Spacelike CMC 1 surfaces in de Sitter 3-space $\mathbf{S}^3_1(1)$: their construction and some examples

Sungwook Lee

Abstract

In this paper, we use a Bryant type representation ([1], [12]) to construct some examples of spacelike CMC 1 surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$. These surfaces are realised in $\mathbb{E}^3_1 \setminus \mathbb{H}^2(-1)$ via the standard stereographic projection.

M.S.C. 2000: 53A10, 53C42, 53C50.
Key words: de Sitter, constant mean curvature, spacelike surface.

1 Introduction

It is well-known that any minimal surface in Euclidean 3-space $\mathbb{E}^3$ can be represented by an integral formula, the so-called *Weiertraß representation formula* given in terms of a meromorphic map g and a holomorphic 1-form ω. The ordered pair (g, ω) is said to be the *Weierstraß data*. Conversely, given any Weierstraß data (g, ω) the Weiertraß representation formula yieds a minimal surface in $\mathbb{E}^3$.

In [4], R. L. Bryant proved a representation formula for constant mean curvature 1 (abbreviated CMC 1) surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ which is analogous to the Weierstraß representation formula for minimal surfaces in $\mathbb{E}^3$. Bryant showed that minimal surfaces in $\mathbb{E}^3$ and CMC 1 surfaces in $\mathbb{H}^3(-1)$ share many properties in common. However, there are also notable differences. For instance, the hyperbolic Gauß map of CMC 1 surfaces in $\mathbb{H}^3(-1)$ may not extend across the ends even if the total Gaußian curvature is finte, unlike the Euclidean case. The total curvature of minimal surfaces in $\mathbb{E}^3$ satisfies a certain quantisation. For instance, the classical Enneper surface and the catenoid have total curvature -4π. (In fact, they are the only complte minimal surfaces in $\mathbb{E}^3$ whose total curvature is -4π.) In general, complete minimal surfaces of finite total curvature in $\mathbb{E}^3$ have total curvature $-4m\pi$, where m is a nonnegative integer. Bryant also showed in [4] that such quantisation does not hold in general for CMC 1 surfaces in $\mathbb{H}^3(-1)$.

The existence of Bryant's representation formula for CMC 1 surfaces in $\mathbb{H}^3(-1)$ is not a coincidence. Its existence was indeed expected due to *Lawson correspondence.* In [10], H. Blaine Lawson showed that there exists a 1 : 1 correspondence between minimal surfaces in $\mathbb{E}^3$ and CMC ±1 surfaces in $\mathbb{H}^3(-1)$ and that the corresponding minimal surfaces in $\mathbb{E}^3$ and CMC ±1 surfaces in $\mathbb{H}^3(-1)$ satisfy the same Gauß-Codazzi equations.

Differential Geometry - Dynamical Systems, Vol.7, 2005, pp. 49-73.

In [15], B. Palmer proved Lawson correspondence between maximal spacelike surfaces in Minkowski 3-space $\mathbb{E}^3_1$ and spacelike CMC ± 1 surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$. Hence, one would expect to have a Bryant type representation formula for spacelike CMC ± 1 surfaces in $\mathbb{S}^3_1(1)$, which is analogous to the Weierstraß type representation formula (4.7) ([9], [14]) for spacelike maximal surfaces in $\mathbb{E}^3_1$. In [11] and [12], the author proved a Bryant type representation formula (3.4) for spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$. This representation formula was already known to R. Aiyama and K. Akutagawa and they introduced the formula without proof in [1]. In [13], the author and S.-D. Yang gave a spinor representation formula for spacelike CMC -1 surfaces in $\mathbb{S}^3_1(1)$. They used this spinor representation formula to construct spacelike CMC -1 trinoid in $\mathbb{S}^3_1(1)$.

The main purpose of this paper is to show how to construct spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ using the representation formula (3.4) and to present some examples of such surfaces.

In section 2, we discuss basic differential geometric settings for spacelike surfaces in $\mathbb{S}^3_1(1)$. In section 3, we introduce the Bryant type representation formula without proof. In section 4, we exhibit an explicit 1 : 1 correspondence between spacelike maximal surfaces in $\mathbb{E}^3_1$ and spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ in terms of the Weierstraß data (g, ω), where g is a holomorphic function[1] mapped into the Poincaré open disk $\mathbb{D}$ and ω is a holomorphic 1-form. In section 5, we discuss a duality of spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$. Here, the hyperbolic Gauß map which is an analogue of the classical Gauß map plays an important role. Finally, in section 6, we construct some examples of spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ such as spacelike Enneper cousin, spacelike catenoid cousin and spacelike helicoid cousin in $\mathbb{S}^3_1(1)$ that are corresponded to spacelike Enneper surface, spacelike catenoid and spacelike helicoid in $\mathbb{E}^3_1$, resp., via the Lawson correspondence. These spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ can be visualised in $\mathbb{E}^3_1 \setminus \mathbb{H}^2(-1)$ via the standard stereographic projection. The graphical images of these projected spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ and their spacelike maximal correspondents in $\mathbb{E}^3_1$ are produced by MAPLE and are exhibited in this section.

There exist spacelike surfaces in $\mathbb{S}^3_1(1)$ that can be regarded as an analogue of *horospheres* in $\mathbb{H}^3(-1)$. The existence of such spacelike surfaces in $\mathbb{S}^3_1(1)$ was proven by K. Akutagawa in [2] and by J. Ramanathan in [16]. We include an example of such horosphere type spacelike surfaces in section 6.

2 Spacelike Surfaces in de Sitter 3-Space $\mathbb{S}^3_1(1)$

Let $\mathbb{E}^4_1$ be Minkowski 4-space with rectangular coordinates $\xi_0, \xi_1, \xi_2, \xi_3$ and the standard flat Lorentzian metric $\langle\ ,\ \rangle$ of signature $(-,+,+,+)$ given by the quadratic form

$$-d\xi_0^2 + d\xi_1^2 + d\xi_2^2 + d\xi_3^2.$$

The de Sitter 3-space $\mathbb{S}^3_1(1)$ is a semi-Riemannian 3-manifold of constant sectional curvature 1 that can be realised as the hyperquadric in $\mathbb{E}^4_1$:

$$\mathbb{S}^3_1(1) = \{(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{E}^4_1 : -(\xi_0)^2 + (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1\}.$$

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[1] Note that g may also be a meromorphic function mapped into $\mathbb{C} \setminus \mathbb{D}$. See [11] or [12] for more details. In this paper, we only consider holomorphic function g mapped into $\mathbb{D}$.

Let M be a Riemann surface and $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ an immersion. The immersion φ is said to be *spacelike* if the induced metric I on M is Riemannian (positive definite). The induced metric I determines a conformal structure $\mathcal{C}_I$ on M.

Let (x, y) be an *isothermal* coordinate system with respect to the conformal structure $\mathcal{C}_I$. Then the first fundamental form I is written in terms of (x, y) as

$$I = e^u\{(dx)^2 + (dy)^2\}.$$

Let $z = x + \sqrt{-1}y$. Then $(z, \bar{z})$ defines a complex coordinate system with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I can also be written in terms of $(z, \bar{z})$ as

$$I = e^u dz \otimes d\bar{z}.$$

We compute

$$\langle \varphi_z, \varphi_z \rangle = \langle \varphi_{\bar{z}}, \varphi_{\bar{z}} \rangle = 0,\ \langle \varphi_z, \varphi_{\bar{z}} \rangle = \frac{1}{2}e^u,$$

where $\varphi_z = \frac{1}{2}(\varphi_x - \sqrt{-1}\varphi_y)$ and $\varphi_{\bar{z}} = \frac{1}{2}(\varphi_x + \sqrt{-1}\varphi_y)$. Let N be a unit normal vector field on M. Then

$$\langle N, N \rangle = -1,\ \langle \varphi_z, N \rangle = \langle \varphi_{\bar{z}}, N \rangle = 0.$$

The quadratic 1-form $Qdz \otimes dz := \langle \varphi_{zz}, N \rangle dz \otimes dz$ is called *Hopf differential*. Abusing the terminology, we simply call the coefficient Q Hopf differential. It is known that $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ has a constant mean curvature (abbreviated CMC) if and only if the Hopf differential Q is holomorphic, i.e., $Q_{\bar{z}} = 0$. (See [12] for more details.)

The second fundamental form II is given by

$$II = -\langle df, dN \rangle = Qdz \otimes dz + He^u dz \otimes d\bar{z} + \bar{Q}d\bar{z} \otimes d\bar{z},$$

where H is the mean curvature. Note that the mean curvature H is computed by $\langle \varphi_{z\bar{z}}, N \rangle = \frac{1}{2}He^u$.

The shape operator S of M derived from N is $S := -dN$. The shape operator S is related to II by

$$II(X, Y) = \langle SX, Y \rangle$$

for all vector fields X, Y on M. The shape operator is also represented by the matrix $II \cdot I^{-1}$. The mean curvature H of M and the Gaußian curvature K of M are given by

$$H = \frac{1}{2}\operatorname{tr} S = \frac{1}{2}\operatorname{tr}(II \cdot I^{-1}), \tag{2.1}$$

$$K = 1 - \det S = 1 - \det(II \cdot I^{-1}) = 1 - H^2 + 4Q\bar{Q}e^{-2u}. \tag{2.2}$$

Let $(\xi_0, \xi_1, \xi_2, \xi_3) \in \mathbb{E}^4_1$. Then $X = (\xi_0, \xi_1, \xi_2, \xi_3)$ can be identified with the 2×2 Hermitian matrix

$$\begin{pmatrix} \xi_0 + \xi_3 & \xi_1 + \sqrt{-1}\xi_2 \\ \xi_1 - \sqrt{-1}\xi_2 & \xi_0 - \xi_3 \end{pmatrix} = \sum_{\alpha=0}^{3} \xi_\alpha \sigma_\alpha, \tag{2.3}$$

where σ_α are the *Pauli spin matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In terms of the corresponding matrices, the inner product of two vectors X and Y is given by

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr}(X\sigma_3 Y^t \sigma_3).$$

In particular,

$$\langle X, X \rangle = -\frac{1}{2} \operatorname{tr}(X\sigma_3 X^t \sigma_3) = -\det X,$$

i.e., the above identification is an isometry.

The complex special linear group $\mathrm{SL}(2;\mathbb{C})$ acts isometrically on $\mathbb{E}^4_1$ via the C^∞ action:

$$\mu : \mathrm{SL}(2;\mathbb{C}) \times \mathbb{E}^4_1 \longrightarrow \mathbb{E}^4_1;\ \mu(g, X) = gXg^*, g \in \mathrm{SL}(2;\mathbb{C}), X \in \mathbb{E}^4_1$$

Moreover, $\mathrm{SL}(2;\mathbb{C})$ acts transitively on $\mathbb{S}^3_1(1)$. The isotropy subgroup of $\mathrm{SL}(2;\mathbb{C})$ at σ_3 is

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : |\alpha| - |\beta| = 1,\ \alpha, \beta \in \mathbb{C} \right\}.$$

Thus, $\mathbb{S}^3_1(1)$ can be represented as the symmetric space

$$\mathbb{S}^3_1(1) = \mathrm{SL}(2;\mathbb{C})/\mathrm{SU}(1,1) = \{g\sigma_3 g^* : g \in \mathrm{SL}(2;\mathbb{C})\}.$$

Let $\mathrm{SO}(3,1)^+$ be the identity component of the special Lorentz group

$$\mathrm{SO}(3,1) = \{\mathcal{A} \in G(4;\mathbb{R}) : \det \mathcal{A} = 1, \langle \mathcal{A}X, \mathcal{A}Y \rangle = \langle X, Y \rangle, X, Y \in \mathbb{E}^4_1\}.$$

Then the action μ induces a representation

$$\rho : \mathrm{SL}(2;\mathbb{C}) \longrightarrow \mathrm{SO}(3,1)^+$$

such that for each $g \in \mathrm{SL}(2,\mathbb{C})$,

$$\rho(g) : \mathbb{E}^4_1 \longrightarrow \mathbb{E}^4_1;\ \rho(g)X = gXg^*, X \in \mathbb{E}^4_1$$

is a Lorentz transformation which preserves parity and time orientation. The representation ρ is a double covering map and $\mathrm{SL}(2;\mathbb{C})$ is a double covering group of $\mathrm{SO}(3,1)^+$.

Let M be a simply connected Riemann surface and $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike immersion with unit normal vector field N. Then we find an orthonormal frame field $\mathcal{F}$ defined by

$$\mathcal{F} = (N, e^{-\frac{u}{2}}\varphi_x, e^{-\frac{u}{2}}\varphi_y, \varphi) : M \longrightarrow \mathrm{SO}(3,1)^+.$$

By using the double covering ρ we can find a lift F (called a coordinate frame) of $\mathcal{F}$ to $\mathrm{SL}(2;\mathbb{C})$:

$$\rho(F)(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = \mathcal{F}.$$

Let M be a Riemann surface and $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike immersion. Then there exists a local framing $F : U \longrightarrow \mathrm{SL}(2;\mathbb{C})$, where U is an oriented and simply connected open set in M, such that

$$\begin{aligned}
e_0 &= \rho(F)(\sigma_0) = F\sigma_0 F^* = FF^* = N,\\
e_1 &= \rho(F)(\sigma_1) = F\sigma_1 F^* = e^{-\frac{u}{2}}\varphi_x,\\
e_2 &= \rho(F)(\sigma_2) = F\sigma_2 F^* = e^{-\frac{u}{2}}\varphi_y,\\
e_3 &= \rho(F)(\sigma_3) = F\sigma_3 F^* = \varphi.
\end{aligned}$$

3 A Bryant Type Representation Formula for Spacelike CMC 1 Surfaces in $\mathbb{S}^3_1(1)$

Definition 1. *Let M be a Riemann surface. Then a map $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is said to be* null *if $F^*(\phi) = 0$ or equivalently $\det(F^{-1}dF) = 0$, where ϕ is the quadratic Cartan-Killing form $\phi = -4\det(g^{-1}dg)$.*

Based upon the geometric settings in section 2, the author proved the following *Bryant type representation formula* for spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ ([11], [12]).

Theorem 1 (Bryant type representation formula). *Let M be a Riemann surface and $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ a holomorphic null immersion. Assume that the pullback metric $e^*(ds^2)$ is nondegenerate, where ds^2 is the induced metric on $\mathbb{S}^3_1(1)$. Then*

$$\varphi := F\sigma_3 F^* : M \longrightarrow \mathbb{S}^3_1(1) \tag{3.4}$$

is a conformal spacelike CMC 1 immersion[2]*.*

Conversely, if M is an oriented open simply connected Riemann surface and $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ a spacelike CMC 1 immersion, then there exists a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that $\varphi = F\sigma_3 F^*$. Moreover, F is unique up to right multiplication by $g \in \mathrm{SU}(1,1)$.

Remark 1. In [4], R. L. Bryant proved a similar representation formula for CMC 1 surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$. That is the reason why our representation formula (3.4) is called a Bryant type representation formula. Note that a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ induces both a CMC 1 surface in $\mathbb{H}^3(-1)$ and a spacelike CMC 1 surface in $\mathbb{S}^3_1(1)$ via the representation formulae. More specifically, by Bryant's representation formula,

$$f := F\sigma_0 F^* = FF^* : M \longrightarrow \mathbb{H}^3(-1)$$

gives rise to a CMC 1 surface f in hyperbolic 3-space $\mathbb{H}^3(-1)$.

Remark 2. In Theorem 1, we have excluded the compact simply connected Riemann surface, i.e., the Riemann sphere $\mathbb{S}^2$, because there is no non-zero non-constant holomorphic 1-form globally defined in $\mathbb{S}^2$. Such a holomorphic 1-form is necessary to prove Theorem 1.

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[2] Since the sign of the mean curvature depends on the orientation of a surface, i.e., the orientation of the unit normal vector field N, the representation formula may as well define a CMC -1 spacelike immersion in $\mathbb{S}^3_1(1)$.

Remark 3. The representation formula (3.4) was first discovered by R. Aiyama and K. Akutagawa. They introduced the formula in [1] without proof.

One can immediately see that the Theorem 1 is a close analogue of Weierstraß type representation formula in [9] and [14] if we replace $\mathbb{S}^3_1(1)$ by $\mathbb{E}^3_1$, $\mathrm{SL}(2;\mathbb{C})$ by $\mathbb{C}^3$, $e_3 : \mathrm{SL}(2;\mathbb{C}) \longrightarrow \mathbb{S}^3_1(1)$ by $\mathrm{Re} : \mathbb{C}^3 \longrightarrow \mathbb{E}^3_1$, the Cartan-Killing form ϕ by the natural complex inner product in $\mathbb{C}^3$, and finally $H = 1$ by $H = 0$ in the Theorem 1.

The existence of the representation formula (3.4) and this analogue are not surprising and they are even expected due to the Lawson correspodence proved by B. Palmer in [15]. In particular, it can be seen that there is a $1:1$ correspondence between CMC ± 1 spacelike surfaces in $\mathbb{S}^3_1(1)$ and maximal spacelike surfaces in $\mathbb{E}^3_1$. These corresponding surfaces in $\mathbb{S}^3_1(1)$ and $\mathbb{E}^3_1$ via the Lawson correspondence are called *cousins* of each other. Note that this is not just a bijective correspondence but these surfaces in different space forms share the same Gauß-Codazzi equations. In [11] and [12], the author showed an explict relationship between CMC 1 surfaces in $\mathbb{S}^3_1(1)$ and maximal surfaces in $\mathbb{E}^3_1$, namely cousin surfaces share the same Weierstraß data for both representation formulae. This is also discussed in the next section.

The existence of Bryant representation formula for CMC 1 surfaces in $\mathbb{H}^3(-1)$ was also expected due to H. Blaine Lawson's original correspondence (see [10]), namely, there is a $1:1$ correspondence between CMC 1 surfaces in $\mathbb{H}^3(-1)$ and minimal surfaces in $\mathbb{E}^3$. For more details about the relationship between CMC 1 surfaces in $\mathbb{H}^3(-1)$ and minimal surfaces in $\mathbb{E}^3$ including their similarities and differences, please see R. Bryant's paper [4].

4 A Correspodence between Spacelike CMC 1 Surfaces in $\mathbb{S}^3_1(1)$ and Spacelike Maximal Surfaces in $\mathbb{E}^3_1$

The hyperbolic 2-space $\mathbb{H}^2(-1)$ can be described as

$$\mathbb{H}^2(-1) = \{\xi \in \mathrm{Herm}(2) : \xi_3 = 0, \det \xi = 1, \xi_0 > 0\}$$

by identifying $\mathbb{E}^4_1$ with the collection $\mathrm{Herm}(2)$ of 2×2 hermitian matrices (2.3). The Lie group $\mathrm{SU}(1,1)$ acts on $\mathbb{H}^2(-1)$ isometrically and transitively via the μ action. The istropy subgroup of $\mathrm{SU}(1,1)$ is the unit circle $\mathrm{U}(1)$ and so, $\mathbb{H}^2(-1)$ is represented as the symmetric space

$$\mathbb{H}^2(-1) = \mathrm{SU}(1,1)/\mathrm{U}(1) = \{hh^* : h \in \mathrm{SU}(1,1)\}.$$

$\mathbb{H}^2(-1)$ is also indentified with the Poincaré open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the metric $ds^2 = \dfrac{4dzd\bar{z}}{(1-|z|^2)^2}$ via the stereographic projection $\Psi : \mathbb{H}^2(-1) \longrightarrow \mathbb{D}$ from $-e_0 = (-1,0,0) \in \mathbb{E}^3_1$: Let $(\xi_0, \xi_1, \xi_2) \in \mathbb{H}^2(-1)$. Then

$$\Psi(\xi_0, \xi_1, \xi_2) = \left(0, \frac{\xi_1}{1+\xi_0}, \frac{\xi_2}{1+\xi_0}\right) \cong \frac{\xi_1 + \sqrt{-1}\xi_2}{1+\xi_0} \in \mathbb{D}.$$

Given $h = \begin{pmatrix} p & \bar{q} \\ q & \bar{p} \end{pmatrix} \in \mathrm{SU}(1,1)$,

$$hh^* = \begin{pmatrix} p\bar{p} + q\bar{q} & 2p\bar{q} \\ 2\bar{p}q & p\bar{p} + q\bar{q} \end{pmatrix} = \begin{pmatrix} \xi_0 & \xi_1 + \sqrt{-1}\xi_2 \\ \xi_1 - \sqrt{-1}\xi_2 & \xi_0 \end{pmatrix}$$

and

$$\Psi(\xi_0, \xi_1, \xi_2) = \Psi(hh^*) = \frac{\bar{q}}{\bar{p}} \in \mathbb{D}.$$

Thus, we see that $\frac{q}{p} \in \mathbb{D}$ as well.

Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion. Then by Theorem 1, $\varphi = F\sigma_3F^* : M \longrightarrow \mathbb{S}^3_1$ defines a spacelike CMC 1 surface. The author showed in [11] and [12] that the holomorphic 1-form can be written locally on a simply connected open set $U \subset M$ as

$$F^{-1}dF = \begin{pmatrix} pq & -p^2 \\ q^2 & -pq \end{pmatrix} \eta = \begin{pmatrix} \frac{q}{p} & -1 \\ \frac{q^2}{p^2} & -\frac{q}{p} \end{pmatrix} p^2\eta. \tag{4.5}$$

Here, η is a 1-form of type $(1,0)$ defined on U such that $ds^2_\varphi = \eta \otimes \eta$ and p, q are smooth functions defined on U such that $\frac{q}{p} : U \longrightarrow \mathbb{D}$ is a holomorphic function and $\begin{pmatrix} p & \bar{q} \\ q & \bar{p} \end{pmatrix} \in \mathrm{SU}(1,1)$. Let $g := \frac{q}{p}$ and $\omega := p^2\eta$. Then the equation (4.5) is rewritten as

$$F^{-1}dF = \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} \omega. \tag{4.6}$$

The induced metric of the local spacelike CMC 1 immersion φ is given in terms of g and ω as

$$ds^2_\varphi = (1 - |g|^2)^2 \omega \otimes \bar{\omega}.$$

Corresponding to this spacelike CMC 1 immersion φ, there exists a local maximal spacelike immersion ψ in $\mathbb{E}^3_1$ via the Weierstraß type representation formula ([9], [14])

$$\psi(\zeta) = \mathrm{Re} \int_{\zeta_0}^{\zeta} \left(2g\omega, (1 + g^2)\omega, -\sqrt{-1}(1 - g^2)\omega\right). \tag{4.7}$$

The induce metric of the maximal spacelike immersion is

$$ds^2_\psi = (1 - |g|^2)^2 \omega \otimes \bar{\omega}.$$

Conversely, any maximal spacelike surface in $\mathbb{E}^3$ can be represented locally by the formula (4.7) with the data (g, ω) ([9], [14]). We consider the following initial value problem:

$$F^{-1}dF = \begin{pmatrix} g & -1 \\ g^2 & -g \end{pmatrix} \omega, \ F(z_0) = \sigma_0.$$

This equation satisfies the integrability condition, i.e., $\Omega = F^{-1}dF$ is a solution of the Maurer-Cartan equation $d\Omega + \Omega \wedge \Omega = 0$. Hence, there exists a unique solution $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$, which is a holomorphic null immersion, to the initial value problem. Theorem 1 then yields a conformal spacelike CMC 1 immersion $\varphi = F\sigma_3F^* : M \longrightarrow \mathbb{S}^3_1(1)$.

5 The Hyperbolic Gauß Map and the Dual Spacelike CMC 1 Surfaces in $\mathbb{S}^3_1(1)$

Let $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ be a spacelike immersion. At each base point $e_3 = \varphi(m) \in \mathbb{S}^3_1(1)$, $e_0 \in T_{e_3}\mathbb{S}^3_1(1)$ is an oriented unit normal vector to the tangent plane $\varphi_*(T_mM)$. The oriented normal geodesic in $\mathbb{S}^3_1(1)$ emanating from e_3, which is tangent to the normal vector $e_0(m)$, asymptotically approaches to the boundary S^2_∞ (the null cone) exactly at two points $[e_0 + e_3], [e_0 - e_3] \in S^2_\infty$. The orientation allows us to name $[e_0 + e_3]$ the initial point and $[e_0 - e_3]$ the terminal point. Define a map $G : M \longrightarrow S^2_\infty$ by $G(m) = [e_0 + e_3](m)$ for each $m \in M$. This map is an analogue of *hyperbolic Gauß map*[3] of surfaces in $\mathbb{H}^3(-1)$. This map is also called the hyperbolic Gauss map in this paper. By identifying S^2_∞ and the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the canonical conformal structure, we can talk about the holomorphicity of the hyperbolic Gauss map. In [12], the author showed that

Proposition 2.
(a) *The hyperbolic Gauß map* $G : M \longrightarrow \mathbb{C}\cup\{\infty\}$ *of a spacelike immersion* $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ *is holomorphic if and only if* φ *has CMC* 1.
(b) *The hyperbolic Gauß map* $G : M \longrightarrow \mathbb{C}\cup\{\infty\}$ *of a spacelike immersion* $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ *is antiholomorphic if and only if* φ *is totally umbilic.*

Remark 4. The holomorphicity and antiholomorphicity of the hyperbolic Gauß map may be interchanged in Proposition 2 depending on the orientation of the hyperbolic Gauß map. See [12] for more details.

The following corollary is an immediate consequence of Proposition 2.

Corollary 3. *Let* $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ *be a spacelike immersion. Then* φ *is totally umbilic and and has CMC* 1 *if and only if the hyperbolic Gauß map of* φ *is constant.*

Let $\varphi : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a spacelike CMC 1 immersion. Then by Theorem 1, there exist a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$. One can easily see that $F^{-1} : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ is also a holomorphic null immersion. By Theorem 1, the map

$$\varphi^\sharp := F^{-1}\sigma_3(F^{-1})^* : M \longrightarrow \mathbb{S}^3_1(1)$$

defines a spacelike CMC 1 immersion in $\mathbb{S}^3_1(1)$. This surface $\varphi^\sharp$ is said to be the *dual CMC* 1 *spacelike surface* of φ. Since $(F^{-1})^{-1} = F$, $(\varphi^\sharp)^\sharp = \varphi$. Such duality was first considered and studied by M. Umehara and K. Yamada for CMC 1 surfaces in $\mathbb{H}^3(-1)$ ([18]).

Let $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ be a holomorphic null immersion with $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$. In [12], it is shown that the hyperbolic Gauß map G can be written locally as

$$G = \frac{F_3 + F_4 g}{F_1 + F_2 g}. \tag{5.8}$$

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[3] The hyperbolic Gauß map was introduced by C. Epstein in [8] and used by R. Bryant to study CMC 1 surfaces in $\mathbb{H}^3(-1)$ of surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ in [4].

Let $G_0 := -(F_1 + F_2 g)$ and $G_1 := -(F_3 + F_4 g)$. Then $G = \dfrac{G_1}{G_0}$ and

$$FdF^{-1} = \begin{pmatrix} G & -1 \\ G^2 & -G \end{pmatrix} W, \tag{5.9}$$

where $W = -(G_0)^2 \omega$.

Theorem 4 ([11] and [12]). *Let* $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ *be a spacelike CMC* 1 *immersion with Weierstraß data* (g, ω) *and Hopf differential* $Q = \omega \otimes dg$*. Then the hyperbolic Gauß map* $G^\sharp$*, Weierstraß data* $(g^\sharp, \omega^\sharp)$ *and Hopf differential* $Q^\sharp$ *of the dual spacelike CMC* 1 *immersion are, respectively, given by*

$$G^\sharp = g, \; g^\sharp = G, \; \omega^\sharp = -\frac{Q}{dG}, \; Q^\sharp = -Q.$$

6 The Construction of Spacelike CMC 1 Surfaces and Some Examples

Given Weierstraß data $(g, \omega = fdz)$, the equation (4.6) can be written as

$$\begin{pmatrix} F_1' & F_2' \\ F_3' & F_4' \end{pmatrix} = \begin{pmatrix} F_1 gf + F_2 g^2 f & -F_1 f - F_2 gf \\ F_3 gf + F_3 g^2 f & -F_3 f - F_3 gf \end{pmatrix}. \tag{6.10}$$

Here, $'$ stands for $\dfrac{d}{dz}$.

The equation (6.10) can be written simply as the following system of coupled first order linear equations:

$$\begin{cases} X' = gfX + g^2 fY, \\ Y' = -fX - gfY. \end{cases} \tag{6.11}$$

¿From this system, one can easily derive the second order linear equation

$$Y'' - \left(\frac{f'}{f}\right) Y' + fg' Y = 0 \tag{6.12}$$

This second order linear equation enables us to find the holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ which is the solution to the equation (6.10). In [17], M. Umehara and K. Yamada used a similar differential equation to (6.12) to study CMC 1 surfaces in $\mathbb{H}^3(-1)$.

Similarly, the equation (5.9) can be written as

$$\begin{pmatrix} F_4' & -F_2' \\ -F_3' & F_1' \end{pmatrix} = \begin{pmatrix} F_4 g^\sharp f^\sharp - F_2 {g^\sharp}^2 f^\sharp & -F_4 f^\sharp + F_2 g^\sharp f^\sharp \\ -F_3 g^\sharp f^\sharp + F_1 {g^\sharp}^2 f^\sharp & F_3 f^\sharp - F_1 g^\sharp f^\sharp \end{pmatrix} \tag{6.13}$$

with the Weierstraß data $(G = g^\sharp, W = f^\sharp dz)$. Again, this equation can be written simply as the system of first order linear equations

$$\begin{cases} X' = Xg^\sharp f^\sharp - Y(g^\sharp)^2 f^\sharp, \\ Y' = Xf^\sharp - Yg^\sharp f^\sharp. \end{cases} \tag{6.14}$$

¿From this system (6.14), we derive the second order linear differential equation

$$Y'' - \left[\frac{(f^\sharp)'}{f^\sharp}\right] Y' + f^\sharp (g^\sharp)' Y = 0. \tag{6.15}$$

By solving this differential equation, one can also construct dual CMC 1 spacelike surfaces in $\mathbb{S}^3_1(1)$ with the Weierstraß data $(g^\sharp, \omega^\sharp = f^\sharp dz)$.

Let us consider the following stereographic projections in order to view the isometric images of spacelike CMC 1 surfaces in $\mathbb{S}^3_1(1)$ into the exterior $\mathrm{Ext}\mathbb{H}^2(-1) = \{(\xi_0, \xi_1, \xi_2) \in \mathbb{E}^3_1 : -\xi_0^2 + \xi_1^2 + \xi_2^2 > -1\}$ of hyperbolic 2-space $\mathbb{H}^2(-1)$[4].

Let $\wp_+ : \mathbb{S}^3_1(1) \setminus \{\xi_3 = -1\} \longrightarrow \mathbb{E}^3_1 \setminus \mathbb{H}^2(-1)$ be the stereographic projection from $-e_3 = (0,0,0,-1)$. Then

$$\wp_+(\xi_0, \xi_1, \xi_2, \xi_3) = \left(\frac{\xi_0}{1+\xi_3}, \frac{\xi_1}{1+\xi_3}, \frac{\xi_2}{1+\xi_3}\right). \tag{6.16}$$

Let $\wp_- : \mathbb{S}^3_1(1) \setminus \{\xi_3 = 1\} \longrightarrow \mathbb{E}^3_1 \setminus \mathbb{H}^2(-1)$ be the stereographic projection from $e_3 = (0,0,0,1)$. Then

$$\wp_-(\xi_0, \xi_1, \xi_2, \xi_3) = \left(\frac{\xi_0}{1-\xi_3}, \frac{\xi_1}{1-\xi_3}, \frac{\xi_2}{1-\xi_3}\right). \tag{6.17}$$

Cut $\mathbb{S}^3_1(1)$ into two halves by the hyperplane $\xi_3 = 0$. Denote by $\mathbb{S}^3_1(1)_+$ ($\mathbb{S}^3_1(1)_-$) the half containing $e_3 = (0,0,0,1)$ ($-e_3 = (0,0,0,-1)$). Then $\wp_+ : \mathbb{S}^3_1(1)_+ \longrightarrow \mathrm{Ext}\mathbb{H}^2(-1)$ and $\wp_- : \mathbb{S}^3_1(1)_- \longrightarrow \mathrm{Ext}\mathbb{H}^2(-1)$.

Example 1 ***(Spacelike Enneper Cousin in** $\mathbb{S}^3_1(1)$**)***.* Let $(g, \omega) = (z, dz)$. Using the Umehara-Yamada type representation (4.6), we consider the following initial value problem:

$$F^{-1}dF = \begin{pmatrix} z & -1 \\ z^2 & -z \end{pmatrix} dz,\ F(0) = \sigma_0.$$

Then this I.V.P. has a unique solution

$$F = \begin{pmatrix} z\sin z + \cos z & -\sin z \\ -z\cos z + \sin z & \cos z \end{pmatrix}$$

which is a holomorphic null immersion into $\mathrm{SL}(2;\mathbb{C})$. Thus, the Bryant type representation formula (3.4) yields a spacelike CMC 1 surface in $\mathbb{S}^3_1(1)$. The resulting surface is a correspondent of spacelike Enneper surface (figure 3) in $\mathbb{E}^3_1$ under the Lawson correspondence. For this reason, the resulting surface is called *spacelike Enneper cousin* in $\mathbb{S}^3_1(1)$.

Figure 2 shows different views of the Enneper cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$.

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[4] Usually, the upper hyperboloid $\{(\xi_0, \xi_1, \xi_2) \in \mathbb{E}^3_1 : -\xi_0^2 + \xi_1^2 + \xi_2^2 = -1,\ \xi_0 > 0\}$ is called the hyperbolic 2-space $\mathbb{H}^2(-1)$. The upper hyperboloid is isometrically diffeomorphic to the Poincaré model of hyperbolic 2-space via the stereographic projection from $-e_0 = (-1,0,0)$ as seen in section 4. In this section, we regard the hyperboloid of two sheets $\{(\xi_0, \xi_1, \xi_2) \in \mathbb{E}^3_1 : -\xi_0^2 + \xi_1^2 + \xi_2^2 = -1\}$ as hyperbolic 2-space $\mathbb{H}^2(-1)$

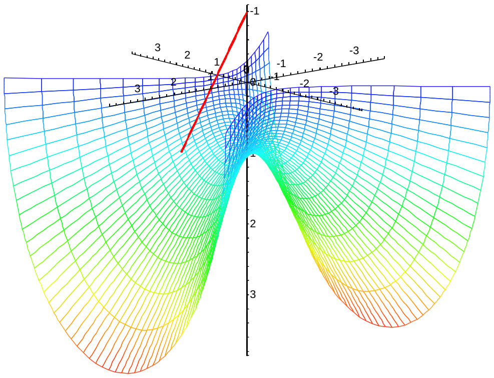

Fig. 1: $\wp_+ : \mathbb{S}^3_1(1)_+ \longrightarrow \mathrm{Ext}\mathbb{H}^2(-1)$

Note that representation formula (5.9) also can be used to construct a spacelike Enneper cousin in $\mathbb{S}^3_1(1)$ as mentioned. Using (5.9), we set up the following initial value problem with data $(G, W) = (z, dz)$:

$$FdF^{-1} = \begin{pmatrix} z & -1 \\ z^2 & -z \end{pmatrix} dz, \ F(0) = \sigma_0.$$

Then this I.V.P. has a unique solution

$$F = \begin{pmatrix} \cos z & \sin z \\ z\cos z - \sin z & \cos z + z\sin z \end{pmatrix}$$

which is a holomorphic null immersion into $\mathrm{SL}(2;\mathbb{C})$. The resulting surface by the formula (3.4) is also spacelike Enneper cousin in $\mathbb{S}^3_1(1)$. Remember that $G = z$ is the hyperbolic Gauß map of the resulting spacelike Enneper cousin in $\mathbb{S}^3_1(1)$.

Figure 4 shows different views of this spacelike Enneper cousin in $\mathbb{S}^3_1(1)$ projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$.

Example 2 ***(Spacelike Catenoid Cousin in*** $\mathbb{S}^3_1(1)$***).*** In this example, we construct a spacelike CMC 1 surface in $\mathbb{S}^3_1(1)$ which is a cousin of maximal catenoid (figure 6) in $\mathbb{E}^3_1$. This surface is called the *spacelike catenoid cousin* in $\mathbb{S}^3_1(1)$. Such a surface can be constructed using the data

$$g = z^\mu, \ \omega = az^{-\mu-1}$$

with $\mu \in \mathbb{R}^-$ and $a \in \mathbb{R}$ ([4],[17]). Here, we consider the case $\mu = -2$ and $a = 1$, i.e., $(g, \omega) = (z^{-2}, z)$. Then the equation (4.6) is

$$F^{-1}dF = \begin{pmatrix} z^{-2} & -1 \\ z^{-4} & -z^{-2} \end{pmatrix} zdz$$

$$= \begin{pmatrix} z^{-1} & -z \\ z^{-3} & -z^{-1} \end{pmatrix} dz.$$

This differential equation has a unique solution F with the initial condition $F(1) = \sigma_0$:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned}
F_{11} &= \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{-(1-\sqrt{3})} + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{-(1+\sqrt{3})} \\
F_{12} &= -(2-\sqrt{3}) \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{1+\sqrt{3}} - (2+\sqrt{3}) \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{1-\sqrt{3}} \\
F_{21} &= -(2+\sqrt{3}) \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{-(1-\sqrt{3})} - (2-\sqrt{3}) \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{-(1+\sqrt{3})} \\
F_{22} &= \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{1-\sqrt{3}} + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{1+\sqrt{3}}.
\end{aligned}$$

Figure 5 shows different views of the resulting spacelike catenoid cousin in $\mathbb{S}^3_1(1)$ from the holomorphic null immersion F into $\mathrm{SL}(2;\mathbb{C})$ by the formula (3.4).

Remark 5. Spacelike catenoid cousin in $\mathbb{S}^3_1(1)$ also can be constructed using the data $(g, \omega) = (-e^{-z}, -e^{z} dz)$.

Now, we construct spacelike catenoid cousin using the equation (5.9). Let $(G, W) = (z^{-2}, z dz)$. Then the equation (5.9) becomes

$$F dF^{-1} = \begin{pmatrix} z^{-1} & -z \\ z^{-3} & -z^{-1} \end{pmatrix} dz.$$

This differential equation has a unique solution F with the initial condition $F(1) = \sigma_0$:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned}
F_{11} &= \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{1+\sqrt{3}} + \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{1-\sqrt{3}}, \\
F_{12} &= (2-\sqrt{3}) \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{1+\sqrt{3}} + (2+\sqrt{3}) \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{1-\sqrt{3}}, \\
F_{21} &= -\frac{\sqrt{3}}{6} z^{-(1-\sqrt{3})} + \frac{\sqrt{3}}{6} z^{-(1+\sqrt{3})}, \\
F_{22} &= \left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) z^{-(1-\sqrt{3})} + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) z^{-(1+\sqrt{3})}.
\end{aligned}$$

Figure 7 shows the resulting spacelike catenoid cousin in $\mathbb{S}^3_1(1)$ from the holomorphic null immersion F into $\mathrm{SL}(2;\mathbb{C})$ by the formula (3.4).

Example 3 ***(Spacelike Helicoid Cousin in** $\mathbb{S}^3_1(1)$**)**. Spacelike helicoid cousin in $\mathbb{S}^3_1(1)$ can be constructed with data

$$g = z^{\mu},\ \omega = a\sqrt{-1}z^{-\mu-1}$$

with $\mu \in \mathbb{R}^-$ and $a \in \mathbb{R}$. Here, we consider the case $\mu = -2$ and $a = 1$, i.e., $(g, \omega) = (z^{-2}, \sqrt{-1}z)$. Then the equation (4.6) is

$$\begin{aligned} F^{-1}dF &= \begin{pmatrix} z^{-2} & -1 \\ z^{-4} & -z^{-2} \end{pmatrix} \sqrt{-1}zdz \\ &= \begin{pmatrix} \sqrt{-1}z^{-1} & -\sqrt{-1}z \\ \sqrt{-1}z^{-3} & -\sqrt{-1}z^{-1} \end{pmatrix} dz. \end{aligned}$$

This differential equation has a unique solution F with the initial condition $F(1) = \sigma_0$:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned} F_{11} = &-\frac{\sqrt{-1}}{5}\sqrt{1+\sqrt{-1}}\Bigg\{\left(1+\frac{\sqrt{-1}}{2}\right)\left(1+\sqrt{1+2\sqrt{-1}}\right)z^{-\square 1-\sqrt{1+2\sqrt{-1}}\square} \\ &-\left(1-\sqrt{1+2\sqrt{-1}}\right)z^{-\square 1+\sqrt{1+2\sqrt{-1}}\square}\Bigg\}, \\ F_{12} = &-\frac{\sqrt{1+2\sqrt{-1}}}{5}\left(1+\frac{\sqrt{-1}}{2}\right)(z^{1+\sqrt{1+2\sqrt{-1}}} - z^{1-\sqrt{1+2\sqrt{-1}}}), \\ F_{21} = &\frac{\sqrt{-1}}{2}\left(1-\frac{3}{5}\sqrt{1+2\sqrt{-1}}+\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) \\ &\left(1+\sqrt{-1}+\sqrt{1+2\sqrt{-1}}\right)z^{-\square 1-\sqrt{1+2\sqrt{-1}}\square} \\ &+\frac{\sqrt{-1}}{2}\left(1+\frac{3}{5}\sqrt{1+2\sqrt{-1}}-\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) \\ &\left(1+\sqrt{-1}-\sqrt{1+2\sqrt{-1}}\right)z^{-\square 1+\sqrt{1+2\sqrt{-1}}\square}, \\ F_{22} = &\frac{1}{2}\left(1-\frac{3}{5}\sqrt{1+2\sqrt{-1}}+\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right)z^{1+\sqrt{1+2\sqrt{-1}}} \\ &+\frac{1}{2}\left(1+\frac{3}{5}\sqrt{1+2\sqrt{-1}}-\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right)z^{1-\sqrt{1+2\sqrt{-1}}}. \end{aligned}$$

Figure 8 shows different views of the resulting sapcelike helicoid cousin from the holomorphic null immersion F into $\mathrm{SL}(2;\mathbb{C})$ by the formula (3.4).

Remark 6. Spacelike helicoid cousin in $\mathbb{S}^3_1(1)$ also can be constructed using the data $(g, \omega) = (-e^{-z}, -\sqrt{-1}e^{z}dz)$.

In order to construct spacelike helicoid cousin in $\mathbb{S}^3_1(1)$ using the equation (5.9), we set up the initial value problem:

$$FdF^{-1} = \begin{pmatrix} \sqrt{-1}z^{-1} & -\sqrt{-1}z \\ \sqrt{-1}z^{-3} & -\sqrt{-1}z^{-1} \end{pmatrix} dz,\ F(1) = \sigma_0$$

with data $(G, W) = (z^{-2}, \sqrt{-1}zdz)$. This I.V.P. has a unique solution $F \in \mathrm{SL}(2;\mathbb{C})$ which is a holomorphic null immersion:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where

$$\begin{aligned}
F_{11} &= \frac{1}{5}\sqrt{1+2\sqrt{-1}}\left(1+\frac{\sqrt{-1}}{2}\right)\Bigg[\left\{1-\sqrt{-1}\left(1+\sqrt{1+2\sqrt{-1}}\right)\right\} \\
&\quad z^{1-\sqrt{1+2\sqrt{-1}}} - \left\{1-\sqrt{-1}\left(1-\sqrt{1+2\sqrt{-1}}\right)\right\} z^{1+\sqrt{1+2\sqrt{-1}}}\Bigg], \\
F_{12} &= \frac{1}{2}\left(1-\frac{3}{5}\sqrt{1+2\sqrt{-1}}+\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) \\
&\quad \left\{1-\sqrt{-1}\left(1+\sqrt{1+2\sqrt{-1}}\right)\right\} z^{1-\sqrt{1+2\sqrt{-1}}} \\
&\quad + \frac{1}{2}\left(1+\frac{3}{5}\sqrt{1+2\sqrt{-1}}-\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) \\
&\quad \left\{1-\sqrt{-1}\left(1-\sqrt{1+2\sqrt{-1}}\right)\right\} z^{1+\sqrt{1+2\sqrt{-1}}}, \\
F_{21} &= -\frac{1}{5}\sqrt{1+2\sqrt{-1}}\left(1+\frac{\sqrt{-1}}{2}\right)\left(z^{-\square\,1-\sqrt{1+2\sqrt{-1}}\,\square} - z^{-\square\,1+\sqrt{1+2\sqrt{-1}}\,\square}\right), \\
F_{22} &= \frac{1}{2}\left(1+\frac{3}{5}\sqrt{1+2\sqrt{-1}}-\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) z^{-\square\,1-\sqrt{1+2\sqrt{-1}}\,\square} \\
&\quad + \frac{1}{2}\left(1-\frac{3}{5}\sqrt{1+2\sqrt{-1}}+\frac{\sqrt{-1}}{5}\sqrt{1+2\sqrt{-1}}\right) z^{-\square\,1+\sqrt{1+2\sqrt{-1}}\,\square}.
\end{aligned}$$

Figure 10 shows different views of the resulting spacelike helicoid cousin from the holomorphic null immersion F by the formula (3.4).

K. Akutagawa ([2]) and J. Ramanathan ([16]) proved the following theorem.

Theorem 5. *Let M be a complete spacelike surface in $\mathbb{S}^3_1(1)$ with constant mean curvature $H = \pm 1$. Then M is a totally umbilic flat surface. Moreover, M is a parabolic type surface of revolution.*

The first part of this theorem can be proven easily using the 1 : 1 correspondence that we discussed in section 4. The following lemma is needed to prove Theorem 5.

Lemma 6 ([5], [6], [9]). *The only complete spacelike maximal surface in $\mathbb{E}^3_1$ is spacelike planes.*

Now, we prove Theorem 5 (only the first part).

Proof. Let $\varphi : M \longrightarrow \mathbb{S}^3_1(1)$ be a complete spacelike CMC 1 surface in $\mathbb{S}^3_1(1)$. Then, by Theorem 1, there exists a holomorphic null immersion $F : M \longrightarrow \mathrm{SL}(2;\mathbb{C})$ such that $\varphi = F\sigma_3F^*$.

The holomorphic null immersion F satisfies the equation (5.9)

$$FdF^{-1} = \begin{pmatrix} G & -1 \\ G^2 & -G \end{pmatrix} W,$$

where G is the hyperbolic Gauß map of φ. The corresponding maximal spacelike surface in $\mathbb{E}^3_1$ is found by the Weierstraß representation formula (4.7) with data (G, W). This maximal spacelike surface is also complete, so by Lemma 6 it is a spacelike plane. Note that G is the projected Gauß map of the maximal spacelike surface. (See [9] or [14] for more details.) Hence, it must be constant and so, by Corollary 3, φ is totally umbilic. The Gaußian curvature of φ is then $K = 0$ by equation (2.2). This completes the proof. □

By Theorem 5 and Corollary 3, the hyperbolic Gauß map of complete spacelike CMC ±1 surfaces is constant. Hence, they can be regarded as an analogue of *horospheres* in $\mathbb{H}^3(-1)$.

Example 4 ***(Horosphere type spacelike surfaces in*** $\mathbb{S}^3_1(1)$***).*** We use the data $(G, W) = (0, dz)$ to construct an example of such horosphere type spacelike surfaces in $\mathbb{S}^3_1(1)$. The I.V.P.

$$FdF^{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} dz, \; F(0) = \sigma_0$$

has a unique solution

$$F = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

and

$$\varphi = F\sigma_3 F^* = \begin{pmatrix} 1 - |z|^2 & -z \\ -\bar{z} & -1 \end{pmatrix} \tag{6.18}$$

is a horosphere type spacelike surface in $\mathbb{S}^3_1(1)$ described in Theorem 5.

Figure 11 shows different views of the horosphere type spacelike surface (6.18).

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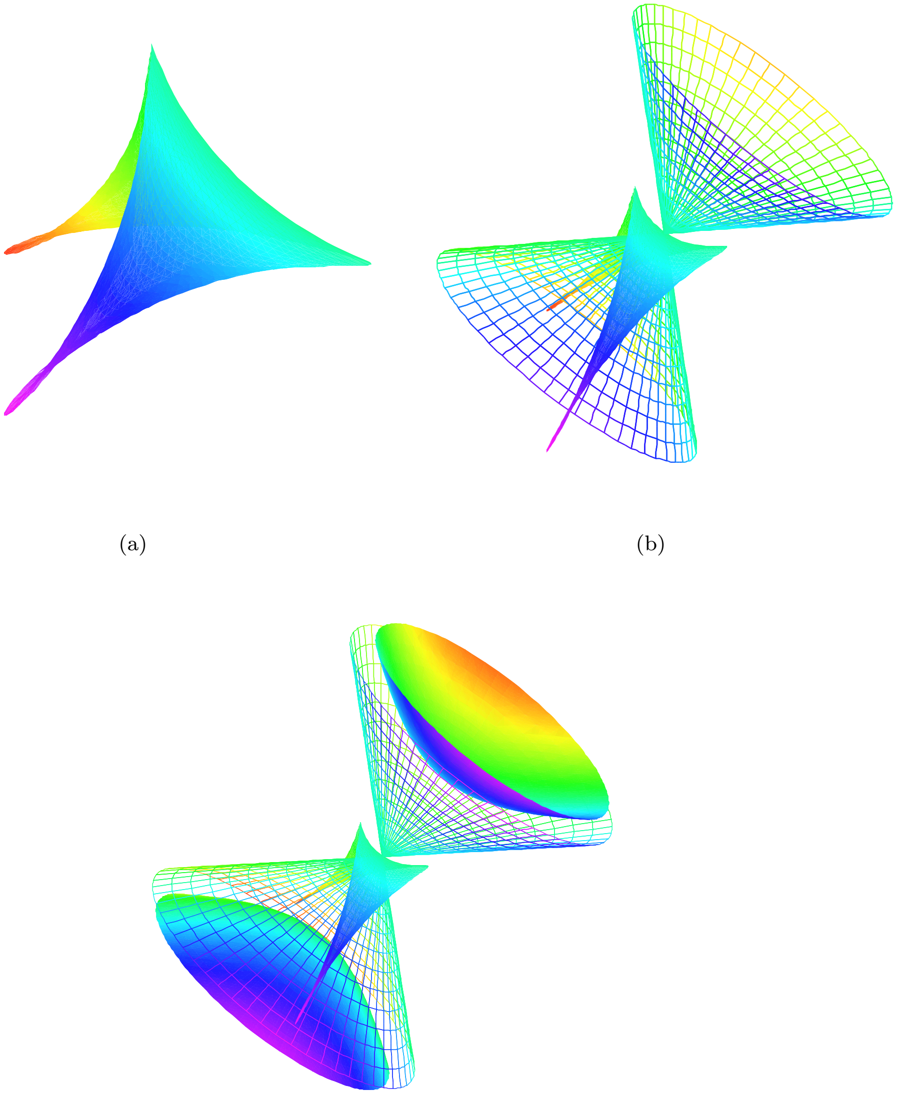
Sungwook Lee
Department of Mathematics,
University of Southern Mississippi,
Hattiesburg, MS 39406-5045, U.S.A.,
email: sunglee@usm.edu

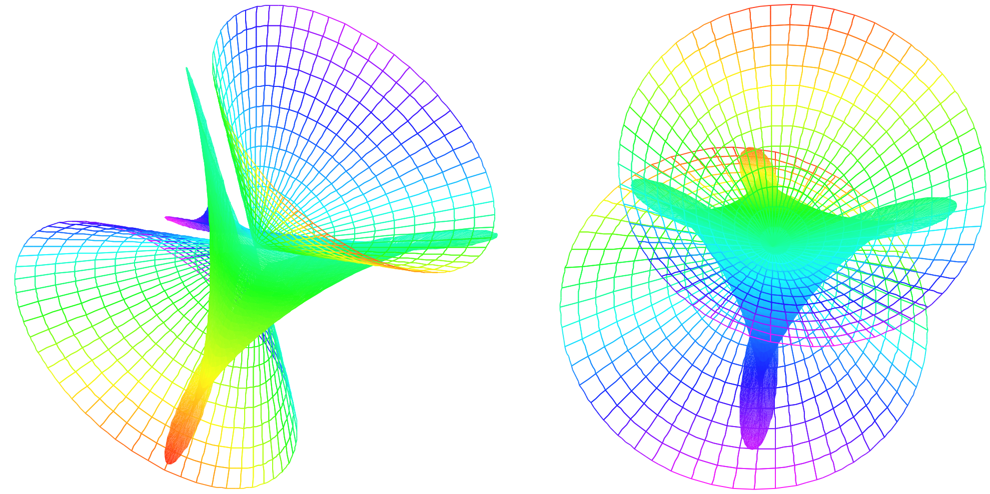


(a)

(b)

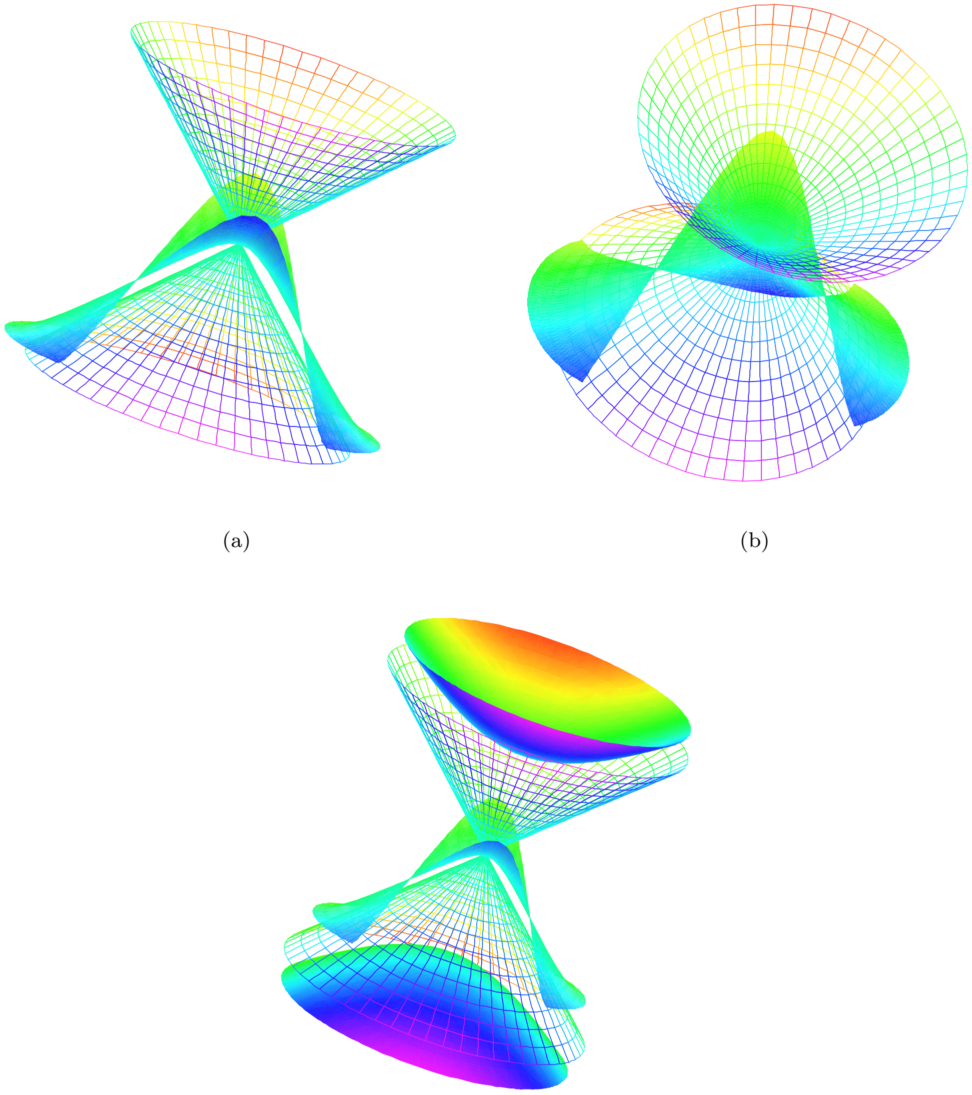
(c)

Fig. 2: Spacelike Enneper cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$ with light cone in $\mathbb{E}^3_1$ and the boundary $\mathbb{H}^2(-1)$



(a) (b)

Fig. 3: Spacelike Enneper surface in $\mathbb{E}^3_1$

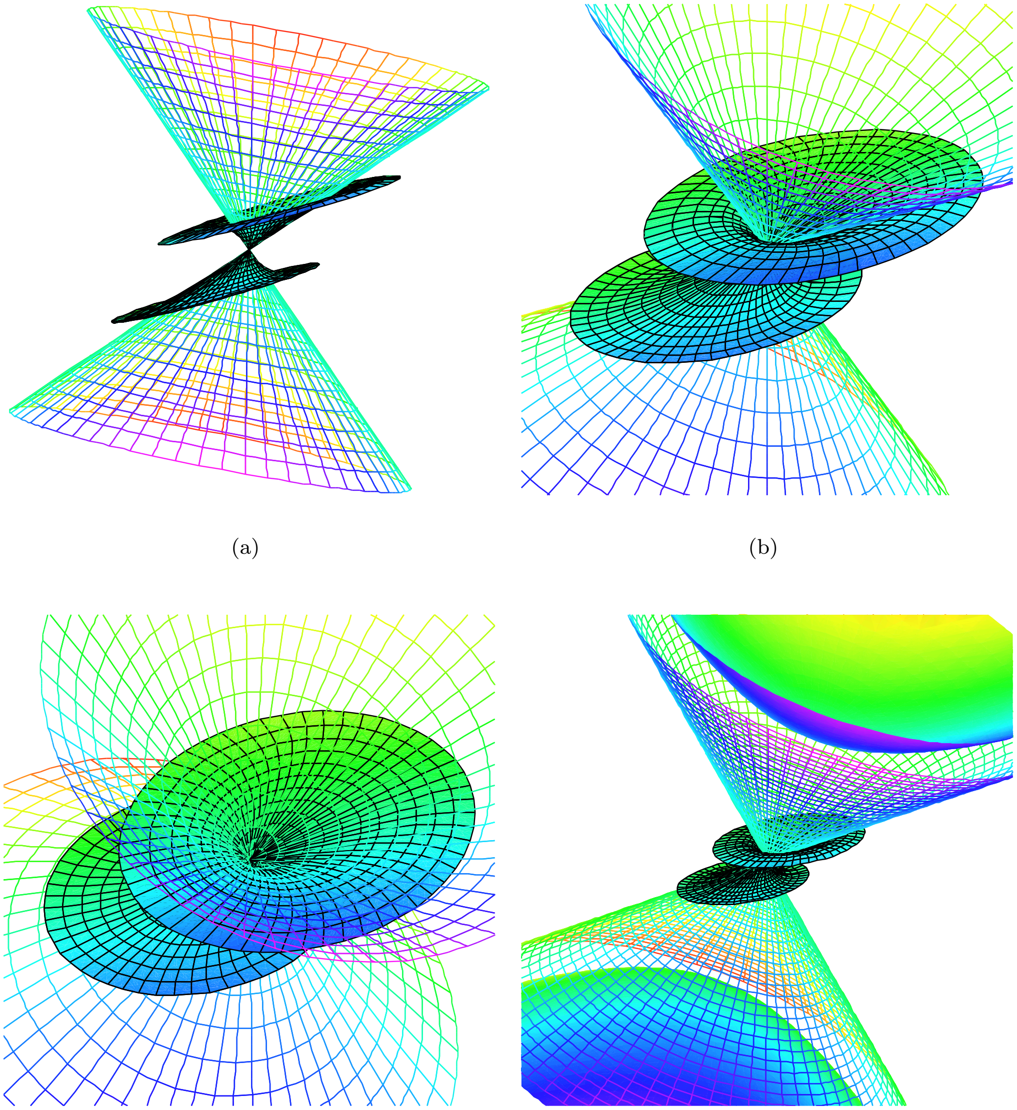


(a)

(b)

(c)

Fig. 4: Spacelike Enneper cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$ with light cone in $\mathbb{E}^3_1$ and the boundary $\mathbb{H}^2(-1)$



(a) (b)

(c) (d)

Fig. 5: Spacelike catenoid cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$ with light cone in $\mathbb{E}^3_1$ and the boundary $\mathbb{H}^2(-1)$

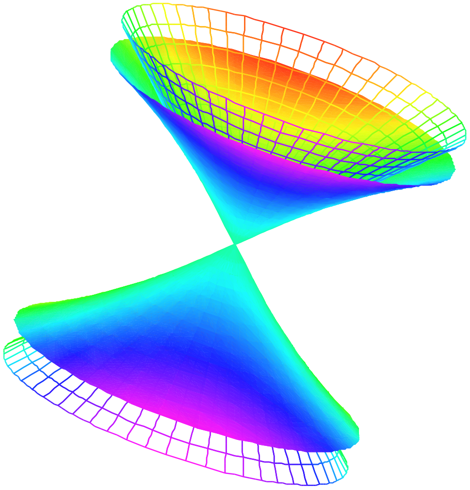
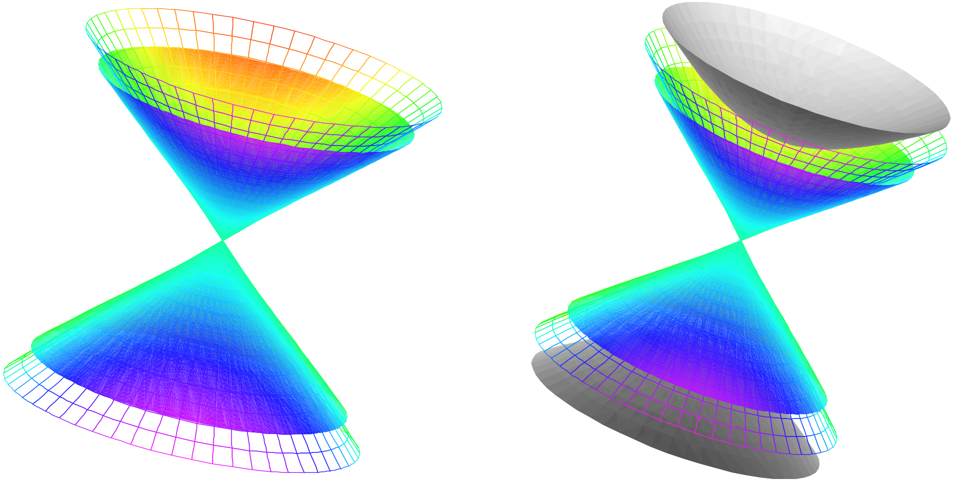
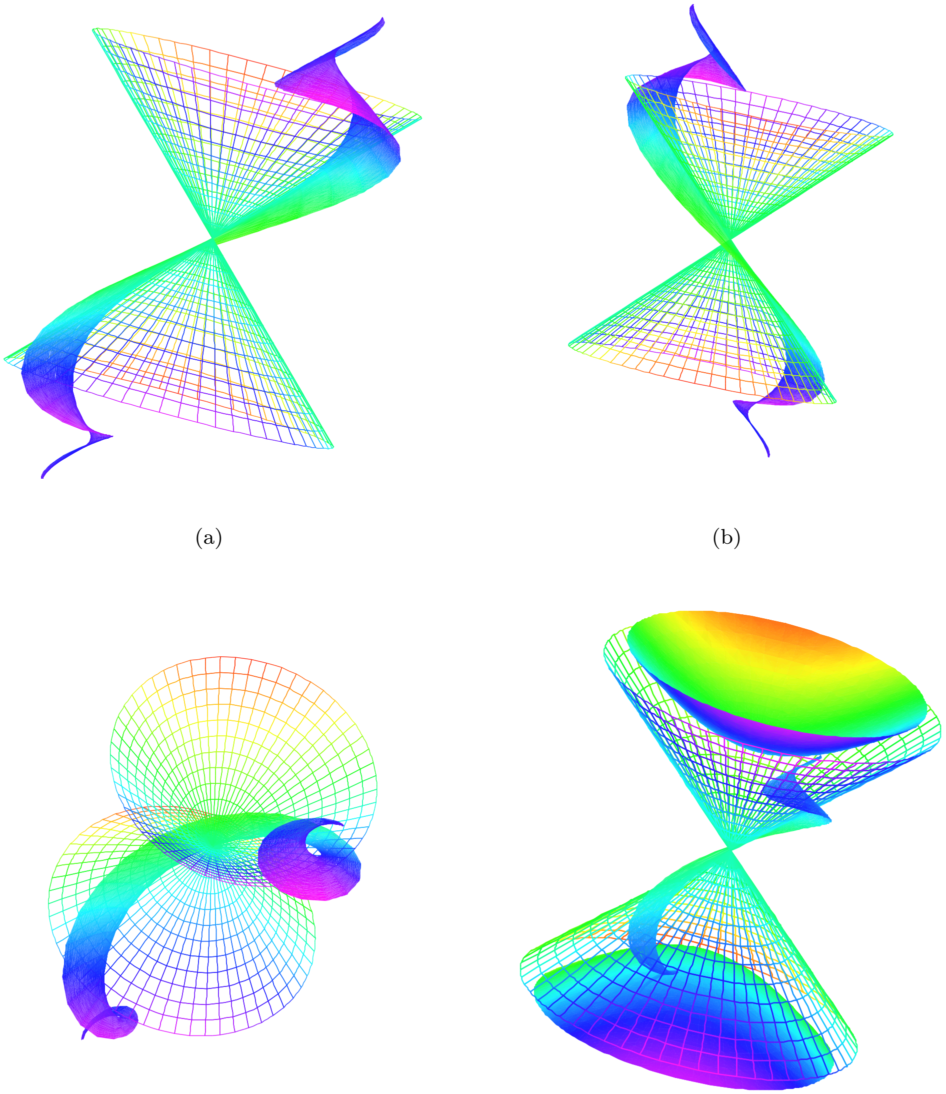


Fig. 6: Spacelike catenoid in $\mathbb{E}^3_1$



(a) (b)

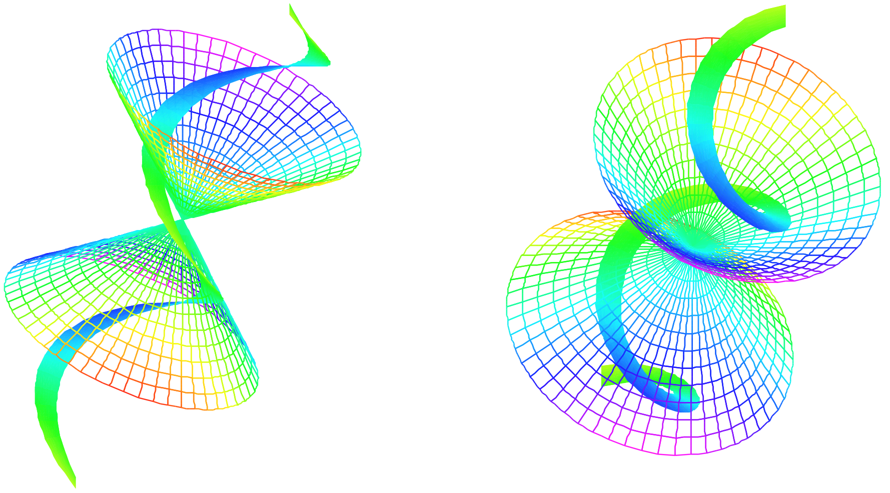
Fig. 7: Spacelike catenoid cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$



(a) (b)

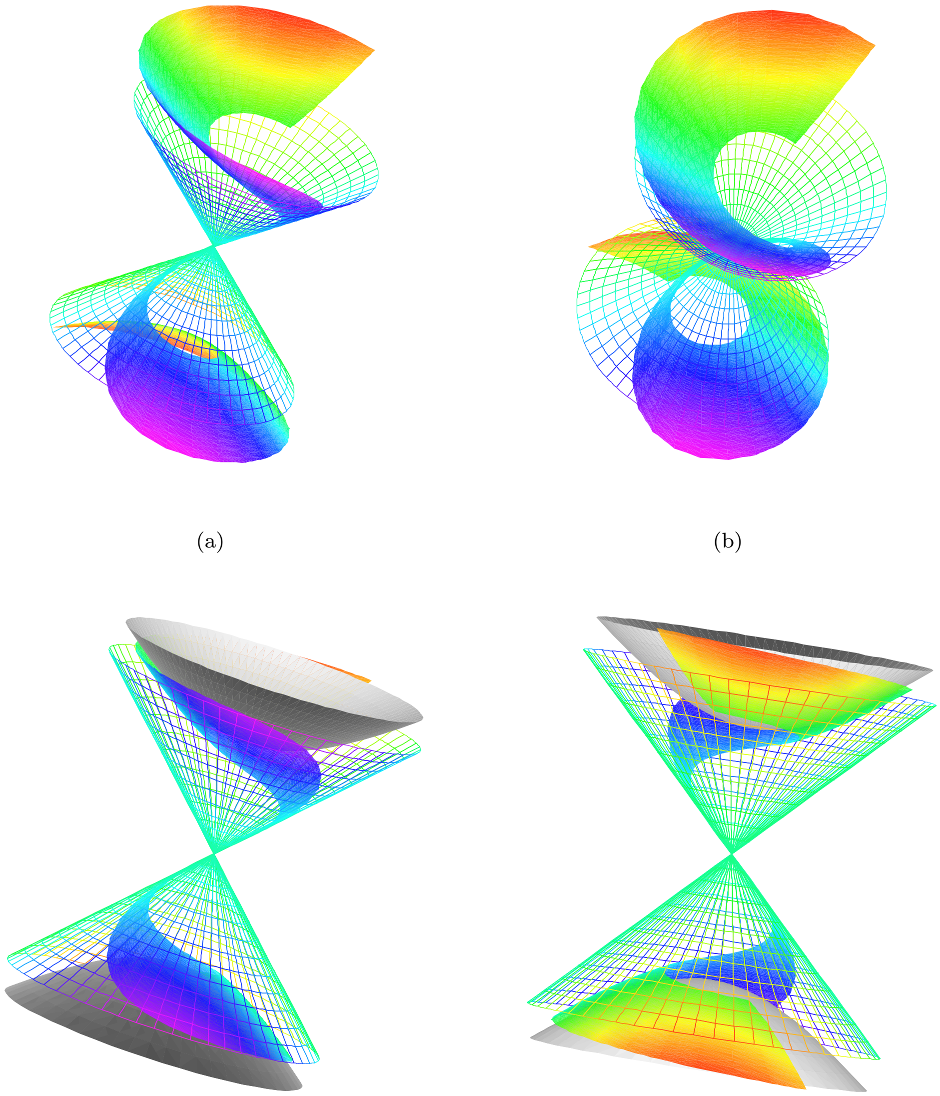
(c) (d)

Fig. 8: Spacelike helicoid cousin projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$



(a) (b)

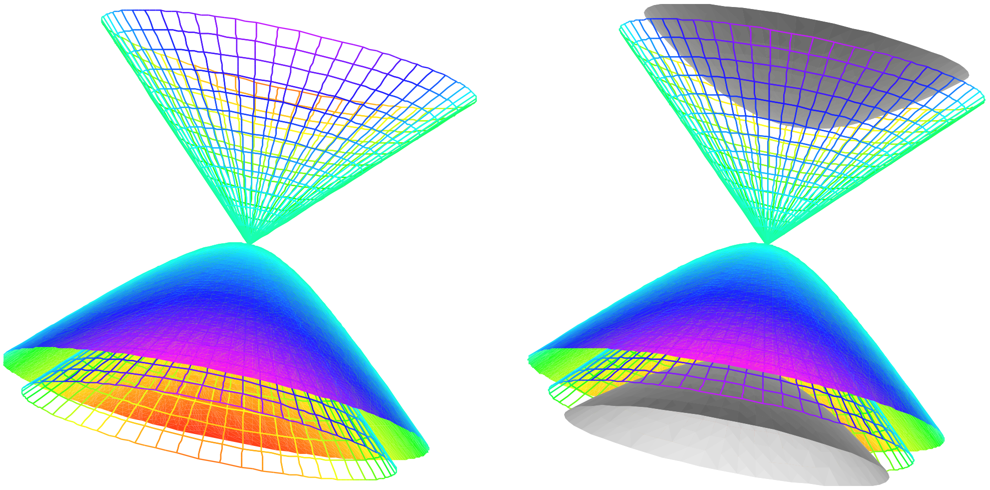
Fig. 9: Spacelike helicoid in $\mathbb{E}^3_1$



(a) (b)

(c) (d)

Fig. 10: Spacelike helicoid cousin projected into Ext$\mathbb{H}^2(-1)$ via $\wp_+$



(a) (b)

Fig. 11: Horosphere type spacelike surface (6.18) projected into $\mathrm{Ext}\mathbb{H}^2(-1)$ via $\wp_+$ with light cone and the boundary $\mathbb{H}^2(-1)$