

On some applications of non-standard analysis in geometry

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Abstract

This paper is a survey on the use of the non-standard methods in the study of certain geometric structures. After a short presentation of the basic notions and principles of Non-Standard Analysis, we recall some of their applications in Geometry - e.g., in the study of Gauss varieties of dimension m with continuous curvature where the use of infinitesimals in problems of local nature is exemplified. Finally, we present the concept of non-standard manifold.

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§1. Principles of Non-Standard Analysis

We use the basic notions, definitions and principles belonging to Non-Standard Analysis ([1], [2], [3], [6], [8], [9]) and the IST approach to Non-Standard Analysis as developed by E. Nelson.

Internal Set Theory (IST) is an axiomatic theory of sets proposed by E. Nelson in 1977 [5]. One considers a classic axiomatization of the theory of sets, for instance ZFC (Zermelo-Fraenkel and Choice) ([10]); we associate to this a language containing in addition to the usual undefined binary predicate " \in " a new undefined unary predicate "standard" (abbreviated "st"). One adds three supplementary axioms: the Transfer Principle (T.P.), the Idealization Principle (I.P.) and the Standardization Principle (S.P.) Nelson proved that IST is a conservative extension of ZFC (i.e., it is non-contradictory - as far as ZFC is non contradictory, and any formula from ZFC is true in IST iff it is true in ZFC).

A formula in IST is called *internal* if it does not contain the predicate *st* and any of its derivatives; otherwise it is called *external*. A set defined by an internal formula is called *internal set*; if it is defined by an external formula it is called *external set*.

Analogously, we can define similar concepts for functions, and relations (which are also sets).

A formula in IST is called *standard* if it is internal, its constants are elements in ZFC and the variables run over sets from ZFC; the other internal formulas are

called *non-standard*. As before, one defines *standard* and *non-standard* sets (functions, relations, etc.)

Only the internal sets can be standard or non-standard.

THE TRANSFER PRINCIPLE (T.P.): For any *standard* formula $F(y)$, we have:

$$\forall x F(x) \Leftrightarrow \forall^{st} x F(x)$$

(or, equivalently, with \exists instead \forall). Here $\forall^{st} x$ means $(\forall x st(x) \wedge)$ and $\exists^{st} x$ means $(\exists x st(x) \rightarrow)$.

THE IDEALIZATION PRINCIPLE (I.P.): For any *internal* formula $B(x, y)$, we have

$$[\forall^{st} Y, Y \text{ finite} \Rightarrow \exists x \forall y \in Y B(x, y)] \Leftrightarrow [\exists x \forall^{st} y B(x, y)].$$

THE STANDARDIZATION PRINCIPLE (S.P.): For *any* formula $F(x)$ (internal or external) we have:

$$\forall^{st} E \exists^{st} S_F \forall^{st} x [x \in S_F \Leftrightarrow x \in E \wedge F(x)].$$

If E is *any* set we call the *shadow* (or the *standardization*) of E the unique standard set having as standard elements the standard elements of E (cf. S.P.); we denote this with ${}^\circ E$ (or ${}^S E$).

In IST the following Rules work:

Rule 1: All the objects of the classical mathematics are standard.

Rule 2: All the classical theorems remain true (so it is not necessary to re-prove them).

Rule 3: Any new theorem proved using the IST Principles which has an internal statement (i.e., can be expressed in classical terms) is automatically true in ZFC.

Using the (T.P.) any standard (usual) statement expressible in a first-order language \mathcal{L} is transferred to a statement (similar) in IST. If A is a set (or a function f) in ZFC we denote by $*A$ (resp. $*f$) the corresponding transferred sets. The same holds true for constants.

If $k \in \mathbf{N}$ (the set of naturals) then $*k \in *\mathbf{N}$.

A binary relation satisfying the left side of (I.P.) is called a *concurrent* relation. So (I.P.) may be used for such binary relations.

Examples: 1. $B(x, y) : (x \in \mathbf{N}) \wedge (y \in \mathbf{N}) \rightarrow (x \geq y)$.

We infer: $(\exists)x \in *\mathbf{N}$ s.t. $x \geq y$, $(\forall) y \in \mathbf{N}$, hence the existence in $*\mathbf{N}$ of *infinite large* numbers.

(I.P.) If we use the same concurrent relation on \mathbb{R} , we infer the existence of $\omega \in *\mathbb{R}$, an *infinite large* real number. Because the language of fields is a first order-language, (T.P.) implies that $*\mathbb{R}$ is a field (because \mathbb{R} is a field). So, if $\omega_0 \in *\mathbb{R}$ is *i*-large, $\varepsilon_0 = \frac{1}{\omega_0} \in *\mathbb{R}$ has the property

$$|\varepsilon_0| < \varepsilon, (\forall) \varepsilon > 0, \varepsilon \in \mathbb{R}$$

and $\varepsilon_0 \neq 0$. These are the *infinitesimals*.

We remark that the property of \mathbb{R} of being *Archimedean* is *not* expressible in a first-order language, so it cannot be transferred using (T.P.) Any ${}^* \mathbb{R}$ is *non-Archimedean*.

Let $(K, |\cdot|)$ be a field endowed with an absolute value (Archimedean or not). Then $({}^*K, {}^*|\cdot|)$ is a normed field as well.

If one considers

$$B_1(x, y) : (x \in K) \wedge (y \in \mathbf{N}) \rightarrow (|x| \geq y).$$

If K is Archimedean, then B_1 is concurrent, so we get elements $\omega \in {}^*K$ s.t. $|\omega| \geq y$ $(\forall) y \in \mathbf{N}$. So $\varepsilon = \omega^{-1}$ satisfies (1). We define the *halo of 0*

$$hal(0) = hal_K(0) := \{x \in {}^*K \mid |x| < \varepsilon, \quad (\forall)\varepsilon > 0, \varepsilon \in \mathbb{R}\}.$$

If $x_0 \in K$, $hal_K(x_0) := \{x \in {}^*K \mid |x - x_0| < \varepsilon, \forall \varepsilon \in \mathbb{R}, \varepsilon > 0\}$. As examples of such K , we have $K = \mathbf{Q}, \mathbb{R}, \mathbf{C}, \dots$ If K is non-Archimedean (for example $K =$ the field of p -adic numbers) we may do similar constructions using $B_2(x, y) : (x \in K) \wedge (y \in K) \rightarrow (|x| \geq |y|)$.

2. Applying (I.P.) to the relation

$$B(x, y) : x \in E \wedge (x \neq y)$$

we conclude that a set E has only standard elements iff E is finite.

3. If $K = \mathbb{R}, \mathbf{C}$, the (S.P) can be used in order to prove that $(\forall)x \in {}^*K, |x| < \infty$, $(\exists!) {}^\circ x \in {}^*K$, ${}^\circ x$ standard such that $|x - {}^\circ x|$ is infinitesimal; ${}^\circ x$ is called *the shadow of x*.

If K is a field, then $|x|_o = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$, $x \in K$ is a norm (absolute value) on

K . It gives the discrete topology on K , so it is called the *discrete* norm; $(K, |\cdot|_o)$ is called a *discrete* field.

The notions of i -large elements or infinitesimals can be defined in any metric space.

Definition 1.1. Let E, F be metric spaces in IST and $f : D(f) \subset E \rightarrow F$ a function. Let ${}^\circ E$ and ${}^\circ F$ be the standardizations of E and F (use (S.P.)).

A standard function ${}^\circ f : D({}^\circ f) \subset {}^\circ E \rightarrow {}^\circ F$ is called *the shadow of f* if

- 1) $(\forall) (x, f(x)) \in E \times F$ nearly standard, then ${}^\circ x \in D({}^\circ f)$.
- 2) $(\forall) x_0 \in D({}^\circ f)$, x_0 standard, $(\forall) x \in D(f)$, we have:

$$x \simeq x_0 \Rightarrow f(x) \simeq {}^\circ f(x_0).$$

Remarks. 1) We write $x \simeq y$ if $dist(x, y)$ is a positive real infinitesimal.

2. $x \in E$ is called *nearly standard* if $(\exists)y \in E$, y standard s.t. $dist(x, y)$ is a real infinitesimal.

Definition 1.2. Let E, F be two standard metric spaces. An internal application (standard or not) $f : D(f) \subset E \rightarrow F$ is called *S-continuous* in $x_0 \in D(f)$ if, for any $x \in D({}^*f) \subset {}^*E$, we have

$$x \simeq x_0 \Rightarrow {}^*f(x) \simeq {}^*f(x_0).$$

Theorem 1.3. Let E, F, f as before. Then there exists the *shadow* of f , ${}^\circ f : D({}^\circ f) \subset E \rightarrow F$ and ${}^\circ D(f) \subset D({}^\circ f)$ and ${}^\circ f$ is uniformly continuous on ${}^\circ D(f)$ iff f is *S-continuous* on $D(f)$.

This Theorem ([9]) is called the *Continuous shadow Theorem*.

Definition 1.4. 1) Let E, F be standard normed spaces and $f : E \rightarrow F$ an internal function (we may consider $f : D(f) \subset E \rightarrow F$), f is called *S-differentiable* in $x_0 \in D(f)$, with x_0 standard with the differential $L = d_{x_0} f : E \rightarrow F$, L a continuous, linear function, if f is defined on $hal(x_0)$ and $(\forall) X \in E, \|X\| < \infty, \forall \varepsilon \simeq 0, \varepsilon \neq 0, (\forall) x \simeq x_0$, we have

$$\frac{f(x + \varepsilon X) - f(x)}{\varepsilon} \simeq L(X).$$

2. f is *S-differentiable* if it is *S-differentiable* in any $x_0 \in D(f)$ standard such that $f(x_0)$ is nearly standard in F .

Theorem 1.5. (of the differentiable shadow): Let E, F be standard normed spaces and $f : D(f) \subset E \rightarrow F$ an internal *S-differentiable* function, $D(f)$ being a standard open set. Then there exists the shadow ${}^\circ f$ of f defined on a standard open set $D({}^\circ f)$, which has the class C^1 .

Definition 1.6. Let $f : D(f) \subset E \rightarrow F$ (E, F as above) a standard function, and $D(f)$ a standard open set. Let $x_0 \in D(f)$ and $A \subset D(f)$. We say that:

1. f is of class S^1 in x_0 if f is *S-continuous* in x_0 and $(\exists) l \in F$ standard such that

$$\frac{f(x) - f(x')}{\|x - x'\|} \simeq l \quad (\forall) x \simeq x', x' \simeq x_0, x \neq x';$$

2. f is of class S^1 on A if it is of class S^1 in any $x_0 \in A$;
3. f is of class S^k and of class S^∞ , inductively.

Let K be a non-discrete, complete, normed field (e.g., $K = \mathbb{R}, \mathbf{C}$, p -adic numbers, etc.) Let's denote by $\Lambda_n := K\{X_1, \dots, X_n\}$ the ring of convergent series over K . If $f \in \Lambda_n$, we can associate to it an analytic function $\bar{f} : D(f) \rightarrow K$ in a natural way, as shown below:

Theorem 1.7. (A. and O. Pășărescu, [8]). Using the previous notations, suppose that \bar{f} is defined and takes only nearly standard values on $hal(0)$. Then:

- 1) \bar{f} is *S-continuous* and of class S^∞ ;
- 2) there exists ${}^\circ \bar{f}$ (the shadow of \bar{f}), which is analytic over K at the origin (hence of class C^∞);
- 3) there exists the shadow of any partial derivative of \bar{f} of standard order and it coincides with the same partial derivative of ${}^\circ \bar{f}$ (i.e. $(\exists) {}^\circ \left(\frac{\partial^{(m)} \bar{f}}{\partial x^m} \right)$ and ${}^\circ \left(\frac{\partial^{(m)} f}{\partial x^m} \right) = \frac{\partial^{(m)} ({}^\circ \bar{f})}{\partial x^m}$, $(\forall) m \in N^n$, st).

This theorem extends from \mathbf{C} to K^n a Theorem of Robinson-Callot; its proof makes use of the analyticity instead of holomorphy.

As an application, we state the following

Theorem 1.8. [7]: Let $f : \mathbf{C}^n \rightarrow \mathbf{C}$ be an entire function. If f is bounded, then f is constant.

§2. Gauss varieties with continuous curvature

Definition 2.1 ([11]). A set $V \subset \mathbb{R}^n$ is called *Gauss variety of dimension m with continuous curvature* if there is a standard application $\mathcal{T} : V \rightarrow \mathcal{A}_m(\mathbb{R}^n)$ (where $\mathcal{A}_m(\mathbb{R}^n)$ is the set of affine hyperplanes of dimension m from \mathbb{R}^n) satisfying:

- 1) $A \in V \Rightarrow A \in \mathcal{T}(A)$;
- 2) for any nearly standard $A \in {}^*V$, the orthogonal projection from *V to $\mathcal{T}(A)$ associates to the points $B \in {}^*V$ with $B \simeq A$ a point $b \in \mathcal{T}(A)$ such that $b \simeq A$; moreover, these are surjective maps.
- 3) for any two nearly standard points $A, B \in {}^*V$, such that $B \simeq A$ with $B - A = t + n$ (where $t \in \mathcal{T}(A)$ and n is normal to $\mathcal{T}(A)$), then $\frac{\|n\|}{\|B - A\|} \simeq 0$.

Remark. 1) imposes to V standard conditions, and 2), 3) impose to *V two non-standard conditions. V is a standard (usual) object.

Theorem 2.2 ([11]). *If $V \subset \mathbb{R}^n$ is a Gauss variety of dimension m with continuous curvature, then V is a differential variety of dimension m and class C^1 (in usual sense).*

Moreover, the parametrizations are given by the local projections onto the tangent spaces.

For the proof one may use the following:

Theorem 2.3 (Stroyan and Luxemburg [11]). *Let $f : U \rightarrow \mathbb{R}^n$, $U \subset \mathbb{R}^m$ be a standard application. The following are equivalent:*

- i) *There is a standard map $Df : U \rightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that if x is nearly standard and $\delta \in \text{hal}_m^{\mathbb{R}}(0)$, there is $\eta \in \text{hal}_n^{\mathbb{R}}(0)$ such that:*

$$f(x + \delta) - f(x) = Df_x(\delta) + |\delta|\eta;$$

- ii) *For any standard point $a \in U$ there exists a finite internal linear (i.e., having a finite sup norm) map $L_a \in {}^*\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, such that:*

$$x \simeq y \simeq a \Rightarrow (\exists) \eta \in \text{hal}_n^{\mathbb{R}}(0)$$

with

$$f(y) - f(x) \simeq L_a(y - x) + |y - x| \cdot \eta;$$

- iii) *U is open and $f \in C^1(U)$ (f is continuously differentiable on U).*

A particular case.

If $m = 1$ and $V \subset \mathbb{R}^3$ is a Gauss variety of dimension 1 with continuous curvature, we call it *curve*. We suppose that V is locally parametrised and take as parameter the length u along the tangent.

If we consider the usual Frenet trihedron $T, N, B = T \times N$, then we can yield in this case the infinitesimal Frenet formulas:

$$\begin{aligned} T(s + \delta) &\simeq T(s) + \delta|k|N, \\ N(s + \delta) &\simeq N(s) - \delta|k|T + \delta|\tau|B, \\ B(s + \delta) &\simeq B(s) - \delta|\tau|N, \end{aligned}$$

where δ is infinitesimal, and k (respectively τ) is the curvature (resp. torsion). For details, see [11].

§3. Non-standard differential geometric structures on non-standard differential manifolds

Definition 3.1. Let I be a standard finite set, M an arbitrary set, $n \in \mathbf{N}$, n standard and $(U_i)_{i \in I}$ a family of S -open subsets in \mathbb{R}^n .

In addition, consider a family $(\varphi_i)_{i \in I}$ of injective maps $\varphi_i : U_i \rightarrow M$ such that $\bigcup_{i \in I} \varphi_i(U_i) = M$, and $\varphi_i^{-1}(\varphi_i(U_i) \cap \varphi_j(U_j)) := U_{ij}$ is always S -open in \mathbb{R}^n for any $i, j \in I$.

If the transition functions $\varphi_{ji} : U_{ij} \rightarrow U_{ji}$ defined by $\varphi_{ji} = \varphi_j^{-1} \circ \varphi_i$ are always standardizable (i.e. $\forall^{st} p \in U_{ij} \exists_q^{st} q \in U_{ji} q \simeq \varphi_{ji}(p)$, S -continuous and S -differentiable with S -continuous derivative), we call $(U_i, \varphi_i)_{i \in I}$ an S^1 -structure on M and $(M, (U_i, \varphi_i)_{i \in I})$ is called an n -dimensional concrete S^1 -manifold.

The mappings φ_i are called *parametrizations*.

Even in classical differential geometry there is another possibility to define a manifold. Instead of starting with a given set M , one considers the sets U_i and U_{ij} together with the transition functions and some given conditions on them. Taking the quotient of the disjoint union of the U_i with respect to the equivalence relation identifying points which are mapped to each other by a transition function, we get back M .

One of the merits of this approach is that it explicitly shows invariance of differential geometric objects with respect to diffeomorphisms.

We shall now present (following [12]) a non-standard manifold concept, based on this version of the classical definition.

Definition 3.2. Let I be an arbitrary standard set, $n \in \mathbf{N}$, n standard (i.e. $n \in {}^\sigma\mathbf{N}$), $(U_i)_{i \in {}^\sigma I}$, $(U_{ij})_{(i,j) \in {}^\sigma I \times {}^\sigma I}$ external families of S -open subsets of \mathbb{R}^n with $U_{ii} = U_i$ and $U_{ij} \subseteq U_i$ for $i, j \in {}^\sigma I$ (${}^\sigma I$ = standard elements of $*I$).

In addition, we suppose that for $i, j \in {}^\sigma I$ we have always a mapping $\varphi_{ji} : U_{ij} \rightarrow U_{ji}$ which is S -continuous, S -differentiable and has continuous derivative. If φ_{ji} are standardizable and for any $i, j, k \in {}^\sigma I$ we have:

- i) $\forall^{st} p \in U_i, \varphi_{ii}(p) \simeq p$;
- ii) $\forall^{st} p \in U_{ij}, (\varphi_{ij} \circ \varphi_{ji})(p) \simeq p$;
- iii) $\forall^{st} p \in U_{ij} \cap U_{ik}, \varphi_{ji}(p) \in U_{jk} \wedge (\varphi_{kj} \circ \varphi_{ji})(p) \approx \varphi_{ki}(p)$

then we call $(U_{ij}, \varphi_{ij})_{(i,j) \in {}^\sigma I \times {}^\sigma I}$ an n -dimensional abstract S^1 -manifold. Speaking of the φ_{ji} , we will again use the term *transition function*.

Remark. Along the same line we could introduce concrete and abstract S^r -manifolds $r \in {}^\sigma\mathbf{N}$ or S -holomorphic manifolds by replacing \mathbb{R}^n with \mathbf{C}^n and \mathbb{R} -differentiability with \mathbf{C} -differentiability.

If we do not want to delimit these cases we will simply speak of a concrete or an abstract S -manifold.

Definition 3.3. Let $(U_{ij}, \varphi_{ij})_{(i,j) \in {}^\sigma I \times {}^\sigma I}$ be an n -dimensional abstract S^r -manifold $r \in {}^\sigma\mathbf{N}, r \geq 2$) or an S -holomorphic manifold. A $2n$ -dimensional abstract S^{r-1} -manifold or abstract S -holomorphic manifold is called an S^{r-1} or S -holomorphic

tangent bundle (or short: by an *S-tangent bundle*) of $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ if there exists an external family $(\pi_i)_{i \in \sigma I}$ of S^{r-1} or *S*-holomorphic mappings $\pi_i : V_i \rightarrow U_i$ satisfying:

- i) $\forall^{st} i, j V_{ij} = U_{ij} \times \mathbf{K}^n$, where \mathbf{K} denotes \mathbb{R} or \mathbb{C} ;
- ii) $\forall^{st} i, \forall^{st} \xi \in V_i, \pi_i(\xi) \simeq pr_1(\xi)$ where (pr_1 denotes the projection operator onto the first component $U_i \times \mathbf{K}^n$);
- iii) $\forall^{st} i, j \forall^{st} \xi \in V_{ij}, (\pi_j \circ \psi_{ji})(\xi) \simeq (\varphi_{ji} \circ \pi_i)(\xi)$;
- iv) $\forall^{st} i, j \forall^{st} \xi \in V_{ij}, (pr_2 \circ \psi_{ji})(\xi) \simeq D\varphi_{ji}(\pi_i(\xi))pr_2(\xi)$,

with pr_2 the projection operator onto the second component of $U_i \times \mathbf{K}^n$.

Proposition 3.4 ([12]). Every abstract *S*-manifold has an *S*-tangent bundle.

Remark. The tangent bundle determined above is not unique.

Definition 3.5. Let V be a standard Euclidian space over \mathbb{R} .

- i) A limited map $f : V \times V \rightarrow \mathbb{R}$ is called *S-bilinear form* if it satisfies for limited elements of V and limited scalar multiples the axioms of a bilinear map up to infinitesimals;
- ii) f *S*-bilinear is called *S-symmetric* if

$$\forall^{st} \xi \in V \forall^{st} \eta \in V, f(\xi, \eta) \simeq f(\eta, \xi) \text{ holds;}$$

- iii) f *S*-bilinear is called *S-nondegenerate* if

$$\forall^{st} \eta \in V (\forall^{st} \xi \in V f(\xi, \eta) \simeq 0 \Rightarrow \eta \simeq 0)$$

and

$$\forall^{st} \eta \in V (\forall^{st} \xi \in V f(\eta, \xi) \simeq 0 \Rightarrow \eta \simeq 0) \text{ hold;}$$

- iv) f *S*-bilinear is called *S-positive definite* if $\forall^{st} \xi \in V (\xi \not\simeq 0 \Rightarrow f(\xi, \xi) > 0$ and $f(\xi, \xi)$ is not an infinitesimal) holds.

Definition 3.6. Let $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ be an abstract S^r -manifold ($r \geq 2$) of dimension n . An *S^r-pseudo Riemannian metric* on $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ is an external family $(g^i)_{i \in \sigma I}$ of mappings g^i giving for each point $p \in U_i$ a map $\langle \cdot, \cdot \rangle_p^i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that for $p \in {}^\sigma U_i, \langle \cdot, \cdot \rangle_p^i$ is an *S*-bilinear, *S*-symmetric, *S*-nondegenerate form and the following conditions are satisfied:

- (i) For $a, b \in \{1, 2, \dots, n\}$ and e_a, e_b the corresponding unit vectors in $\mathbb{R}^n, g_{ab}^i(p) := \langle e_a, e_b \rangle_p^i$, defines a map of class S^r on U_i ;
- ii) If $(U_{ij} \times \mathbb{R}^n, \psi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ is an *S*-tangent bundle for $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$, we have

$$\forall^{st} i, j \forall^{st} p \in U_{ji}, g_{ab}^i(p) \simeq \langle \psi_{ij}(p, e_a), \psi_{ij}(p, e_b) \rangle_{\varphi_{ij}(p)}^i.$$

Then $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ together with $(g^i)_{i \in \sigma I}$ is called an *S-pseudo - Riemannian manifold*. If, in addition $\langle \cdot, \cdot \rangle_p^i$ is always *S*-positive definite for $p \in {}^\sigma U_i$, we speak of an *S^r-Riemannian metric* and an *S^r-Riemann manifold*, respectively.

Proposition 3.7. Let $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$ be an abstract *S*-manifold, $(g^i)_{i \in \sigma I}$ an *S*-pseudo-Riemannian metric on it. Then $(g^i)_{i \in \sigma I}$ is a (classical) pseudo-Riemannian

metric on the standardization of $(U_{ij}, \varphi_{ij})_{(i,j) \in \sigma I \times \sigma I}$. If $(g^i)_{i \in \sigma I}$ is an S -Riemannian metric, then ${}^\circ(g^i)_{i \in \sigma I}$ is Riemannian. For more details, see [12].

This geometrical approach may be extended to more general structures, like non-standard Finsler spaces, for instance.

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