Torseforming vector field in a 3-dimensional trans-Sasakian manifold

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Abstract. The object of the present paper is to study a torseforming vector field in a 3-dimensional trans-Sasakian manifold. Here we proved that the torseforming vector field in a 3-dimensional trans -Sasakian manifold under the condition $\phi(grad\alpha) = grad\beta$, is a concircular vector field.

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Introduction. The contact manifolds are odd dimensional manifolds with specified contact structure. One can obtain different structures like Sasakian, Quasi-Sasakian, Kenmotsu and trans-Sasakian by providing additional conditions. The geometry of these manifolds is extensively studied by [2] to [11]. Now the Torseforming vector field in a Riemannian manifold has been introduced by K. Yano in 1944 [14]. In the present paper we consider a Torseforming vector field in a 3-dimensional trans-Sasakian manifold under the condition $\phi(grad\alpha) = grad\beta$, and proved that it is of concircular type.

1 Preliminaries.

Let M be an $(2m+1)$ dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is(1,1) tensor field, ξ is a vector field, η is a 1-form and g is the associated Riemannian metric such that [5],

$$\begin{aligned}\phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta o \phi = 0, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),\end{aligned} \tag{1.1}$$

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X), \quad \forall X, Y \in TM, \tag{1.2}$$

An almost contact metric structure (ϕ, ξ, η, g), on M is called a trans-Sasakian structure [10], if $(M \times R, J, G)$ belongs to the class W_4 [9], where J is the almost complex structure on $M \times R$ defined by $J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt)$ for all

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vector fields X on M and smooth functions f on $M \times R$. This may be expressed by the condition [6],

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{1.3}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).
From (1.3) it follows that

$$\begin{aligned} \nabla_X \xi &= -\alpha\phi X + \beta(X - \eta(X)\xi) \\ (\nabla_X \eta)Y &= -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \end{aligned} \tag{1.4}$$

On trans-Sasakian manifolds we have the following results [7]:

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi(X) \\ &\quad -\eta(X)\phi(Y)) + (Y\alpha)\phi X - (X\alpha)\phi Y \\ &\quad +(Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{1.5}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), \tag{1.6}$$

$$2\alpha\beta + \xi\alpha = 0, \tag{1.7}$$

$$S(X,\xi) = (2m(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2m-1)X\beta - (\phi X)\alpha, \tag{1.8}$$

$$Q\xi = (2m(\alpha^{-}\beta^2) - \xi\beta)\xi - (2m-1)grad\beta + \phi(grad\alpha). \tag{1.9}$$

When, $\phi(grad\alpha) = (2m-1)grad\beta$, (1.8) and (1.9) reduce to

$$S(X,\xi) = 2m(\alpha^2 - \beta^2)\eta(X) \tag{1.10}$$

$$Q\xi = 2m(\alpha^2 - \beta^2)\xi. \tag{1.11}$$

Also under the above condition $\phi(grad\alpha) = (2m-1)grad\beta$ the expressions for Ricci tensor and the scalar curvature in a 3-dimensional trans-Sasakian manifold are given respectively by [13]:

$$S(X,Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] g(X,Y) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\eta(Y) \tag{1.12}$$

and

$$\begin{aligned} R(X,Y)Z &= \left[\frac{r}{2} - 2(\alpha^2 - \beta^2)\right] [g(Y,Z)X - g(X,Z)Y] \\ &\quad - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\ &\quad - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{1.13}$$

A vector field ρ defined by $g(X,\rho) = w(X)$ for any vector field X is said to be a torseforming vector field [12, 14] if

$$(\nabla_X w)(Y) = kg(X,Y) + \pi(X)w(Y) \tag{1.14}$$

where k is a non-zero scalar and π is a non-zero 1-form.

2 Torseforming vector field in a 3-dimensional Trans-Sasakian Manifold

We consider a unit torseforming vector field $\widetilde{\rho}$ corresponding to the vector field ρ. Suppose $g(X, \widetilde{\rho}) = T(X)$. Then

$$T(X) = \frac{w(X)}{\sqrt{w(\rho)}}. \tag{2.1}$$

From (1.14) we get

$$\frac{(\nabla_X w)(Y)}{\sqrt{w(\rho)}} = \frac{k}{\sqrt{w(\rho)}} g(X,Y) + \frac{\pi}{\sqrt{w(\rho)}} w(Y)$$

Using (2.1) in the above, we obtain

$$(\nabla_X T)(Y) = \lambda g(X,Y) + \pi(X)T(Y) \tag{2.2}$$

where $\lambda = \frac{k}{\sqrt{w(\rho)}}$.
Putting $Y = \widetilde{\rho}$ in (2.2), we obtain

$$(\nabla_X T)(\widetilde{\rho}) = \lambda g(X, \widetilde{\rho}) + \pi(X)T(\widetilde{\rho}) \tag{2.3}$$

As $T(\widetilde{\rho}) = g(\widetilde{\rho}, \widetilde{\rho}) = 1$, equation (2.3) reduces to

$$\pi(x) = -\lambda T(X) \tag{2.4}$$

and hence (2.2) can be written in the form

$$(\nabla_X T)(Y) = \lambda \left[g(X,Y) - T(X)T(Y)\right] \tag{2.5}$$

which implies T is closed.
Taking covariant differential of (2.5) and using Ricci identity we get

$$\begin{aligned} -T(R(X,Y)Z) \;=\; & (X\lambda)\left[g(Y,Z) - T(Y)T(Z)\right] - (Y\lambda)\left[g(X,Z) - T(X)T(Z)\right] \\ & + \lambda^2 \left[g(Y,Z)T(X) - g(X,Z)T(Y)\right] \end{aligned} \tag{2.6}$$

Putting $Z = \xi$ in (2.6) and using (1.3), we obtain

$$\begin{aligned} -T\left((\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y)\right) \;=\; & (X\lambda)\left[\eta(Y) - T(Y)T(\xi)\right] \\ & -(Y\lambda)[\eta(X) - T(X)T(\xi)] \\ & +\lambda^2[\eta(Y)T(X) - \eta(X)T(Y)] \end{aligned} \tag{2.7}$$

Since $T(\xi) = g(\xi, \widetilde{\rho}) = \eta(\widetilde{\rho})$, (2.7) reduces to

$$\begin{aligned} & \left[\lambda^2 + (\alpha^2 - \beta^2)\right] \; [\eta(Y)T(X) - \eta(X)T(Y)] + [X(\lambda)\eta(Y) - (Y\lambda)\eta(X)] \\ & + \eta(\widetilde{\rho})\left[(Y\lambda)T(X) - (X\lambda)T(Y)\right] = 0 \end{aligned} \tag{2.8}$$

Putting $X = \widetilde{\rho}$ in (2.8) and as $T(\widetilde{\rho}) = g(\widetilde{\rho}, \widetilde{\rho}) = 1$, we get

$$\left[\lambda^2 + (\alpha^2 - \beta^2) + \widetilde{\rho}\lambda\right] \; [\eta(Y) - \eta(\widetilde{\rho})T(Y)] = 0. \tag{2.9}$$

Thus we have the following:

Lemma 1 *If a 3-dimensional trans-Sasakian manifold admits a torseforming vector field, then the following cases occur:*

$$\eta(Y) - \eta(\widetilde{\rho})T(Y) = 0 \tag{2.10}$$

$$\lambda^2 + (\alpha^2 - \beta^2) + \widetilde{\rho}\lambda = 0 \tag{2.11}$$

We first consider the case where (2.10) holds good. From (2.10) we get

$$\eta(Y) = \eta(\widetilde{\rho})T(Y)$$

Now $Y = \xi$ *implies*

$$1 = (\eta(\widetilde{\rho}))^2$$

and thus

$$\eta(\widetilde{\rho}) = \pm 1.$$

So,

$$\eta(Y) = \pm T(Y). \tag{2.12}$$

Using (2.12) in (1.4) in view of (2.5) we have

$$-\alpha g(\phi X, Y) + \beta\,[g(X,Y) - T(X)T(Y)] = \pm\lambda\,[g(X,Y) - T(X)T(Y)]$$

This implies $\lambda = \pm\beta$.
Hence (2.4) reduces to

$$\pi(X) = \pm\beta T(X) \tag{2.13}$$

Since T is closed, π is also closed.

Hence we can state:

Lemma 2 *The equation (2.10) implies that the vector field* $\widetilde{\rho}$ *is a concircular vector field.*

We next assume the case (2.11). Then

$$\eta(Y) - \eta(\widetilde{\rho})T(Y) \neq 0. \tag{2.14}$$

From (2.6), we get

$$-\,T(QX) = 2(X\lambda) - (X\lambda) + (\widetilde{\rho}\lambda)T(X) + 2\lambda^2 T(X). \tag{2.15}$$

where $g(QX,Y) = S(X,Y)$.
Put $X = \xi$ in (2.15) and using (1.11), we obtain

$$\xi\lambda = -\eta(\widetilde{\rho})[\lambda^2 + (\alpha^2 - \beta^2)]. \tag{2.16}$$

Putting $Y = \xi$ in (2.8), in virtue of (2.16) and $T(\xi) = \eta(\widetilde{\rho})$ we get

$$X\lambda = -\left\{\lambda^2 + (\alpha^2 - \beta^2)\right\}T(X). \tag{2.17}$$

From (2.4)it follows that

$$Y\pi(X) = -\left[(Y\lambda)T(X) + \lambda(YT(X))\right]$$

Using (2.17) in the above equation, we get

$$Y\pi(X) = -\left\{-\left[\lambda^2 + (\alpha^2 - \beta^2)\right]T(Y)T(X) + \lambda\left[YT(X)\right]\right\} \tag{2.18}$$

Also

$$Y\pi(Y) = -\left\{-\left[\lambda^2 + (\alpha^2 - \beta^2)\right]T(X)T(Y) + \lambda\left[XT(Y)\right]\right\} \tag{2.19}$$

and

$$\pi([X,Y]) = -\lambda T([X,Y]). \tag{2.20}$$

From (2.18), (2.19) and (2.20), we obtain

$$d\pi(X,Y) = -\lambda\left[(dT)(X,Y)\right]. \tag{2.21}$$

Since T is closed, π is also closed. Thus we have

Lemma 3 *The equation (2.11) implies that the vector field $\widetilde{\rho}$ is a concircular vector field. Thus from Lemma 2 and Lemma 3, we can state the following:*

Theorem 2.1 *A torseforming vector field in a trans-Sasakian manifold is a concircular vector field.*

From (1.4) it follows that in a trans-Sasakian manifold ξ is a torseforming vector field. Hence from Theorem 2.1, we can state the following:

Theorem 2.2 *A trans-Sasakian manifold admits a proper concircular vector field.*

Conformally flat trans-Sasakian manifold has been studied by C.S.Bagewadi and Venkatesha [4]. It is known [1] that if a Conformally flat manifold M (whose dimension n is greater than 3) admits a proper Concircular vector field, then the manifold is a subprojective manifold in the sense of Kagan. Since the trans-Sasakian manifold admits a proper concircular vector field, namely the vector field ξ, we can state as follows:

Theorem 2.3 *A Conformally flat trans-Sasakian manifold of dimension greater than 3, is a subprojective manifold in the sense of Kagan.*

References

[1] T.Adati, *On subprojective spaces III,* Tohoku Math. J., 3 (1951), 343 - 358.

[2] C.S.Bagewadi and E.Grish Kumar, *Note on Trans-Sasakian Manifolds,* Tensor.N.S., 65 (1)(2004), 80-88.

[3] C.S.Bagewadi and E.Grish Kumar, *Generalised Reccurent Manifolds,* Tensor.N.S., 65 (3)(2004), 261 - 271.

[4] C.S.Bagewadi and Venkatesha, *Some curvature tensors on a Trans-Sasakian Manifold,* (Communicated)

[5] D.E.Blair, *Contact manifolds in Riemannian Geometry,* Lecturer Notes in Mathematics, 509, Springer - Verlag, Berlin., (1976).

[6] D.E.Blair and J.A.Oubina *Conformal and related changes of metric on the product of two almost contact metric manifolds,* Publ.Mathematiques., 34 (1990), 199 - 207.

[7] U.C.De and M.M.Tripathi, *Ricci tensor in 3-dimensional trans-Sasakian manifolds,* Kyungpook Math.J., 43 (2)(2003), 247-255.

[8] N.B.Gatti and C.S.Bagewadi, *On irrotational Quasi conformal curvature tensor,* Tensor.N.S., 64 (3)(2003), 248-258.

[9] A.Gray and L.M.Hervella, *The Sixteen classes of almost Hermitian manifolds and their Linear invariants,* Ann.Mat.Pura. Appl., 123 (4)(1980), 35-58.

[10] J.A.Oubina, *New Classes of almost contact metric structures,* Publ.Math. Debrecen., 32 (1985), 187 - 193.

[11] B.Ravi and C.S.Bagewadi, *Invariant submanifolds of a conformal $K-$contact Riemannian manifold,* Ind.Jou. Pure and Appl. Math., 20 (11)(1989), 1119-1125.

[12] J.A.Schouten, *Ricci calculus,* Springer - Verlag, Berlin, 1954.

[13] Venkatesha and C.S.Bagewadi, *On 3-dimensional trans-Sasakian manifolds,* (Communicated)

[14] K.Yano, *On the torseforming direction in Riemannian spaces,* Proc. Imp. Acad. Tokyo., 20 (1994), 340 - 345.

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