On the singly periodic genus one helicoid

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Abstract. The main objective of this paper is to introduce the singly periodic genus one helicoid. We will exhibit some Weierstrass representations and a uniqueness theorem for it.

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1 Introduction and preliminaries

One of the fundamental problems in classical theory of minimal surfaces is the existence of complete examples in the three dimensional Euclidean space. Complete minimal surfaces of finite total curvature have some very special properties that are not shared by general minimal surfaces. Osserman began the study of this family of surfaces. He proved that a complete immersed minimal surface whose total curvature is finite must have finite topology (see [9]). Therefore a such surface,

$$X = (X_1, X_2, X_3) : M \longrightarrow \mathbb{R}^3 ,$$

is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured in a finite number of points $P_1, \ldots, P_r$:

$$M = \overline{M} \setminus \{P_1, \ldots, P_r\} .$$

These points are called the ends of M.

1.1 The Weierstrass representation

We denote by $g : M \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the stereographically projected Gauss map of X and by Φ_3 the holomorphic differential defined as

$$\Phi_3 = dX_3 + \mathrm{i} dX_3^* ,$$

where X_3^* denotes the harmonic conjugate function of X_3.

In this setting, the pair (g, Φ_3) is usually called the Weierstrass data of the minimal surface, and X can be expressed up to translations as

$$X = \mathrm{Re} \int^z \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{\mathrm{i}}{2}\left(\frac{1}{g} + g\right), 1 \right) \Phi_3 , \tag{1.1.1}$$

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where Re stands for real part and z is a conformal parameter on M. The pair (g, Φ_3) satisfies certain compatibility conditions:

(A) The zeros of Φ_3 coincide with the poles and zeros of g, and the order of a point as zero of Φ_3 is the same than its order as zero or pole of g;

(B) Given a closed curve $\gamma \subset M$, then

$$\overline{\int_\gamma g\Phi_3} = \int_\gamma \frac{\Phi_3}{g} , \quad \mathrm{Re} \int_\gamma \Phi_3 = 0 .$$

The condition (B) is usually called the period problem. Conversely, we can construct minimal surfaces by the following way:

Theorem 1.1 *Let M be a Riemann surface, $g : M \to \overline{\mathbb{C}}$ a meromorphic function and Φ_3 a holomorphic 1-form on M satisfying the compatibility conditions (A) and (B). Then the map $X : M \to \mathbb{R}^3$ defined as* (1.1.1) *is a conformal minimal immersion. Moreover, the pair (g, Φ_3) corresponds with the Weierstrass data of X.*

1.2 Periodic minimal surfaces

A classical example of minimal surface is the helicoid (Figure 1.1), which was discovered in 1776 by Meusnier. Its Weierstrass representation is given by

$$M = \mathbb{C}^* , \quad g = z , \quad \Phi_3 = \frac{\mathrm{i}}{z} \, dz .$$

The helicoid is connected and invariant under a vertical translation. We say that a minimal surface is periodic if it is connected and invariant under a group of isometries that acts freely on $\mathbb{R}^3$.

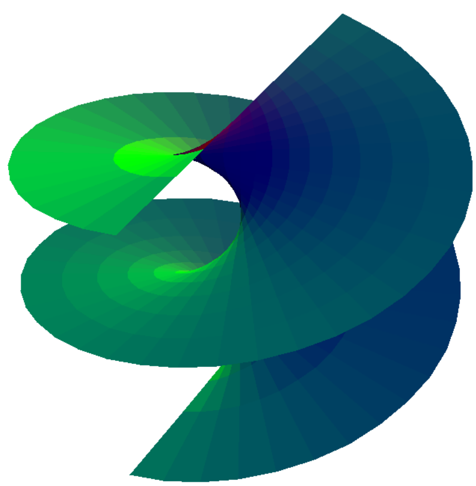

Figure 1.1: The helicoid

An end of a complete surface of finite topology is necessarily annular. That is, it is homeomorphic to a punctured disk. On the other hand, an annular end of a properly embedded periodic minimal surface of finite topology has to be asymptotic to an end of a helicoid, to a flat annulus or to a plane (see [3], [7]). We shall say that the end is a helicoidal end, a Scherk end, or a planar end, respectively.

2 The singly periodic genus one helicoid

In the last few years, one of the most active focus in the study of minimal surfaces has been the genus one helicoid. The existence of such a surface was proved by Hoffman, Karcher and Wei in [4], and it was the first example of an embedded minimal surface with infinite total curvature and finite topology.

One important step in the discovery of the genus one helicoid was the construction by Hoffman, Karcher and Wei of the singly periodic genus one helicoid (see [5]), which will be represented as $\mathcal{H}_1$.

2.1 Geometric description

$\mathcal{H}_1$ is a complete embedded singly periodic minimal surface which is invariant under a vertical translation so that the quotient has genus one and two ends. Moreover, $\mathcal{H}_1$ contains a vertical line, its ends in the quotient by the vertical translation are helicoidal ends, and the quotient has two parallel horizontal lines that are different in half of the period and each meeting the vertical line in a single point. You can see three fundamental pieces of $\mathcal{H}_1$ and their lines in Figure 2.1.

As a consequence of the existence of such lines, the Schwarz Reflection Principle guarantees that the nontrivial symmetries of the surface, modulo vertical translations, consist of:

- 180 rotation about the vertical line.
- 180 rotation about the horizontal lines, the symmetry does not depend on the line because they are different in half of the period.
- The composition of the two previous ones, which is a 180 rotation about a line orthogonal to all the lines on the surface and passing through their point of intersection.

Now, it is easy to check that the total curvature of the quotient surface is $C = -8\pi$. In fact, since the surface is a torus punctured twice, its Euler characteristic is $\chi = -2$ and since both ends are helicoidal ends we obtain that its total winding number is $W = 2$. Therefore, taking into account that $C = -2\pi(W - \chi)$ we conclude that $C = -8\pi$.

2.2 Determination of the Weierstrass data

Hoffman, Karcher and Wei proved in [5, Theorem 1] that this geometric description leads to an explicit two-parameter family of Weierstrass data:

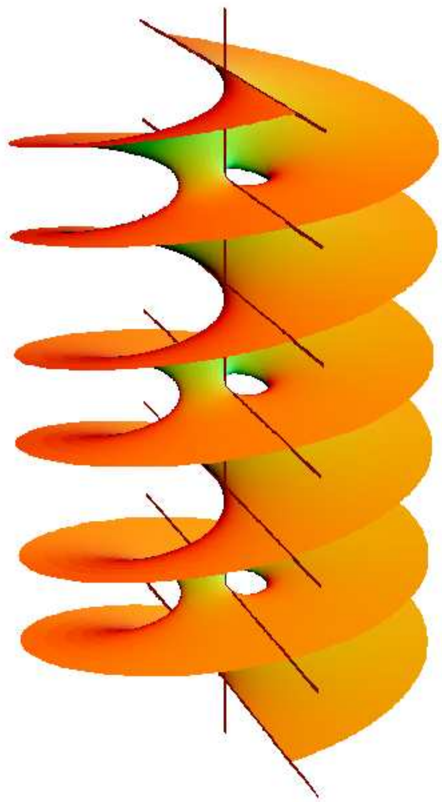

Figure 2.1: $\mathcal{H}_1$ and its lines

Theorem 2.1 *Any surface satisfying the previous geometric description is representable with Weierstrass data of the form*

$$g(x,y) = \frac{y - r\sqrt{\mathrm{i}}}{y + r\sqrt{\mathrm{i}}}\,, \quad \Phi_3(x,y) = \frac{x - \mathrm{i}\lambda}{x - \frac{\mathrm{i}}{\lambda}}\,\frac{\sqrt{\mathrm{i}}}{x^2 - 1 + 2\mathrm{i}x\cos\theta}\,\frac{dx}{y}\,, \tag{2.2.1}$$

on the rombic torus

$$T = \left\{ (x,y) \in \overline{\mathbb{C}}^2 \mid y^2 = \frac{-2x\sin\theta}{x^2 - 1 + 2\mathrm{i}x\cos\theta} \right\}\,,$$

punctured at the two points verifying $x = \mathrm{i}/\lambda$, *where* r *is given by*

$$\lambda + \frac{1}{\lambda} = -2\cos\theta + \frac{2\sin\theta}{r^2}\,, \tag{2.2.2}$$

with $\lambda \in (0,1)$ *and* $\theta \in (0,\pi)$.

They also proved (see [5, Theorem 2]) that there exist $\lambda \in (0,1)$ and $\theta \in (0,\pi)$ such that the Weierstrass data (2.2.1) satisfy the period problem. By this way, they obtained the existence of the singly periodic genus one helicoid.

2.3 A first approach to the uniqueness of $\mathcal{H}_1$

In order to give a uniqueness result for $\mathcal{H}_1$, Ferrer and Martn have obtained the following result (see [2, Remark 4]):

Theorem 2.2 *Any complete, periodic, minimal surface containing a vertical line, whose quotient by vertical translations has genus one, contains two parallel horizontal lines, has two helicoidal ends and total curvature -8π is $\mathcal{H}_1$.*

The contribution of Ferrer and Martn consists of giving another approach to the proof of the uniqueness of the period problem, they proved that there is only one pair of this parameter that solves the period problem. In order to do it, they show an alternative Weierstrass representation for $\mathcal{H}_1$. They proved that there exist $b \in]0,1[$ and $\rho \in]0,\pi[$ such that $\mathcal{H}_1$ is representable by the data

$$\widetilde{g}(u,v) = \frac{u+b^2}{b^2u+1}\ , \quad \widetilde{\Phi}_3(u,v) = \frac{1}{2u}\frac{v-c_1\mathrm{i}}{v+c_1\mathrm{i}}\frac{du}{v}\ , \tag{2.2.3}$$

on the torus

$$\widetilde{T} = \left\{(u,v) \in \overline{\mathbb{C}}^2 \mid v^2 = u + \frac{1}{u} - 2\cos\rho\right\}$$

punctured at the two points with $v = -\mathrm{i}c_1$, where

$$c_1 = \frac{1}{b}\sqrt{b^4 + 1 + 2b^2\cos\rho}\ .$$

2.4 One more step on the uniqueness of $\mathcal{H}_1$

After this, we have proved the following uniqueness theorem for $\mathcal{H}_1$ that improves the aforementioned one (see [1, Theorem 2]). The improvement consists of removing one symmetry:

Theorem 2.3 *Any properly embedded, singly periodic minimal surface that is symmetric respect to a vertical line, whose quotient by a vertical translation has genus one, two helicoidal ends and total curvature -8π is $\mathcal{H}_1$.*

In this work, we use another Weierstrass representation for $\mathcal{H}_1$. We consider suitables $a \in (0,1)$, $\eta \in (0,2\pi)$ and we have the Weierstrass data

$$\widehat{g}(\alpha,\beta) = \alpha\ , \quad \widehat{\Phi}_3(\alpha,\beta) = \frac{\beta+\sqrt{a}}{\beta-\sqrt{a}}\frac{\mathrm{i}}{(\alpha-e^{\mathrm{i}\eta})(\alpha-e^{-\mathrm{i}\eta})}\frac{d\alpha}{\beta}\ , \tag{2.2.4}$$

on the torus

$$\widehat{T} = \left\{(\alpha,\beta) \in \overline{\mathbb{C}}^2 \mid \beta^2 = \frac{(\alpha-a)(a\alpha-1)}{(\alpha-e^{\mathrm{i}\eta})(\alpha-e^{-\mathrm{i}\eta})}\right\}$$

punctured at the two points satisfying $\beta = \sqrt{a}$.

3 Appendix

In this section we will show that all the Weierstrass representations exhibited along this paper are equivalent.

Proposition 3.1 *The Weierstrass representations given by tha data (2.2.3) on the torus $\widetilde{T}$ and by (2.2.4) on $\widehat{T}$ are aquivalent.*

Proof. Take $a = b^2$ and $\eta \in (0, 2\pi)$ such that

$$e^{\mathrm{i}\eta} = \frac{e^{\mathrm{i}\rho} + b^2}{b^2 e^{\mathrm{i}\rho} + 1} .$$

Now, the map $\mathcal{F} : \widehat{T} \to \widetilde{T}$ defined as

$$\mathcal{F}(\alpha, \beta) = \left(\frac{b^2 - \alpha}{b^2\alpha - 1} , \frac{-\mathrm{i}bc_1}{\beta} \right) ,$$

is a conformal biholomorphism which identifies the Weierstrass data (2.2.3) with (2.2.4), up to multiplying $\widehat{\Phi}_3$ by a real constant. Therefore, taking into account (1.1.1) we conclude the proof of the proposition. □

Proposition 3.2 *The Weierstrass representations given by the data (2.2.1) on the torus T and by (2.2.3) on $\widetilde{T}$ are aquivalent.*

Proof. Choose $\theta = \rho$ and consider the torus

$$\overline{T} = \left\{ w^2 = (z-1)(z+1) \left(z - \mathrm{i}\frac{\sin\rho}{1+\cos\rho} \right) \left(z + \mathrm{i}\frac{\sin\rho}{1+\cos\rho} \right) \right\} ,$$

and the map $\mathcal{F}_1 : \overline{T} \to T$ given by

$$\mathcal{F}_1(z, w) = \left(\mathrm{i}(1+\cos\rho) \left(w + z^2 - \frac{\cos\rho}{1+\cos\rho} \right) , \sqrt{\mathrm{i}\frac{\sin\rho}{1+\cos\rho}\frac{1}{z}} \right) .$$

Then, $\mathcal{F}_1$ is a conformal biholomorphism which gets the Weierstrass data (2.2.1) on to the data

$$g^1(z,w) = \frac{1 - r\sqrt{\frac{1+\cos\rho}{\sin\rho}}z}{1 + r\sqrt{\frac{1+\cos\rho}{\sin\rho}}z} , \quad \Phi_3^1(z,w) = \frac{w + z^2 - \frac{\lambda+\cos\rho}{1+\cos\rho}}{w + z^2 - \frac{1/\lambda+\cos\rho}{1+\cos\rho}}\tau^1 , \tag{3.3.1}$$

on the torus $\overline{T}$, where τ^1 is a holomorphic 1-form on $\overline{T}$.

In the same way, the map $\mathcal{F}_2 : \overline{T} \to \widetilde{T}$ defined as

$$\mathcal{F}_2(z, w) = \left(\frac{1-z}{1+z} , -2\mathrm{i}\cos(\rho/2)\frac{w}{z^2-1} \right)$$

is a conformal biholomorphism which identifies the data (2.2.3) with the Weierstrass data on $\overline{T}$

$$g^2(z,w) = \frac{1-\frac{1-b^2}{1+b^2}z}{1+\frac{1-b^2}{1+b^2}z}\,, \quad \Phi_3^2(z,w) = \frac{\frac{1}{2}\frac{c_1}{\cos(\rho/2)}(z^2-1)+w}{\frac{1}{2}\frac{c_1}{\cos(\rho/2)}(z^2-1)-w}\tau^2\,, \tag{3.3.2}$$

where τ^2 is a real multiple of τ^1. Therefore, choosing

$$r = \frac{1-b^2}{1+b^2}\sqrt{\frac{\sin\rho}{1+\cos\rho}}\,,$$

and using (2.2.2), it is a straightforward computation to check that $g^1 = g^2$ and $\Phi_3^1 = \lambda\frac{\tau^1}{\tau^2}\Phi_3^2$. Hence, taking into account (1.1.1) we obtain that the Weierstrass data (3.3.1) and (3.3.2) are equivalent. This fact finishes the proof. □

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