On the NEM method in thermoelasticity

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Abstract

This paper implements the NEM method in linear thermoelasticity. It presents the equations of the linear thermoelasticity within the frame of a relatively new meshless method, Natural Element Method. The paper discuss the main features of Delauney tesselation and of the Voronoi triangulation in order to implement them in the analysis of the linear thermoelasticity problem.

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§1. Introduction

In the last seven years, meshless methods like: smooth particle hydrodynamics, reproducing kernel particle methods, hp-clouds, natural element method and element free Galerkin method, received more attention. Their main advantage consists in the fact that they don't use a mesh in order to assemble the system of equations. Mesh free methods are of great interest in the study of problems that involve discontinuous fields, such as crack problems or phase changes, and adaptive refinement (T. Belytschko, [2]).

In this paper, we will study the implementation of the Natural Element Method (NEM) in linear thermoelasticity. This notion was proposed by Braun and Sambridge [4] and by Traversoni [21]. The latter called it Natural Neighbour Finite Elements Method. First, NEM was used in modelling geophysical phenomena (Alfred [1], Watson [23], Brown [6], Traversoni [21]). This interpolation method hasn't been used on a large scale, in comparison with other schemes such as Shepard functions (Shepard [17], Lancaster MLS [11]), Hardy's multiquadrics ([9]). The basic idea behind Natural Neighbor Interpolation belongs to computational geometry, more specifically the Voronoi diagram ([22]). Delauney tesselations ([7]) is the topological dual of the Voronoi diagram. In the interpolation process natural neighbor $(n-n)$ coordinates (Braun & al. [5]) are used as the interpolation functions.

The outline of this paper is the following: in Section 2 we will provide the linear thermoelasticity equations; in Section 3 we will present the main features of Voronoi diagrams and Delauney tessellations; in Section 4 we will discuss natural neighbor interpolation techniques and we will discuss the Galerkin-type formulation of the NEM problem; in Section 5 we shall conclude and formulate future possible results.

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§2. Basic equations

Let D be a bounded domain in the two dimensional Euclidian space. Suppose that the domain D is filled by an isotropic and homogenous medium. As in Iesan ([10]) or Prasad ([14]), the basic equations of equilibrium of the linear thermoelasticity are: the equilibrium equations

$$t_{\beta\alpha,\beta} + \rho_0 f_\alpha = 0 \text{ on } D; \tag{2.1}$$

the energy equation

$$q_{\alpha,\alpha} = \text{ - s on } D; \tag{2.2}$$

the constitutive equations

$$t_{\alpha\beta} = 2\mu\varepsilon_{\alpha\beta} + \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} - \beta\theta\delta_{\alpha\beta} \tag{2.3}$$

$$q_\alpha = k\theta_{,\alpha}\,; \tag{2.4}$$

the strain-displacement relations

$$\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} \text{ on } D, \tag{2.5}$$

where, u_α are the components of the displacement vector, $t_{\alpha\beta}$ are the components of the stress tensor, $\varepsilon_{\alpha\beta}$ are the components of the strain tensor, f_α are the components of the specific body force, s is the specific heat supplied, θ is the temperature measured from a constant reference temperature $\theta_0, \lambda, \mu, \beta, k$ are constants, characteristics of the material. We shall attach the following boundary conditions:

$$u_\alpha = \overline{u}_\alpha \text{ on } \Gamma_u,\ t_{\beta\alpha}n_\beta = \overline{t}_\alpha \text{ on } \Gamma_t \tag{2.6}$$

$$\theta = \overline{\theta} \text{ on } \Gamma_\theta,\ q_\alpha n_\alpha = \overline{q} \text{ on } \Gamma_q, \tag{2.7}$$

where $\overline{u}_\alpha, \overline{t}_\alpha, \overline{\theta}, \overline{q}$ are continuous functions given on the specified boundary parts, and

$$\overline{\Gamma}_u \bigcup \Gamma_t = \overline{\Gamma}_\theta \bigcup \Gamma_q = \partial D,\ \Gamma_u \bigcap \Gamma_t = \Gamma_\theta \bigcap \Gamma_q = \Phi.$$

Thus the boundary value problem is to find u_α, θ which satisfy (2.1)-(2.6) and the boundary conditions (2.7) and (2.8). These lead to the following equations:

$$\mu\ u_{\alpha,\beta\beta} + (\lambda+\mu)u_{\beta,\beta\alpha} + \rho_0 f_\alpha = \beta\theta_{,\alpha} \tag{2.8}$$

$$k\theta_{,\alpha\alpha} = -s. \tag{2.9}$$

We consider the transformed boundary conditions attached to these equations

$$u_\alpha = \overline{u}_\alpha \text{ on } \Gamma_u,\ \lambda u_{\gamma,\gamma}n_\alpha + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})n_\beta = \overline{t}_\alpha + \beta\theta\ n_\alpha \quad \text{ on } \Gamma_t \tag{2.10}$$

$$\theta = \overline{\theta} \text{ on } \Gamma_\theta,\ k\theta_{,\alpha}\, n_\alpha = \overline{q} \quad \text{ on } \Gamma_q. \tag{2.11}$$

In the following, we will take $\beta = 0$, and thus we will consider the uncoupled equations

$$\mu\ u_{\alpha,\beta\beta} + (\lambda+\mu)u_{\beta,\beta\alpha} + \rho_0 f_\alpha = 0 \tag{2.12}$$

$$k\theta_{,\alpha\alpha} = -s. \tag{2.13}$$

The attached boundary conditions become

$$u_\alpha = \overline{u}_\alpha \text{ on } \Gamma_u,\ \lambda u_{\gamma,\gamma} n_\alpha + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}) n_\beta = \bar{t}_\alpha \quad \text{on } \Gamma_t \tag{2.14}$$

$$\theta = \overline{\theta} \text{ on } \Gamma_\theta,\ k\theta_{,\alpha}\, n_\alpha = \overline{q} \quad \text{on } \Gamma_q. \tag{2.15}$$

After straightforward computations, we obtain the following weak form of the equations above:

$$\int_D \left[\lambda u_{\beta,\beta}(\delta v_\alpha)_{,\alpha} - \mu(u_{\varepsilon,\beta} + u_{\beta,\alpha})(\delta v_\alpha)_{,\beta}\right] dD = \int_D \rho_0 f_\alpha \delta v_\alpha dD + \int_{\Gamma_t} \bar{t}_\alpha (\delta v_\alpha) d\Gamma \tag{2.16}$$

$$\int_D k\theta_{,\alpha}\,(\delta\varphi)_{,\alpha}\, dD = \int_D s\delta\varphi_\alpha dD + \int_{\Gamma_q} \overline{q}_\alpha \delta\varphi_\alpha d\Gamma, \tag{2.17}$$

where $u, \theta \in H^1$ and $\delta v, \delta\varphi \in H^1$ are test functions.

§3. Voronoi diagrams and Delaunay tessellations

These techniques are specific to computational geometry. In this section we shall follow Sukumar's ideas [19] to summarize the main features of Voronoi diagrams and Delaunay tessellations. In order to simplify the presentation we shall consider the bidimensional case. For a set of M distinct nodes in $\mathbb{R}^2$

$$N = \{n_1, n_2, ..., n_M\}, \tag{3.1}$$

the first order Voronoi diagram of the set N is a subdivision of the plane into regions T_I (Sibson [18]):

$$T_I = \{x \in R^2 | d(x, x_I) < d(x, x_J), \forall J \neq I\}, \tag{3.2}$$

where $d(x_I, x_J)$ is the distance between x_I and x_J. The plane is divided by the Voronoi diagram of a set of nodes into subdomains corresponding to the nodes. In a specific subdomain corresponding to a generic node x, any point is closer to x than to any other node. Voronoi diagrams are very well discussed in Boots ([3]) and Okabe & al. ([13]).

The Delaunay tessellation is the dual of the Voronoi diagram; it is constructed by connecting the nodes whose Voronoi cells have common boundaries. Among the remarkable properties of the Delaunay triangles, fundamental to computational geometry, we remind: the empty circle criterion (Lawson [12]), and the property that states that the Delaunay triangles minimize the minimum angle, among all other triangles (Lawson [12]).

§4. Numerical implementation. Natural neighbor interpolation

Because Natural Element Method is essentially a meshless method, the implementation of the NEM is quite similar to EFG method. The main difference consists in the way of computation of the shape functions and their derivates. As it was very well pointed out in (Farin [8]), the natural neighbor interpolant or the Sibson's interpolant

is a real interpolant in the case of a convex domain (like in the finite element method case). When the domain is non-convex, like in the case of cracks or material interfaces, one can assume the linearity of the approximation between adjacent boundary nodes (Sukumar [20]), but only with a certain discretization of the boundary. If we consider ourselves in the case mentioned above, we can impose the essential boundary conditions directly on the nodes, like in the case of the finite element method and in contrast to the case of the meshless methods.

We consider for the unknowns the following approximations:

$$u^h = \sum_{I=1}^{M} \Phi_I(x) u_I, \quad \theta^h = \sum_{I=1}^{M} \Phi_I(x) \theta_I \tag{4.1}$$

and for the test functions, similar types of approximation. Considering (4.1) in (2.13) and (2.14), one can obtain

$$\begin{aligned} Ad &= b \\ B\theta &= g, \end{aligned} \tag{4.2}$$

where,

$$A_{IJ} = \int_D K_I^T R K_J dD, \quad b_I = \int_D \rho_0 \Phi_I f dD + \int_{\Gamma_t} \Phi_I \bar{t} d\Gamma \tag{4.3}$$

$$B_I = k\Phi_{I,\alpha}\, \Phi_{I,\alpha}\,, \quad b_I = \int_D \Phi_I s dD + \int_{\Gamma_t} \Phi_I \bar{q} d\Gamma \tag{4.4}$$

$$R = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \text{ for plain strain,} \tag{4.5}$$

and K_I is the matrix containing the derivatives of the shape functions, as in the EFG method (Raducanu [15]). In order to compute the natural neighbor shape functions, one can use the algorithm proposed by Watson ([24]). One can notice that the stiffness matrix is symmetric and sparse. There are two main ways for numerical computations: one proposed by Traversoni ([21]) and one inspired from the EFG computations, which uses a background composed by Delaunay triangles. Some numerical examples, which will compare the solution obtained through this NEM method with the ones existing in literature (analytical and numerical) will follow in another paper.

§5. Conclusions

This paper proposes the implementation of NEM method in linear thermoelasticity. After 1995, the method was developed by a great number of scientists, who proposed new meanings and interpretations. Meshless methods have to be developed in the future, especially regarding the computational cost which presently is too high. This paper represents the first step in implementing NEM in thermoelasticity. It is presented the case of linear uncoupled thermoelasticity, and an algorithm for numerical implementation was proposed as well. A possible slight different approach of the matter could consist in the case when the linearity of the approximation between adjacent boundary nodes is not a hypothesis. In this case a non-conformal approach is needed (Raducanu [16]). Some numerical examples, which will compare the present solution with the ones existing in literature will follow in another paper.

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