Linear connections on the cotangent bundle of order two $(T^{*2}M, \pi^{*2}, M)$

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Abstract. In the recent theory on the Hamilton space of the order two (see R.Miron [9], [10], where $k = 2$, and R. Miron and his partners [11]), a linear connection has the smallest possible number of sets of components, namely three, given by $\left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$, $(a, b, c = 1, ..., n;\ n = \dim M)$. In this paper a linear connection in the differential geometry of the 2-cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$, needed in the study of the Hamilton spaces of the order two, will have nine sets of coefficients. The whole calculus: the parallelism of vector fields, the torsion, the curvature, the structure equations, the Ricci and Bianchi identities, etc., can be based of the set of these nine coefficients (see [6]).

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1 The manifold $T^{*2}M$.Nonlinear connection.

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinate system by $(U, \varphi), \varphi(x) = (x^a)$. The indices $a, b, ...$ run over set $\{1, ..., n\}$ and Einstein convention of summarizing is adapted all over this work. Let (TM, π, M) be the tangent bundle of the manifold M and (T^*M, π^*, M) its cotangent bundle, [11], [15].

Definition 1.1 ([11]) We call the dual bundle of the 2−tangent bundle (T^2M, π^2, M), the differentiable bundle $(T^{*2}M, \pi^{*2}, M)$ whose total space is

$$T^{*2}M = TM \ \times_M \ T^*M. \tag{1.1}$$

Sometimes we denote $\left(T^{*2}M, \pi^{*2}, M\right)$ by $T^{*2}M$.A point $u \in T^{*2}M$ will be denoted by $u = (x, y, p)$ having the local coordinates (x^a, y^a, p_a). The projection $\pi^{*2}(u) = \pi^{*2}(x, y, p) = x$. Evidently, we take the projections on the factors of the fibered products (1.1):$\pi_1^{*2} : T^{*2}M \longrightarrow TM$, $\pi : TM \longrightarrow M$ as being $\pi_1^{*2}(x, y, p) = (x, y)$ and $\pi^*(x, y) = x$; also, $\overline{\pi}^* : T^{*2}M \longrightarrow T^*M$ is given by $\overline{\pi}^*(u) = \overline{\pi}^*(x, y, p) = (x, p)$.

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The change of local coordinates on the manifold $T^{*2}M$ is:

$$\left\{\begin{array}{l} \widetilde{x}^{a}=\widetilde{x}^{a}\left(x^{1},...,x^{n}\right)\ ,\ \det\left(\dfrac{\partial\widetilde{x}^{a}}{\partial x^{b}}\right)\neq 0, \\ \widetilde{y}^{a}=\dfrac{\partial\widetilde{x}^{a}}{\partial x^{b}}y^{b}, \\ \widetilde{p}_{a}=\dfrac{\partial x^{b}}{\partial\widetilde{x}^{a}}p_{b}. \end{array}\right. \tag{1.2}$$

The dimension of the manifold $T^{*2}M$ is $3n$.

The null section $0:M\rightarrow T^{*2}M$ of the projection π^{*2} is defined by $0:(x)\in M\rightarrow (x,0,0)\in T^{*2}M$ we denote by $\widetilde{T^{*2}M}=T^{*2}M\setminus\{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*2}M$, $\left(TT^{*2}M,\tau^{*2},T^{*2}M\right)$, where τ^{*2} is the cannonical projection and the vertical distribution $V:u\in T^{*2}M\longrightarrow V(u)\subset T_{u}T^{*2}M$,, generated by the vector fields $\left\{\dfrac{\partial}{\partial y^{a}}|_{u},\dfrac{\partial}{\partial p_{a}}|_{u}\right\}$, $\forall u\in T^{*2}M$. As usually, the natural basis, let us denote

$$\partial_{a}=\frac{\partial}{\partial x^{a}},\ \dot{\partial}_{a}=\frac{\partial}{\partial y^{a}},\ \dot{\partial}^{a}=\frac{\partial}{\partial p_{a}}. \tag{1.3}$$

By means of (1.2), we can consider the following subdistributions of V :

$$V_{1}:u\in T^{*2}M\longrightarrow V_{1}(u)\subset T_{u}T^{*2}M, \tag{1.4}$$

and

$$W_{2}:u\in T^{*2}M\longrightarrow W_{2}(u)\subset T_{u}T^{*2}M, \tag{1.4'}$$

locally generated by the vector fields $\left\{\dot{\partial}_{a}\ |_{u},u\in T^{*2}M\right\}$ and $\left\{\dot{\partial}^{a}\ |_{u},u\in T^{*2}M\right\}$, respectively. Clearly,we have

$$V(u)=V_{1}(u)\oplus W_{2}(u)\,,\forall u\in T^{*2}M. \tag{1.5}$$

Let us consider the following forms

$$\omega=p_{a}dx^{a}\ \text{(Liouville 1-form)}, \tag{1.6}$$

$$\theta=d\omega=dp_{a}\wedge dx^{a}. \tag{1.7}$$

Theorem 1.1 ([11]) 1°.*The differential forms ω and θ are globally defined on the manifold $T^{*2}M$.*

2°. *The 2-form θ is closed and rank θ is $2n$.*

3°.*θ is a presymplectic structures on $T^{*2}M$.*

Also, the following $\mathcal{F}\left(T^{*2}M\right)$ -linear mapping

$$J : \mathcal{X}\left(T^{*2}M\right) \to \mathcal{X}\left(T^{*2}M\right),$$

defined by

$$J\left(\partial_a\right) = \dot{\partial}_a,\ J\left(\dot{\partial}_a\right) = 0,\ J\left(\dot{\partial}^a\right) = 0, \forall u \in \widetilde{T^{*2}M}, \tag{1.8}$$

has geometrical meaning. It is not difficult to prove:

Theorem 1.2 *1°. J is a tensor field of type $(1,1)$ on manifold $T^{*2}M$.*
2°. J is a tangent structure on $T^{*2}M$, i.e. $J_0 J = 0$.
3°. J is a integrable structure.
4°. $J_0 J = J^2 = 0$.
5°. $Ker J = V_1 \oplus W_2,\ \mathrm{Im} J = V_1$.

With these object fields we can construct the geometry of the manifold $T^{*2}M$.

Now, we extend the classical definition [14], of the nonlinear connection on the total space of the dual bundle $\left(T^{*2}M, \pi^{*2}, M\right)$.

Definition 1.2 A nonlinear connection of the manifold $T^{*2}M$ is a regular distribution N on $T^{*2}M$, supplimentary to the vertical distribution V, i.e.

$$T_u T^{*2}M = N(u) \oplus V(u), \forall u \in T^{*2}M. \tag{1.9}$$

Taking into account (1.5) it follows that the distribution N has the property

$$T_u T^{*2}M = N(u) \oplus V_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{1.10}$$

Therefore, the main geometrical objects on $T^{*2}M$ will be reported to the direct sum (1.10) of vector spaces.

We denote by

$$\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial p_a}\right\}, (a = 1, ..., n), \tag{1.11}$$

a local adapted basis to N, V_1, W_2. Clearly, we have

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b_{\ a}\frac{\partial}{\partial y^b} + N_{ab}\frac{\partial}{\partial p_b}. \tag{1.12}$$

The sistem of functions $\left(N^b_{\ a}(x,y,p), N_{ab}(x,y,p)\right)$ are the coefficients of the nonlinear connection N.

With respect to the coordinate transformations (1.2), we have the rule:

$$\frac{\delta}{\delta x^a} = \frac{\partial \widetilde{x}^b}{\partial x^a}\frac{\delta}{\delta \widetilde{x}^b}, \frac{\partial}{\partial y^a} = \frac{\partial \widetilde{x}^b}{\partial x^a}\frac{\partial}{\partial \widetilde{y}^b}, \frac{\partial}{\partial p_a} = \frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\partial}{\partial \widetilde{p}_b}. \tag{1.12'}$$

Theorem 1.3 *With respect to (1.2) the coefficients $(N^a_{\ b}, N_{ab})$ of a nonlinear connection N on $T^{*2}M$ obey the rule*

$$\begin{aligned} \widetilde{N}^{a}_{\ c}\frac{\partial\widetilde{x}^{c}}{\partial x^{b}} &= N^{c}_{\ b}\frac{\partial\widetilde{x}^{a}}{\partial x^{c}}-\frac{\partial\widetilde{y}^{a}}{\partial x^{b}}, \\ \widetilde{N}_{ab} &= \frac{\partial x^{c}}{\partial\widetilde{x}^{a}}\frac{\partial x^{d}}{\partial\widetilde{x}^{b}}N_{cd}+p_{c}\frac{\partial^{2}x^{c}}{\partial\widetilde{x}^{a}\partial\widetilde{x}^{b}}. \end{aligned} \tag{1.13}$$

Conversely, if the system of functions $(N^{a}_{\ b}, N_{ab})$ *are given on the every domain of local chart of the manifold* $T^{*2}M$, *such that the equations* (1.13) *hold, then* $(N^{a}_{\ b}, N_{ab})$ *are the coefficients of a nonlinear connection on* $T^{*2}M$.

Assuming that the manifold M is paracompact it follows that the manifold $T^{*2}M$ is paracompact, too. Let $\gamma_{ab}\left(x\right),\ x\in M$ be a Riemannian metric on M and $\gamma^{a}_{bc}\left(x\right)$ its Christoffel symbols. Setting

$$f_{b}=\gamma^{a}_{bc}\left(x\right)p_{a}y^{c}, \tag{1.14}$$

then, the system of functions

$$N^{a}_{\ b}=\dot{\partial}^{a}f_{b},\ N_{ab}=\dot{\partial}_{b}f_{a}, \tag{1.15}$$

are geometrical object fields on $T^{*2}M$, having the rules of transformations (1.13), with respect to the changing of local coordinates (1.2). Hence:

Theorem 1.4 *If the base manifold* M *is paracompact, then there exist nonlinear connection on the manifold* $T^{*2}M$.

From now we denote the basis (1.11) by:

$$\left\{\delta_{a},\dot{\partial}_{a},\dot{\partial}^{a}\right\}. \tag{1.11$'$}$$

The dual basis of the adapted basis (1.11) is given by

$$\left\{dx^{a},\delta y^{a},\delta p_{a}\right\}, \tag{1.16}$$

where

$$\delta y^{a}=dy^{a}+N^{a}_{\ b}dx^{b},\ \delta p_{a}=dp_{a}-N_{ba}dx^{b}. \tag{1.16$'$}$$

With respect to (1.2), the covector fields (1.16) are transformed by the rules:

$$d\widetilde{x}^{a}=\frac{\partial\widetilde{x}^{a}}{\partial x^{b}}dx^{b},\ \delta\widetilde{y}^{a}=\frac{\partial\widetilde{x}^{a}}{\partial x^{b}}\delta y^{b},\ \delta\widetilde{p}_{a}=\frac{\partial x^{b}}{\partial\widetilde{x}^{a}}\delta p_{b}, \tag{1.16$''$}$$

2 The Algebra of distinguished tensor fields.

Let N be a nonlinear connection on $T^{*2}M$. It determines, at every point $u\in T^{*2}M$, the direct sum (1.10) of the linear space $T_{u}\left(T^{*2}M\right)$.

A vector field $X\in\chi\left(T^{*2}M\right)$ and an one form $\omega\in\chi^{*}\left(T^{*2}M\right)$ uniquely can be written in the form:

$$\begin{aligned} X &= X^H + X^{V_1} + X^{W_2}, \\ \omega &= \omega^H + \omega^{V_1} + \omega^{W_2}. \end{aligned} \tag{2.1}$$

Clearly if h, v_1, w_2 are the projectors determined by the direct decomposition (1.10), we have

$$\begin{aligned} X^H &= hX,\ X^{V_1} = v_1 X,\ X^{W_2} = w_2 X, \\ \omega^H &= \omega \circ h,\ \omega^{V_1} = \omega \circ v_1,\ \omega^{W_2} = \omega \circ w_2. \end{aligned} \tag{2.2}$$

Definition 2.1 ([10]) A distinguished tensor field (briefly, $d-$ tensor field) on the manifold $T^{*2}M$ is a $d-$tensor field T of type (r,s) on $T^{*2}M$, with the property:

$$\begin{aligned} &T\left(\overset{1}{\omega}, ..., \overset{r}{\omega},\ \underset{1}{X}, ..., \underset{s}{X}\right) = T\left(\overset{1}{\omega}{}^H, ..., \overset{r}{\omega}{}^{W_2},\ \underset{1}{X}{}^H, ..., \underset{s}{X}{}^{W_2}\right), \\ &\forall \overset{1}{\omega}, ..., \overset{r}{\omega} \in \chi^*\left(T^{*2}M\right),\ \forall \underset{1}{X}, ..., \underset{s}{X} \in \chi\left(T^{*2}M\right). \end{aligned} \tag{2.3}$$

For instance, every components X^H, X^{V_1}, X^{W_2} of a vector field X is a $d-$tensor field of type $(1,0)$, and every components $\omega^H, \omega^{V_1}, \omega^{W_2}$ of a $1-$form ω is a $d-$tensor field of type $(0,1)$.

In the adapted basis $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$ and its dual basis $(dx^a, \delta y^a, \delta p_a)$ a $d-$tensor field T of type (r,s) can written in the form:

$$T = T^{\ a_1 \dots a_r}_{b_1 \dots b_s}(x,y,p)\, \delta_{a_1} \otimes \dots \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes \dots \otimes \delta p_{a_r}, \tag{2.4}$$

where

$$T^{\ a_1 \dots a_r}_{b_1 \dots b_s}(x,y,p) = T\left(dx^{b_1}, ..., \delta p_{a_r}, \delta_{a_1}, ..., \dot{\partial}^{b_s}\right).$$

It follows that the set $\left\{1, \delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}$ generates **the algebra** of the $d-$tensor fields over the ring of functions $\mathcal{F}\left(T^{*2}M\right)$, (see R. Miron [9], [10]).

With respect to the transformation of the coordinates on $T^{*2}M$, the local coefficients $T^{\ a_1 \dots a_r}_{b_1 \dots b_s}$ of T are transformed by classical rule:

$$\widetilde{T}^{\ c_1 \dots c_r}_{d_1 \dots d_s} = \frac{\partial \widetilde{x}^{c_1}}{\partial x^{a_1}} \cdots \frac{\partial \widetilde{x}^{c_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \widetilde{x}^{d_1}} \cdots \frac{\partial x^{b_s}}{\partial \widetilde{x}^{d_s}} T^{\ a_1 \dots a_r}_{b_1 \dots b_s}. \tag{2.5}$$

Exemples

1. If $f \in \mathcal{F}\left(T^{*2}M\right)$, then every set of functions $\delta_a f, \dot{\partial}_a, \dot{\partial}^a f$ is a $d-$covector field, and $\dot{\partial}^a f$ is a $d-$vector field.

2. Let us consider a Riemann structure $\mathbb{G}$ on $T^{*2}M$ and assume that the distributions N, V_1, W_2 are orthogonal in pairs, with respect to $\mathbb{G}$:

$$\begin{array}{lll}\mathbb{G}\left(X^{H},Y^{V_1}\right) & =\mathbb{G}\left(X^{H},Y^{W_2}\right)=\mathbb{G}\left(X^{V_1},Y^{W_2}\right)=0, \\ \forall X,Y & \in\chi\left(T^{*2}M\right).\end{array} \tag{2.6}$$

In this case $\mathbb{G}$ can be uniquely written as a sum of $d-$tensors:

$$\mathbb{G}=\mathbb{G}^{H}+\mathbb{G}^{V_1}+\mathbb{G}^{W_2}, \tag{2.7}$$

where, for any $X,Y\in T^{*2}M$, we have

$$\mathbb{G}^{H}\left(X,Y\right)=\mathbb{G}\left(X^{H},Y^{H}\right),\mathbb{G}^{V_1}\left(X,Y\right)=\mathbb{G}\left(X^{V_1},Y^{V_1}\right),\mathbb{G}^{W_2}\left(X,Y\right)=\mathbb{G}\left(X^{W_2},Y^{W_2}\right). \tag{2.7'}$$

Consequently, in the adapted cobasis, $\mathbb{G}$ can be uniquely written as

$$\mathbb{G}=\underset{(0)}{g}{}_{ab}dx^{a}\otimes dx^{b}+\underset{(1)}{g}{}_{ab}\delta y^{a}\otimes\delta y^{b}+\underset{(2)}{g}{}^{ab}\delta p_{a}\otimes\delta p_{b}, \tag{2.8}$$

where

$$\underset{(\alpha)}{g}{}_{ab}\left(x,y,p\right)=\underset{(\alpha)}{g}{}_{ba}\left(x,y,p\right),\parallel\underset{(\alpha)}{g}{}_{ab}\parallel=\parallel\underset{(\alpha)}{g}{}^{ab}\parallel^{-1}, \tag{2.8'}$$

$$rank\parallel\underset{(\alpha)}{g}{}_{ab}\parallel=n,\left(\alpha=0,1,2\right). \tag{2.8''}$$

The quantities $\underset{(\alpha)}{g}{}_{ab},\left(\alpha=0,1,2\right)$, are $d-$tensors of type $\left(0,2\right)$ on $T^{*2}M$, and $\underset{(\alpha)}{g}{}^{ab},\left(\alpha=0,1,2\right)$, are $d-$tensors of type $\left(2,0\right)$ on $T^{*2}M$.

3 The almost contact structure $\mathbb{F}$

The nonlinear connection N being fixed we have the decomposition (1.9), (1.10) and the corresponding adapted basis (1.11).

Let us consider the $\mathcal{F}\left(T^{*2}M\right)-$linear mapping:

$$\mathbb{F}:\chi\left(T^{*2}M\right)\longrightarrow\chi\left(T^{*2}M\right),$$

determined by

$$\mathbb{F}\left(\delta_{a}\right)=-\dot{\partial}_{a},\ \mathbb{F}\left(\dot{\partial}_{a}\right)=\delta_{a},\ \mathbb{F}\left(\dot{\partial}^{a}\right)=0. \tag{3.1}$$

Then, we deduce

Theorem 3.1 *The mapping* $\mathbb{F}$ *has the following properties:*

1°. It is globally defined on $\widetilde{T^{*2}M}$.

2°. $\mathbb{F}$ *is a tensor field of type* $(1,1)$.
3°. $Ker\ \mathbb{F} = W_2,\ \mathrm{Im}\mathbb{F} = N \oplus V_1$.
4°. $rank\ \mathbb{F} = 2n$.
5°. $\mathbb{F}^3 + \mathbb{F} = 0$.

Proof. See [11] pg.259.

We can say that $\mathbb{F}$ is a **natural almost contact structure** determined by the nonlinear connection N.

Remark 3.1 In the case when we have a Riemann structure $\mathbb{G}$ on $T^{*2}M$ and the following relation holds

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y), \forall X, Y \in \chi\left(\widetilde{T^{*2}M}\right), \tag{3.2}$$

then the pair $(\mathbb{G}, \mathbb{F})$ is a **Riemann almost contact structure** on $\widetilde{T^{*2}M}$. The manifold $T^{*2}M$ endowed with these structure gives us the geometrical **model** $\left\{T^{*2}M, \mathbb{G}, \mathbb{F}\right\}$ for this space.

By direct calculus, we get:

Theorem 3.2 *If* $\mathbb{G}$ *is a Riemann structure on* $T^{*2}M$ *given in the adapted cobasis* (1.15) *by*

$$\mathbb{G} = \underset{(00)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(01)}{g}{}_{ab} dx^a \otimes \delta y^a + \underset{(02)}{g}{}_a{}^b dx^a \otimes \delta p_b + ... + \underset{(22)}{g}{}^{ab} \delta p_a \otimes \delta p_b,$$

then, the pair $(\mathbb{G}, \mathbb{F})$ *is a Riemann almost structure on* $\widetilde{T^{*2}M}$ *if and only if in the adapted basis determined by* N *and* V *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G} = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^a \otimes \delta y^b + h^{ab} \delta p_a \otimes \delta p_b. \tag{3.3}$$

where $g_{ab}(x,y,p), h^{ab}(x,y,p)$ *have the propetries* $(2.8')$ *and* $(2.8'')$.

Corollary 3.1 *With respect to the Riemann structure (3.3) the distributions* N, V_1, W_2 *are orthogonal in pairs.*

4 $N-$linear connections on $T^{*2}M$

A linear connection on $T^{*2}M$ is an application

$$D : \chi\left(T^{*2}M\right) \times \chi\left(T^{*2}M\right) \longrightarrow \chi\left(T^{*2}M\right),\ (X, Y) \longmapsto D_X Y,$$

with the properties:

1. $D_{X_1+X_2}Y = D_{X_1}Y + D_{X_2}Y$,
   $D_{fX}Y = fD_XY,\ \forall f \in \mathcal{F}\left(T^{*2}M\right),\ \forall X, X_1, X_2, Y \in \chi\left(T^{*2}M\right)$.
2. $D_X(Y_1 + Y_2) = D_XY_1 + D_XY_2,\ \forall X, Y_1, Y_2 \in \chi\left(T^{*2}M\right)$.
3. $D_X(fY) = (Xf)Y + fD_XY,\ \forall X, Y \in \chi\left(T^{*2}M\right),\ \forall f \in \mathcal{F}\left(T^{*2}M\right)$.

We consider $X, Y \in \chi\left(T^{*2}M\right)$. With respect to decomposition of type (2.1), we have

$$D_X Y = \sum_{\alpha=0}^{2} \left(D_{X^H} Y^{V_\alpha} + D_{X^{V_1}} Y^{V_\alpha} + D_{X^{W_2}} Y^{V_\alpha} \right), \tag{4.1}$$

where $V_0 = H$ and $V_2 = W_2$.

The components $D_{X^H} Y^{V_\alpha},\ D_{X^{V_1}} Y^{V_\alpha},\ D_{X^{W_2}} Y^{V_\alpha},\ (V_0 = H,\ V_2 = W_2)$, are vector fields not necessary distinguished.

The linear connection D on $T^{*2}M$ is uniquely determined by its 27 coefficients, written in the adapted basis .

To work with these 27 coefficients is not imposible, but is laborious.

We will use in continuation the $N-$linear connections whose coefficients are much easy to shunt.

Let N be a nonlinear connection on $T^{*2}M$.

Definition 4.1 A linear connection D on $T^{*2}M$ is called an N-linear connection if it preserves by parallelism the horizontal and vertical distributions N, V_1 and W_2 on $T^{*2}M$.

By the general theory of connections on manifolds, the horizontal and vertical distributions are preserves by parallelilsm if for any $X \in \chi\left(T^{*2}M\right)$, D_X carries the horizontal vector fields to the horizontal vector fields and the vertical vector fields to the vertical vector fields. Thus $D_X Y^H$ is always an horizontal vector field, and $D_X Y^{V_\beta}$ are verticals, $(\beta = 1, 2;\ V_2 = W_2)$.

Obviously, the local description of an $N-$linear connection $D\Gamma(N)$ on $T^{*2}M$ is given by **nine** unique adapted **coefficients**:

$$D\Gamma(N) := \square\ \underset{(00)}{H}{}^{a}{}_{bc},\ \underset{(10)}{H}{}^{a}{}_{bc},\ \underset{(20)}{H}{}_{a}{}^{b}{}_{c},\ \underset{(01)}{C}{}^{a}{}_{bc},\ \underset{(11)}{C}{}^{a}{}_{bc},\ \underset{(21)}{C}{}_{a}{}^{b}{}_{c},\ \underset{(02)}{C}{}^{a}{}_{b}{}^{c},\ \underset{(12)}{C}{}^{a}{}_{b}{}^{c},\ \underset{(22)}{C}{}_{a}{}^{bc}\ \square\ . \tag{4.2}$$

We have

Theorem 4.1 1°. *An $N-$linear connection D on $T^{*2}M$ can be uniquely represented in the adapted basis* $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$ *in the form*

$$\begin{cases} D_{\delta_c}\delta_b = \underset{(00)}{H}{}^{a}{}_{bc}\delta_a,\ D_{\delta_c}\dot{\partial}_b = \underset{(10)}{H}{}^{a}{}_{bc}\dot{\partial}_a,\ D_{\delta_c}\dot{\partial}^b = -\underset{(00)}{H}{}_{a}{}^{b}{}_{c}\dot{\partial}^a, \\ D_{\dot{\partial}_c}\delta_b = \underset{(01)}{C}{}^{a}{}_{bc}\delta_a,\ D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(11)}{C}{}^{a}{}_{bc}\dot{\partial}_a,\ D_{\dot{\partial}_c}\dot{\partial}^b = -\underset{(21)}{C}{}_{a}{}^{b}{}_{c}\dot{\partial}^a, \\ D_{\dot{\partial}^c}\delta_b = \underset{(02)}{C}{}^{a}{}_{b}{}^{c}\delta_a,\ D_{\dot{\partial}^c}\dot{\partial}_b = \underset{(12)}{C}{}^{a}{}_{b}{}^{c}\dot{\partial}_a,\ D_{\dot{\partial}^c}\dot{\partial}^b = -\underset{(22)}{C}{}_{a}{}^{bc}\dot{\partial}^a. \end{cases} \tag{4.3}$$

2°. *With respect to the coordinate transformation (1.2), the coefficients* $\underset{(\alpha 0)}{H}{}^{a}{}_{bc}, \left(\alpha = 0, 1, 2;\ \underset{(20)}{H}{}^{a}{}_{bc} := \underset{(20)}{H}{}_{b}{}^{a}{}_{c}\right)$ *obey the rule of transformation:*

$$\underset{(\alpha 0)}{\widetilde{H}}{}^{a}{}_{de}\frac{\partial \widetilde{x}^d}{\partial x^b}\frac{\partial \widetilde{x}^e}{\partial x^c} = \frac{\partial \widetilde{x}^a}{\partial x^e}\underset{(\alpha 0)}{H}{}^{e}{}_{bc} - \frac{\partial^2 \widetilde{x}^a}{\partial x^b \partial x^c}. \tag{4.4}$$

3°.*The system of functions*

$$\underset{(\alpha 1)}{C}{}^{a}{}_{bc},\ \underset{(\alpha 2)}{C}{}^{a}{}_{b}{}^{c},\ \left(\alpha = 0, 1, 2;\ \underset{(21)}{C}{}^{a}{}_{bc} := \underset{(21)}{C}{}_{b}{}^{a}{}_{c};\ \underset{(22)}{C}{}^{a}{}_{b}{}^{c} := \underset{(22)}{C}{}_{b}{}^{ac}\ \right)$$

are $d-$tensor fields of type $(1,2)$ *and* $(2,1)$, *respectively.*

We have the following theorem of existence of $N-$linear connection on $T^{*2}M$.

Theorem 4.2 *If the manifold* M *is paracompact and* N *is a nonlinear connection on* $T^{*2}M$ *with coefficients* N^a_b, N_{ab}, *then there exists an* $N-$*linear connection on* $T^{*2}M$.

Proof. Since M is paracompact, there exists a linear connection on M of local coefficients, say $\Gamma^a_{bc}(x)$ .Let $N^a_b(x,y,p)$ and $N_{ab}(x,y,p)$ be the local coefficients of the nonlinear connection N. We set $\underset{(00)}{H}{}^a{}_{bc} = \Gamma^a_{bc}(x)$, $\underset{(10)}{H}{}^a{}_{bc} = \dot{\partial}_b N^a_c$, $\underset{(20)}{H}{}^a{}_{bc} = \dot{\partial}^a N_{bc}$. Thus, we obtain three set of function which transform, with respect to (1.2), by (4.4). It result that $D\Gamma(N)$ given by

$$D\Gamma(N) = \left(\Gamma^a_{bc}(x), \dot{\partial}_b N^a{}_c, \dot{\partial}^a N_{bc}, 0,0,0,0,0,0\right), \tag{4.5}$$

defines an $N-$linear connection on $T^{*2}M$.

5 The $h_\alpha-, v_{1\alpha}-$ and $w_{2\alpha}-$covariant derivatives in the local adapted basis, $(\alpha = 0,1,2)$

The $N-$linear connection $D\Gamma(N)$ induces the linear connections on the $d-$tensors set of the 2$-$cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$, in a natural way. Thus, starting with a $d-$vector field X and a $d-$tensor field T, locally expressed by

$$X = X^{(0)a}\delta_a + X^{(1)a}\dot{\partial}_a + \underset{(2)}{X}{}_a\dot{\partial}^a, \tag{5.1}$$

$$T = T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}(x,y,p)\,\delta_{a_1}\otimes\ldots\otimes\dot{\partial}^{b_s}\otimes dx^{b_1}\otimes\ldots\otimes\delta p_{a_r}, \tag{5.2}$$

we can define the covariante derivative

$$\begin{aligned} D_X T &= \Big\{X^{(0)d}T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s|\alpha d} + \\ &\quad + X^{(1)d}T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|_{\alpha d} + \underset{(2)}{X}{}_d T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|^{\alpha d}\Big\}\,\delta_{a_1}\otimes\ldots\otimes\delta p_{a_r}, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} & T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s|\alpha d} = \delta_d T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s} + \underset{(\alpha 0)}{H}{}^{a_1}{}_{cd}T^{\,ca_2\ldots a_r}_{\,b_1\ldots b_s} + \ldots + \underset{(\alpha 0)}{H}{}^{a_r}{}_{cd}T^{\,a_1\ldots c}_{\,b_1\ldots b_s} - \\ & - \underset{(\alpha 0)}{H}{}^{c}{}_{b_1 d}T^{\,a_1\ldots a_r}_{\,cb_2\ldots b_s} - \ldots - \underset{(\alpha 0)}{H}{}^{c}{}_{b_s d}T^{a_1\ldots a_r}_{b_1\ldots c}, \end{aligned} \tag{5.4$_0$}$$

$$\begin{aligned} & T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|_{\alpha d} = \dot{\partial}_d T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s} + \underset{(\alpha 1)}{C}{}^{a_1}{}_{cd}T^{\,ca_2\ldots a_r}_{\,b_1\ldots b_s} + \ldots + \underset{(\alpha 1)}{C}{}^{a_r}{}_{cd}T^{\,a_1\ldots c}_{\,b_1\ldots b_s} - \\ & - \underset{(\alpha 1)}{C}{}^{c}{}_{b_1 d}T^{\,a_1\ldots a_r}_{\,cb_2\ldots b_s} - \ldots - \underset{(\alpha 1)}{C}{}^{c}{}_{b_s d}T^{\,a_1\ldots a_r}_{\,b_1\ldots c}, \end{aligned} \tag{5.4$_1$}$$

$$(5.4_2)\qquad \begin{aligned} & T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|^{\alpha d} = \dot{\partial}^d T^{\,a_1\ldots a_r}_{\,b_1\ldots b_r} + \underset{(\alpha2)}{C}{}_c{}^{a_1 d}T^{\,ca_2\ldots a_r}_{\,b_1\ldots b_s} + \ldots + \underset{(\alpha2)}{C}{}_c{}^{a_r d}T^{\,a_1\ldots c}_{\,b_1\ldots b_s} - \\ & - \underset{(\alpha2)}{C}{}_{b_1}{}^{cd}T^{\,a_1\ldots a_r}_{\,cb_2\ldots b_s} - \ldots - \underset{(\alpha2)}{C}{}_{b_s}{}^{cd}T^{\,a_1\ldots a_r}_{\,b_1\ldots c}. \end{aligned}$$

Definition 5.1 The local derivative operators " ${}_{\shortmid\alpha d}$", " $|_{\alpha d}$ " and " $|^{\alpha d}$ " are called the $\mathbf{h}_\alpha$-, $\mathbf{v}_{1\alpha}$- and $\mathbf{w}_{2\alpha}$- covariant derivatives of $D\Gamma\left(N\right), (\alpha = 0, 1, 2)$.

Remark 5.1 (i) In the particular case if T is a function $f(x, y, p)$ on $T^{*2}M$, the preceding covariant derivatives reduce to

$$(5.5)\qquad \begin{aligned} & f_{\shortmid\alpha d} = \delta_d f = \partial_d f - N^c{}_d \dot{\partial}_c, \\ & f\,|_{\alpha d} = \dot{\partial}_d f, f\,|^{\alpha d} = \dot{\partial}^d f, \forall f \in \mathcal{F}\left(T^{*2}M\right). \end{aligned}$$

(ii) Considering the $d-$tensor $T = Y$ like a $d-$tensor on $T^{*2}M$, locally expressed by

$$(5.6)\qquad Y = Y^{(0)a}\delta_a + Y^{(1)a}\dot{\partial}_a + \underset{(2)}{Y}{}_a\dot{\partial}^a,$$

the following expressions of local covariant derivatives of $D\Gamma\left(N\right)$ hold good:

$$(5.7)\qquad \begin{aligned} & Y^{(0)a}{}_{\shortmid\alpha c} = \delta_c Y^{(0)a} + Y^{(0)b}\underset{(\alpha0)}{H}{}^a{}_{bc}, Y^{(1)a}{}_{\shortmid\alpha c} = \delta_c Y^{(1)a} + Y^{(1)b}\underset{(\alpha0)}{H}{}^a{}_{bc}, \\ & \underset{(2)}{Y}{}_{b\shortmid\alpha c} = \delta_c \underset{(2)}{Y}{}_b - \underset{(2)}{Y}{}_a\underset{(\alpha0)}{H}{}^a{}_{bc}, \\ & Y^{(0)a}\,|_{\alpha c} = \dot{\partial}_c\, Y^{(0)a} + Y^{(0)b}\underset{(\alpha1)}{C}{}^a{}_{bc}, Y^{(1)a}\,|_{\alpha c} = \dot{\partial}_c\, Y^{(1)a} + Y^{(1)b}\underset{(\alpha1)}{C}{}^a{}_{bc}, \\ & \underset{(2)}{Y}{}_b\,|_{\alpha c} = \dot{\partial}_c \underset{(2)}{Y}{}_b - \underset{(2)}{Y}{}_a\underset{(\alpha1)}{C}{}^a{}_{bc}, \\ & Y^{(0)a}\,|^{\alpha b} = \dot{\partial}^b\, Y^{(0)a} + Y^{(0)c}\underset{(\alpha2)}{C}{}_c{}^{ab}, Y^{(1)a}\,|^{\alpha b} = \dot{\partial}^b\, Y^{(1)a} + Y^{(1)c}\underset{(\alpha2)}{C}{}_c{}^{ab}, \\ & \underset{(2)}{Y}{}_b\,|^{\alpha b} = \dot{\partial}^b \underset{(2)}{Y}{}_c - \underset{(2)}{Y}{}_a\underset{(\alpha2)}{C}{}_c{}^{ab}. \end{aligned}$$

Proposition 5.1 *The quantities* $T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s\shortmid\alpha d}$, $T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|_{\alpha d}$, $T^{\,a_1\ldots a_r}_{\,b_1\ldots b_s}\,|^{\alpha d}$, $(\alpha = 0, 1, 2)$ *are* $d-$*tensor fields on* $T^{*2}M$. *The firsts six are of type* $\left(r, s+1\right)$, *the last three are of type* $\left(r+1, s\right)$.

6 Metric $N-$linear connections

Definition 6.1 A metric structure on the manifold $T^{*2}M$ is a symmetric covariant tensor field $\mathbb{G}$ of type (0,2) which is non degenerate at each point $u \in T^{*2}M$ and of constant signature on $T^{*2}M$. If $\mathbb{G}$ is positive definite we say that it defines a Riemann structure on $T^{*2}M$.

Let us consider a metric structure $\mathbb{G}$ on $T^{*2}M$ for which the distributions N, V_1, W_2 are much general then (3.3), namely it have the decomposition:

$$\mathbb{G}\left(X,Y\right)=\mathbb{G}\left(X^{H},Y^{H}\right)+\mathbb{G}\left(X^{V_1},Y^{V_1}\right)+\mathbb{G}\left(X^{W_2},Y^{W_2}\right),\forall X,Y\in T^{*2}M. \tag{6.1}$$

With the other words, $\mathbb{G}$ decomposes in a sum of three $d-$tensor fields:

(0) $\mathbb{G}^{H}$ of type $(0,2)$ defined by $\mathbb{G}^{H}\left(X,Y\right)=\mathbb{G}\left(X^{H},Y^{H}\right)$,
(1) $\mathbb{G}^{V_1}$ of type $(0,2)$ defined by $\mathbb{G}^{V_1}\left(X,Y\right)=\mathbb{G}\left(X^{V_1},Y^{V_1}\right)$,
(2) $\mathbb{G}^{W_2}$ of type $(0,2)$ defined by $\mathbb{G}^{W_2}\left(X,Y\right)=\mathbb{G}\left(X^{W_2},Y^{W_2}\right)$.

Locally, these $d-$tensor fields can be written as

$$\begin{cases} \mathbb{G}^{H}=\underset{(0)}{g}_{ab}dx^{a}\otimes dx^{b}, & \mathbb{G}^{V_1}=\underset{(1)}{g}_{ab}\delta y^{a}\otimes\delta y^{b}, \\ & \mathbb{G}^{W_2}=\underset{(2)}{g}^{ab}\delta p_{a}\otimes\delta p_{b}, \end{cases} \tag{6.2}$$

where

$$\underset{(0)}{g}_{ab}=\mathbb{G}\left(\delta_a,\delta_b\right),\ \underset{(1)}{g}_{ab}=\mathbb{G}\left(\dot{\partial}_a,\dot{\partial}_b\right),\ \underset{(2)}{g}^{ab}=\mathbb{G}\left(\dot{\partial}^{a},\dot{\partial}^{b}\right), \tag{6.2$'$}$$

$$rank\parallel\underset{(\alpha)}{g}_{ab}\parallel=n,\left(\alpha=0,1,2\right),\ \parallel\underset{(2)}{g}_{ab}\parallel=\parallel\underset{(2)}{g}^{ab}\parallel^{-1}. \tag{6.3}$$

Thus the decomposition (6.1) looks locally as following:

$$\mathbb{G}=\underset{(0)}{g}_{ab}dx^{a}\otimes dx^{b}+\underset{(1)}{g}_{ab}\delta y^{a}\otimes\delta y^{b}+\underset{(2)}{g}^{ab}\delta p_{a}\otimes\delta p_{b}. \tag{6.4}$$

Definition 6.2 An $N-$linear connection D on $T^{*2}M$ endowed with a metric structure $\mathbb{G}$ is said to be a metric $\mathbf{N}-$linear connection if $D_X\mathbb{G}=0$ for every $X\in T^{*2}M$.

Let $\mathbb{G}$ be a metric structure on $T^{*2}M$ given by (6.4)

We have

Proposition 6.2 *An $N-$linear connection on $T^{*2}M$ is a metric $N-$linear connection with respect to* $\mathbb{G}$ given by (6.4) *if and only if*

$$\underset{(\alpha)}{g}_{ab\shortmid\alpha c}=0,\ \underset{(\alpha)}{g}_{ab}\,|_{\alpha c}=0,\ \underset{(\alpha)}{g}^{ab}\,|^{\alpha c}=0, \tag{6.5}$$

where $\parallel\underset{(\alpha)}{g}^{ab}\parallel=\parallel\underset{(\alpha)}{g}_{ab}\parallel^{-1},\ \left(\alpha=0,1,2\right)$.

Remark 6.1 The conditions (6.5) are equivalent with the conditions

$$\underset{(\alpha)}{g}^{ab}{}_{\shortmid\alpha c}=0,\ \underset{(\alpha)}{g}^{ab}\,|_{\alpha c}=0,\ \underset{(\alpha)}{g}_{ab}\,|^{\alpha c}=0,\ \left(\alpha=0,1,2\right). \tag{6.5$'$}$$

We shall now discuss the existence of metric $N-$linear connection on $T^{*2}M$.

By a straightforward calculus we get

Theorem 6.1 *If the manifold $T^{*}M$ is endowed with the metric structure $\mathbb{G}$ given by (6.4), then there exists on $T^{*2}M$ a metric $N-$linear connection, depending only on $\mathbb{G}$. Its local coefficients:*

$$D\Gamma\left(N\right):=\left(\underset{(00)}{H}{}^{a}{}_{bc},\underset{(10)}{H}{}^{a}{}_{bc},\underset{(20)}{H}{}_{a}{}^{b}{}_{c},\underset{(01)}{C}{}^{a}{}_{bc},\underset{(11)}{C}{}^{a}{}_{bc},\underset{(21)}{C}{}_{a}{}^{b}{}_{c},\underset{(02)}{C}{}^{a}{}_{b}{}^{c},\underset{(12)}{C}{}^{a}{}_{b}{}^{c},\underset{(22)}{C}{}_{a}{}^{bc}\right),$$

are as follows

$$(6.6)\qquad\begin{aligned}
&\underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(0)}{g}{}^{ad}\left(\delta_{c}\underset{(0)}{g}{}_{bd}+\delta_{b}\underset{(0)}{g}{}_{dc}-\delta_{d}\underset{(0)}{g}{}_{bc}\right),\\
&\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc}=\underset{(11)}{B}{}^{a}{}_{cb}+\frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\delta_{c}\underset{(1)}{g}{}_{bd}-\underset{(11)}{B}{}^{f}{}_{cb}\underset{(1)}{g}{}_{fd}-\underset{(11)}{B}{}^{f}{}_{cd}\underset{(1)}{g}{}_{bf}\right),\\
&\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}=-\underset{(22)}{B}{}^{a}{}_{bc}+\frac{1}{2}\underset{(2)}{g}{}^{ad}\left(\delta_{c}\underset{(2)}{g}{}_{bd}+\underset{(22)}{B}{}^{f}{}_{bc}\underset{(2)}{g}{}_{fd}+\underset{(22)}{B}{}^{f}{}_{dc}\underset{(2)}{g}{}_{bf}\right),\\
&\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(0)}{g}{}^{ad}\dot{\partial}_{c}\underset{(0)}{g}{}_{bd},\\
&\underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\dot{\partial}_{c}\underset{(1)}{g}{}_{bd}+\dot{\partial}_{b}\underset{(1)}{g}{}_{dc}-\dot{\partial}_{d}\underset{(1)}{g}{}_{bc}\right),\\
&\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(2)}{g}{}^{ad}\dot{\partial}_{c}\underset{(2)}{g}{}_{bd},\\
&\underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(0)}{g}{}_{ad}\dot{\partial}_{c}\underset{(0)}{g}{}^{bd},\\
&\underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(1)}{g}{}_{ad}\dot{\partial}^{c}\underset{(1)}{g}{}^{bd},\\
&\underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(2)}{g}{}_{ad}\left(\dot{\partial}^{c}\underset{(2)}{g}{}^{bd}+\dot{\partial}^{b}\underset{(2)}{g}{}^{dc}-\dot{\partial}^{d}\underset{(2)}{g}{}^{bc}\right).
\end{aligned}$$

Definition 6.3 The metric $N-$linear connection given by (6.6) will be called the canonical $N-$linear connection associated with $\mathbb{G}$.

Let

$$D\overset{*}{\Gamma}\left(N\right)=\left(\underset{(00)}{\overset{*}{H}}{}^{a}{}_{bc},\underset{(10)}{\overset{*}{H}}{}^{a}{}_{bc},\underset{(20)}{\overset{*}{H}}{}^{a}{}_{bc},\underset{(01)}{\overset{*}{C}}{}^{a}{}_{bc},\underset{(11)}{\overset{*}{C}}{}^{a}{}_{bc},\underset{(21)}{\overset{*}{C}}{}^{a}{}_{bc},\underset{(02)}{\overset{*}{C}}{}_{a}{}^{bc},\underset{(12)}{\overset{*}{C}}{}_{a}{}^{bc},\underset{(22)}{\overset{*}{C}}{}_{a}{}^{bc}\right)$$

be an $N-$linear connection on $T^{*2}M$ which is endowed with a metric structure $\mathbb{G}$.

If we denote by $\overset{*}{\shortmid}_{\alpha c},\overset{*}{|}_{\alpha c},\overset{*}{|}{}^{\alpha c}$ the $h_{\alpha}-$,$v_{1\alpha}-$ and $w_{2\alpha}-$ covariant derivations with respect to $D\overset{*}{\Gamma}\left(N\right)$, then by a direct calculation one checks that the $N-$linear connection whose local coefficients are given by

$$\begin{aligned}
&\underset{(\alpha 0)}{H}{}^{a}{}_{bc} = \underset{(\alpha 0)}{\overset{*}{H}}{}^{a}{}_{bc} + \frac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db\overset{*}{|}\alpha c},\\
(6.7)\qquad &\underset{(\alpha 1)}{C}{}^{a}{}_{bc} = \underset{(\alpha 1)}{\overset{*}{C}}{}^{a}{}_{bc} + \frac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db}\overset{*}{|}_{\alpha c},\\
&\underset{(\alpha 2)}{C}{}_{a}{}^{bc} = \underset{(\alpha 2)}{\overset{*}{C}}{}_{a}{}^{bc} - \frac{1}{2}\underset{(\alpha)}{g}{}_{ad}\underset{(\alpha)}{g}{}^{db}\overset{*}{|}{}^{\alpha c}.
\end{aligned}$$

is a metrical $N-$linear connection .

This method of metrisation of an $N-$linear connection is called the Kawaguchi metrisation process, [1], [11].

Let us associate to $\mathbb{G}$ the following operators of Obata type:

$$(6.8)\qquad \underset{1}{\overset{\alpha}{O}}{}^{cd}_{ab} = \frac{1}{2}\left(\delta^c_a\delta^d_b - \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}\right), \ \underset{2}{\overset{\alpha}{O}}{}^{cd}_{ab} = \frac{1}{2}\left(\delta^c_a\delta^d_b + \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}\right), \quad (\alpha = 0,1,2)\,.$$

Theorem 6.2 *The set of all metric $N-$linear connections with respect to $\mathbb{G}$ on the manifold $T^{*2}M$ is given by*

$$\begin{aligned}
&\underset{(\alpha 0)}{H}{}^{a}{}_{bc} = \underset{(\alpha 0)}{\overset{c}{H}}{}^{a}{}_{bc} + \underset{1}{\overset{\alpha}{O}}{}^{fa}_{bd}\overset{\alpha}{X}{}^{d}{}_{fc},\\
(6.9)\qquad &\underset{(\alpha 1)}{C}{}^{a}{}_{bc} = \underset{(\alpha 1)}{\overset{c}{C}}{}^{a}{}_{bc} + \underset{1}{\overset{\alpha}{O}}{}^{fa}_{bd}\overset{\alpha}{Y}{}^{d}{}_{fc},\\
&\underset{(\alpha 2)}{C}{}_{a}{}^{bc} = \underset{(\alpha 2)}{\overset{c}{C}}{}_{a}{}^{bc} + \underset{1}{\overset{\alpha}{O}}{}^{bd}_{fa}\overset{\alpha}{Z}{}_{d}{}^{fc},\\
&\qquad\qquad (\alpha = 0,1,2)\,,
\end{aligned}$$

where $\left(\underset{(\alpha 0)}{\overset{c}{H}}{}^{a}{}_{bc}, \underset{(\alpha 1)}{\overset{c}{C}}{}^{a}{}_{bc}, \underset{(\alpha 2)}{\overset{c}{C}}{}_{a}{}^{bc}\right)$ *is the canonical $N-$linear connection (6.6), $\overset{\alpha}{X}{}^{a}{}_{bc}, \overset{\alpha}{Y}{}^{a}{}_{bc}$ are arbitrary $d-$tensor fields of type $(1,2)$ and $\overset{\alpha}{Z}{}_{a}{}^{bc}$ are arbitrary $d-$tensor fields of type $(2,1)$.*

For demonstration we can see V. Cruceanu, R. Miron [7], V. Oproiu [12], [13].

7 $N-$linear connections of Miron type on $T^{*2}M$

An important particular case of $N-$linear connection on $T^{*2}M$ is given by following definition.

Definition 7.1 An $N-$linear connection D on $T^{*2}M$ is called **MN-**linear connection ($N-$linear connection of Miron type) if:

1°. The 1−tangent structure J is absolute parallel with respect to D.

2°. The presymplectic structure θ is absolute parallel with respect to D.

By straightforward calcul we get

Theorem 7.1 *An $MN-$linear connection on $T^{*2}M$ is characterized by the coefficients $MD\Gamma\left(N\right)$ given by (4.2) where*

$$\begin{aligned}
&\underset{(00)}{H}{}^{a}{}_{bc}=\underset{(10)}{H}{}^{a}{}_{bc}=\underset{(20)}{H}{}^{a}{}_{bc}=:H^{a}{}_{bc},\\
&\underset{(01)}{C}{}^{a}{}_{bc}=\underset{(11)}{C}{}^{a}{}_{bc}=\underset{(21)}{C}{}^{a}{}_{bc}=:C^{a}{}_{bc},\\
&\underset{(02)}{C}{}^{a}{}_{b}{}^{c}=\underset{(12)}{C}{}^{a}{}_{b}{}^{c}=\underset{(22)}{C}{}^{a}{}_{b}{}^{c}=:C^{a}{}_{b}{}^{c}.
\end{aligned} \tag{7.1}$$

Also, we obtain

Proposition 7.1 *For any $MN-$linear connection $MD\Gamma\left(N\right)=\left(H^{a}{}_{bc},C^{a}{}_{bc},\right.$ $\left.C_{a}{}^{bc}\right)$ we have*

$$D_{X}\mathbb{F}=0,\ \forall X\in\chi\left(T^{*2}M\right). \tag{7.2}$$

Remark 7.1 1°.We have $\left\{MD\Gamma\left(N\right)\right\}\subset\left\{D\Gamma\left(N\right)\right\}.$

2°.For any $MN-$linear connection, the $h_{\alpha}-,v_{1\alpha}-$ and $w_{2\alpha}-$covariant derivatives, $\left(\alpha=0,1,2\right)$, one reduce to $h-,v_{1}-$ and $w_{2}-$ covariant derivatives. Also " $_{\shortmid \alpha c}$ ", $\left(\alpha=0,1,2\right)$ one reduce to " $_{\shortmid c}$ " only, " $|_{\alpha c}$ ", $\left(\alpha=0,1,2\right)$ one reduce to " $|_{c}$ " and " $|^{\alpha c}$ ", $\left(\alpha=0,1,2\right)$ one reduce to " $|^{c}$ ", $\left(c=1,...,n\right)$, respectively.

Whole these correspond of Acad. R. Miron theory on the Hamilton spaces of higher order recently achieved (see R.Miron [9], [10]).

From this pharagraf we clearly see how the results of this paper generalize the works remarked before. To work with nine coefficients for a linear connection on $T^{*2}M$ (replaced three) is an advantage in the physical applications, because, the torsion, the curvature, remarkable identities, etc., are much more substantial (see [6]).

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