Some inequalities for warped products in cosymplectic space forms

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Abstract

In this article, we investigate the inequality between the warping function of a warped product submanifold isometrically immersed in a cosymplectic space form of constant φ-sectional curvature and the squared mean curvature. Furthermore, some applications are derived.

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§1. Introduction

Let M_1 and M_2 be Riemannian manifolds of positive dimension n_1 and n_2, equipped with Riemannian metrics g_1 and g_2, respectively. Let f be a positive function on M_1. The warped product $M_1 \times_f M_2$ is defined to be the product manifold $M_1 \times M_2$ with the warped metric: $g = g_1 + f^2 g_2$ (see, for instance [3]).

It is well-known that the notion of warped products plays some important role in differential geometry as well as in physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3].

Let $\phi : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\tilde{M}(c)$ with constant sectional curvature c. We denote by h the second fundamental form of ϕ and $H_i = \frac{1}{n_i}\text{trace } h_i$, where trace h_i is the trace of h restricted to M_i. We call H_i ($i = 1, 2$) the partial mean curvature vectors. The immersion ϕ is said to be *mixed totally geodesic* if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2 respectively.

Recently, in [6] B. Y. Chen established the following sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\tilde{M}(c)$ and the squared mean curvature $||H||^2$.

Theorem 1.1 ([6]). *Let $\phi : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of a warped product into a Riemannian m-manifold of constant sectional curvature c. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2}||H||^2 + n_1 c, \tag{1.1}$$

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where Δ *is the Laplacian operator of* M_1.

As an immediate application, he obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form.

On the other hand, for the above related researches B. Y. Chen investigated the inequality (1.1) of a warped product submanifold into complex hyperbolic space ([5]) and complex projective space form ([2]). Also, K. Matsumoto and I. Mihai ([8]) studied the inequality (1.1) of a warped product submanifold into Sasakian space form of constant φ-sectional curvature.

In this paper, we prove similar inequality for warped product submanifolds of cosymplectic space forms of constant φ- sectional curvature c.

§2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact structure (φ, ξ, η), that is, φ is a (1,1) tensor field, ξ is a vector field and η is a 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$

Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalent, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in \tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g). An almost contact metric manifold is *cosymplectic* ([1]) if $\tilde{\nabla}_X \varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g. From the formula $\tilde{\nabla}_X \varphi = 0$ it follows that $\tilde{\nabla}_X \xi = 0$.

A plane section π in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a *φ-section* if $\pi \perp \xi$ and $\varphi(\pi) = \pi$. $\tilde{M}$ is of *constant φ-sectional curvature* if sectional curvature $\tilde{K}(\pi)$ dose not depend on the choice of the φ-section π of $T_p\tilde{M}$ and the choice of a point $p \in \tilde{M}$. A cosymplectic manifold $\tilde{M}$ is of constant φ-sectional curvature c if and only if its curvature tensor $\tilde{R}$ is of the form ([7])

$$\begin{aligned} 4\tilde{R}(X, Y, Z, W) = c\{ & g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \\ & - 2g(X, \varphi Y)g(Z, \varphi W) \\ & - g(X, W)\eta(Y)\eta(Z) + g(X, Z)\eta(Y)\eta(W) \\ & - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z)\}. \end{aligned} \tag{2.1}$$

Let M be an n-dimensional submanifold of a manifold $\tilde{M}$ equipped with a Riemannian metric g. The Gauss and Wiengarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad \tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}, \nabla$ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}, M$ and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to the shape operator A by $g(h(X, Y), N) = g(A_N X, Y)$.

For any vector X tangent to M we put $\varphi X = PX + FX$, where PX and FX are the tangential and the normal components of φX, respectively. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of M, we define the squared norm of P by

$$||P||^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j)$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$. We put

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$$

where $\{e_{n+1}, \ldots, e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$ and $r = n+1, \ldots, 2m+1$. A submanifold M is *totally geodesic* in $\tilde{M}$ if $h = 0$, and *minimal* if $H = 0$.

On the other hand, M is said to be a *totally real submanifold* if P is identically zero, that is, $\varphi X \in T_p^{\perp}M$ for any $X \in T_pM, p \in M$.
For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$. For any orthornormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined by to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{2.2}$$

§3. Some inequality for warped product submanifolds

We give the following lemma for later use.

Lemma 3.1 ([4]). *Let $a_1, \ldots, a_n, a_{n+1}$ be $n+1$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + a_{n+1}\right). \tag{3.1}$$

Then, $2a_1a_2 \geq a_{n+1}$, with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

We investigate warped product submanifolfs tangent to the structure vector field ξ in a cosymplectic space form $\tilde{M}(c)$.

Theorem 3.2. *Let $\phi : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional warped product into a $(2m+1)$-dimensional cosymplectic space form of constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Then, we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + \frac{c}{4}(n_1 + 2), \tag{3.1}$$

where $n_i = dimM_i, i = 1, 2,$ and Δ is the Laplacian operator of M_1.

Proof. Let $M_1 \times_f M_2$ be a warped product submanifold of a cosymplectic space form $\tilde{M}(c)$ with constant φ-sectional curvature c whose structure vector field ξ is tangent to M_1. Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{3.2}$$

for any vector fields X, Z tangent to M_1, M_2, respectively. If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{3.3}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_{n_1} = \xi$ are tangent to M_1, $e_{n_1+1}, \ldots, e_n$ are tangent to M_2 and e_{n+1} is parallel to H. Then, using (3.3) we obtain

$$\frac{\Delta f}{f} = \sum_{j=1}^{n_1} K(e_j \wedge e_s), \tag{3.4}$$

for each $s \in \{n_1 + 1, \ldots, n\}$.
From the equation of Gauss, we obtain

$$2\tau = \{n(n-1) + 3||P||^2 - 2n + 2\}\frac{c}{4} + n^2||H||^2 - ||h||^2. \tag{3.5}$$

We denote

$$\delta = 2\tau - \{n(n-1) + 3||P||^2 - 2n + 2\}\frac{c}{4} - \frac{n^2}{2}||H||^2. \tag{3.6}$$

Substituting (3.5) in (3.6), we have

$$n^2||H||^2 = 2(\delta + ||h||^2). \tag{3.7}$$

With respect to the above orthonormal basis, (3.7) takes the following form:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left(\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i \neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right),$$

which implies

$$\begin{aligned}\left(\sum_{i=1}^{3} a_i\right)^2 =& 2\Bigg\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{1 \leq i \neq j \leq n}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ & - \sum_{2 \leq j \neq k \leq n_1} h_{jj}^{n+1} h_{kk}^{n+1} - \sum_{n_1+1 \leq s \neq t \leq n} h_{ss}^{n+1} h_{tt}^{n+1}\Bigg\},\end{aligned} \tag{3.8}$$

where $a_1 = h_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1} h_{ii}^{n+1}$ and $a_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}$.

Applying Lemma 3.1 to (3.8) yields

$$\begin{aligned}&\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\\ &\ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n}(h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}^{r})^2,\end{aligned} \tag{3.9}$$

with equality holding if and only if we have

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.10}$$

On the other hand, (2.5) and (3.4) imply

$$\begin{aligned} n_2\frac{\Delta f}{f} &= \tau - \sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t)\\ &= \tau - \frac{n_1(n_1-1)c}{8} - \frac{3c}{4}\sum_{1\le j<k\le n_1} g^2(Pe_j,e_k) - \frac{c}{4}(1-n_1)\\ &\quad - \sum_{r=n+1}^{2m+1}\sum_{1\le j<k\le n_1}(h_{jj}^r h_{kk}^r - (h_{jk}^r)^2) - \frac{n_2(n_2-1)c}{8}\\ &\quad - \frac{3c}{4}\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t) - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le s<t\le n}(h_{ss}^r h_{tt}^r - (h_{st}^r)^2).\end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11) and taking account of (3.4), we have

$$\begin{aligned} n_2\frac{\Delta f}{f} \le &\tau - \frac{n(n-1)}{8}c + \frac{c}{4}n_1n_2 - \frac{\delta}{2} - \frac{c}{4}(1-n_1)\\ &- \frac{3c}{4}\sum_{1\le j<k\le n_1} g^2(Pe_j,e_k) - \frac{3c}{4}\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t).\end{aligned} \tag{3.12}$$

By (3.6), the inequality (3.12) reduces to

$$\begin{aligned}\frac{\Delta f}{f} &\le \frac{n^2}{4n_2}||H||^2 + (n_1-1)\frac{c}{4} + \frac{3c}{4n_2}\sum_{\substack{1\le j\le n_1\\ n_1+1\le t\le n}} g^2(Pe_j,e_t)\\ &\le \frac{n^2}{4n_2}||H||^2 + (n_1-1)\frac{c}{4} + \frac{3c}{4}\min\left\{\frac{n_1}{n_2},1\right\}.\end{aligned} \tag{3.13}$$

We distinguish two cases:
(a) $n_1 \le n_2$, in this case the inequality (3.13) implies (3.1).
(b) $n_1 > n_2$, in this case (3.13) also becomes (3.1). It completes the proof. □

Corollary 3.3. *Let $\phi : M = M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional totally real warped product into a $(2m+1)$-dimensional a cosymplectic space form whose the structure vector field ξ is tangent to M_1. Then, we have*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2}||H||^2 + (n_1-1)\frac{c}{4}, \tag{3.14}$$

where, $n_i = dim M_i, i = 1, 2,$ and Δ is the Laplacian operator of M_1.

Moreover, the equality case of (3.14) holds if and only if ϕ is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where, $H_i, i = 1, 2$ are the partial mean curvatures.

Proof. By (3.13), we can easily obtain the inequality (3.14). Also, we see that the equality sign of (3.13) holds if and only if

$$h^r_{jt} = 0, \quad 1 \le j \le n_1, \quad n_1 + 1 \le t \le n, \quad n + 1 \le r \le 2m + 1, \tag{3.15}$$

and

$$\sum_{i=1}^{n_1} h^r_{ii} = \sum_{t=n_1+1}^{n} h^r_{tt} = 0, \quad n + 2 \le r \le 2m + 1. \tag{3.16}$$

Obviously (3.15) is equivalent to the mixed totally geodesic of the warped product M and (3.10) and (3.16) imply $n_1 H_1 = n_2 H_2$. The converse statement is straightforward. □

Corollary 3.4. *Let $M_1 \times_f M_2$ be a totally real warped product in a cosymplectic space form $\tilde{M}(c)$ whose the structure vector ξ is tangent to $M_1 (n_1 > 1)$ and a warping function f is a harmonic. Then, $M_1 \times_f M_2$ admits no minimal totally real immersion into a cosymplectic space form $\tilde{M}(c)$ with $c < 0$.*

Proof. Assume f is a harmonic function on M_1 and $M_1 \times_f M_2$ admits a minimal totally real immersion in a cosymplectic space form $\tilde{M}(c)$. Then, the inequality (3.14) becomes $c \ge 0$. □

Corollary 3.5. *Let $M_1 \times_f M_2$ be a totally real warped product in a cosymplectic space form $\tilde{M}(c)$ whose the structure vector ξ is tangent to $M_1 (n_1 > 0)$. If the warping function f of a warped product $M_1 \times_f M_2$ is an eigenfunction of the Laplacian on M_1 with corresponding eigenvalue $\lambda > 0$, then $M_1 \times_f M_2$ dose not admit a minimal totally real immersion into a cosymplectic space form $\tilde{M}(c)$ with $c \le 0$.*

Proof. If f is an eigenfunction of the Laplacian on M_1 with eigenvalue $\lambda > 0$. Then inequality (3.14) implies that $(n_1 - 1)\frac{c}{4} \ge \lambda > 0$. Therefore, we have Corollary 3.5. □

Corollary 3.6. *Let $M_1 \times_f M_2$ be a compact totally real warped product in a cosymplectic space form $\tilde{M}(c)$ such that the structure vector ξ is tangent to $M_1 (n_1 > 1)$ and $c \le 0$. Then $M_1 \times_f M_2$ is a Riemannian product.*

Theorem 3.7. *Let $\phi : M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional warped product into a $(2m + 1)$-dimensional cosymplectic space form of constant φ-sectional curvature c whose structure vector field ξ is tangent to M_2. Then, we have*

$$\frac{\Delta f}{f} \le \frac{n^2}{4n_2} ||H||^2 + \left(3 + n_1 - \frac{n_1}{n_2}\right) \frac{c}{4},$$

where $n_i = dim M_i, i = 1, 2,$ and Δ is the Laplacian operator of M_1.

Corollary 3.8. *Let $\phi : M = M_1 \times_f M_2 \longrightarrow \tilde{M}(c)$ be an isometric immersion of an n-dimensional totally real warped product into a $(2m+1)$-dimensional a cosymplectic* *space form whose the structure vector field* ξ *is tangent to* M_2. *Then, we have*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}||H||^2 + \left(n_1 - \frac{n_1}{n_2}\right)\frac{c}{4}, \tag{3.17}$$

where, $n_i = dim M_i, i = 1, 2$, *and* Δ *is the Laplacian operator of* M_1.

Moreover, the equality case of (3.17) holds if and only if ϕ *is a mixed totally geodesic immersion and* $n_1 H_1 = n_2 H_2$, *where,* $H_i, i = 1, 2$ *are the partial mean curvatures.*

Corollary 3.9. *Let* $M_1 \times_f M_2$ *be a totally real warped product in a cosymplectic space form* $\tilde{M}(c)$ *whose the structure vector* ξ *is tangent to* $M_2(n_2 > 1)$ *and a warping function* f *is a harmonic. Then,* $M_1 \times_f M_2$ *admits no minimal totally real immersion into a cosymplectic space form* $\tilde{M}(c)$ *with* $c < 0$.

Corollary 3.10. *Let* $M_1 \times_f M_2$ *be a totally real warped product in a cosymplectic space form* $\tilde{M}(c)$ *whose the structure vector* ξ *is tangent to* $M_2(n_2 > 0)$. *If the warping function* f *of a warped product* $M_1 \times_f M_2$ *is an eigenfunction of the Laplacian on* M_1 *with corresponding eigenvalue* $\lambda > 0$, *then* $M_1 \times_f M_2$ *dose not admit a minimal totally real immersion into a cosymplectic space form* $\tilde{M}(c)$ *with* $c \leq 0$.

Corollary 3.11. *Let* $M_1 \times_f M_2$ *be a compact totally real warped product in a cosymplectic space form* $\tilde{M}(c)$ *such that the structure vector* ξ *is tangent to* $M_2(n_2 > 1)$ *and* $c \leq 0$. *Then* $M_1 \times_f M_2$ *is a Riemannian product.*

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