb{R}^4$ on $\mathcal{M}$ such as $\varphi(\{y^3=c_3x^3+c_2x^2+c_1x+c_0\})=(c_3,c_2,c_1,c_0)$. Then, we describe the action of G on the manifold $\mathcal{M}$. We have

Theorem 3. *If $\varphi(\mathcal{C})=(c_3,c_2,c_1,c_0)$ and $\alpha_1,\cdots,\alpha_8$ are arbitrary numbers and α_4 be small as enough, then*

$$\begin{aligned}
TA_{\alpha_1,0,0}(\mathcal{C}) &\mapsto \varphi\Big(c_3\,,\,3\alpha_1^2c_3+2\alpha_1c_2+c_1\,,\,3\alpha_1c_3+c_2\,,\\
&\qquad\alpha_1^3c_3+\alpha_1^2c_2+\alpha_1c_1+c_0\Big)\\
TA_{0,\alpha_2,0}(\mathcal{C}) &\mapsto \varphi\Big(\alpha_2^3c_3\,,\,\alpha_2^2c_2\,,\,\alpha_2c_1\,,\,c_0\Big)\\
TA_{0,0,\alpha_3}(\mathcal{C}) &\mapsto \varphi\Big(\alpha_3^{-1}c_3\,,\,\alpha_3^{-1}c_2\,,\,\alpha_3^{-1}c_1\,,\,c_0\Big)\\
TB_{\alpha_4,0,0,0,0}(\mathcal{C}) &\mapsto \varphi\Big(6c_3-6\alpha_4c_2+6\alpha_4^2c_1-6\alpha_4^3c_0\,,\,2c_2\\
&\qquad-4\alpha_4c_1+6\alpha_4c_0\,,\,-3\alpha_4c_0+c_1\,,\,c_0\Big)\\
TB_{0,\alpha_5,0,0,0}(\mathcal{C}) &\mapsto \varphi\Big(6c_3\,,\,2c_2\,,\,c_1\,,\,c_0-2\alpha_5\Big)\\
TB_{0,0,\alpha_6,0,0}(\mathcal{C}) &\mapsto \varphi\Big(6c_3\,,\,2c_2\,,\,c_1-3\alpha_6\,,\,c_0\Big)\\
TB_{0,0,0,\alpha_7,0}(\mathcal{C}) &\mapsto \varphi\Big(6c_3\,,\,2c_2-6\alpha_7\,,\,c_1\,,\,c_0\Big)\\
TB_{0,0,0,0,\alpha_8}(\mathcal{C}) &\mapsto \varphi\Big(6c_3-18\alpha_8\,,\,2c_2\,,\,c_1\,,\,c_0\Big)
\end{aligned} \tag{4.1}$$

and G is generated by the 8 elements (4.2). Further, The action of G on $\mathcal{M}$ is transitive.

Proof: By applying the formulas (3.20) and (3.22), we find out the first fact. In order to show the second fact, we consider that, if we set $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = -c_0/2$, $\alpha_5 = -c_1/3$, $\alpha_5 = -c_2/6$ and $\alpha_5 = -c_3/18$ in the equations (4.2) and assuming

$$\begin{aligned} T &= TB_{0,0,0,0,\alpha_8} \circ TB_{0,0,0,\alpha_7,0} \circ TB_{0,0,\alpha_6,0,0} \\ &\quad \circ TB_{0,\alpha_5,0,0,0} \circ TB_{\alpha_4,0,0,0,0} \circ TA_{0,0,\alpha_3} \\ &\quad \circ TA_{0,\alpha_2,0} \circ TA_{\alpha_1,0,0} \end{aligned} \tag{4.2}$$

then, we find that $T(\{y^3 = 0\}) = \{y^3 = c_3x^3 + c_2x^2 + c_1x + c_0\}$, and the proof is complete. □

§5. Classification

In this section, with a view to the above facts, we find the invariants of action A on $\mathcal{M}$, to classifying the curves in the form (1.1). For this, we calculate the Killing vector fields of the action of A on $\mathbb{R}^2$, by the infinitesimal method.

Theorem 4. *By the above assumptions, if $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ are the set of Killing vector fields corresponding to A and B respectively, and $\partial_{c_i} := \frac{\partial}{\partial c_i}$, then $\tilde{\mathcal{A}} = \mathrm{span}_{\mathbb{R}}\{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$, $\tilde{\mathcal{B}} = \mathrm{span}_{\mathbb{R}}\{\tilde{X}_4, \tilde{X}_5, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8\}$, and*

$$\begin{array}{ll} \tilde{X}_1 = 3c_3\partial_{c_2} + 2c_2\partial_{c_1} + c_1\partial_{c_0} & \tilde{X}_5 = \partial_{c_0} \\ \tilde{X}_2 = 3c_3\partial_{c_3} + 2c_2\partial_{c_2} + c_1\partial_{c_1} & \tilde{X}_6 = \partial_{c_1} \\ \tilde{X}_3 = c_3\partial_{c_3} + c_2\partial_{c_2} + c_1\partial_{c_1} - c_0\partial_{c_0} & \tilde{X}_7 = \partial_{c_2} \\ \tilde{X}_4 = c_2\partial_{c_3} + 2c_1\partial_{c_2} + 3c_0\partial_{c_1} & \tilde{X}_8 = \partial_{c_3} \end{array}$$

Proof: In accordance with the definition of the such vector fields (page 56 of [3]), if $\tilde{X}$ be the corresponding Killing vector field to X, then $\tilde{X} = \dfrac{d}{dt}\mathcal{F}^X \cdot \{y^3 = c_3x^3 + c_2x^2 + c_1x + c_0\}\Big|_{t=0}$. Where, $\mathcal{F}^X$ is the flow of X. We proceed the computations by the parts (b) and (e) of Theorem 2. □

The next step is to find the invariants of A on $\mathcal{M}$.

Theorem 5. *The action of A on $\mathcal{M}$ has only one functionally independent invariant:*

$$I := \frac{27c_0c_3^2 - 9c_1c_3|c_2| + 2|c_2|^3}{(3c_1c_3 - c_2^2)^3} \tag{5.1}$$

Proof: If $F(c_3, c_2, c_1, c_0)$ be the an arbitrary invariant of the action A on $\mathcal{M}$, then (by the page 62 of [3]) $\tilde{X}_i(F) = 0$, for $i = 1, 2, 3$. In this manner we have a system of partial differential equations:

$$\begin{array}{l} 3c_3F_{c_2} + 2c_2F_{c_1} + c_1F_{c_0} = 0 \\ 3c_3F_{c_3} + 2c_2F_{c_2} + c_1F_{c_1} = 0 \\ c_3F_{c_3} + c_2F_{c_2} + c_1F_{c_1} - c_0F_{c_0} = 0 \end{array}$$

Solving this system, leads that F must be a function of I in (5.1). □

Theorem 6. *If $c_3 \neq 0$ and $3c_1c_3 - c_2^2 \neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3 \neq 0$, $3\bar{c}_1\bar{c}_3 - \bar{c}_2^2 \neq 0$ and*

$$\frac{27c_0c_3^2 - 9c_1c_3|c_2| + 2|c_2|^3}{(3c_1c_3 - c_2^2)^3} = \frac{27\bar{c}_0\bar{c}_3^2 - 9\bar{c}_1\bar{c}_3|\bar{c}_2| + 2|\bar{c}_2|^3}{(3\bar{c}_1\bar{c}_3 - \bar{c}_2^2)^3}. \tag{5.2}$$

Proof: In view that the necessary condition for $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{M}$ be equivalent with respect to the action of A (that is, have a same $A-$orbit), is that $I(\mathcal{C}_1) = I(\mathcal{C}_2)$, the necessity part is proved. For the sufficiency part, we find a transformation $T_{\alpha,\beta,\gamma} \in \mathcal{A}$ such that $T_{\alpha,\beta,\gamma}(\{y^3 = \sum_{i=0}^3 c_ix^i\}) = (\{y^3 = \sum_{i=0}^3 \bar{c}_ix^i\})$. Then, it is necessary that

$$\gamma^3\bar{c}_0 = c_3\alpha^3 + c_2\alpha^2 + c_1\alpha + c_0 \tag{5.3}$$
$$\gamma^3\bar{c}_1 = 3c_3\alpha^2\beta + 2c_2\alpha\beta + c_1\beta \tag{5.4}$$
$$\gamma^3\bar{c}_2 = 3c_3\alpha\beta + c_2\beta^2 \tag{5.5}$$
$$\gamma^3\bar{c}_3 = c_3\beta^3 \tag{5.6}$$

By the (5.3) we have $\gamma = \beta\sqrt[3]{\frac{c_3}{\bar{c}_3}}$. Then, by the (5.5) we conclude that $\alpha = \frac{c_3\bar{c}_2\beta - c_2\bar{c}_3}{3c_3\bar{c}_3}$, and by the (5.4), we have $\beta = \frac{\bar{c}_3}{c_3}\sqrt{\frac{3c_1c_3 - c_2^2}{3\bar{c}_1\bar{c}_3 - \bar{c}_2^2}}$. If in the case $3\bar{c}_1\bar{c}_3 - \bar{c}_2^2 = 0$, then by the equations (5.3) to (5.6), it is necessary that $3c_1c_3 - c_2^2 = 0$, and two denominators are zero. Now, by letting this values in the (5.3), we conclude the equation (5.2). □

We further consider the case $c_3 = 0$, and prove the following

Theorem 7.

a) *If $c_3 = 0$ and $c_2 \neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3 = 0$ and $\bar{c}_2 \neq 0$.*

b) *If $c_3 = c_2 = 0$ and $c_1 \neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3 = \bar{c}_2 = 0$ and $\bar{c}_1 \neq 0$.*

c) *If $c_3 = c_2 = c_1 = 0$ and $c_0 \neq 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3 = \bar{c}_2 = \bar{c}_1 = 0$ and $\bar{c}_0 \neq 0$.*

d) *If $c_3 = c_2 = c_1 = c_0 = 0$, then the necessary and sufficient condition for the two curves $\mathcal{C}$ and $\bar{\mathcal{C}}$ are congruent up to a projective transformation is that $\bar{c}_3 = \bar{c}_2 = \bar{c}_1 = \bar{c}_0 = 0$.*

Proof: In each case, we find a transformation $T_{\alpha,\beta,\gamma} \in \mathcal{A}$ such that $T_{\alpha,\beta,\gamma}(\{y^3 = \sum_{i=0}^3 c_ix^i\}) = (\{y^3 = \sum_{i=0}^3 \bar{c}_ix^i\})$.

In the case (a) we assume that

$$\gamma = \left(\frac{\bar{c}_2}{c_2}\alpha\right)^{2/3} \quad , \quad b = \frac{c_2\bar{c}_1 - c_1\bar{c}_2}{2c_2\bar{c}_2}\alpha \quad , \quad \alpha = \frac{\bar{c}_2}{c_2}\sqrt{\frac{4c_0c_2 - c_1^2}{4\bar{c}_0\bar{c}_2 - \bar{c}_1^2}} \tag{5.7}$$

and the condition $4c_0c_2 - c_1^2 = 0$ is equivalence with $4\bar{c}_0\bar{c}_2 - \bar{c}_1^2$.

In the case (b) we assume that

$$\gamma = \bar{c}_3^3 \quad , \quad b = \bar{c}_0 c_2^2 - \frac{\bar{c}_0}{\bar{c}_1} \quad , \quad \alpha = c_1^2 \bar{c}_1 \tag{5.8}$$

and the condition $c_1 = 0$ is equivalent with $\bar{c}_1$. In the case (c), the two curves are $y = 1$ and, in the case (d), the two curves are $y = 0$. □

Now the main theorem is a conclusion of the Theorems 6 and 7.

References

[1] E. Cartan, *La Theorie des Groups Finis et Continus et la Geometrie differentielle Traitees par la Methode du Repere Mobile*, Cahiers Scientifiques, Vol. 18, Gauthier-Villars, Paris, 1937.

[2] M. Fels, and P. J. Olver, *Moving coframes. I : A practical algorithm*, Acta Appl. Math., Math. 51 (1998) 161-213.

[3] P. J. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge University Press, 1995.

[4] P. J. Olver, *Application of Lie Groups to Differential Equations*,GTM vol. 107, Springer Verlag, New Yourk, 1993.

Author's address:

Mehdi Nadjafikhah
Iran University of Science and Technology,
Faculty of Mathematics,
Department of Pure Mathematics,
Narmak-16, Tehran, Iran
E-mail: m_nadjafikhah@iust.ac.ir