

The connections of 3–types in fiberings over centred Grassmann manifolds

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Abstract. The centred Grassman manifold (a family of planes, passing through one point) is considered in the projective space. The principal fiber bundle which arises has as typical fiber the stationarity subgroup of a centred plane. A fundamental–group connection is given in this fibering. The curvature object of the connection is introduced. It is shown, that this object is a tensor containing certain subtensors. It is proved, that Bortolotti’s equipment [3] of the centred Grassman manifold induces connections of 3–types in associated fibering. The conditions of their coincidence and their interdependence are obtained.

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Let P_n be the n –dimensional real projective space referred to the mobile frame $\{A, A_I\}$ ($I, J, K, \dots = \overline{1, n}$), with infinitesimal displacements defined by:

$$dA = \theta A + \omega^I A_I, dA_I = \theta A_I + \omega_I^J A_J + \omega_I A. \quad (1)$$

The forms $\omega^I, \omega_I^J, \omega_I$ satisfy the structural equations of the projective group $GP(n)$:

$$D\omega^I = \omega^J \wedge \omega_J^I, D\omega_J^I = \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I, D\omega_I = \omega_I^J \wedge \omega_J. \quad (2)$$

In the projective space P_n we shall consider the centred Grassman manifold $V^* = Gr^*(m, n)$, i.e. the family of all m –dimensional planes L_m , passing through a fixed point. We specialize a mobile frame $\{A, A_a, A_\alpha\}$ ($a, b, c, \dots = \overline{1, m}; \alpha, \beta, \gamma, \dots = \overline{m+1, n}$), putting the top A in the given point, and tops A_a — on the centred plane $L_m^* = [A, A_a]$. While fixing a point A , we obtain the identity $\omega^I = 0$. From (1) follows, that the equations $\omega_a^\alpha = 0$ are conditions of stationarity of the plane L_m^* , i.e. the forms ω_a^α are main, and for the centred Grassman manifold — basic ($\dim V^* = m(n-m)$). From (2) it follows that these satisfy the structural equations

$$D\omega_a^\alpha = \omega_b^\beta \wedge (\delta_a^b \omega_\beta^\alpha - \delta_\beta^a \omega_a^b). \quad (3)$$

The exterior differentials of the fibre forms look as follows:

$$D\omega_b^a = \omega_b^c \wedge \omega_c^a + \omega_b^\alpha \wedge \omega_\alpha^a, D\omega_\beta^\alpha = \omega_\beta^\gamma \wedge \omega_\gamma^\alpha + \omega_\beta^a \wedge \omega_a^\alpha, D\omega_\alpha^a = \omega_\alpha^b \wedge \omega_b^a + \omega_\alpha^\beta \wedge \omega_\beta^a; \quad (4)$$

$$D\omega_a = \omega_a^b \wedge \omega_b + \omega_a^\alpha \wedge \omega_\alpha, D\omega_\alpha = \omega_\alpha^\beta \wedge \omega_\beta + \omega_\alpha^\beta \wedge \omega_\beta.$$

A principal fiber bundle $G(V^*)$ is constructed over the manifold V^* whose fiber is the subgroup G of stationary centred planes $L_m^* \in V^*$. The fibering $G(V^*)$ contains the affine factor fibering $H(V^*)$ with the structural equations (3), (4). In the principal fiber bundle $G(V^*)$ we shall set the fundamental-group connection by means of Laptev's method [4]. We shall consider the transformation of secondary forms with the help of the basic forms of the manifold V^* :

$$\begin{aligned} \tilde{\omega}_b^a &= \omega_b^a - \Gamma_{ba}^c \omega_c^\alpha, \tilde{\omega}_\beta^\alpha = \omega_\beta^\alpha - \Gamma_{\beta\gamma}^{\alpha a} \omega_a^\gamma, \tilde{\omega}_\alpha^a = \omega_\alpha^a - \Gamma_{\alpha\beta}^b \omega_b^\beta, \\ \tilde{\omega}_a &= \omega_a - \Gamma_{a\alpha}^b \omega_b^\alpha, \tilde{\omega}_\alpha = \omega_\alpha - \Gamma_{\alpha\beta}^a \omega_b^\beta. \end{aligned}$$

A connection in the principal fiber bundle is given with the help of a field of connection object Γ on base V^* whose components should satisfy the equations:

$$\begin{aligned} \Delta \Gamma_{ba}^{ac} + \delta_b^c \omega_\alpha^a &= \Gamma_{ba\beta}^{ace} \omega_e^\beta, \Delta \Gamma_{\beta\gamma}^{\alpha a} - \delta_\gamma^\alpha \omega_\beta^a = \Gamma_{\beta\gamma\mu}^{\alpha ab} \omega_b^\mu, \\ \Delta \Gamma_{\alpha\beta}^{ab} - \Gamma_{c\beta}^{ab} \omega_\alpha^c + \Gamma_{\alpha\beta}^{\gamma b} \omega_\gamma^a &= \Gamma_{\alpha\beta\gamma}^{abc} \omega_c^\gamma, \Delta \Gamma_{a\alpha}^b + \Gamma_{a\alpha}^{cb} \omega_c + \delta_a^b \omega_\alpha = \Gamma_{a\alpha\beta}^{bc} \omega_c^\beta, \\ \Delta \Gamma_{\alpha\beta}^a + \Gamma_{\alpha\beta}^{ba} \omega_b - \Gamma_{b\beta}^a \omega_\alpha^b + \Gamma_{\alpha\beta}^{\gamma a} \omega_\gamma &= \Gamma_{\alpha\beta\gamma}^{ab} \omega_b^\gamma, \end{aligned} \quad (5)$$

where the operator Δ acts in usual manner (see, e.g., [1]).

The structural equations of the connection forms are

$$\begin{aligned} D\tilde{\omega}_b^a &= \tilde{\omega}_b^c \wedge \tilde{\omega}_c^a + R_{b\alpha\beta}^{acd} \omega_c^\alpha \wedge \omega_d^\beta, D\tilde{\omega}_\beta^\alpha = \tilde{\omega}_\beta^\gamma \wedge \tilde{\omega}_\gamma^\alpha + R_{\beta\gamma\mu}^{\alpha ab} \omega_a^\gamma \wedge \omega_b^\mu, \\ D\tilde{\omega}_\alpha^a &= \tilde{\omega}_\alpha^b \wedge \tilde{\omega}_b^a + \tilde{\omega}_\alpha^\beta \wedge \tilde{\omega}_\beta^a + R_{\alpha\beta\gamma}^{abc} \omega_b^\beta \wedge \omega_c^\gamma, D\tilde{\omega}_a = \tilde{\omega}_a^b \wedge \tilde{\omega}_b + R_{a\alpha\beta}^{bc} \omega_b^\alpha \wedge \omega_c^\beta, \\ D\tilde{\omega}_\alpha &= \tilde{\omega}_\alpha^a \wedge \tilde{\omega}_a + \tilde{\omega}_\alpha^\beta \wedge \tilde{\omega}_\beta + R_{\alpha\beta\gamma}^{ab} \omega_a^\beta \wedge \omega_b^\gamma, \end{aligned}$$

where the components of the curvature object R of the group connection Γ are expressed by the formulas:

$$\begin{aligned} R_{b\alpha\beta}^{acd} &= \Gamma_{b[\alpha\beta]}^{[cd]} - \Gamma_{b[\alpha}^e \Gamma_{e\beta]}^{ad}, R_{\beta\gamma\mu}^{\alpha ab} = \Gamma_{\beta[\gamma\mu]}^{\alpha [ab]} - \Gamma_{\beta[\gamma}^\eta \Gamma_{\eta\mu]}^{\alpha b]}, \\ R_{\alpha\beta\gamma}^{abc} &= \Gamma_{\alpha[\beta\gamma]}^a [bc] + \Gamma_{e[\beta}^a \Gamma_{\alpha\gamma]}^{ec]} - \Gamma_{\alpha[\beta}^\mu \Gamma_{\mu\gamma]}^{ac]}, \\ R_{a\alpha\beta}^{bc} &= \Gamma_{a[\alpha\beta]}^b [c] - \Gamma_{a[\alpha}^e \Gamma_{e\beta]}^c], \\ R_{\alpha\beta\gamma}^{ab} &= \Gamma_{\alpha[\beta\gamma]}^a [b] - \Gamma_{\alpha[\beta}^c \Gamma_{c\gamma]}^b] - \Gamma_{\alpha[\beta}^\mu \Gamma_{\mu\gamma]}^b]. \end{aligned}$$

They satisfy the following relations on the module of basic forms ω_a^α :

$$\begin{aligned} \Delta R_{b\alpha\beta}^{acd} &\equiv 0, \Delta R_{\beta\gamma\mu}^{\alpha ab} \equiv 0, \Delta R_{\alpha\beta\gamma}^{abc} - R_{e\beta\gamma}^{abc} \omega_\alpha^e + R_{\alpha\beta\gamma}^{\mu bc} \omega_\mu^a \equiv 0, \\ \Delta R_{a\alpha\beta}^{bc} + R_{a\alpha\beta}^{bc} \omega_e &\equiv 0, \Delta R_{\alpha\beta\gamma}^{ab} + R_{\alpha\beta\gamma}^{\mu ab} \omega_\mu + R_{\alpha\beta\gamma}^{cab} \omega_c - R_{c\beta\gamma}^{ab} \omega_\alpha^c \equiv 0. \end{aligned}$$

Theorem 1. *The object of curvature R is a tensor, containing 2 elementary subtensors ([5]): $R_{b\alpha\beta}^{acd}$, $R_{\beta\gamma\mu}^{\alpha ab}$.*

The Bortolotti's equipment of the centred Grassman manifold V^* is set. It consists in addition to each centred plane L_m^* of an $(n - m - 1)$ -dimensional plane P_{n-m-1} ,

which has no common points with the plane L_m^* . The equipping of the plane P_{n-m-1} is defined by the sum total of points $B_\alpha = A_\alpha + \lambda_\alpha^a A_a + \lambda_\alpha A$. The conditions of stationarity of the equipping plane P_{n-m-1} are:

$$\Delta\lambda_\alpha^a + \omega_\alpha^a = \lambda_{\alpha\beta}^{ab}\omega_b^\beta, \Delta\lambda_\alpha + \lambda_\alpha^a\omega_a + \omega_\alpha = \lambda_{\alpha\beta}^a\omega_a^\beta. \quad (6)$$

Theorem 2. *The Bortolotti's equipment setting by a field of quasitensor $\lambda = \{\lambda_\alpha^a, \lambda_\alpha\}$ on the manifold V^* , induces the 1-st type connection*

$$\Gamma = \{\Gamma_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^{0a}, \Gamma_{a\alpha}^b, \Gamma_{\alpha\beta}^{ab}, \Gamma_{\alpha\beta}^{01}\}$$

in the fibering $G(V^*)$.

The first encompassing of the components of the connection object by the components of equipping quasitensor is made with the help of the hypotheses and looks as follows:

$$\Gamma_{b\alpha}^{ac} = \delta_b^c\lambda_\alpha^a, \Gamma_{\beta\gamma}^{0a} = -\delta_\gamma^a\lambda_\beta^a, \Gamma_{a\alpha}^b = \delta_a^b\lambda_\alpha^a, \quad (7)$$

$$\Gamma_{\alpha\beta}^{01} = -\lambda_\beta^a\lambda_\alpha^b, \Gamma_{\alpha\beta}^{01} = -\lambda_\alpha^a\lambda_\beta^b. \quad (8)$$

With the help of the object of the group connection Γ , we shall define the concept of covariant differential and of covariant derivative of the quasitensor λ regarding to this connection. By defining the connection forms $\tilde{\omega}$ in (6), we obtain the covariant differentials of the components of the object λ

$$\nabla\lambda_\alpha^a = d\lambda_\alpha^a + \lambda_\alpha^b\tilde{\omega}_b^a - \lambda_\beta^a\tilde{\omega}_\alpha^\beta + \tilde{\omega}_\alpha^a, \nabla\lambda_\alpha = d\lambda_\alpha + \lambda_\alpha^a\tilde{\omega}_a - \lambda_\beta\tilde{\omega}_\alpha^\beta + \tilde{\omega}_\alpha \quad (9)$$

and the covariant derivatives

$$\nabla_\beta^b\lambda_\alpha^a = \lambda_{\alpha\beta}^{ab} - \lambda_\alpha^c\Gamma_{c\beta}^{ab} + \lambda_\gamma^a\Gamma_{\alpha\beta}^{\gamma b} - \Gamma_{\alpha\beta}^{ab}, \nabla_\beta^a\lambda_\alpha = \lambda_{\alpha\beta}^a - \lambda_\alpha^b\Gamma_{b\beta}^a + \lambda_\gamma\Gamma_{\alpha\beta}^{\gamma a} - \Gamma_{\alpha\beta}^a. \quad (10)$$

Theorem 3. *The covariant derivatives (10) of the equipping quasitensor λ in the group connection Γ are tensors, containing subtensors.*

Proof. We examine the components of the covariant derivatives, which have the form:

$$\Delta\nabla_\beta^b\lambda_\alpha^a \equiv 0, \Delta\nabla_\beta^a\lambda_\alpha + \nabla_\beta^a\lambda_\alpha^b\omega_b \equiv 0.$$

With the help of the structural equations, we shall find the exterior differentials from the covariant differentials (9) of the components of equipping quasitensor λ

$$D\nabla\lambda_\alpha^a = \nabla\lambda_\alpha^b \wedge \tilde{\omega}_b^a - \nabla\lambda_\beta^a \wedge \tilde{\omega}_\alpha^\beta + T_{\alpha\beta\gamma}^{abc}\omega_b^\beta \wedge \omega_c^\gamma,$$

$$D\nabla\lambda_\alpha = \nabla\lambda_\alpha^a \wedge \tilde{\omega}_a - \nabla\lambda_\beta \wedge \tilde{\omega}_\alpha^\beta + T_{\alpha\beta\gamma}^{ab}\omega_a^\beta \wedge \omega_b^\gamma,$$

where

$$T_{\alpha\beta\gamma}^{abc} = R_{\alpha\beta\gamma}^{abc} + \lambda_\alpha^d R_{d\beta\gamma}^{abc} - \lambda_\mu^a R_{\alpha\beta\gamma}^{\mu bc}, T_{\alpha\beta\gamma}^{ab} = R_{\alpha\beta\gamma}^{ab} + \lambda_\alpha^c R_{c\beta\gamma}^{ab} - \lambda_\mu R_{\alpha\beta\gamma}^{\mu ab}.$$

Thus, the exterior differentiation of the covariant differential contains an object of relative motion ([6]) whose components are combinations of the components of curvature tensor of the group connection having as coefficients the components of the

equipping quasitensor. The given object $T = \{T_{\alpha\beta\gamma}^{abc}, T_{\alpha\beta\gamma}^{ab}\}$ is a tensor, including the subtensor $T_{\alpha\beta\gamma}^{abc}$, with the following differential relations on the components:

$$\Delta T_{\alpha\beta\gamma}^{abc} \equiv 0, \Delta T_{\alpha\beta\gamma}^{ab} + T_{\alpha\beta\gamma}^{cab} \omega_c \equiv 0.$$

The condition of vanishing of the covariant derivatives (10) of the components of equipping quasitensor λ is invariant (see Theorem 3). Setting them to zero, we shall find the expressions of the components $\Gamma_{\alpha\beta}^a, \Gamma_{\alpha\beta}^{ab}$ of the connection object Γ through the components $\Gamma_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^{\alpha a}, \Gamma_{a\alpha}^b$. We get a bunch of connections of 2-nd type. Taking into account the encompassing (7), we have

$$\Gamma_{\alpha\beta}^a \stackrel{02}{=} \lambda_{\alpha\beta}^a - 2\lambda_{\alpha}^a \lambda_{\beta}, \Gamma_{\alpha\beta}^{ab} \stackrel{02}{=} \lambda_{\alpha\beta}^{ab} - 2\lambda_{\beta}^a \lambda_{\alpha}^b. \quad (11)$$

Theorem 4. *The Bortolotti's equipment of the centred Grassman manifold V^* induces a set of connections of 2-nd type in the associated fibering $G(V^*)$ from which a unique connection of 2-nd type is inferred*

$$\Gamma = \{\Gamma_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^{\alpha a}, \Gamma_{a\alpha}^b, \Gamma_{\alpha\beta}^{ab}, \Gamma_{\alpha\beta}^a\}.$$

We further construct the third encompassing of the components of connection object Γ . We shall take into account the continued differential relations of the components of equipping quasitensor λ , the differential equations (5) which are satisfied by the components of connection object Γ , and the encompassing (7). Then the following formulas hold true

$$\Gamma_{\alpha\beta}^{ab} \stackrel{03}{=} -\lambda_{\alpha\beta}^{ab}, \Gamma_{\alpha\beta}^a \stackrel{03}{=} -\lambda_{\alpha\beta}^a. \quad (12)$$

Theorem 5. *Bortolotti's equipment induces a connection of the third type*

$$\Gamma = \{\Gamma_{b\alpha}^{ac}, \Gamma_{\beta\gamma}^{\alpha a}, \Gamma_{a\alpha}^b, \Gamma_{\alpha\beta}^{ab}, \Gamma_{\alpha\beta}^a\}.$$

A condition of coincidence of the three constructed encompassings is the equality

$$\lambda_{\alpha\beta}^{ab} = \lambda_{\beta}^a \lambda_{\alpha}^b, \lambda_{\alpha\beta}^a = \lambda_{\alpha}^a \lambda_{\beta}.$$

Theorem 6. *The connection of the 1-st type is the average ([5]) of the connections of 2-nd and 3-d types, i.e. $\Gamma = \frac{01}{2}(\Gamma + \Gamma)$.*

Proof. Straightforward, using the formulas (8), (11), (12), which express the connection components.

Remark. *The similar theorem is proved in [2] for the noncentred Grassman manifold.*

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