Conformally flat almost Hermitian manifolds

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Abstract

Some conformally flat almost Hermitian manifolds with pointwise constant holomorphic (bi)sectional curvature are studied.

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§1. Introduction

Let $M := (M, \Phi, g)$ be a compact symplectic manifold with compatible metric g. The associated almost complex structure J is defined by $\Phi(X, Y) = g(X, JY)$. We denote the scalar curvature and $*$-scalar curvature of (M, Φ, g) by S and S^*, respectively.

D. E. Blair and S. Ianus [1] proved that an associated metric g on a compact symplectic manifold (M, Φ) is a critical point of the functional $A(g) = \int_M SdV$ or $K(g) = \int_M (S - S^*)dV$ if and only if the Ricci operator is J-invariant.

T. Draghici [3] proved that when (M, Φ) is a compact symplectic 4-manifold, if there exists a compatible metric g with J-invariant Ricci tensor and non-negative scalar curvature, then J is integrable. In [3] the author defined a Ricci form on an almost Kähler manifold by decomposing the Ricci tensor ρ into J-invariant and J-anti invariant parts.

M. Falcitelli, A. Farinola, G. Ganchev and O. T. Kassabov [4], [5], [6] studied almost Hermitian manifolds of pointwise constant antiholomorphic sectional curvature, which have J-invariant the Ricci tensor. Thus it is worthwhile to study the following:

PROBLEM 1 "*Study almost Hermitian manifolds of dimension* $2n(\geq 4)$, *equipped with* J*-invariant or* J*-anti invariant Ricci tensor*".

It is well known that on any almost Hermitian manifold we can define $*$-Ricci tensor, which is an analogue of the Ricci tensor but involving also the almost Hermitian structure. On a Kähler manifold, the Ricci tensor and the $*$-Ricci tensor coincide. Thus the following question arises:

QUESTION "*Do there exist non-Kähler manifolds whose Ricci tensor and the* $*$*-Ricci tensor or (the more general condition) the scalar curvature and the* $*$*scalar curvature do not coincide ?*

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A. Gray and L. Vanhecke [9] constructed examples of Hermitian manifolds with pointwise constant holomorphic sectional curvature which are not of constant sectional curvature. The almost Hermitian manifolds of pointwise constant sectional curvature have been studied by many authors ([2], [8], [11], [12], [15]) in low dimension. Thus it is interesting to study the following:

PROBLEM 2 "*Study almost Hermitian manifolds of dimension* $2n(\geq 4)$, *with pointwise constant holomorphic sectional curvature, whose the classification is still an open problem*".

In [16], Ph. J. Xenos and the author proved that a conformally flat almost Hermitian manifold of dimension $2n$ satisfies $S^* = S/(2n-1)$. If in addition, this manifold satisfies $\rho + \rho^* = 0$, then it is flat.

In the present paper we deal with the above mentioned question and problems. We shall apply them for some classes of $2n$-dimensional conformally flat almost Hermitian manifolds.

§2. Preliminaries

Let $M^{2n} := (M^{2n}, g, J)$ be a $2n$-dimensional $(n \geq 2)$ almost Hermitian manifold equipped with the almost Hermitian structure (g, J) and Φ the Kähler form of M^{2n} defined by $\Phi(X, Y) = g(X, JY)$ for $X, Y \in \mathcal{X}(M^{2n})$. Here $\mathcal{X}(M^{2n})$ denotes the Lie algebra of all smooth vector fields on M^{2n}. We denote by ∇, R, ρ, Q and S the Riemannian connection, the curvature tensor $(R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]})$, the Ricci tensor, the Ricci operator $(\rho(X, Y) = g(QX, Y))$ and the scalar curvature of M^{2n}, respectively.

We denote by ρ^* the $*$-Ricci tensor of M^{2n} defined by $\rho^*(x, y) = \frac{1}{2}\text{trace } (z \mapsto R(x, Jy)Jz)$ for $x, y, z \in T_p(M^{2n})$, $p \in M^{2n}$. We denote by S^* the $*$-*scalar curvature* of M^{2n}, which is the trace of the linear endomorphism Q^* defined by $g(Q^*x, y) = \rho^*(x, y)$, for $x, y \in T_p(M^{2n})$, $p \in M^{2n}$. If M^{2n} is a Kähler manifold, then $\rho = \rho^*$ (and therefore $S = S^*$).

A. Gray [8] considered almost Hermitian manifolds whose curvature tensor has a certain degree of resemblance to that of a Kähler manifold. For a given class $\mathbb{AH}$ of almost Hermitian manifolds, he determined the subclasses $\mathbb{AH}_i$ of manifolds whose curvature operator satisfies identity (i), where

(1) $R(X, Y, Z, W) = R(X, Y, JZ, JW)$,

(2) $R(X, Y, Z, W) = R(JX, JY, Z, W) + R(JX, Y, JZ, W) + R(JX, Y, Z, JW)$,

(3) $R(X, Y, Z, W) = R(JX, JY, JZ, JW)$,

where $R(X, Y, Z, W) = g(R(X, Y)W, Z)$ for all $X, Y, Z, W \in \mathcal{X}(M^{2n})$.

It is known ([8]) the inclusion relations $\mathbb{AH}_1 \subset \mathbb{AH}_2 \subset \mathbb{AH}_3 \subset \mathbb{AH}$ hold.

C. C. Hsiung, W. Yang and L. Friedland [10] determined the class $\mathbb{AH}_1^{'} (\subset \mathbb{AH})$ containing the almost Hermitian manifolds whose curvature operator satisfies

$(1^{'})$ $R(X, Y, Z, W) + R(X, Y, JZ, JW) = 0$.

We can easily prove that $\mathbb{AH}_1^{'} \subset \mathbb{AH}_3$.

On a Riemannian manifold M^{2n} the *Weyl tensor* W is defined by

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY +$$

$$+\rho(Y,Z)X - \rho(X,Z)Y] + \frac{S}{2(2n-1)(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

for all $X, Y, Z, W \in \mathcal{X}(M^{2n})$.

A Riemannian manifold M^{2n} is *conformally flat* if and only if $W(X,Y)Z \equiv 0$, or equivalently,

$$R(X,Y)Z = \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY + \rho(Y,Z)X - \tag{2.1}$$

$$-\rho(X,Z)Y] - \frac{S}{2(2n-1)(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

The *bisectional curvature* of an almost Hermitian manifold (M,g,J) is defined by

$$k(X,Y,Z,W) = \frac{R(X,Y,Z,W)}{\sqrt{\|X\|^2\|Y\|^2 - g(X,Y)^2}\sqrt{\|Z\|^2\|W\|^2 - g(Z,W)^2}},$$

for all vector fields X, Y, Z and W on M^{2n}. Thus we can obtain the following relations: the *sectional curvature* $K(X,Y) = k(X,Y,X,Y)$, for $X, Y \in \mathcal{X}(M^{2n})$, the *holomorphic sectional curvature* $H(X) = k(X,JX,X,JX)$, for $X \in \mathcal{X}(M^{2n})$ and the *holomorphic bisectional curvature* $h(X,Y) = k(X,JX,Y,JY)$, $\forall X, Y \in \mathcal{X}(M^{2n})$.

The bisectional curvature [7] depends only on the plane sections p and q spanned by (X,Y) and (Z,W) respectively (*i.e.*, $p := (X,Y)$ and $q := (Z,W)$).

If the holomorphic sectional curvature $H(x)$ is constant $f(p)$ for all $x \in T_p(M^{2n})$ at each point p of M^{2n}, then M^{2n} is said to be of *pointwise constant holomorphic sectional curvature*. Further, if f is constant whole on M^{2n}, then M^{2n} is said to be of *constant holomorphic sectional curvature*.

Using the the definition of the conformal flatness of a Riemannian manifold we can prove the following.

THEOREM A

Let (M,g) be a conformally flat Riemannian manifold. Then, we have

(i) *If (M,g) is Ricci-flat ($\rho = 0$), then it is flat.*

(ii) *(M,g) is an Einstein manifold if and only if it has constant sectional curvature.*

Throughout this paper, we assume that all manifolds are connected and smooth and further that all quantities on manifolds are smooth, unless otherwise specified.

§3. Conformally flat almost Hermitian manifolds with pointwise constant holomorphic sectional curvature

In what follows by *CF-manifold* we mean conformally flat almost Hermitian manifold of dimension $2n \geq 4$, by a *Q-manifold*, an almost Hermitian manifold of dimension $2n \geq 4$ with J-invariant Ricci tensor ($QJ = JQ$) and by a *$(-Q)$-manifold*, an almost Hermitian manifold of dimension $2n \geq 4$ with J-anti invariant Ricci tensor ($QJ + JQ = 0$).

It is known that Kähler manifolds of constant sectional curvature are flat. We shall extend this argument.

Proposition 1 *Let M^{2n} be an almost Hermitian manifold with constant sectional curvature. If $S = S^*$, then M^{2n} is flat.*

Proof. *Let $\{e_i\}_{i=1}^{2n} = \{e_1, ..., e_n, Je_1, ..., Je_n\}$ be a local orthonormal frame on M^{2n} . Then*

$$S - S^* = \sum_{i,j=1}^{2n} [R(e_i, e_j, e_i, e_j) - R(e_i, e_j, Je_i, Je_j)].$$

Since M^{2n} is of constant sectional curvature $c \in \mathbf{R}$, we have

$$R(X, Y, Z, W) = c\ [g(X, Z)g(Y, W) - g(Y, Z)g(X, W)],$$

for all $X, Y, Z, W \in \mathcal{X}(M^{2n})$.

Then, the above relations imply $n(n-1)c = 0$ and because of $n \neq 0, 1$, we have $c = 0$. □

Using Proposition 1, we extend Theorem 4.2 of [10] in the following way:

Corollary 1 *If M^{2n} is an almost Hermitian manifold of constant sectional curvature and belongs to $\mathbb{AH}_1 \cup \mathbb{AH}_1'$ then M^{2n} is flat.*

Theorem 1 *A CF-manifold with pointwise constant holomorphic sectional curvature is a weakly $*$-Einstein manifold.*

Proof. Let M^{2n} be a CF-manifold with pointwise constant holomorphic sectional curvature f. Using (2.1) and the pointwise constancy of the holomorphic sectional curvature we have

$$f\,\|X\|^2 = \frac{1}{2(n-1)}\{\rho(X, X) + \rho(JX, JX)\} - \frac{S\,\|X\|^2}{2(2n-1)(n-1)},\ X \in \mathcal{X}(M^{2n}).$$

Polarization of the above equation leads us to

$$\rho(X, Y) + \rho(JX, JY) = [2(n-1)f + \frac{S}{2n-1}]g(X, Y),\ X, Y \in \mathcal{X}(M^{2n}). \tag{3.1}$$

Choose $X = Y = e_i$ in (3.1) and summing it up with respect to $i = 1, ..., n$, we have

$$S = 2n(2n-1)f. \tag{3.2}$$

Using (3.1) and (3.2) we obtain

$$\rho(X, Y) + \rho(JX, JY) = 2(2n-1)fg(X, Y). \tag{3.3}$$

It is known ([14]) that on an almost Hermitian manifold M^{2n} with pointwise constant holomorphic sectional curvature we have for all $X, Y \in \mathcal{X}(M^{2n})$:

$$\rho(X,Y) + \rho(JX,JY) + 3[\rho^*(X,Y) + \rho^*(JX,JY)] = 4(n+1)fg(X,Y). \tag{3.4}$$

Using the definition of the $*$-Ricci tensor we can easily prove that for arbitrary vector fields X, Y on a CF-manifold M^{2n}, we have

$$\rho^*(X,Y) = \frac{1}{2(n-1)}[\rho(X,Y) + \rho(JX,JY)] - \frac{S}{2(n-1)(2n-1)}g(X,Y).$$

The last two relations and (3.2) imply

$$\rho^*(X,Y) = fg(X,Y), \qquad X, Y \in \mathcal{X}(M^{2n}).$$

Thus M^{2n} is a weakly $*$-Einstein manifold with $*$-scalar curvature $S^* = 2nf$. □

Theorem 2 *A CF-Q-manifold with pointwise constant holomorphic sectional curvature has constant sectional curvature.*

Proof. Let M^{2n} be a CF-Q-manifold with pointwise constant holomorphic sectional curvature f. From the J-invariance of the Ricci curvature and (3.1) we have

$$\rho(X,Y) = (2n-1)fg(X,Y), \quad X, Y \in \mathcal{X}(M^{2n}),$$

by the Schur's lemma, M^{2n} is Einstein and hence it is of constant curvature by virtue of Theorem A. □

It is well known that every manifold belonging to $\mathbb{AH}_3$ has J-invariant Ricci tensor. Thus by using Theorem 2 we obtain

Corollary 2 *A CF-manifold belonging to $\mathbb{AH}_3$ and having pointwise constant holomorphic sectional curvature has constant sectional curvature.*

Using Proposition 1 and Theorem 2 we can prove.

Corollary 3 *A CF-Q-manifold of pointwise constant holomorphic sectional curvature with $S = S^*$ is flat.*

Using Proposition 1, Theorem 2 and [13] we obtain.

Corollary 4 *An almost Kähler CF-Q-manifold of pointwise constant holomorphic sectional curvature is flat.*

Proposition 1, Theorem 2 and [13] imply:

Corollary 5 *Let M^{2n} be a CF-Q-manifold of pointwise constant holomorphic sectional curvature. If $M^{2n} \in \mathbb{AH}_1 \cup \mathbb{AH}_1'$ then it is flat.*

Theorem 3 *On every CF-(Q)-manifold the holomorphic bisectional curvature vanishes.*

Proof. On a CF-manifold we have

$$\begin{aligned} R(X,JX,Y,JY) &= \tfrac{1}{2(n-1)}[g(X,Y)\rho(JX,JY) - g(JX,Y)\rho(X,JY) + \\ &+\rho(X,Y)g(X,Y) - \rho(JX,Y)g(X,JY)] - \\ &-\tfrac{S}{2(n-1)(2n-1)}[g(X,Y)^2 + \Phi(X,Y)^2], \end{aligned}$$

for all $X, Y \in \mathcal{X}(M^{2n})$.

Because of $QJ + JQ = 0$, the above relation takes the form

$$R(X,JX,Y,JY) = -\frac{S}{2(n-1)(2n-1)}[g(X,Y)^2 + \Phi(X,Y)^2], \tag{3.5}$$

therefore

$$H(X) = -\frac{S}{2(n-1)(2n-1)}, \quad X \in \mathcal{X}(M^{2n}).$$

Since S is a fuction on M^{2n} which does not depend on X, the holomorphic sectional curvature $H(X)$ is pointwise constant. We denote by the holomorphic sectional curvature function by f as before. The equation (3.3) can be written as

$$g(QX,Y) + g(QJX,JY) = 2(2n-1)fg(X,Y), \tag{3.6}$$

Next, the condition $QJ + JQ = 0$ implies $Q = JQJ$. Inserting this into (3.6), we get

$$2(2n-1)fg(X,Y) = 0,$$

for all $X, Y \in \mathcal{X}(M)$. Thus we obtain $f = 0$. The pointwise constancy of $H(X)$ of M^{2n} implies (3.2) and above relation yields $S = 0$. Therefore, from (3.4) we obtain the required result. □

Theorem 4 *Every CF-manifold with pointwise constant holomorphic bisectional curvature is flat.*

Proof. Let M^{2n} be a CF-manifold.Then, from (2.1) for arbitrary vector fields X, Y on M^{2n} we have

$$R(X,Y,JX,JY) = \frac{1}{2(n-1)}\Phi(X,Y)[\rho(X,JY) - \rho(JX,Y)] - \frac{S}{2(n-1)(2n-1)}\Phi(X,Y)^2,$$

$$R(X,JY,JX,Y) = -\frac{1}{2(n-1)}g(X,Y)[\rho(X,Y) + \rho(JX,JY)] + \frac{S}{2(n-1)(2n-1)}g(X,Y)^2.$$

From the first Bianchi identity we have

$$R(X,JX,Y,JY) = R(X,Y,JX,JY) - R(X,JY,JX,Y).$$

The above three equations yield

$$\begin{aligned} R(X,JX,Y,JY) &= \tfrac{1}{2(n-1)}\{\Phi(X,Y)[\rho(X,JY) - \rho(JX,Y)] + \\ &+g(X,Y)[\rho(X,Y) + \rho(JX,JY)]\} - \\ &-\tfrac{S}{2(n-1)(2n-1)}[g(X,Y)^2 + \Phi(X,Y)^2]. \end{aligned} \tag{3.7}$$

If the holomorphic bisectional curvature of M^{2n} is pointwise constant f, then (3.7) for $X = Y$ implies

$$[f + \frac{S}{2(n-1)(2n-1)}]\,\|X\|^2 = \frac{1}{2(n-1)}[\rho(X,X) + \rho(JX,JX)].$$

If $X \in \{e_i\}_{i=1}^{2n}$, then the above equation takes the form

$$f = \frac{1}{2n(2n-1)}S. \tag{3.8}$$

Using (3.8) the relation (3.7) implies

$$\frac{S}{n(2n-1)}\{(n-1)\,\|X\|^2\,\|Y\|^2 + n[g(X,Y)^2 + \Phi(X,Y)^2]\}$$
$$= g(X,Y)[\rho(X,Y) + \rho(JX,JY)] + \Phi(X,Y)[\rho(X,JY) - \rho(JX,Y)].$$

If $X = e_i$, $Y = e_j$, $(i \neq j)$ then the above equation yields $S = 0$.

Thus, the equation (2.1) can be written as

$$\begin{aligned} R(X,Y,Z,W) &= \tfrac{1}{2(n-1)}[g(X,Z)\rho(Y,W) - g(Y,Z)\rho(X,W) + \\ &\quad + g(Y,W)\rho(X,Z) - g(X,W)\rho(Y,Z)]. \end{aligned}$$

If $X, Y, Z, W \in \{e_i\}_{i=1}^{2n}$, then the above equation yields $\rho = 0$. This conclusion and the theorem A imply the result. □

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